The worst case for real options*

Magdalena Trojanowska[†]

University of Antwerp

Peter M. Kort[‡]

Tilburg University and University of Antwerp

Abstract

The purpose of this paper is to derive an optimal decision rule for investment in a finite life project, where we impose that the firm is ambiguous about the development of profit over time in the sense that it has no perfect information about parameter values governing the profit dynamics. Assuming that the firm is ambiguity averse, the analysis is based on Knight's distinction between risk and uncertainty and is carried out by dynamic programming in continuous time.

We find that ambiguity aversion affects the value of waiting equivocally and thus may accelerate investment. Yet, in the long run a large degree of Knightian uncertainty results in foregoing investment with greater probability than in the absence of Knightian uncertainty.

Keywords: real options, Knightian uncertainty, ambiguity aversion.

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[†]Department of Economics, University of Antwerp, Prinsstraat 13, 2000 Antwerp 1, Belgium; tel. +32 3 220 40 51; *email:* Magdalena.Trojanowska@ua.ac.be.

[‡]Department of Econometrics & Operations Research and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands; tel: +31 13 466 2062; *email:* Kort@uvt.nl and Department of Economics, University of Antwerp, Prinsstraat 13, 2000 Antwerp 1, Belgium.

1 Introduction

We depart from the real options approach to investment under uncertainty in the spirit of McDonald and Siegel (1986) and Dixit and Pindyck (1994). A first change is that we consider a finite life-time project rather than a perpetual one. This seems to be more realistic in today's knowledge economy with its fast-changing technology environment (see also Gryglewicz et al. (2005) and Sarkar (2003)).

When the firm faces an investment opportunity in a stochastic environment, it is very unlikely that market uncertainty is time invariant and can be captured by a simple stochastic process like e.g. geometric Brownian motion. However, this is regularly assumed in the standard real option literature. We try to modify this unsatisfying situation by studying a framework where market uncertainty may be perturbed by some noise process. That is, in addition to risk, represented by geometric Brownian motion in our model, the firm is aware of the presence of additional disturbance, where the latter has an unspecified distribution. We employ the Knightian specification of uncertainty (called also ambiguity in the literature) to model this probabilistic phenomenon. Mathematically, this form of uncertainty arises if the uncertainty is not reducible to a single probability measure and thus needs to be characterized by a spectrum of probability measures (cf. Knight (1921/2002)).

Most of the real options contributions assume that the market is complete and frictionless, which implies that the firm's decisions are made in a preference-free environment. However, if the firm is not perfectly certain about the dynamics of the state variable (the profit flow in our model), the firm's investment decision must be based on priors with respect to the future form of uncertainty. Following Chen and Epstein (2002), the firm's priors are represented by a set of probability measures. Furthermore, we impose that the firm is ambiguity averse, which in fact means that the firm is pessimistic about future profits. Having assumed the Knightian specification of uncertainty, the optimal decision is taken as if the state variable was governed by the worst-case probability measure among the measures considered. Consequently, the optimal investment problem is set in a maximin decision framework.

We derive the optimal investment trigger for the investment problem using the dynamic programming approach. We show that ambiguity aversion leads to an erosion of the value of the investment opportunity, but this does not necessarily bring about investment acceleration. This distinguishes our findings from Nishimura and Ozaki (2003), who established only the first result for the case of a perpetual project and showed in this way that the impact of ambiguity on optimal investment behavior is drastically different than the impact of risk (volatility). The latter result that ambiguity may speed up or delay option exercise links our paper to Miao and Wang (2006). Their contribution builds on a comparison of the effects of different degrees of ambiguity aversion on continuation payoffs and termination payoffs. If uncertainty is fully resolved after the investment is undertaken, then an increase in ambiguity accelerates investment. However, if the firm is ambiguous about the termination payoff, then the firm may delay the investment if this ambiguity dominates ambiguity about continuation payoff.

In our paper we assume that undertaking the investment does not change the level of ambiguity after the investment has been carried out. Still, there are two contradictory effects that govern the investment timing. First, there is option value effect. It captures the negative impact of the Knightian uncertainty on the hysteresis factor. Second, there is present value effect linked with subjective discount rate. We find that the higher the degree of ambiguity the lower is the net present value of the project under the worst case scenario. Combining these effects implies non-monotonicity of the investment trigger and an equivocal relationship in the value of waiting to invest.

We further examine the probability of investment. We find that the reduced value of waiting makes that in the short-run Knightian uncertainty may be investment enhancing. The opposite effect holds in the long-run: the ambiguity averse firm refrains from investment with a larger probability than a firm facing no ambiguity.

The remainder of the paper is organized as follows. Section 2 describes two cases. First, it presents a standard investment problem, where the project's value follows a geometric Brownian motion process and the project has finite life-time. Second, it develops a model where the firm is uncertain about the correct process generating future revenues. The analytical solutions for both cases are provided. Section 3 examines the sensitivity of the results to the key parameters and presents probability calculations. The last section concludes.

