Entry and Exit Decisions under Uncertainty in a Symmetric Duopoly

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Abstract

In this paper, we analyze entry and exit decisions in a symmetric duopoly setting. The model is based on Dixit (1989), which is a standard model in this area. To incorporate competitive nature into the output price, we use an inverse demand function. We aim to present an equilibrium strategy of entry and exit decisions in a symmetric duopoly. To do so, we consider four states of two firms and pay attention to setting the value-matching and smooth-pasting conditions. As a result, the relation between four optimal thresholds is shown. Moreover, we show that a competitive exit threshold has different characteristics from others.

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1 Introduction

Models of entry and exit decisions under uncertainty are main tools in real options theory. Brennan and Schwartz (1985) generalized the theory and provided the production schedule

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of a copper mine. Especially, Dixit (1989), which is used as a baseline model in this paper, finds 'hysteresis' is significant even with small sunk costs.

As the new trend in real options theory, competitive nature is discussed in many researches. Although the standard real options approach assumes a monopoly setting implicitly, there must be competitors in the real world. Dixit and Pindyck (1994, Ch.9) incorporated competitive nature into real options approach properly at the earliest date. While they did not derive the equilibrium by game theory, Grenadier (1996) develops an equilibrium framework for strategic options excercise games, and provides an explanation for development cascades and overbuilding in real estate markets.

The basic enrty-exit model has been extended in several directions, e.g., initial construction and final scrapping, the number of switches, diminishing production capacity, and time lags. However, there are no literature which discusses in a duopoly setting.

The objective of this paper is to present an equilibrium strategy of entry and exit decisions in a symmetric duopoly. We introduce an inverse demand function for a duopoly setting. And we pay attention to setting the value-matching and smooth-pasting conditions in order to solve the problem.

2 The Model

We consider two identical firms which are labeled 1 and 2. By index i we refer to an arbitrary firm and by j to the 'other firm.' A firm can become active by paying some cost K, then it can produce a unit flow of output at the variable cost C. Moreover, the operation can be suspended by paying an exit cost E, and it can be restarted by paying K again at some future time.

The demand function will be subject to continuous shocks. It is of the following form:

$$P_t = D(Q_t)X_t, \tag{1}$$

where P_t denotes the output price at time t, Q_t denotes the supply of the product in the market, and $D(\cdot)$ is a differential function with $D'(\cdot) < 0$, which ensures the first mover's advantage. This market is characterized by evolving uncertainty in the state of demand. X_t represents a multiplicative demand shock, and follows a geometric Brownian motion:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t, \quad X_0 = x, \tag{2}$$

where μ is the instantaneous expected growth rate of X_t, σ (> 0) is the instantaneous volatility of X_t, and W_t is a standard Brownian motion.

There are two state variables, the current price X_t and discrete variable Y_i that indicates whether firm i is active (1) or not (0). Theorefore, the state of both firms should count toward the value function of firm i, which represents for $V_{Y_iY_j}(x)$.

Firm i maximizes its value by choosing the stopping time to invest:

$$\tau_{Y_iY_j} = \inf\{t > 0 : X_t \geq X^*_{Y_iY_j}\}, \tag{3}$$

where $X^*_{Y_iY_j}$ denotes the corresponding optimal threshold, e.g., τ_{1Y_j} and τ_{0Y_j} are the time to suspend and restart, respectively. In this paper, we assume the following:

Assumption 1 *The simultaneous investment is eliminated and if each tries to invest first, one will randomly win the race with probability* 1/2 *because of the symmetry.*

Assumption 1 reflects the symmetry of the both firms and follows Grenadier (1996).

3 Equilibria

First, we define the value fucntions for four cases. From Assumption 1, for the case of two idle firms, we have

$$V_{00}(x) = \sup_{\tau_{00}\in\mathcal{T}} \mathbb{E}\left[\mathrm{e}^{-\rho\tau_{00}}\frac{1}{2}\big(V_{10}(X_{\tau_{00}}) - K + V_{01}(X_{\tau_{00}})\big)\right], \tag{4}$$

where $\mathcal{T}$ denotes the collection of admissible stopping times and ρ ($> \mu$) denotes a discount rate.

