Pricing Real Options under the CEV Diffusion

José Carlos Dias*

Department of Finance, ISCAC Business School, Coimbra, Portugal

João Pedro Nunes

Department of Finance, ISCTE Business School, Lisbon, Portugal

February 2008

*Corresponding author: José Carlos Dias, Department of Finance, ISCAC Business School, Quinta Agrícola, Bencanta, 3040-316 Coimbra, Portugal. Tel: +351 239802185, e-mail: jdias@iscac.pt.

Pricing Real Options under the CEV Diffusion

Abstract

Much of the work on real options assumes that the underlying state variable follows a geometric Brownian motion with constant volatility. This paper uses a more general assumption for the state variable process which may better capture the empirical observations found in the financial economics literature. We use the so-called constant elasticity of variance (CEV) diffusion model where the volatility is a function of the underlying asset price and provide analytical solutions for perpetual American-style call and put options under the CEV diffusion. When the constant risk-free interest rate r is different from the dividend yield q, the perpetual American option price is based on an infinite series of terms involving confluent hypergeometric functions. For $r = q$, the computation of the perpetual American option formula involves the use of modified Bessel functions. We demonstrate the implications of the correct specification of the underlying state variable process for the valuation of real assets and show that a firm that uses the standard geometric Brownian motion assumption is exposed to significant errors of analysis which may lead to non-optimal investment and disinvestment decisions.

Keywords: real options, CEV diffusion

JEL Classification: G13; G31; D81; D92; C61.

1 Introduction

The so-called real options approach for valuing investment decisions under uncertainty highlights the importance of uncertainty for the value of a project and for determining the solution for the whether-and-when to invest paradigm. The valuation of real investment projects and the rule for determining the optimal timing of investment depend significantly on the stochastic process assumed for the relevant state variable of the project. Much of the academic literature on real options draws upon the work of McKean (1965) and Merton (1973) and assumes that the underlying state variable (e.g., spot asset price, output price, net operating value or cash flow stream) follows a lognormal diffusion process (a geometric Brownian motion), usually with constant drift and variance parameters [e.g., Brennan and Schwartz (1985) and McDonald and Siegel (1986)].[1]

The form of this stochastic process (a random walk model in a continuous time framework) is based on the conjecture that the state variable has a lognormal-stationary distribution. The geometric Brownian motion assumption provides the highest degree of analytical tractability and may be a reasonable approximation for the stochastic behaviour of some real and financial asset prices. However, the accumulated empirical evidence indicates that this distributional assumption is not reach enough to capture the empirical observations. An important empirical feature is that equity option prices exhibit pronounced implied volatility smiles or volatility skew.[2] As documented by Jackwerth and Rubinstein (2001), these volatility smiles contradict the assumption of geometric Brownian motion which would imply a flat line.

To overcome this issue, the academic research has proceeded in two directions. One has been to specify alternative stochastic processes, which in turn imply alternative risk-neutral densities. In the case of the real options literature, some authors have considered a different assumption regarding the stochastic process governing the underlying state variable by assuming a mean-reverting type diffusion [e.g., Bhattacharya (1978), Gibson

[1]The results for the basic case in which the state variable follows a geometric Brownian motion are extensively reviewed in the seminal book of Dixit and Pindyck (1994).

[2]A volatility smile describes implied volatilities that are largely convex and monotonically decreasing functions of strike prices.

and Schwartz (1990), Brennan (1991), Gibson and Schwartz (1991), Laughton and Jacoby (1993), Bjerksund and Ekern (1995), Metcalf and Hassett (1995), Schwartz (1997), Biekpe et al. (2003) and Klumpes and Tippett (2004)], birth-and-death processes [e.g., Baldwin (1982) and Pinto et al. (2006)] or including a jump into the process [e.g., McDonald and Siegel (1986)].[3] The other possibility followed by the financial economics literature is to develop procedures for backing out implied risk-neutral density functions from observed option prices [see Rubinstein (1994) and Jackwerth and Rubinstein (1996)]. This stream of the literature has found that implied risk-neutral probability densities tend to be heavily skewed to the left and highly leptokurtic relative to the Black and Scholes (1973) and Merton (1973) lognormal presumption.

The so-called constant elasticity of variance (CEV) model of Cox (1975)[4] allows the instantaneous conditional variance of asset return to depend on the asset price level. The CEV model has the advantage that the volatility of the underlying asset is linked to its price level, thus exhibiting an *implied volatility smile* (or *implied volatility skew*) that is a convex and monotonically decreasing function of exercise price, similar to the volatility smile curves observed in practice [see, for example, Dennis and Mayhew (2002)]. The CEV framework is also consistent with the so-called *leverage effect* (i.e., the existence of a negative correlation between stock returns and realized stock volatility) as documented for instance in Bekaert and Wu (2000).

Although the empirical research on real options is not so well developed and documented, there is also evidence supporting the use of a CEV diffusion type model. Gibson and Schwartz (1991) present a methodology to value long term real and financial assets whose payoffs are contingent upon the spot price of crude oil. Their empirical results suggest that the volatility of oil price relative changes was negatively related to the oil price level and that the constant elasticity of variance parameter β had an average level of -0.72 which is significantly less than 2, thus suggesting that a CEV diffusion process might be better suited than the lognormal one to describe the evolution of the spot price

[3] Dixit and Pindyck (1994, Chapter 5) also examine the value of the investment opportunity and the optimal investment rule of a given project assuming its value follows these alternative stochastic processes.

[4] A brief summary of the original manuscript was published in Cox (1996) in a special issue paying tribute to Fischer Black.

of oil.

Given that the lognormal assumption with constant volatility does not capture the implied volatility smile effect observed across a wide range of markets and underlying assets, our main purpose is to extend the standard real options model to incorporate this feature using a CEV diffusion type model. The CEV diffusion process have been extensively used to obtain the solutions of several financial option pricing problems. Closed-form solutions for pricing vanilla options under the CEV process are derived by Cox (1975) for $\beta < 2$, Emanuel and MacBeth (1982) for $\beta > 2$ and Schroder (1989). In these papers, the closed-form CEV option pricing formula can be expressed in terms of the standard complementary gamma distribution function and the complementary noncentral chi-square distribution function. Davydov and Linetsky (2003) also consider the problem of pricing vanilla options under the CEV process but using the eigenfunction expansion approach. The problem of pricing single-barrier and double-barrier options under the CEV diffusion process is examined by Boyle and Tian (1999) in the numerical trinomial lattice framework, by Davydov and Linetsky (2001) using the Laplace transform method and by Davydov and Linetsky (2003) using eigenfunction expansions. Analytical solutions for lookback options under the CEV diffusion are obtained by Davydov and Linetsky (2001) using Laplace trsnsforms and by Linetsky (2004) using the spectral expansion approach. Andersen and Andreasen (2000) consider extensions of the Libor market model to markets with volatility skews in observable option prices and derive closed-form expressions for cap and swaption prices. More recently, Carr and Linetsky (2006) develop a reduced-form credit risk modelling framework based on a jump to default extended CEV process and Nunes (2008) uses the CEV diffusion to price American options.