2 The model

2.1 The project with finite lifetime

Let time be continuous and indexed by $t \geq 0$. Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{Q})$. At each point of time the firm is facing an investment opportunity yielding a stochastic profit. Throughout this subsection the firm is assumed to be perfectly certain about the future process governing the profit (this conventional real options assumption is relaxed in the next subsection). The profit is modelled as a geometric Brownian motion and its dynamics under $\mathcal{Q}$ is given by

$$d\pi_t = \pi_t(\mu dt + \sigma dB_t), \tag{1}$$

with $\pi_0 > 0$ and where μ and σ are positive constants denoting drift and instantaneous standard deviation, respectively, while dB_t is the increment of a Wiener process. The firm is looking for the optimal time to pay a sunk cost I in return for a project whose value is stochastic.[1]

Suppose that the investment is undertaken at time t. The value of the project installed, W_t, equals the expected present value of the profit flow generated by the project during its life-time τ:

$$W(\pi_t) = \mathbb{E}^{\mathcal{Q}}\left[\int_t^{t+\tau} e^{-r(s-t)}\pi_s ds \middle| \mathcal{F}_t\right] = \frac{\pi_t(1-e^{-(r-\mu)\tau})}{r-\mu} =: W_t, \tag{2}$$

where $r > \mu$ is the (exogenous) interest rate. Applying Itô's lemma to $W(\pi_t)$ implies that the dynamics of the value of the project under $\mathcal{Q}$ satisfies

$$dW_t = W_t(\mu dt + \sigma dB_t) \quad \text{with} \quad W_0 = \pi_0 \frac{1-e^{-(r-\mu)\tau}}{r-\mu}.$$

The value of the opportunity to invest at time t, F_t, equals

$$F_t = \max_{t'\geq t} \mathbb{E}^{\mathcal{Q}}\left[\int_{t'}^{t'+\tau} e^{-r(s-t)}\pi_s ds - e^{-r(t'-t)} I \middle| \mathcal{F}_t\right],$$

[1] We assume that the project has no salvage value, i.e. I is totally sunk. As claimed in Dobbs (2004), second hand markets are of great importance for a non-growing steady state economy. The fast-changing technology environment constitutes the complementary case where the second hand value is negligible.

which, because the model is stationary, leads to the following Bellman equation:

$$F(W_t) = \max\left\{W_t - I, \mathbb{E}^{\mathbb{Q}}[dF_t|\mathcal{F}_t] + F(W_t) - rF(W_t)dt\right\}.$$

In the continuation region it holds that $rF(W_t)dt = \mathbb{E}^{\mathbb{Q}}[dF_t|\mathcal{F}_t]$, which by Itô's lemma simplifies to

$$\frac{1}{2}\sigma^2 W_t^2 F''(W_t) + \mu W_t F'(W_t) - rF(W_t) = 0.$$

The equation above must be solved subject to appropriate boundary conditions (value-matching, smooth-pasting, and absorbing-barrier condition, respectively):

$$F(W^*) = W^* - I \qquad F'(W^*) = 1 \qquad F(0) = 0.$$

The main result of this subsection is summarized in the proposition below.

Proposition 2.1 *Let $r > \mu$ and let β denote the positive root of the fundamental quadratic $\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r$. The firm optimally exercises the investment option the first time the process W_t exceeds the threshold $W^* = \frac{\beta}{\beta-1}I$. The corresponding value function equals*

$$F(W_t) = \begin{cases} (W^* - I)\left(\dfrac{W_t}{W^*}\right)^{\beta} & \text{if} \quad W_t \leq W^* \\ W_t - I & \text{if} \quad W_t > W^*. \end{cases}$$

The corresponding trigger on the process of profit flows equals $\pi^* = \frac{\beta}{\beta-1} I \frac{r-\mu}{1-\exp(-(r-\mu)\tau)}$.

2.2 The optimal investment policy under ambiguity aversion

In contrast to the previous section, we assume that the firm has no perfect confidence about the process governing the stochastic state variable (profit). Thus instead of assuming that future market uncertainty is driven by a single stochastic process with time invariant parameters, the profit flow is perceived as a stochastic process perturbed with noise of an unspecified distribution. This perturbation does not vanish over time, but persists in the independently and indistinguishably distributed manner (IID) according to terminology from Epstein and Schneider (2003). Formally, the noise is represented by a process θ over the range $[-\kappa, \kappa]$. The ambiguity level (called also the degree of Knightian uncertainty) is thus represented by the parameter κ. Due to the probabilistic model misspecification for θ, the arising form of uncertainty is not reducible to a single probability measure and thus needs to be characterized by a spectrum of probability measures.

The process θ serves itself as density generator, so the ambiguity in continuous time is defined as an expansion of the measure $\mathcal{Q}$ with the set of density generators. Consequently, the uncertainty is represented by a family of filtered probability spaces indexed by the set of density generators θ: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{Q}^\theta)$. For a formal construction the reader is referred to Appendix B.

It follows from the representation of ambiguity and Girsanov's theorem that a stochastic process

$$dB_t^\theta = dB_t + \theta_t dt$$

is a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{Q}^\theta)$. This fact allows us incorporate the noise process affecting the profit flows and state the general representation of the profit flow as follows:

$$\forall \theta \quad d\pi_t = (\mu - \sigma\theta_t)\pi_t dt + \sigma\pi_t dB_t^\theta.$$

This construction restricts the analysis to the set of measures equivalent to a reference probability measure. Measures are defined by their density generators and the model uncertainty is made equivalent with parameter uncertainty after the change of measure. Thus all processes used to describe the profit evolution are equivalent and ambiguity is captured by the spectrum of the subjective drift terms.

Furthermore we assume that the firm is pessimistic about future profits, which is identified as the extremal form of investor's ambiguity aversion. That is, the firm assigns the lowest value of growth rate to the project, such that

$$d\pi_t = (\mu - \kappa\sigma)\pi_t dt + \sigma\pi_t dB_t^\kappa$$

With this form of pessimistic beliefs the optimal decisions are taken as if the state variable was governed by the worst-case probability measure among the elements in the candidate set. That leads to the maximin decision framework.