When only one firm is active, if X_t rises enough then the idle firm will restart the operation, and if X_t declines then the active firm will suspend it. Theorefore, we have

$$V_{01}(x) = \sup_{\tau_{01}\in\mathcal{T}} \mathbb{E}\left[\mathbf{1}_{\{\tau_{01}<\tau_{10}\}}\mathrm{e}^{-\rho\tau_{01}}\big(V_{11}(X_{\tau_{01}}) - K\big) + \mathbf{1}_{\{\tau_{01}\geq\tau_{10}\}}\mathrm{e}^{-\rho\tau_{10}}V_{00}(X_{\tau_{10}})\right], \tag{5}$$

and

$$\begin{aligned} V_{10}(x) = \sup_{\tau_{10}\in\mathcal{T}} \mathbb{E}\Bigg[\int_0^{\tau_{10}\wedge\tau_{01}} &\mathrm{e}^{-\rho t}(D(1)X_t - C)\mathrm{d}t \\ &+ \mathbf{1}_{\{\tau_{10}<\tau_{01}\}}\mathrm{e}^{-\rho\tau_{10}}\big(V_{00}(X_{\tau_{10}}) - E\big) + \mathbf{1}_{\{\tau_{10}\geq\tau_{01}\}}\mathrm{e}^{-\rho\tau_{01}}V_{11}(X_{\tau_{01}})\Bigg]. \end{aligned} \tag{6}$$

Note that τ_{10} in equation (5) and τ_{01} in equation (6) denote the stopping time of firm j, because of the symmetry.[1] For the case of two active firms, we have

$$V_{11}(x) = \sup_{\tau_{11}\in\mathcal{T}} \mathbb{E}\left[\int_0^{\tau_{11}} \mathrm{e}^{-\rho t}(D(2)X_t - C)\mathrm{d}t + \mathrm{e}^{-\rho\tau_{11}}\frac{1}{2}\big(V_{01}(X_{\tau_{11}}) - E + V_{10}(X_{\tau_{11}})\big)\right]. \tag{7}$$

[1]Let $\tilde{\tau}_{Y_jY_i}$ denote the stopping time of firm j. $\tilde{\tau}_{01}$ means the case that firm j is inactive and firm i is active, where the stopping time of firm i is τ_{10}. Because of the symmetry, $\tilde{\tau}_{01} = \tau_{10}$.

Secondly, based on Dixit (1989), we can find the following forms for each value function in the continuation region:

$$V_{00}(x) = A_{00}x^{\alpha}, \tag{8}$$
$$V_{01}(x) = A_{01}x^{\alpha} + B_{01}x^{\beta}, \tag{9}$$
$$V_{10}(x) = \frac{D(1)x}{\rho - \mu} - \frac{C}{\rho} + A_{10}x^{\alpha} + B_{10}x^{\beta}, \tag{10}$$
$$V_{11}(x) = \frac{D(2)x}{\rho - \mu} - \frac{C}{\rho} + B_{11}x^{\beta}, \tag{11}$$

where α (> 1) and β (< 0) are the positive and negative root of the following characteristic equation:

$$\frac{1}{2}\sigma^2\gamma(\gamma - 1) + \mu\gamma - \rho = 0, \tag{12}$$

respectively. Note that $B_{01}x^{\beta}$ and $A_{10}x^{\alpha}$ are *not* firm i's options. These result from firm j's action, e.g., $B_{01}x^{\beta}$ is the part which changes the value function into V_{00} at the moment that active firm j will suspend the operation.

Finally, we need the following boundary conditions in order to find the optimal thresholds $X^*_{Y_iY_j}$ and unknown coefficients $A_{Y_iY_j}$, $B_{Y_iY_j}$:

$$V_{00}(X^*_{00}) = \frac{1}{2}\big(V_{10}(X^*_{00}) - K + V_{01}(X^*_{00})\big), \tag{13}$$
$$V'_{00}(X^*_{00}) = \frac{1}{2}\big(V'_{10}(X^*_{00}) + V'_{01}(X^*_{00})\big), \tag{14}$$
$$V_{01}(X^*_{01}) = V_{11}(X^*_{01}) - K, \tag{15}$$
$$V'_{01}(X^*_{01}) = V'_{11}(X^*_{01}), \tag{16}$$
$$V_{01}(X^*_{10}) = V_{00}(X^*_{10}), \tag{17}$$
$$V_{10}(X^*_{10}) = V_{00}(X^*_{10}) - E, \tag{18}$$
$$V'_{10}(X^*_{10}) = V'_{00}(X^*_{10}), \tag{19}$$
$$V_{10}(X^*_{01}) = V_{11}(X^*_{01}), \tag{20}$$
$$V_{11}(X^*_{11}) = \frac{1}{2}\big(V_{01}(X^*_{11}) - E + V_{10}(X^*_{11})\big), \tag{21}$$
$$V'_{11}(X^*_{11}) = \frac{1}{2}\big(V'_{01}(X^*_{11}) + V'_{10}(X^*_{11})\big). \tag{22}$$