The contributions of the present paper are twofold. Firstly, we derive solutions for pricing real options under a CEV diffusion process. In Proposition 1 we review the analytical formulae to price perpetual American options that is standard in the real options literature when one assumes the geometric Brownian motion assumption. Proposition 2 gives explicitly analytical formulae for the perpetual American option pricing problem under the CEV diffusion and for the case where the constant risk-free interest rate (r) is different from the dividend yield of the asset (q), based on an infinite series of terms involving confluent hypergeometric functions. Proposition 3 specializes the perpetual American option pricing formulae for the case where $r = q$, giving explicitly analytical formulae which involves the use of modified Bessel functions. These two new resulting option pricing formulae can be easily programmed in *Mathematica* and *Maple* since these software packages include all the required special functions as built-in functions.

Secondly, we use the option pricing formulae to carry out a comparative static analysis. More specifically, we demonstrate the implications of the correct specification of the underlying state variable process for the valuation of real assets. Our results indicate that a firm that uses the standard geometric Brownian motion assumption is exposed to significant errors of analysis which may lead to non-optimal investment and disinvestment decisions.

The remainder of this paper is organized as follows. In section 2 we develop the general setup of the model. The basic premise is that the stochastic component of the instantaneous cash flow of a firm is characterized by a CEV diffusion process. In section 3 we extend the real options valuation approach to a prototypical CEV-type diffusion process which may be economically more relevant for valuing real options than other stochastic processes usually used in the real options literature. Section 4 provides the computational results. Section 5 concludes the paper. Proofs are collected in the appendix.

2 General Setup of the Model

The valuation of American options are generally explored in the context of a stochastic intertemporal economy with continuous trading on the time interval $[t_0, T]$, for some fixed time $T > t_0$, where uncertainty is represented by a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. Throughout the paper, $\mathbb{Q}$ denotes the martingale probability measure obtained when the numéraire of the underlying economy is taken to be a money market account B_t, whose dynamics is governed by the following ordinary differential equation:

$$dB_t = rB_t dt \tag{1}$$

where r denotes the instantaneous riskless interest rate, which is assumed to be constant.

In this paper we model asset prices based on the following one-dimensional diffusion process:

$$\frac{dV_t}{V_t} = (r - q)\, dt + \sigma(t, V)\, dW_t^{\mathbb{Q}} \tag{2}$$

where $r \geq 0$ represents the constant risk-free interest rate, $q \geq 0$ represents the dividend yield for the asset price, $\sigma(t, V)$ corresponds to the instantaneous volatility per unit of time of the asset returns and $dW_t^{\mathbb{Q}} \in \mathbb{R}$ is a standard Brownian motion under $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$.

Let us now specialize our discussion to the constant elasticity of variance (CEV) process of Cox (1975). We assume that under the risk-neutral measure $\mathbb{Q}$ the underlying asset price $\{V_t,\, t \geq 0\}$ is governed by a CEV diffusion process described by the following stochastic differential equation:

$$dV_t = (r - q)\, V_t\, dt + \delta\, V_t^{\frac{\beta}{2}}\, dW_t^{\mathbb{Q}} \tag{3}$$

with a local volatility function given by:

$$\sigma(t, V) = \delta\, V_t^{\frac{\beta}{2}-1} \tag{4}$$

for $\delta,\, \beta \in \mathbb{R}$.

The CEV specification given by equation (3) nests the lognormal assumption of Black and Scholes (1973) and Merton (1973) ($\beta = 2$) and the absolute diffusion ($\beta = 0$) and the square-root diffusion ($\beta = 1$) models of Cox and Ross (1976) as special cases. For $\beta < 2$ ($\beta > 2$) the local volatility given by equation (4) is a decreasing (increasing) function of the asset price. The two model parameters β and δ can be interpreted, respectively, as the elasticity of the local volatility function and the scale parameter fixing the initial instantaneous volatility at time $t = 0$, $\sigma_0 = \sigma(0, V_0) = \delta\, V_0^{\beta/2-1}$. Cox (1975) originally studied the case $\beta < 2$ and Emanuel and MacBeth (1982) extended the analysis for $\beta > 2$. Cox (1975) restricted the β parameter to the range $0 \leq \beta \leq 2$. However, Jackwerth and Rubinstein (2001) document that typical values of β implicit in the post-crash of 1987

S&P 500 stock index option prices are as low as $\beta = -6$. They call the model with $\beta < 0$ the unrestricted CEV.

According to Feller's classification of boundaries for one-dimensional diffusions[5], for $\beta < 2$ infinity is a natural boundary for the CEV diffusion process; attracting for $\mu = r - q > 0$ and non-attracting for $\mu = r - q \leq 0$. For $1 \leq \beta < 2$, the origin is an exit boundary. For $\beta < 1$, the origin is a regular boundary point and is specified as a killing boundary by adjoining a killing boundary condition (i.e., the process is sent to the cemetery or, in financial terms, to the bankruptcy state ∂ at the first hitting time of zero, $\mathcal{T}_0 = \inf\{t \geq 0 : V_t = 0\}$). For $\beta = 2$ (the lognormal assumption), both zero and infinity are natural boundaries. For $\beta > 2$, the origin is a natural boundary and infinity is an entrance boundary.[6]

In general terms, the underlying asset of the CEV diffusion can be thought of as a stock, a stock index, an exchange rate, or a financial futures contract, so long as the parameter q is understood as, respectively, a dividend yield, an average dividend yield, the foreign default-free interest rate, or the domestic risk-free interest rate. In this paper, we want to specialize the CEV general setup to analyze the investment/disinvestment decision on real assets faced by a firm under conditions of uncertainty. Our stochastic variable is assumed to be the present value of the expected future net cash flows. Thus, the dividend yield parameter q should be interpreted as the so-called *rate of return shortfall* of McDonald and Siegel (1984) or the *convenience yield* of Brennan and Schwartz (1985).

3 Pricing Perpetual American Options under the CEV Diffusion

Hereafter, $\Theta_t(V, X, T; \phi)$ will denote the time-t (≥ 0) value of an American option on the asset price V, with strike price X, and with maturity date T ($\geq t$), where $\phi = -1$ for an American call or $\phi = 1$ for an American put. For the perpetual case, it is well known -

[5]For a complete description of the boundary classification for one-dimensional diffusions see Karlin and Taylor (1981, chap. 15) and Borodin and Salminen (2002, chap. 2).

[6]For additional details regarding the boundary conditions of the CEV diffusion process see Andersen and Andreasen (2000) and Davydov and Linetsky (2001, 2003).

see, for instance, Shreve (2004, Definition 8.3.1) - that the American option price can be represented by the following *Snell envelope*:

$$\lim_{T\uparrow\infty}\Theta_t\left(V,X,T;\phi\right)=\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau-t)}\left(\phi X-\phi V_\tau\right)^+\mathbb{I}_{\{\tau<\infty\}}\middle|\,\mathcal{F}_t\right], \tag{5}$$

where $\mathcal{T}$ is the set of all stopping times for the filtration $\mathbb{F}$ generated by the underlying price process, and $\mathbb{I}_A$ denotes the indicator function of the set A. Moreover, it is also well known that the critical asset price that separates the *stopping* and *continuation* regions of the perpetual American option is a time-invariant constant that will be denoted by V_∞.