As proved in Appendix B, the project value under the worst case scenario is described by the following proposition.

Proposition 2.2 *Under the worst case scenario, the present value at t of a cash flow stream generated by the project during its lifetime τ equals*

$$W(\pi_t, t) = \min_{\mathcal{Q}\in\mathcal{P}^\Theta} \mathbb{E}^{\mathcal{Q}}\left[\int_t^{t+\tau} e^{-r(s-t)}\pi_s ds \middle| \mathcal{F}_t\right] = \pi_t \frac{1-e^{-(r-\mu+\kappa\sigma)\tau}}{r-\mu+\kappa\sigma},$$

and its dynamics under the worst case measure, $\mathcal{Q}^\kappa$, is given by

$$dW_t = W_t(\mu-\kappa\sigma)dt + W_t\sigma dB_t^\kappa \quad \text{with} \quad W_0 = \pi_0\frac{1-e^{-(r-\mu+\kappa\sigma)\tau}}{r-\mu+\kappa\sigma}.$$

The corresponding value of the investment opportunity equals

$$F_t = \max_{t'\geq t}\min_{\mathcal{Q}\in\mathcal{P}^\Theta} \mathbb{E}^{\mathcal{Q}}\left[\int_{t'}^{t'+\tau} e^{-r(s-t)}\pi_s ds - e^{-r(t'-t)}I \middle| \mathcal{F}_t\right]$$

and satisfies the following Bellman equation (see Appendix B):

$$F(W_t) = \max\left\{W_t - K, \min_{\theta\in\Theta}\mathbb{E}^{\mathcal{Q}^\theta}[dF_t|\mathcal{F}_t] + F(W_t) - rF(W_t)dt\right\}.$$

In the continuation region it holds that

$$rF(W_t)dt = \min_{\theta\in\Theta}\mathbb{E}^{\mathcal{Q}^\theta}[dF_t|\mathcal{F}_t],$$

which by Itô's lemma simplifies to

$$\frac{1}{2}\sigma^2 W_t^2 F''(W_t) + (\mu-\kappa\sigma)W_t F'(W_t) - rF(W_t) = 0.$$

The equation above must be solved subject to standard boundary conditions (value-matching, smooth-pasting and absorbing barrier condition, respectively):

$$F(W^*) = W^* - I \qquad F'(W^*) = 1 \qquad F(0) = 0.$$

The main result is summarized in the following proposition.

Proposition 2.3 *Let the firm be ambiguity averse and let $\kappa > 0$ represent the firm's degree of Knightian uncertainty. Let $r > \mu - \kappa\sigma$ and let β denote the positive root of the fundamental quadratic $\frac{1}{2}\sigma^2\beta(\beta-1) + (\mu-\kappa\sigma)\beta - r$. The firm optimally exercises the investment option the first time the process W_t exceeds the threshold $W^* = \frac{\beta}{\beta-1}I$. The corresponding value function equals*

$$F(W_t) = \begin{cases} (W^*-I)\left(\dfrac{W_t}{W^*}\right)^\beta & \text{if} \quad W_t \leq W^* \\ W_t - I & \text{if} \quad W_t > W^*. \end{cases}$$

In order to provide a complete characterization of the optimal investment rule, we state an analogous proposition for profit flows.

Proposition 2.4 *Let the firm be ambiguity averse and let $\kappa > 0$ represent the firm's degree of Knightian uncertainty. Let $r > \mu - \kappa\sigma$ and let β denote the positive root of the fundamental quadratic $\frac{1}{2}\sigma^2\beta(\beta-1) + (\mu - \kappa\sigma)\beta - r$. The firm optimally exercises the investment option the first time the process π_t exceeds the threshold $\pi^* = \frac{\beta}{\beta-1} I \frac{\delta}{1-\exp(-\delta\tau)}$, where $\delta = r - \mu + \kappa\sigma$. The corresponding value function is*

$$F(\pi_t) = \begin{cases} \left(\pi^* \dfrac{1-\exp(-\delta\tau)}{\delta} - I\right)\left(\dfrac{\pi_t}{\pi^*}\right)^{\beta} & \text{if} \quad \pi_t \leq \pi^* \\ \pi_t \dfrac{1-\exp(-\delta\tau)}{\delta} - I & \text{if} \quad \pi_t > \pi^*. \end{cases}$$

3 Main implications of the model

In this section we examine the impact of ambiguity aversion and finite life-time on the project value and optimal investment timing. As a reference point we consider the model without Knightian uncertainty.

3.1 No ambiguity

To examine the effect of an increase in risk (volatility) of the project in the absence of Knightian uncertainty we derive the following partial derivatives of triggers:

$$\frac{\partial W^*}{\partial \sigma} = -\frac{W^*}{\beta(\beta-1)}\frac{\partial \beta}{\partial \sigma} > 0 \quad \text{and} \quad \frac{\partial \pi^*}{\partial \sigma} = \frac{\partial W^*}{\partial \sigma}\frac{r-\mu}{1-\exp(-(r-\mu)\tau)} > 0.$$

Both critical values W^* and π^* increase when σ increases. Thus a greater degree of risk σ results in an increase of the value of the option to wait. This is a well-known result from the real options literature.