Equations (13) and (14) are the value-matching and smooth-pasting conditions respectively, when both idle firms try to restart the operation. The right-hand side reflects Assumption 1. Similarly, equations (21) and (22) are the value-matching and smooth-pasting conditions respectively, when both active firms try to suspend the operation. Equations (15) and (16) are the value-matching and smooth-pasting conditions respectively, when

Table 1: The parameter values for the base case.

parameter	value
μ	0.02
σ	0.2
ρ	0.04
$D(1)$	2
$D(2)$	1
C	5
K	10
E	5

Table 2: The values of optimal thresholds for the base case.

threshold	value
X_{10}^*	1.456
X_{00}^*	3.937
X_{11}^*	4.516
X_{01}^*	6.408

an idle firms try to restart the operation. Similarly, equations (18) and (19) are the value-matching and smooth-pasting conditions respectively, when an active firms try to suspenc the operation.

Note that equations (17) and (20) mean that firm j's action will change the value of firm i. Therefore, these equations need not to associate the smooth-pasting conditions. Since equations (13)–(22) are the system of ten non-linear equations, we must find the thresholds and coefficients numerically.

4 Numerical Examples

In this section, we use the basic parameter values in Table 1. Then, we have the optimal thresholds in Table 2.

X_{00}^* and X_{01}^* are entry thresholds. Especially, X_{00}^* is a competitive entry threshold because both firms will try to entry there. Since X_{00}^* is less than X_{01}^*, competition forces firms to entry earlier than without competition. Similarly, a competitive exit threshold

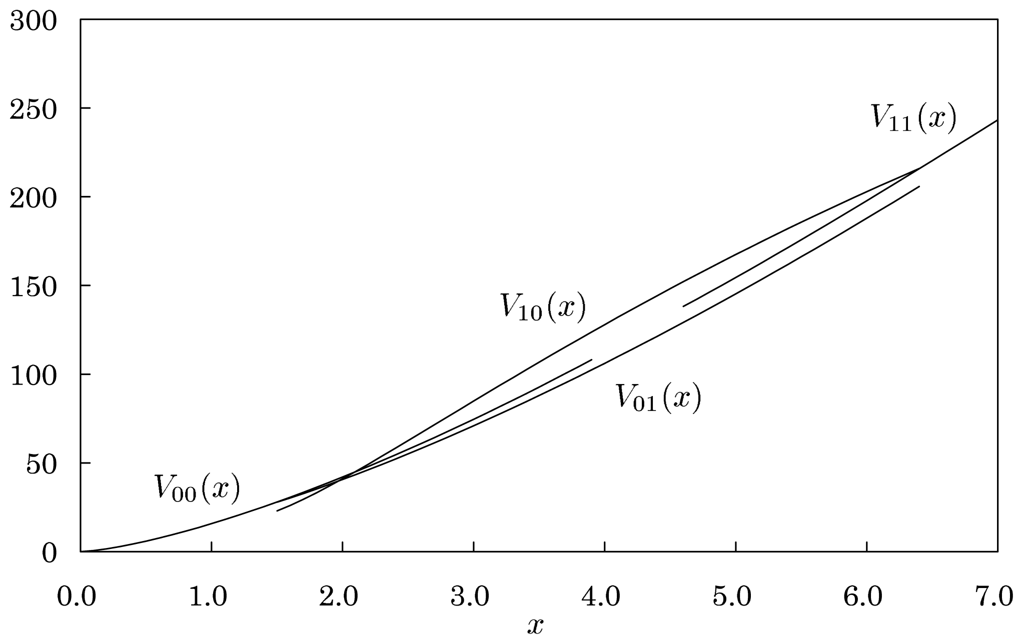

Figure 1: The value functions.

X_{11}^* is larger than X_{10}^*, so competition accelerates firms' exit.