Representing the first passage time of the underlying asset price to the constant early exercise boundary by

$$\tau^*:=\inf\left\{u>t:V_u=V_\infty\right\} \tag{6}$$

and considering that the American option is still alive at the valuation date (i.e., $\phi V_t>\phi V_\infty$), equation (5) can then be restated as:

$$\begin{aligned}\lim_{T\uparrow\infty}\Theta_t\left(V,X,T;\phi\right) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau^*-t)}\left(\phi X-\phi V_{\tau^*}\right)^+\mathbb{I}_{\{\tau^*<\infty\}}\middle|\,\mathcal{F}_t\right]\\ &= \phi\left(X-V_\infty\right)\mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau^*-t)}\mathbb{I}_{\{\tau^*<\infty\}}\middle|\,\mathcal{F}_t\right]. \end{aligned} \tag{7}$$

Based on Nunes (2008, Proposition 4), the following propositions provide a closed-form solution for the perpetual American option contract, under both the Merton (1973) and the CEV models.

Proposition 1 *Under the geometric Brownian motion assumption, i.e., for $\sigma(t,V)=\sigma$ in equation (2), the perpetual American option price is equal to:*

$$\lim_{T\uparrow\infty}\Theta_t\left(V,X,T;\phi\right)=\phi\left(X-V_\infty\right)\left(\frac{V_\infty}{V_t}\right)^{\gamma(\phi)}, \tag{8}$$

where $\phi V_t>\phi V_\infty$, and V_∞ denotes the constant exercise boundary which is defined by the following closed-form solution:

$$V_\infty:=\frac{\gamma(\phi)}{\gamma(\phi)+1}X, \tag{9}$$

with

$$\gamma(\phi) := \frac{r - q - \frac{\sigma^2}{2} + \phi\sqrt{(r - q - \frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2}, \tag{10}$$

$\phi = -1$ *for an American call and* $\phi = +1$ *for an American put.*

Proof. See Appendix A. ■

Proposition 1 gives the analytical formulae to price perpetual American calls and puts that is standard in the real options literature when one assumes the geometric Brownian motion stochastic process [e.g., McDonald and Siegel (1986) and Dixit and Pindyck (1994)]. It is quite straightforward to establish the relationship between the solution given by Proposition 1 and the formulae usually used in the classic real options model. For $\phi = -1$, $\gamma(\phi)$ is the positive root a (with $a > 1$) that solves the quadratic equation $1/2\,\sigma^2\,x\,(x-1) + (r-q)\,x - r = 0$, V_∞ is the critical value $\overline{V}$ at which it is optimal to invest and X represents the investment costs $\overline{X}$. For $\phi = +1$, $\gamma(\phi)$ is the negative root b (with $b < 0$) that solves the fundamental quadratic equation, V_∞ is the threshold level $\underline{V}$ at which it is optimal to disinvest and X is interpreted as the disinvestment proceeds $\underline{X}$.

Proposition 2 *Under the CEV diffusion model and for* $r \neq q$*, the perpetual American option price is equal to:*

$$\begin{aligned}\lim_{T\uparrow\infty} \Theta_t\,(V, X, T; \phi) \quad &= \quad \phi\,(X - V_\infty)\left(\frac{V_t}{V_\infty}\right)^{\eta(\phi)} \exp\{\eta(\phi)[x(V_t) - x(V_\infty)]\} \\ &\quad \frac{M_{\phi(\beta-2)}\left[\eta(\phi) + (-1)^{\eta(\phi)}\,\alpha, \frac{\beta-1-2\eta(\phi)}{\beta-2}, (-1)^{\eta(\phi)}\,x(V_t)\right]}{M_{\phi(\beta-2)}\left[\eta(\phi) + (-1)^{\eta(\phi)}\,\alpha, \frac{\beta-1-2\eta(\phi)}{\beta-2}, (-1)^{\eta(\phi)}\,x(V_\infty)\right]},\end{aligned} \tag{11}$$

where

$$\eta(\phi) := \begin{cases} \mathbb{I}_{\{r>q\,\wedge\,\beta<2\}} & \Leftarrow \quad \phi = +1 \\ 1 - \mathbb{I}_{\{r>q\,\wedge\,\beta>2\}} & \Leftarrow \quad \phi = -1 \end{cases}, \tag{12}$$

$$\alpha := \frac{r}{(\beta - 2)(r - q)}, \tag{13}$$

$$x(V) := \frac{2(r-q)}{\delta^2(\beta-2)} V^{2-\beta}, \tag{14}$$

and

$$M_\lambda(a,b,z) := \begin{cases} M(a,b,z) & \Leftarrow \quad \lambda > 0 \\ U(a,b,z) & \Leftarrow \quad \lambda < 0 \end{cases}, \tag{15}$$

with $M(a,b,z)$ and $U(a,b,z)$ representing the confluent hypergeometric Kummer's functions, as defined by Abramowitz and Stegun (1972, Equations 13.1.2 and 13.1.3)

Proof. See Appendix B. ■

Proposition 3 *Under the CEV diffusion model and for $r = q$, the perpetual American option price is equal to:*

$$\lim_{T\uparrow\infty} \Theta_t\left(V, X, T; \phi\right) = \phi\left(X - V_\infty\right)\sqrt{\frac{V_t}{V_\infty}}\,\frac{I_{\frac{1}{|\beta-2|};\phi(\beta-2)}\left[\varepsilon(V_t)\sqrt{2r}\right]}{I_{\frac{1}{|\beta-2|};\phi(\beta-2)}\left[\varepsilon(V_\infty)\sqrt{2r}\right]}, \tag{16}$$

where

$$\varepsilon(V) := \frac{2V^{1-\beta/2}}{\delta|\beta-2|}, \tag{17}$$

and

$$I_{v;\lambda}(z) := \begin{cases} I_v(z) & \Leftarrow \quad \lambda > 0 \\ K_v(z) & \Leftarrow \quad \lambda < 0 \end{cases}, \tag{18}$$

with $I_v(z)$ and $K_v(z)$ representing the modified Bessel functions, as defined by Abramowitz and Stegun (1972, p. 375).

Proof. See Appendix C. ■

As usual, the constant exercise boundary is obtained, under the CEV model, after maximizing equations (11) or (16) with respect to V_∞.

4 Numerical Analysis

Armed with the perpetual option pricing formulae, we are now ready to analyze the option value and the optimal investment/disinvestment rule of a project under different assumptions regarding the stochastic process assumed for the underlying state variable. The benchmark is the standard real options model given by Proposition 1 which is based on the assumption that the state variable follows a geometric random walk. For comparative purposes, we use the CEV perpetual option pricing formulae given by Propositions 2 and 3 both of which are quite easy to compute in *Mathematica* and *Maple* since these software packages include all the required special functions as built-in functions. We use *Mathematica* 5.0 running on a Pentium IV PC for all the calculations in this paper.