3.2 Comparative statics results under ambiguity

First, we focus on the effect of ambiguity aversion on the time interval before the investment takes place. The threshold level of the present value of the project that triggers investment equals

$$W^* = \frac{\beta}{\beta-1} I,$$

where $\beta = \frac{-(\mu-\kappa\sigma)+\frac{1}{2}\sigma^2+\sqrt{\left((\mu-\kappa\sigma)-\frac{1}{2}\sigma^2\right)^2+2r\sigma^2}}{\sigma^2} > 1$. For this trigger the comparative statics result is easy to establish. Increasing ambiguity (κ) erodes the drift and raises β. This implies that the hysteresis factor $\frac{\beta}{\beta-1}$ decreases, and so does the investment threshold W^*. Hence, higher ambiguity implies that, compared to the standard problem from section 2.1, less precise information about the project value becomes available over time, which makes waiting with investment less valuable. Thus higher ambiguity erodes the subjective value of the investment opportunity. This result is in contrast with the impact of risk (volatility) in the standard real options literature. On the other hand, the hysteresis factor $\frac{\beta}{\beta-1}$ remains above 1, thus invalidation of the simple static NPV rule still takes place.

To further investigate the impact of ambiguity on investment, we present comparative statics results for the trigger π^*. This allows us to incorporate the impact of ambiguity on the profit received by the firm after the investment has taken place. The partial derivative of trigger π^* is

$$\frac{\partial \pi^*}{\partial \kappa} = \underbrace{\frac{\partial \pi^*}{\partial \beta}\frac{\partial \beta}{\partial \kappa}}_{\text{option value effect(-)}} + \underbrace{\frac{\partial \pi^*}{\partial \delta}\frac{\partial \delta}{\partial \kappa}}_{\text{present-value effect (+)}}.$$

The above decomposition reveals two effects that govern the comparative statics results. The first term is the option value effect. It describes the impact of the change in the option value due to the change in the degree of Knightian uncertainty. Similarly as for the trigger W^*, the increase in ambiguity lowers the growth rate of the project. This raises β and decreases the factor $\frac{\beta}{\beta-1}$ and thus lowers the investment threshold. Consequently, this effect accelerates the investment. The second effect is the present-value effect. This measures the change in the trigger caused by the change in the (subjective) discount rate. An increase in the degree of ambiguity lowers the trend $\mu - \kappa\sigma$ and thus reduces the net present value of the project. This makes investment less profitable and consequently raises the trigger. Thus the present-value effect is contradictory to the option value effect. Note that in case of no Knightian uncertainty the present-value effect is absent. Combining both effects yields that the relationship investment trigger-Knightian uncertainty is equivocal for π^*. It is worth stressing that the origin of the non-monotonicity result in the investment trigger here is different than in Miao and

Wang (2006). Their findings are related to the different levels of ambiguity regarding continuation and termination value whereas in our model the level of ambiguity remains unchanged after the investment has been undertaken.

Figure 1 illustrates the difference in the impact of the Knightian uncertainty on the triggers W^* and π^*.

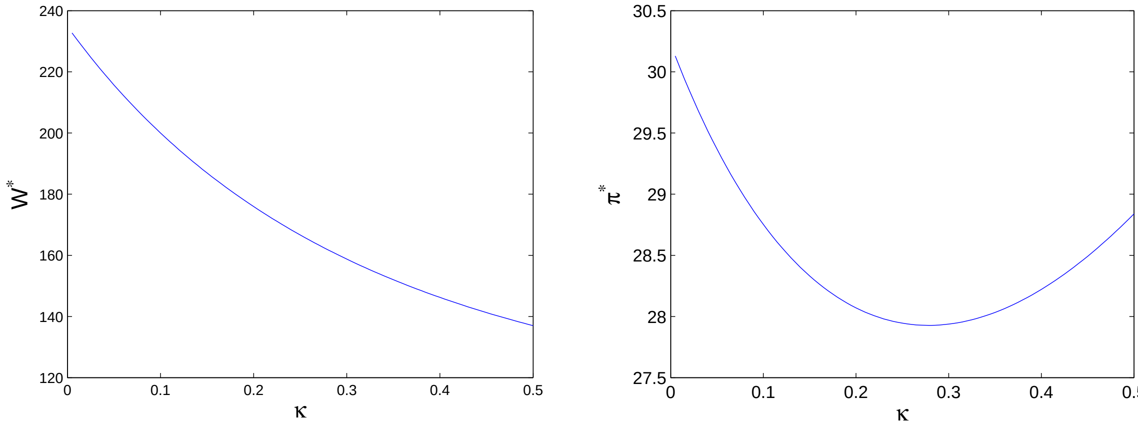

Figure 1: Impact of ambiguity aversion on triggers W^* and π^*. The parameters are $(r, \mu, \sigma, \tau, I) = (0.2, 0.1, 0.2, 15, 100)$.

Let us now turn to the effect of the finite lifetime of the project. It is easy to show that the trigger π^* decreases at an exponential rate with τ. The larger τ, the larger the time that the project generates profits, so that the profit flow can be smaller for investing to be optimal. Figure 2 depicts the convergence of the optimal threshold to the case of the perpetual one and also illustrates the decay in the magnitude of the option value effect relative to the present value effect when the lifetime of the project increases. For the limiting case corresponding to the perpetual project, the comparative statics results for triggers W^* and π^* exhibits monotonicity but in opposite directions: the former is decreasing while the latter is increasing in κ (cf, Nishimura and Ozaki (2003)). This extreme case reflects domination of the net present value effect over the option value effect (see Appendix C).