Figure 1 displays all the four value functions. Beginning with $V_{00}(x)$, both firms try to entry at the moment that X_t reaches X_{00}^*. Note that the value matching condition (13) and the smooth pasting condition (14) are satisfied at X_{00}^*. Then, with probability 1/2, firm i restarts production a output and the value rises up to $V_{10}(X_{00}^*)$, and firm j still suspends it and the value falls down to $V_{01}(X_{00}^*)$. Next, firm i suspends production again at X_{10}^*, and the value of both firms changes into $V_{00}(X_{10}^*)$. On the other hand, firm j restart production at X_{01}^*, and the value of both firms changes into $V_{11}(X_{01}^*)$. Note that firm i does not incur any cost at the moment.

After reaching X_{01}^*, both firms try to exit at X_{11}^*. Then, with probability 1/2, firm i suspends production and the value falls down to $V_{01}(X_{11}^*)$, and firm j still produces a output and the value rises up to $V_{10}(X_{11}^*)$. Moreover, firm i restarts production again at X_{01}^*, and the value of both firms changes into $V_{11}(X_{01}^*)$. On the other hand, firm j suspends production at X_{10}^*, and the value of both firms changes into $V_{00}(X_{10}^*)$. Both firms repeat the above actions, which is the equilibrium strategy.

Figure 2 displays the comparative statics of the thresholds with respect to σ. For all σ, X_{01}^* is the maximum threshold and X_{10}^* is the minimum threshold. Competitive thresholds X_{00}^* and X_{11}^* are at the intermediate level. However, X_{00}^* is less for small σ and greater

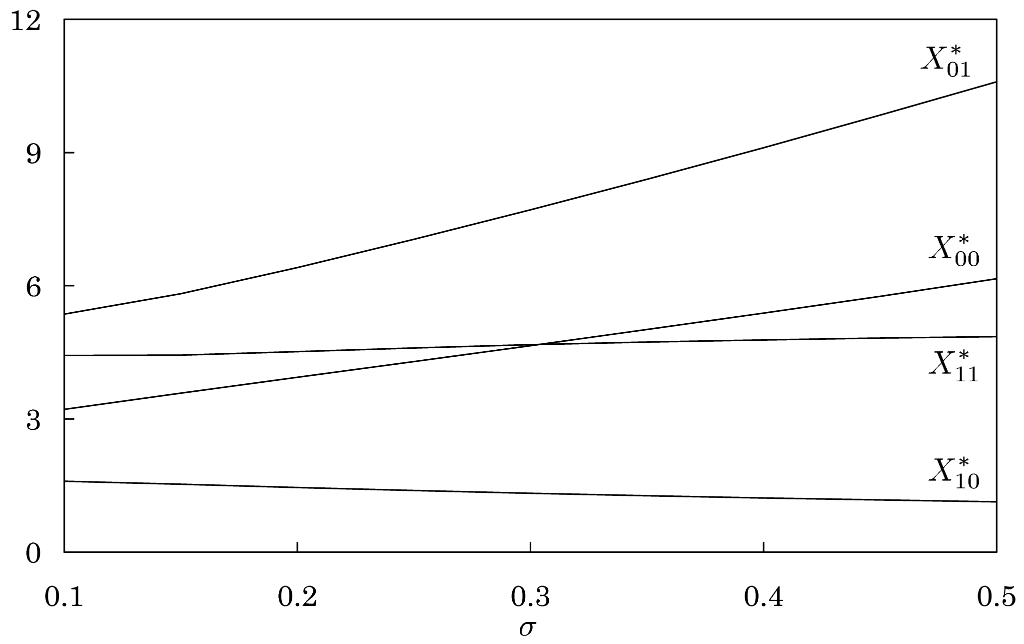

Figure 2: The comparative statics of the thresholds with respect to σ.

for large σ than X_{11}^*.

Entry thresholds X_{00}^* and X_{01}^* increase with σ, which is a usual property in a real options model. And it is also usual that exit threshold X_{10}^* decreases with σ. However, it is unusual that competitive exit threshold X_{11}^* increases with σ. The reason is that uncertainty usually defers investment decision, so that an exit threshold is expected to decrease with σ.

To confirm that, we investigate the difference between X_{01}^* and X_{11}^* in Figure 3. The difference implicates the time required after an idle firm restarts the operation until the firm will suspend again. For all parameter sets, the differences increase with σ. Therefore, competitive exit threshold X_{11}^* increases only apparently and both active firms prefer to defer exit actually. It is usual in a real options model.