4.1 Perpetual American CEV Call Option

Following the standard real options model of McDonald and Siegel (1986), we study the investment decision of a firm that is considering the possibility of undertaking an investment opportunity. The firm can pay a lump-sum cost of X to install an investment project, for which the expected future net cash flows (conditional on exercising the option to invest) have a present value V_0.

We adopt the following parameters for our numerical analysis: the initial asset value is $V_0 = \$100$, the investment cost (i.e., the strike) is $X = \overline{X} = \$80, \$90, \$100, \$110, \$120$ (this allow us to value *out-of-the-money*, *at-the-money*, and *in-the-money* options), the instantaneous volatility at time $t = 0$ is $\sigma_0 = 20\%$ per annum (i.e., the scale parameter δ is selected so that $\sigma_0 = \sigma(0, V_0) = \delta\, V_0^{\beta/2-1} = 0.20$ when $V_0 = \$100$), the risk-free interest rate is 6% per annum ($r = 0.06$) and the dividend yield of the project is 5% per annum ($q = 0.05$). Then we will also consider the cases where $r = q = 6\%$ and $r < q = 8\%$ as well as $\sigma_0 = 40\%$ for comparative purposes.

We employ seven different values of β to show its effect on option values and on the corresponding critical thresholds: $\beta = 3, 2, 1, 0, -2, -4, -6$. The constant volatility case ($\beta = 2$) corresponds to the standard real options model based on the lognormal assumption.[7] In the case of the lognormal model, the standard formulae given by Proposition 1 is

[7]Elasticity values of $\beta < 2$ are characteristic of stock index options and values of $\beta > 2$ are characte-

used. Following MacBeth and Merville (1980), Boyle and Tian (1999) and Davydov and Linetsky (2001, 2003), the scale parameter δ is selected so that the initial instantaneous volatility is the same across different models. Let σ_0 be the instantaneous volatility for the lognormal model. Then the value of δ to be used for the CEV models with different β values is adjusted to be $\delta = \sigma_0 \, V_0^{1-\beta/2}$.

The numerical examples of Table 1 illustrate the computational results for a perpetual American call option under the CEV diffusion process.

[Insert Table 1 Here]

Given that both the lognormal and the CEV models are calibrated so that the instantaneous volatility at the initial project value V_0 is the same across different models, these differences are purely the effect of the inverse relationship between volatility and the asset price level.

incomplete...

4.2 Perpetual American CEV Put Option

Once again, the volatility parameter is adjusted to be $\delta = \sigma_0 \, V_0^{1-\beta/2}$, which ensures that the percentage change in the price of the underlying asset is the same across different β values.

incomplete...

5 Conclusions

Much of the work on real options assumes that the underlying state variable follows a geometric Brownian motion with constant volatility. This paper uses a more general assumption for the state variable process which may better capture the empirical observations found in the financial economics literature. We use the so-called CEV diffusion

ristic of some commodity futures options with upward sloping implied volatility smiles [see Davydov and Linetsky (2001)].

model where the volatility is a function of the underlying asset price and provide analytical solutions for perpetual American-style call and put options under the CEV diffusion. For $r \neq q$, the perpetual American option price is based on an infinite series of terms involving confluent hypergeometric functions. For $r = q$, the computation of the perpetual American option formula involves the use of modified Bessel functions.

We demonstrate the implications of the correct specification of the underlying state variable process for the valuation of real assets and show that a firm that uses the standard geometric Brownian motion assumption is exposed to significant errors of analysis which may lead to non-optimal investment and disinvestment decisions. Our results strongly highlights the case for moving beyond the simplistic real options models based on the lognormal assumption to more realistic models incorporating volatility smile effects.

Appendix A: Proof of Proposition 1

Equation (8) is simply a generalization of the Merton (1973, p. 174) solution for a non-zero dividend yield. Solving the stochastic differential equation (2) for $\sigma(t,V)=\sigma$, the optimal stopping time τ^* can be redefined as

$$\begin{aligned}\tau^* &= \inf\left\{u>t: V_u = V_\infty\right\} \\ &= \inf\left\{u>t: -\frac{\phi}{\sigma}\left(r-q-\frac{\sigma^2}{2}\right)(u-t)-\phi\int_t^u dW_v^{\mathbb{Q}} = \frac{\phi}{\sigma}\ln\left(\frac{V_t}{V_\infty}\right)\right\}. \quad \text{(A.1)}\end{aligned}$$

Applying Shreve (2004, Theorem 8.3.2), the Laplace transform of the first passage time contained in the right-hand side of equation (7) can be written as

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau^*-t)}\mathbb{I}_{\{\tau^*<\infty\}}\middle|\mathcal{F}_t\right] \\ =\ & \exp\left\{-\frac{\phi}{\sigma}\ln\left(\frac{V_t}{V_\infty}\right)\left[\frac{\phi}{\sigma}\left(r-q-\frac{\sigma^2}{2}\right)+\sqrt{\frac{1}{\sigma^2}\left(r-q-\frac{\sigma^2}{2}\right)^2+2r}\right]\right\} \\ =\ & \left(\frac{V_\infty}{V_t}\right)^{\gamma(\phi)}, \quad \text{(A.2)}\end{aligned}$$

where $\gamma(\phi)$ is given by equation (10). Combining equations (7) and (A.2), equation (8) follows immediately.

Finally, equation (9) arises after maximizing equation (8) with respect to V_∞. ■

Appendix B: Proof of Proposition 2

Based on Leblanc and Scaillet (1998, Property 2), Davydov and Linetsky (2001, equations 2 and 4), or Borodin and Salminen (2002, pages 18-19), and assuming that the perpetual American option is still alive on the valuation date (i.e., $\phi V_t > \phi V_\infty$), then

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau^*-t)}\,\mathbb{I}_{\{\tau^*<\infty\}}\middle|\mathcal{F}_t\right] = \begin{cases}\frac{\psi_r(V_t)}{\psi_r(V_\infty)} \Leftarrow \phi=-1 \\ \frac{\varphi_r(V_t)}{\varphi_r(V_\infty)} \Leftarrow \phi=+1\end{cases}, \quad \text{(B.1)}$$

where $\psi_r(V_t)$ and $\varphi_r(V_t)$ are, respectively, the fundamental increasing and decreasing solutions of the following ordinary differential equation:

$$\frac{1}{2}\delta^2 V^\beta \frac{d^2u}{dV^2} + (r-q)\, V \frac{du}{dV} - ru = 0, \tag{B.2}$$

for any $u = u(V) \in C^2$, and with $V_t = V \in [0, +\infty)$. Davydov and Linetsky (2001, Proposition 5) are able to convert the ordinary differential equation (B.2) into the Whittaker's form of the confluent hypergeometric equation - see, for instance, Abramowitz and Stegun (1972, Equation 13.1.31) - and, therefore, show that