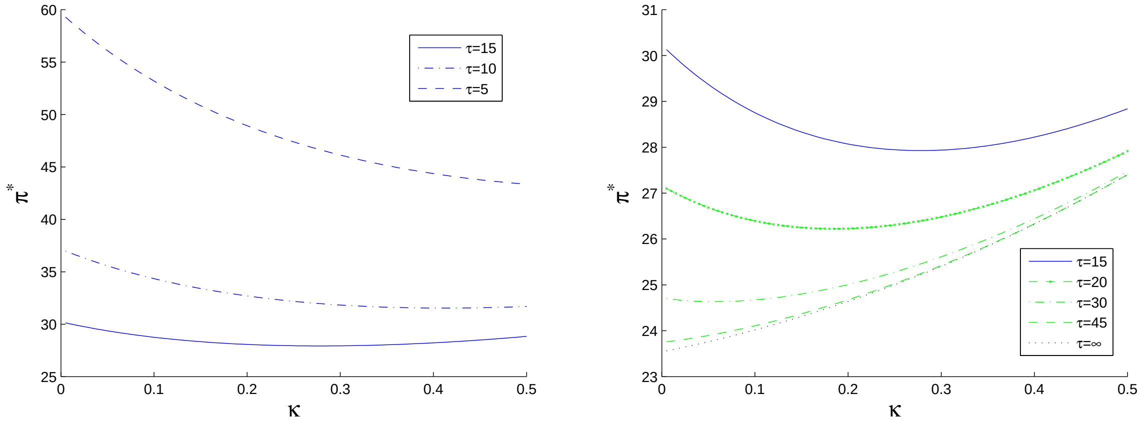

Figure 2: The finite life-time effect in the optimal investment trigger. The parameters are $(r, \mu, \sigma, I) = (0.2, 0.1, 0.2, 100)$.

Concluding, we can summarize the comparative statics results for trigger π^* in the following proposition.

Proposition 3.1 *If the project lifetime is finite, the impact of ambiguity on the investment trigger π^* is equivocal.*

Simulations show that short life-time of projects and low degree of Knightian uncertainty is investment enhancing, while the opposite is true in case of high levels of ambiguity and larger life-times.

3.3 Probability of investment

In the previous section we presented comparative statics results. We showed that investment triggers W^* and π^* exhibit different comparative statics results despite that processes W_t and π_t are governed by the same dynamics (see Figure 1). To investigate this issue further, we determine the impact of ambiguity aversion on the length of the period before the investment takes place. This allows us to establish how powerful the non-monotonic effect in π^* is.

We examine the probability that investment takes place over a H-period horizon. Since the future evolution of π (and W) is unknown, the time of the investment is a random variable. Following Harrison (1985), the cumulative distribution of the first passage time can be written as:

$$\mathcal{Q}^{\kappa}\Big(\sup_{0\leq t\leq H} W_t \geq W^*\Big) = \Phi(d_1) + \Phi(d_2)\Big(\frac{W^*}{W_0}\Big)^{\frac{2\hat{\mu}}{\sigma^2}},$$

where $d_1 = \frac{-\ln(\frac{W^*}{W_0})+\hat{\mu}H}{\sigma\sqrt{H}}$, $d_2 = \frac{-\ln(\frac{W^*}{W_0})-\hat{\mu}H}{\sigma\sqrt{H}}$, $\hat{\mu} = \mu - \kappa\sigma - \frac{\sigma^2}{2}$, and $\Phi(\cdot)$ is the standard normal cumulative distribution function. Before we turn to the analysis of the results let us recall that a change in the degree of ambiguity aversion affects the initial condition for the process of the present values of project, W_0 (see Proposition 2.2), whereas the starting value of the profit flow, π_0, is not sensitive to changes in κ. While marking both triggers down we obtain

$$\frac{W^*}{W_0} = \frac{\pi^*}{\pi_0}. \tag{3}$$

Due to this remark and because both W_t and π_t are geometric Brownian motions with identical parameters, the probability of investment is the same for the processes under question. Therefore, without loss of generality in the remainder of this section we only consider profit flows.

There are two important observations related to the probability of investment. Figure 3 shows the probability of investment for different time horizons and for different levels of ambiguity aversion. The panel for low values of H illustrates the case where the reduced value of waiting makes that in the short-run Knightian uncertainty may be investment enhancing. On the other hand, for relatively high values of H the opposite effect holds: the ambiguous firm is less eager to invest than the firm facing no ambiguity. In fact such a behavior of the probability of investment reflects two opposite effects acting on this measure. First, the investment trigger is decreasing in low values of Knightian uncertainty so that investment occurs earlier. Second, a large degree of Knightian uncertainty lowers the subjective drift of the project. This implies that approaching the zero-absorbing barrier for the process is more likely. Combining these effects imply that, even if the trigger π^* is non-monotonic in the degree of the Knightian uncertainty, the probability of investment appear to be monotonically decreasing in the level of Knightian uncertainty for a sufficiently large time horizon.