Figure 4 displays the comparative statics of the thresholds with respect to $D(1)$. For all $D(1)$, X_{01}^* is the maximum threshold and X_{10}^* is the minimum threshold. Competitive thresholds X_{00}^* and X_{11}^* are at the intermediate level. However, X_{11}^* is less for small $D(1)$ and greater for large $D(1)$ than X_{00}^*.

Entry thresholds X_{00}^* and X_{01}^* decrease with $D(1)$, because the firm prefers to earn a higher profit. On the other hand, exit threshold X_{10}^* decreases with $D(1)$ because of the same reason. However, the sensitivity of exit threshold X_{11}^* is ambiguous.

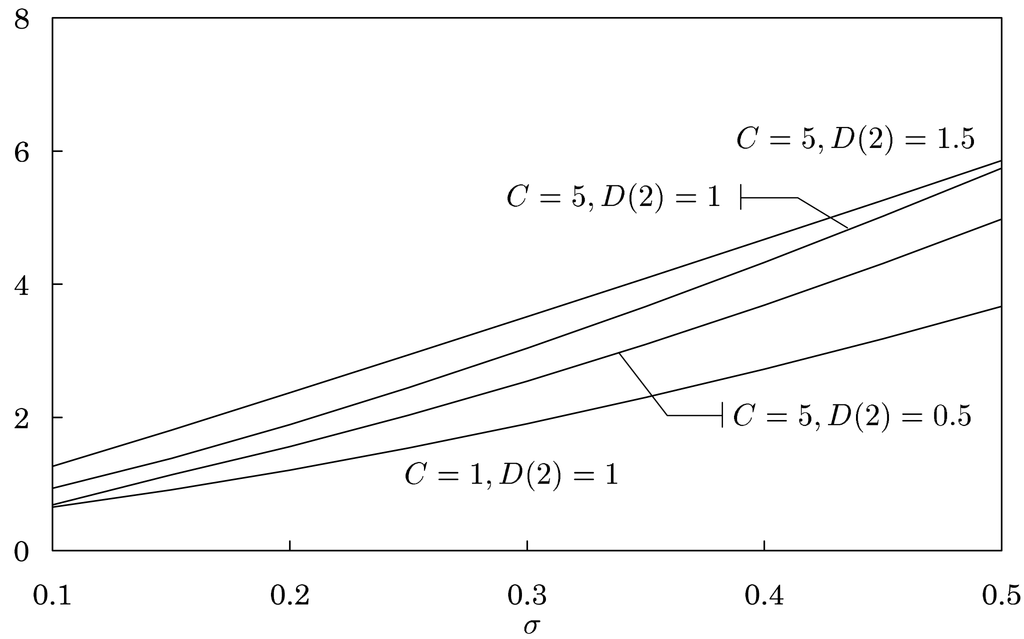

Figure 3: The comparative statics of the difference between X_{01}^* and X_{11}^* with respect to σ.