$$\psi_r(V_t) = \begin{cases} V_t^{\frac{\beta-1}{2}} e^{\frac{\epsilon y(V_t)}{2}} M_{k,m}[y(V_t)] \Leftarrow \beta < 2, r \neq q \\ V_t^{\frac{\beta-1}{2}} e^{\frac{\epsilon y(V_t)}{2}} W_{k,m}[y(V_t)] \Leftarrow \beta > 2, r \neq q \end{cases}, \tag{B.3}$$

and

$$\varphi_r(V_t) = \begin{cases} V_t^{\frac{\beta-1}{2}} e^{\frac{\epsilon y(V_t)}{2}} W_{k,m}[y(V_t)] \Leftarrow \beta < 2, r \neq q \\ V_t^{\frac{\beta-1}{2}} e^{\frac{\epsilon y(V_t)}{2}} M_{k,m}[y(V_t)] \Leftarrow \beta > 2, r \neq q \end{cases}, \tag{B.4}$$

where $M_{k,m}(y)$ and $W_{k,m}(y)$ are the Whittaker functions, as defined by Abramowitz and Stegun (1972, equations 13.1.32 and 13.1.33),

$$\epsilon := \begin{cases} +1 \Leftarrow (r-q)(\beta-2) > 0 \\ -1 \Leftarrow (r-q)(\beta-2) < 0 \end{cases}, \tag{B.5}$$

$$y(V_t) := \frac{|r-q|}{\frac{\delta^2}{2}|\beta-2|} V_t^{2-\beta}, \tag{B.6}$$

$$k := \epsilon\left(\frac{1}{2} + \frac{1}{2\beta-4}\right) - \frac{r}{|(r-q)(\beta-2)|}, \tag{B.7}$$

and

$$m := \frac{1}{2|\beta-2|}. \tag{B.8}$$

The following four cases must be considered:

1. $\phi = -1$ and $\beta > 2$, or $\phi = 1$ and $\beta < 2$.

   Under these conditions,

$$m = -\phi \frac{1}{2(\beta - 2)}, \tag{B.9}$$

while equations (B.1), (B.3), and (B.4) yield

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[e^{-r(\tau^* - t)} \mathbb{I}_{\{\tau^* < \infty\}} \middle| \mathcal{F}_t \right] \\
= & \frac{V_t^{\frac{\beta-1}{2}} e^{\frac{\epsilon y(V_t)}{2}} W_{k,m} \left[y(V_t) \right]}{V_\infty^{\frac{\beta-1}{2}} e^{\frac{\epsilon y(V_\infty)}{2}} W_{k,m} \left[y(V_\infty) \right]} \\
= & \frac{V_t^{\frac{\beta-1}{2}} e^{\frac{\epsilon y(V_t)}{2}} e^{-\frac{y(V_t)}{2}} \left[y(V_t) \right]^{\frac{1}{2}+m} U \left[\frac{1}{2} + m - k, 1 + 2m; y(V_t) \right]}{V_\infty^{\frac{\beta-1}{2}} e^{\frac{\epsilon y(V_\infty)}{2}} e^{-\frac{y(V_\infty)}{2}} \left[y(V_\infty) \right]^{\frac{1}{2}+m} U \left[\frac{1}{2} + m - k, 1 + 2m; y(V_\infty) \right]},
\end{aligned} \tag{B.10}$$

where the last line follows from Abramowitz and Stegun (1972, equation 13.1.33).

(a) $r - q > 0$.

In this case, equations (B.5) to (B.7) imply that

$$\epsilon = -\phi, \tag{B.11}$$

$$y(V_t) = -\phi \frac{r - q}{\frac{\delta^2}{2}(\beta - 2)} V_t^{2-\beta}, \tag{B.12}$$

and

$$k := -\phi \left(\frac{1}{2} + \frac{1}{2\beta - 4} \right) + \phi \frac{r}{(r - q)(\beta - 2)}. \tag{B.13}$$

Combining equations (B.9) through (B.13), then

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau^*-t)}\mathbb{I}_{\{\tau^*<\infty\}}\middle|\mathcal{F}_t\right] \\
= \; & \exp\left\{\mathbb{I}_{\{\phi=1\}}\left[y\left(V_\infty\right)-y\left(V_t\right)\right]\right\} \\
& \frac{V_t^{\frac{\beta-1}{2}}\left(V_t^{2-\beta}\right)^{\frac{1}{2}-\frac{\phi}{2(\beta-2)}}U\left[\frac{1+\phi}{2}-\phi\frac{r}{(r-q)(\beta-2)},1-\frac{\phi}{\beta-2};y\left(V_t\right)\right]}{V_\infty^{\frac{\beta-1}{2}}\left(V_\infty^{2-\beta}\right)^{\frac{1}{2}-\frac{\phi}{2(\beta-2)}}U\left[\frac{1+\phi}{2}-\phi\frac{r}{(r-q)(\beta-2)},1-\frac{\phi}{\beta-2};y\left(V_\infty\right)\right]} \\
= \; & \left(\frac{V_t}{V_\infty}\right)^{\frac{1+\phi}{2}}\exp\left\{\phi\mathbb{I}_{\{\phi=1\}}\left[x\left(V_t\right)-x\left(V_\infty\right)\right]\right\} \\
& \frac{U\left[\frac{1+\phi}{2}-\phi\alpha,1-\frac{\phi}{\beta-2};-\phi x\left(V_t\right)\right]}{U\left[\frac{1+\phi}{2}-\phi\alpha,1-\frac{\phi}{\beta-2};-\phi x\left(V_\infty\right)\right]},
\end{aligned}
\tag{B.14}
$$

where the last line follows from equations (13) and (14). Equations (7) and (B.14) are equivalent to the general solution (11) for $\{\phi=-1,\beta>2,r>q\}$ or $\{\phi=1,\beta<2,r>q\}$.

(b) $r-q<0$.

In this case, equations (B.5) to (B.7) imply that

$$\epsilon=\phi, \tag{B.15}$$

$$y\left(V_t\right)=\phi\frac{r-q}{\frac{\delta^2}{2}\left(\beta-2\right)}V_t^{2-\beta}, \tag{B.16}$$

and

$$k:=\phi\left(\frac{1}{2}+\frac{1}{2\beta-4}\right)-\phi\frac{r}{\left(r-q\right)\left(\beta-2\right)}. \tag{B.17}$$

Therefore, combining equations (B.9), (B.10), (B.15), (B.16) and (B.17),

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau^*-t)} \mathbb{I}_{\{\tau^*<\infty\}} \middle| \mathcal{F}_t \right] \qquad \text{(B.18)} \\
= \; & \exp\left\{ \mathbb{I}_{\{\phi=-1\}} \left[y\left(V_\infty\right) - y\left(V_t\right) \right] \right\} \\
& \frac{V_t^{\frac{\beta-1}{2}} \left(V_t^{2-\beta} \right)^{\frac{1}{2}-\frac{\phi}{2(\beta-2)}} U\left[\frac{1-\phi}{2} - \frac{\phi}{\beta-2} + \phi \frac{r}{(r-q)(\beta-2)}, 1 - \frac{\phi}{\beta-2}; y\left(V_t\right) \right]}{V_\infty^{\frac{\beta-1}{2}} \left(V_\infty^{2-\beta} \right)^{\frac{1}{2}-\frac{\phi}{2(\beta-2)}} U\left[\frac{1-\phi}{2} - \frac{\phi}{\beta-2} + \phi \frac{r}{(r-q)(\beta-2)}, 1 - \frac{\phi}{\beta-2}; y\left(V_\infty\right) \right]}.
\end{aligned}
$$