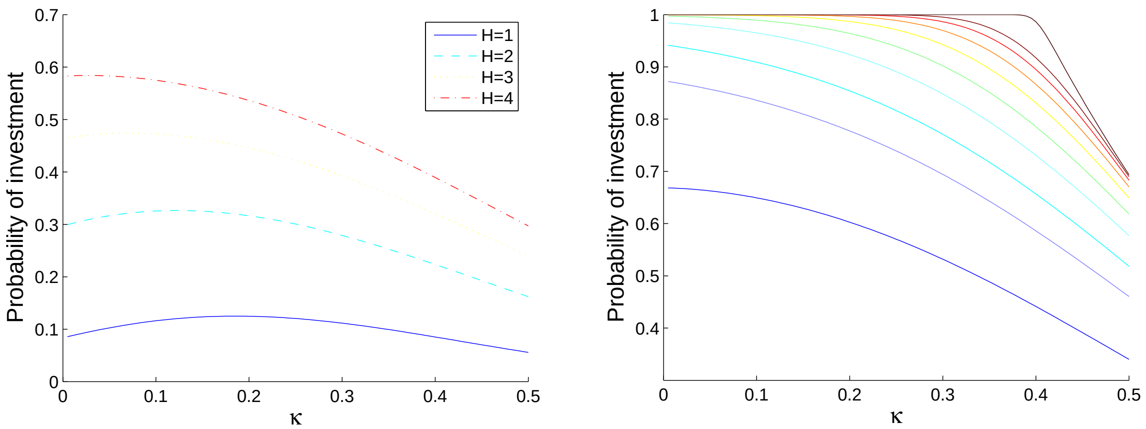

Figure 3: Impact of ambiguity aversion on probability of investment. The left panel for low H, the right panel for $H = (5, 10, 15, 25, 40, 65, 105, 170, 275, 1000)$. The parameters are $(r, \mu, \sigma, \tau, I, \pi_0) = (0.2, 0.1, 0.2, 15, 100, 20)$.

In order to complete the discussion, we focus on the limiting case as H tends to infinity. Following Harrison (1985, p. 43), we can write:

$$\mathcal{Q}^{\kappa}\Big(\sup_{0\leq t\leq\infty} \pi_t \geq \pi^*\Big) = \begin{cases} 1 & \text{if} \quad \frac{2\hat{\mu}}{\sigma^2} \geq 0 \Leftrightarrow \kappa \leq \frac{\mu - \frac{\sigma^2}{2}}{\sigma} \\ \left(\frac{\pi^*}{\pi_0}\right)^{\frac{2\hat{\mu}}{\sigma^2}} & \text{if} \quad \frac{2\hat{\mu}}{\sigma^2} < 0 \Leftrightarrow \kappa > \frac{\mu - \frac{\sigma^2}{2}}{\sigma}. \end{cases}$$

This confirms the intuitive result that high uncertainty regarding the growth rate of future prospect lowers the likelihood of the investment, as summarized in the following proposition.

Proposition 3.2 *The firm with low degree of ambiguity aversion ($\kappa \leq \frac{\mu - \frac{\sigma^2}{2}}{\sigma}$) invests with probability 1, whereas the firm with large ambiguity aversion ($\kappa > \frac{\mu - \frac{\sigma^2}{2}}{\sigma}$) refrains from investment with positive probability if the decision horizon is infinite.*

3.4 Ironing the investment trigger

So far, we investigated the problem where the life-time of the project is given exogenously and does not reflect any characteristics of market uncertainty. The project lasts exactly τ years and then suddenly stops functioning. However, while a firm forms beliefs about the dynamics of profits at the same time a firm may calculate the estimated life-time of the project reflecting the firm's attitude towards uncertainty. In this section we provide comparative statics results helpful in answering the simplified version of this problem.

Namely, we derive the life-time $\tau(\kappa)$ of the project that guarantees that profits at the moment of investment under the worst case scenario are the same as for the case without ambiguity. The trigger corresponding to this subjective project's life-time is denoted by $\pi^*(\kappa, \tau(\kappa))$. The necessary condition to be met is stated in the proposition below.

Proposition 3.3 *If* $\pi^*(0, \tau(0)) > \pi^*(\kappa, \infty)$ *and* $\tau(\kappa) = \frac{1}{\delta(\kappa)} \ln \left(1 - \frac{\pi^*(\kappa,\infty)}{\pi^*(0,\tau(0))}\right)^{-1}$ *then* $\pi^*(\kappa, \tau(\kappa)) \equiv \pi^*(0, \tau(0))$*, where* $\delta(\kappa) = r - \mu + \kappa\sigma > 0$*, and* $\pi^*(\kappa, \infty) := \lim_{\tau\to\infty} \pi^*(\kappa, \tau)$*.*

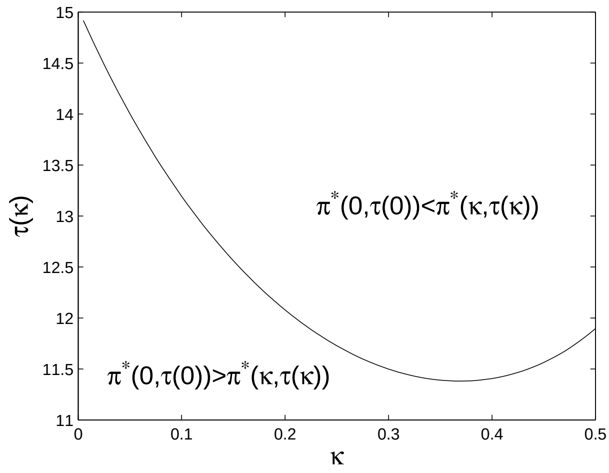

Figure 4: The life-time of the project, $\tau(\kappa)$.

Figure 4 depicts this criterion. If the firm's pessimistic estimate of the project's lifetime is below $\tau(\kappa)$, then the profit earned at the moment of investment exceeds profit under no ambiguity. The opposite holds for values of project's subjective life-times above $\tau(\kappa)$. This region expands with increase of the project's life-time under no Knightian uncertainty.