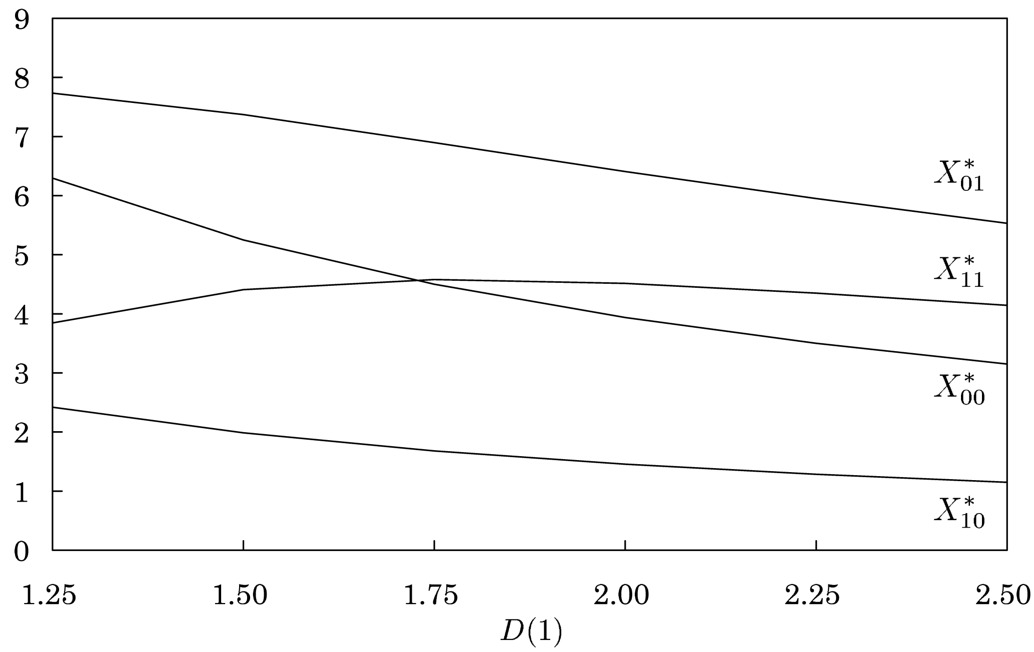

Figure 4: The comparative statics of the thresholds with respect to $D(1)$.

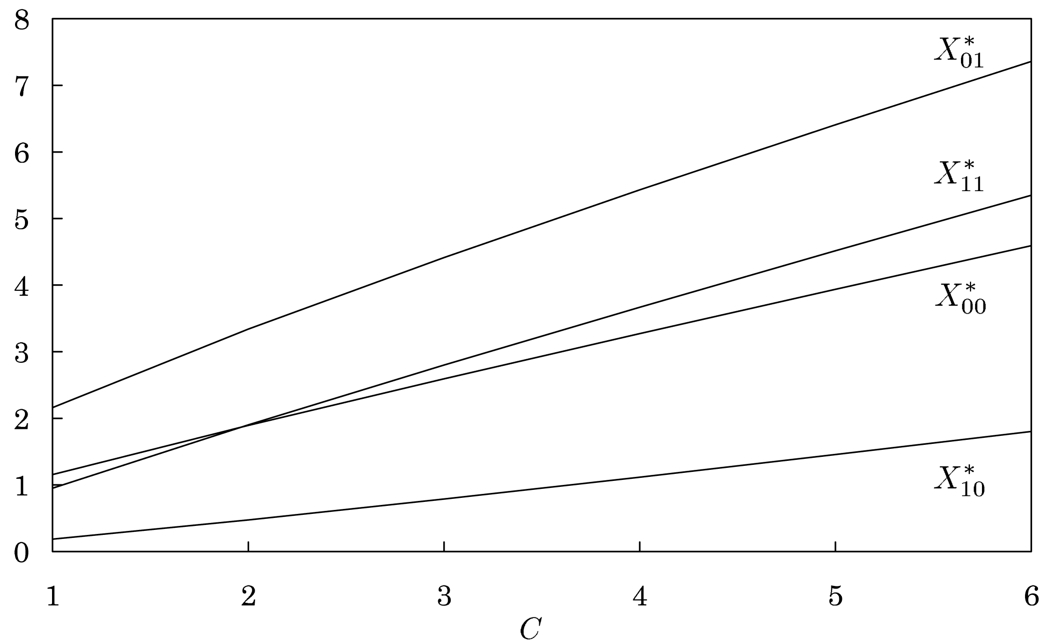

Figure 5: The comparative statics of the thresholds with respect to C.

Figure 5 displays the comparative statics of the thresholds with respect to the variable cost C. For all C, X^*_{01} is the maximum threshold and X^*_{10} is the minimum threshold. Competitive thresholds X^*_{00} and X^*_{11} are at the intermediate level. However, X^*_{11} is less for small C and greater for large C than X^*_{00}. All the thresholds increase with C, because a higher variable cost decrease the oprating profit, so that the firm prefer late entry and early exit.

5 Conclusion

In this paper, we have analyzed entry and exit decisions in a symmetric duopoly setting. As a result, the relation between four optimal thresholds has been shown. Moreover, we have shown that a competitive exit threshold has different characteristics from others. In particular, exit at the cometitive threshold may not optimal, so that the sensitivity is unusual.

Several issues are challenges for the future. First, we will analyze the competitive exit threshold. Second, we could assume another criterion for the simultaneous investment. Or we could consider an asymmetric case like Takashima et al. (2008). And for simplifying the numerical calculation, we could use a discount factor approach proposed by Sødal (2006).

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