Using the Kummer transformation offered by Abramowitz and Stegun (1972, equation 13.1.29), as well as equations (13) and (14), equation (B.18) can be rewritten as

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau^*-t)} \mathbb{I}_{\{\tau^*<\infty\}} \middle| \mathcal{F}_t \right] \\
= \; & \left(\frac{V_t}{V_\infty} \right)^{\frac{1+\phi}{2}} \exp\left\{ \phi \mathbb{I}_{\{\phi=-1\}} \left[x\left(V_\infty\right) - x\left(V_t\right) \right] \right\} \\
& \frac{\left(V_t^{2-\beta} \right)^{\frac{\phi}{\beta-2}} U\left[\frac{1-\phi}{2} + \phi\alpha, 1 + \frac{\phi}{\beta-2}; \phi x\left(V_t\right) \right]}{\left(V_\infty^{2-\beta} \right)^{\frac{\phi}{\beta-2}} U\left[\frac{1-\phi}{2} + \phi\alpha, 1 + \frac{\phi}{\beta-2}; \phi x\left(V_\infty\right) \right]} \\
= \; & \left(\frac{V_t}{V_\infty} \right)^{\frac{1-\phi}{2}} \exp\left\{ \phi \mathbb{I}_{\{\phi=-1\}} \left[x\left(V_\infty\right) - x\left(V_t\right) \right] \right\} \qquad \text{(B.19)} \\
& \frac{U\left[\frac{1-\phi}{2} + \phi\alpha, 1 + \frac{\phi}{\beta-2}; \phi x\left(V_t\right) \right]}{U\left[\frac{1-\phi}{2} + \phi\alpha, 1 + \frac{\phi}{\beta-2}; \phi x\left(V_\infty\right) \right]},
\end{aligned}
$$

which agrees with equation (11) for $\{\phi = -1, \beta > 2, r < q\}$ or $\{\phi = 1, \beta < 2, r < q\}$.

2. $\phi = -1$ and $\beta < 2$, or $\phi = 1$ and $\beta > 2$.

   Under these conditions,

$$
m = \phi \frac{1}{2\left(\beta - 2\right)}, \qquad \text{(B.20)}
$$

   whereas equations (B.1), (B.3), and (B.4) yield

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau^*-t)}\mathbb{I}_{\{\tau^*<\infty\}} \middle| \mathcal{F}_t \right] \\
= & \frac{V_t^{\frac{\beta-1}{2}} e^{\frac{\epsilon y(V_t)}{2}} M_{k,m}\left[y\left(V_t\right)\right]}{V_\infty^{\frac{\beta-1}{2}} e^{\frac{\epsilon y(V_\infty)}{2}} M_{k,m}\left[y\left(V_\infty\right)\right]} \\
= & \frac{V_t^{\frac{\beta-1}{2}} e^{\frac{\epsilon y(V_t)}{2}} e^{-\frac{y(V_t)}{2}} \left[y\left(V_t\right)\right]^{\frac{1}{2}+m} M\left[\frac{1}{2}+m-k, 1+2m; y\left(V_t\right)\right]}{V_\infty^{\frac{\beta-1}{2}} e^{\frac{\epsilon y(V_\infty)}{2}} e^{-\frac{y(V_\infty)}{2}} \left[y\left(V_\infty\right)\right]^{\frac{1}{2}+m} M\left[\frac{1}{2}+m-k, 1+2m; y\left(V_\infty\right)\right]},
\end{aligned} \tag{B.21}
$$

where the last line follows from Abramowitz and Stegun (1972, equation 13.1.32).

(a) $r-q>0$.

In this case, equations (B.5) to (B.7) imply that ϵ, $y(V_t)$, and k are again given by equations (B.15), (B.16) and (B.17), respectively. Hence, equation (B.21) can be simplified into

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau^*-t)}\mathbb{I}_{\{\tau^*<\infty\}} \middle| \mathcal{F}_t \right] \\
= & \exp\left\{\mathbb{I}_{\{\phi=-1\}}\left[y\left(V_\infty\right)-y\left(V_t\right)\right]\right\} \\
& \frac{V_t^{\frac{\beta-1}{2}}\left(V_t^{2-\beta}\right)^{\frac{1}{2}+\frac{\phi}{2(\beta-2)}} M\left[\frac{1-\phi}{2}+\phi\frac{r}{(r-q)(\beta-2)}, 1+\frac{\phi}{\beta-2}; y\left(V_t\right)\right]}{V_\infty^{\frac{\beta-1}{2}}\left(V_\infty^{2-\beta}\right)^{\frac{1}{2}+\frac{\phi}{2(\beta-2)}} M\left[\frac{1-\phi}{2}+\phi\frac{r}{(r-q)(\beta-2)}, 1+\frac{\phi}{\beta-2}; y\left(V_\infty\right)\right]} \\
= & \left(\frac{V_t}{V_\infty}\right)^{\frac{1-\phi}{2}} \exp\left\{\phi\mathbb{I}_{\{\phi=-1\}}\left[x\left(V_\infty\right)-x\left(V_t\right)\right]\right\} \\
& \frac{M\left[\frac{1-\phi}{2}+\phi\alpha, 1+\frac{\phi}{\beta-2}; \phi x\left(V_t\right)\right]}{M\left[\frac{1-\phi}{2}+\phi\alpha, 1+\frac{\phi}{\beta-2}; \phi x\left(V_\infty\right)\right]},
\end{aligned} \tag{B.22}
$$

where the last line follows from equations (13) and (14). Equations (7) and (B.22) are equivalent to the general solution (11) for $\{\phi=-1,\beta<2,r>q\}$ or $\{\phi=1,\beta>2,r>q\}$.

(b) $r-q<0$.