Eventually, the impact of ambiguity on investment policy can be analyzed by examining the change in the probability of investment due to the change in the degree of Knightian uncertainty and the adjusted to it (subjective) life-time of the project. Figure 5 depicts the case where the subjective life-time of the project equals $\tau(\kappa)$. For the set of parameters under question $(r, \mu, \sigma, \tau(0), I, \pi_0) = (0.2, 0.1, 0.2, 15, 100, 20)$ the likelihood of investment is eroded by the presence of Knightian uncertainty. However, establishing this result in full generality remains an open problem.

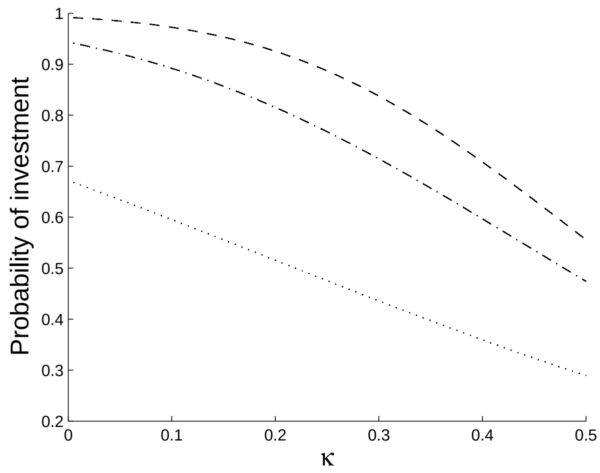

Figure 5: Probability of investment over a H-period horizon for $H = 5, 15, 30$.

4 Conclusions

This paper analyzes the impact of both risk and uncertainty on investment timing. While the standard analysis shows that risk increases the value of waiting in the absence of Knigthian uncertainty, ambiguity erodes the subjective value of the investment opportunity.

Our main focus is on the impact of ambiguity on the investment timing. We identify two factors that critically influence the results: the finite life-time of the project and whether option exercise solves ambiguity or not. We found that if the investor remains ambiguous about the profit flows after the investment is undertaken then under a short life-time of the project ambiguity may be investment enhancing. In general, we obtain that a combination of ambiguity and finite lifetime of the project affect the value of waiting equivocally.

To investigate the magnitude of the latter effect we examine the probability of investment over a finite horizon. We found that in the short-run ambiguity aversion may be investment enhancing. However, it was also found that the non-monotonic effect in the value of waiting is eroded when the time horizon is increased. In the long-run a high degree of ambiguity aversion results in refraining from investment with a substantial probability.

Appendix A: Finite τ

This appendix contains calculations from section 2.1 skipped in the text.

The profit stream solving equation (1) is

$$\pi_s = \pi_t \exp\Big(\int_t^s (\mu - \frac{1}{2}\sigma^2)du + \int_t^s \sigma dB_u\Big), \tag{4}$$

where π_0 is given and the indices are ordered as $0 \le t \le s$.

By Fubini's theorem and substitution of π_t from (4) we get

$$\begin{aligned}
W(t,\pi_t) =& \mathbb{E}^{\mathcal{Q}}\Big[\int_t^{t+\tau} e^{-r(s-t)}\pi_s ds\Big|\mathcal{F}_t\Big] = \int_t^{t+\tau} \mathbb{E}^{\mathcal{Q}}\big[e^{-r(s-t)}\pi_s\Big|\mathcal{F}_t\big]ds = \\
=& \int_t^{t+\tau} \pi_t e^{-r(s-t)} \exp\Big(\int_t^s (\mu - \frac{1}{2}\sigma^2)du\Big)\mathbb{E}^{\mathcal{Q}}\Big[\exp\big(\int_t^s \sigma dB_u\big)\Big|\mathcal{F}_t\Big]ds = \\
=& \pi_t \int_t^{t+\tau} \exp\big((\mu - r - \frac{\sigma^2}{2})(s-t)\big)\mathbb{E}^{\mathcal{Q}}\Big[\exp\big(\sigma(B_s - B_t)\big)\Big|\mathcal{F}_t\Big]ds = \\
=& \pi_t \int_t^{t+\tau} \exp\big((\mu - r - \frac{\sigma^2}{2})(s-t)\big)\exp\big(\frac{\sigma^2(s-t)}{2}\big)ds = \\
=& \pi_t \int_t^{t+\tau} \exp\big((\mu - r)(s-t)\big)ds = \frac{\pi_t(1 - e^{-(r-\mu)\tau})}{r-\mu}.
\end{aligned}$$

Thus $W(\pi_t, t)$ is independent of time and follows the process denoted by W_t:

$$W(\pi_t, t) = \mathbb{E}^{\mathcal{Q}}\Big[\int_t^{t+\tau} e^{-r(s-t)}\pi_s ds\Big|\mathcal{F}_t\Big] = \frac{\pi_t(1 - e^{-(r-\mu)\tau})}{r-\mu} := W_t. \tag{5}$$

The value function is derived by splitting the decision between the immediate investment and waiting for a short time interval $t + dt$ as follows:

$$\begin{aligned}
F_t =& \max_{t' \ge t} \mathbb{E}^{\mathcal{Q}}\Big[\int_{t'}^{t'+\tau} e^{-r(s-t)}\pi_s ds - e^{-r(t'-t)}I\Big|\mathcal{F}_t\Big] = \\
=& \max\Big\{\mathbb{E}^{\mathcal{Q}}\Big[\int_t^{t+\tau} e^{-r(s-t)}\pi_s ds\Big|\mathcal{F}_t\Big] - I, \max_{t' \ge t+dt} \mathbb{E}^{\mathcal{Q}}\Big[\int_{t'}^{t'+\tau} e^{-r(s-t)}\pi_s ds - e^{-r(t'-t)}I\Big|\mathcal{F}_t\Big]\Big\}.
\end{aligned}$$