In this case, equations (B.5) to (B.7) imply that ϵ, $y(V_t)$, and k are now given by equations (B.11), (B.12) and (B.13), respectively. Hence, equation (B.21) can be written as

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau^*-t)}\mathbb{I}_{\{\tau^*<\infty\}}\middle|\mathcal{F}_t\right] && \text{(B.23)}\\
= \ & \exp\left\{\mathbb{I}_{\{\phi=1\}}\left[y\left(V_\infty\right)-y\left(V_t\right)\right]\right\}\\
& \frac{V_t^{\frac{\beta-1}{2}}\left(V_t^{2-\beta}\right)^{\frac{1}{2}+\frac{\phi}{2(\beta-2)}}M\left[\frac{1+\phi}{2}+\frac{\phi}{\beta-2}-\phi\frac{r}{(r-q)(\beta-2)},1+\frac{\phi}{\beta-2};y\left(V_t\right)\right]}{V_\infty^{\frac{\beta-1}{2}}\left(V_\infty^{2-\beta}\right)^{\frac{1}{2}+\frac{\phi}{2(\beta-2)}}M\left[\frac{1+\phi}{2}+\frac{\phi}{\beta-2}-\phi\frac{r}{(r-q)(\beta-2)},1+\frac{\phi}{\beta-2};y\left(V_\infty\right)\right]}.
\end{aligned}
$$

Using the Kummer transformation offered by Abramowitz and Stegun (1972, equation 13.1.27), as well as equations (13) and (14), equation (B.23) can be rewritten as

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau^*-t)}\mathbb{I}_{\{\tau^*<\infty\}}\middle|\mathcal{F}_t\right]\\
= \ & \left(\frac{V_t}{V_\infty}\right)^{\frac{1-\phi}{2}}\exp\left\{\phi\mathbb{I}_{\{\phi=1\}}\left[x\left(V_t\right)-x\left(V_\infty\right)\right]\right\}\\
& \frac{e^{-\phi x(V_t)}M\left[\frac{1-\phi}{2}+\phi\alpha,1+\frac{\phi}{\beta-2};\phi x\left(V_t\right)\right]}{e^{-\phi x(V_\infty)}M\left[\frac{1-\phi}{2}+\phi\alpha,1+\frac{\phi}{\beta-2};\phi x\left(V_\infty\right)\right]}\\
= \ & \left(\frac{V_t}{V_\infty}\right)^{\frac{1-\phi}{2}}\exp\left\{\phi\left(\mathbb{I}_{\{\phi=1\}}-1\right)\left[x\left(V_t\right)-x\left(V_\infty\right)\right]\right\} && \text{(B.24)}\\
& \frac{M\left[\frac{1-\phi}{2}+\phi\alpha,1+\frac{\phi}{\beta-2};\phi x\left(V_t\right)\right]}{M\left[\frac{1-\phi}{2}+\phi\alpha,1+\frac{\phi}{\beta-2};\phi x\left(V_\infty\right)\right]},
\end{aligned}
$$

which agrees with equation (11) for $\{\phi=-1,\beta<2,r<q\}$ or $\{\phi=1,\beta>2,r<q\}$. ■

Appendix C: Proof of Proposition 3

For $r=q$, the ordinary differential equation (B.2) can be reduced into the modified Bessel ODE - see, for instance, Abramowitz and Stegun (1972, equation 9.6.1) - and, therefore, Davydov and Linetsky (2001, Proposition 5) show that

$$
\psi_r\left(V_t\right)=\begin{cases} V_t^{\frac{1}{2}}I_{\frac{1}{|\beta-2|}}\left[\varepsilon\left(V_t\right)\sqrt{2r}\right] \Leftarrow \beta<2,r=q\\ V_t^{\frac{1}{2}}K_{\frac{1}{|\beta-2|}}\left[\varepsilon\left(V_t\right)\sqrt{2r}\right] \Leftarrow \beta>2,r=q\end{cases}, \tag{C.1}
$$

and

$$\varphi_r\left(V_t\right) = \begin{cases} V_t^{\frac{1}{2}} K_{\frac{1}{|\beta-2|}}\left[\varepsilon\left(V_t\right)\sqrt{2r}\right] \Leftarrow \beta < 2, r = q \\ V_t^{\frac{1}{2}} I_{\frac{1}{|\beta-2|}}\left[\varepsilon\left(V_t\right)\sqrt{2r}\right] \Leftarrow \beta > 2, r = q \end{cases}. \tag{C.2}$$

Two cases must now be distinguished:

1. $\phi = -1$ and $\beta > 2$, or $\phi = 1$ and $\beta < 2$.

   Under these conditions, equations (B.1) and (C.1) yield

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau^*-t)}\mathbb{I}_{\{\tau^*<\infty\}}\middle|\,\mathcal{F}_t\right] = \frac{V_t^{\frac{1}{2}} K_{\frac{1}{|\beta-2|}}\left[\varepsilon\left(V_t\right)\sqrt{2r}\right]}{V_\infty^{\frac{1}{2}} K_{\frac{1}{|\beta-2|}}\left[\varepsilon\left(V_\infty\right)\sqrt{2r}\right]}, \tag{C.3}$$

   which agrees with equation (16) for $\{\phi = -1, \beta > 2, r = q\}$ or $\{\phi = 1, \beta < 2, r = q\}$.

2. $\phi = -1$ and $\beta < 2$, or $\phi = 1$ and $\beta > 2$.

   In this case, equations (B.1) and (C.1) imply that

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau^*-t)}\mathbb{I}_{\{\tau^*<\infty\}}\middle|\,\mathcal{F}_t\right] = \frac{V_t^{\frac{1}{2}} I_{\frac{1}{|\beta-2|}}\left[\varepsilon\left(V_t\right)\sqrt{2r}\right]}{V_\infty^{\frac{1}{2}} I_{\frac{1}{|\beta-2|}}\left[\varepsilon\left(V_\infty\right)\sqrt{2r}\right]}, \tag{C.4}$$

   which is equivalent to equation (16) for $\{\phi = -1, \beta < 2, r = q\}$ or $\{\phi = 1, \beta > 2, r = q\}$. ■

Table 1: Perpetual American call options under the Merton (1973) and CEV models, with $V_0 = \$100$ and $\sigma_0 = 20\%$ pa.

Option	Investment	β							Maximum
parameters	cost	3	2	1	0	-2	-4	-6	% difference
	80	36.6216	36.1894	36.0754	36.0095	35.4041	34.5074	33.6035	7.15
		(216.0992)	(189.2820)	(174.3245)	(164.3996)	(151.4603)	(143.3312)	(137.7537)	27.22
	90	34.2660	33.1998	32.5674	32. 0591	30.7762	29.4048	28.1581	15.19
		(256.2467)	(212.9423)	(191.2256)	(177.7353)	(161.3924)	(151.8322)	(145.6637)	31.59
$r = 0.06$	100	32.4355	30.7354	29.5970	28.6662	26.7635	24.9961	23.5117	23.50
$q = 0.04$		(299.6180)	(236.6025)	(207.9356)	(191.0820)	(171.8662)	(161.5160)	(155.6419)	34.22
	110	30.9759	28.6640	27.0513	25.7366	23.3088	21.2697	19.7046	31.26
		(346.3529)	(260.2628)	(224.4870)	(204.4672)	(182.9528)	(172.6142)	(167.9242)	35.48
	120	29.7876	26.8951	24.8475	23.1950	20.3535	18.1807	16.6643	38.04
		(396.6033)	(283.9230)	(240.9047)	(217.9077)	(194.6767)	(185.0487)	(181.6070)	36.04

Note. Perpetual American call options and investment thresholds under the Merton (1973) and CEV models with elasticities $\beta = 3, 2, 1, 0, -2, -4, -6$. The value of the investment threshold is given in parentheses underneath the corresponding option value. The utmost right column gives the absolute value of the percentage difference between the Merton (1973) option value (investment threshold) and the CEV option value (investment threshold) with $\beta = -6$ relative to the Merton (1973) option value (investment threshold). The investment costs (i.e., the strike price X) vary as indicated in the second column. The first column indicates the constant risk-free interest rate (r) and the dividend yield for the asset price (q) used in each case. Parameters used in the calculations: $V_0 = \$100$ and $\sigma_0 = 20\%$ pa.

References

Abramowitz, M. and Stegun, I. A. (1972). *Handbook of Mathematical Functions*. Dover, New York.

Andersen, L. and Andreasen, J. (2000). Volatility skews and extensions of the Libor market model. *Applied Mathematical Finance*, 7:1–32.

Baldwin, C. Y. (1982). Optimal sequential investment when capital is not readily reversible. *Journal of Finance*, 37(3):763–782.

Bekaert, G. and Wu, G. (2000). Asymetric volatility and risk in equity markets. *Review of Financial Studies*, 13(1):1–42.

Bhattacharya, S. (1978). Project valuation with mean-reverting cash flow streams. *Journal of Finance*, 33(5):1317–1331.

Biekpe, N., Klumpes, P., and Tippett, M. (2003). Analytic solutions for the value of the option to (dis)invest. In Paxson, D. A., editor, *Real R&D Options*, pages 67–90. Butterworth-Heinemann, Oxford.

Bjerksund, P. and Ekern, S. (1995). Contingent claims evaluation of mean-reverting cash flows in shipping. In Trigeorgis, L., editor, *Real Options in Capital Investment: Models, Strategies, and Applications*, pages 207–219. Praeger Publishers, Westport, Connecticut.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654.

Borodin, A. N. and Salminen, P. (2002). *Handbook of Brownian Motion - Facts and Formulae*. Birkhauser, Basel, 2nd edition.

Boyle, P. P. and Tian, Y. (1999). Pricing lookback and barrier options under the CEV process. *Journal of Financial and Quantitative Analysis*, 34(2):241–264.

Brennan, M. J. (1991). The price of convenience and the valuation of commodity contingent claims. In Lund, D. and Øksendal, B., editors, *Stochastic Models and Option Values: Applications to Resources, Environment and Investment Problems*, pages 33–71. Elsevier, Amsterdam, Netherlands.

Brennan, M. J. and Schwartz, E. S. (1985). Evaluating natural resource investments. *Journal of Business*, 58(2):135–157.

Carr, P. and Linetsky, V. (2006). A jump to default extended CEV model: An application of Bessel processes. *Finance and Stochastics*, 10:303–330.

Cox, J. C. (1975). Notes on option pricing I: Constant elasticity of variance diffusions. Working Paper, Stanford University.

Cox, J. C. (1996). The constant elasticity of variance option pricing model. *Journal of Portfolio Management*, Special Issue:15–17.

Cox, J. C. and Ross, S. A. (1976). The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3:145–166.

Davydov, D. and Linetsky, V. (2001). Pricing and hedging path-dependent options under the CEV process. *Management Science*, 47(7):949–965.

Davydov, D. and Linetsky, V. (2003). Pricing options on scalar diffusions: An eigenfunction expansion approach. *Operations Research*, 51(2):185–209.

Dennis, P. and Mayhew, S. (2002). Risk-neutral skewness: Evidence from stock options. *Journal of Financial and Quantitative Analysis*, 37(3):471–493.

Dixit, A. K. and Pindyck, R. S. (1994). *Investment Under Uncertainty.* Princeton University Press, Princeton, New Jersey.

Emanuel, D. C. and MacBeth, J. D. (1982). Further results on the constant elasticity of variance call option pricing model. *Journal of Financial and Quantitative Analysis*, 17(4):533–554.

Gibson, R. and Schwartz, E. S. (1990). Stochastic convenience yield and the pricing of oil contingent claims. *Journal of Finance*, 45(3):959–976.

Gibson, R. and Schwartz, E. S. (1991). Valuation of long term oil-linked assets. In Lund, D. and Øksendal, B., editors, *Stochastic Models and Option Values: Applications to Resources, Environment and Investment Problems*, pages 73–101. Elsevier, Amsterdam, Netherlands.

Jackwerth, J. C. and Rubinstein, M. (1996). Recovering probability distributions from option prices. *Journal of Finance*, 51(5):1611–1631.

Jackwerth, J. C. and Rubinstein, M. (2001). Recovering stochastic processes from option prices. Working Paper, London Business School.

Karlin, S. and Taylor, H. M. (1981). *A Second Course in Stochastic Processes*. Academic Press, San Diego.

Klumpes, P. and Tippett, M. (2004). A modified 'square root' process for determining the value of the option to (dis)invest. *Journal of Business Finance and Accounting*, 31(9 & 10):1449–1481.

Laughton, D. G. and Jacoby, H. D. (1993). Reversion, timing options, and long-term decision-making. *Financial Management*, 22(3):225–240.

Leblanc, B. and Scaillet, O. (1998). Path dependent options on yields in the affine term structure model. *Finance and Stochastics*, 2:349–367.

Linetsky, V. (2004). Lookback options and diffusion hitting times: A spectral expansion approach. *Finance and Stochastics*, 8:373–398.

MacBeth, J. D. and Merville, L. J. (1980). Tests of the Black-Scholes and Cox call option valuation models. *Journal of Finance*, 35(2):285–301.

McDonald, R. L. and Siegel, D. R. (1984). Option pricing when the underlying asset earns a below-equilibrium rate of return: A note. *Journal of Finance*, 39(1):261–265.

McDonald, R. L. and Siegel, D. R. (1986). The value of waiting to invest. *Quarterly Journal of Economics*, 101(4):707–728.

McKean, Jr., H. P. (1965). Appendix: A free boundary problem for the heat equation arising from a problem of mathematical economics. *Industrial Management Review*, 6(2):32–39.

Merton, R. C. (1973). Theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4(1):141–183.

Metcalf, G. E. and Hassett, K. A. (1995). Investment under alternative return assumptions: Comparing random walks and mean reversion. *Journal of Economic Dynamics and Control*, 19:1471–1488.

Nunes, J. P. (2008). Pricing American options under the constant elasticity of variance model and subject to bankruptcy. *Journal of Financial and Quantitative Analysis*. forthcoming.

Pinto, H., Howell, S., and Paxson, D. (2006). Modelling the number of customers as a birth and death process. Working Paper.

Rubinstein, M. (1994). Implied binomial trees. *Journal of Finance*, 49(3):771–818.

Schroder, M. (1989). Computing the constant elasticity of variance option pricing formula. *Journal of Finance*, 44(1):211–219.

Schwartz, E. S. (1997). The stochastic behavior of commodity prices: Implications for valuation and hedging. *Journal of Finance*, 52(3):923–973.

Shreve, S. E. (2004). *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer, New York.