Plugging W_t from (5) and applying the tower property, yield

$$\begin{aligned}
F_t =& \max\Big\{W_t - I, e^{-rdt} \max_{t' \ge t+dt} \mathbb{E}^{\mathcal{Q}}\Big[\mathbb{E}^{\mathcal{Q}}\big[\int_{t'}^{t'+\tau} e^{-r(s-t-dt)}\pi_s ds - e^{-r(t'-t-dt)}I|\mathcal{F}_{t+dt}\big]\Big|\mathcal{F}_t\Big]\Big\} = \\
=& \max\Big\{W_t - I, e^{-rdt}\mathbb{E}^{\mathcal{Q}}\Big[dF_{t+dt}\Big|\mathcal{F}_t\Big]\Big\}.
\end{aligned}$$

By the first order Taylor approximation of e^{-rdt} we obtain

$$F(W_t) = \max\big\{W_t - I, \mathbb{E}^{\mathcal{Q}}[dF_t|\mathcal{F}_t] + F(W_t) - rF(W_t)dt\big\}.$$

Appendix B: Knightian uncertainty in continuous time, finite τ and ambiguity aversion

4.1 Knightian uncertainty in continuous time

This subsection outlines the concept of Knightian uncertainty in continuous time as proposed by Chen and Epstein (2002). Some necessary definitions used in further subsections to model multiple priors are also provided here.[2]

Let $\mathcal{L}$ be the set of real-valued, measurable and $(\mathcal{F}_t)$-adapted stochastic processes on $(\Omega, \mathcal{F}_T, \mathcal{Q})$ with index set $[0, T]$. Let $\mathcal{L}^2$ be a subset of $\mathcal{L}$:

$$\mathcal{L}^2 = \left\{ (\theta_t)_{0 \le t \le T} \in \mathcal{L} \middle| \int_0^T \theta_t^2 dt < \infty \quad \mathcal{Q} - a.s. \right\}.$$

Given $\theta = (\theta_t) \in \mathcal{L}^2$, define a stochastic process $(z_t^\theta)_{0 \le t \le T}$ by

$$dz_t^\theta = -z_t^\theta \theta_t dB_t \quad \text{with} \quad z_0^\theta = 1.$$

A stochastic process $(\theta_t) \in \mathcal{L}^2$ is called a density generator if (z_t^θ) is a $(\mathcal{F}_t)$ martingale. The most commonly used sufficient condition for (z_t^θ) to be a $(\mathcal{F}_t)$ martingale is Novikov's condition:

$$\mathbb{E}^{\mathcal{Q}}\left[\exp\left(\frac{1}{2} \int_0^T \theta_s^2 ds \right) \right] < \infty$$

Let θ be a density generator. Define $\mathcal{Q}^\theta$ by

$$\forall_{A \in \mathcal{F}_T} \quad \mathcal{Q}^\theta(A) = \mathbb{E}^{\mathcal{Q}}[z_T^\theta \mathbf{1}_A]. \tag{6}$$

$\mathcal{Q}^\theta$ is a probability measure equivalent to $\mathcal{Q}$. Any equivalent to $\mathcal{Q}$ probability measure can be obtained by a density generator as described above.

Let Θ be a set of density generators. The set of probability measures $\mathcal{P}^\Theta$ on $(\Omega, \mathcal{F}_T)$ generated by Θ is denoted by

$$\mathcal{P}^\Theta = \{\mathcal{Q}^\theta | \theta \in \Theta\},$$

where $\mathcal{Q}^\theta$ is derived from $\mathcal{Q}$ according to (6). This completes the construction of multiple priors. Thus, firm's beliefs are operationally captured by the set of probability measures equivalent to $\mathcal{Q}$ in the multiple-priors set-up proposed by Chen and Epstein (2002).

[2]For a more mathematically elaborated discussion one can refer to Chen and Epstein (2002), Nishimura and Ozaki (2003) and Asano (2005).

Now, by Girsanov's Theorem, $B_t^\theta = B_t + \int_0^t \theta_s ds$ is Brownian motion relative to $\mathcal{Q}^\theta$, and the set of SDEs capturing uncertain profit turns out to be

$$\begin{aligned} d\pi_t =& \mu\pi_t dt + \sigma\pi_t dB_t \\ =& \mu\pi_t dt + \sigma\pi_t(dB_t^\theta - \theta_t dt) \\ =& (\mu - \sigma\theta_t)\pi_t dt + \sigma\pi_t dB_t^\theta \end{aligned} \tag{7}$$

for any $\theta \in \Theta$. Consequently, the multiplicity of measures in $\mathcal{P}$ can be interpreted as *modelling ambiguity about the drift of the driving process.*

Definition 4.1 *A set of density generators is called* strongly rectangular *if there exists a nonempty compact subset $\mathcal{K}$ of $\mathbb{R}$ and a compact-valued, convex-valued, measurable correspondence $K : [0, T] \twoheadrightarrow \mathcal{K}$ such that $0 \in K_t$ and*

$$\Theta^{K_t} = \{(\theta_t) \in \mathcal{L}^2 | \theta_t(\omega) \in K_t \quad m \otimes \mathcal{Q} - a.s.\},$$

where m is the Lebesgue measure restricted on $\mathcal{B}([0, T])$.

In the main part of our paper we focus on the case of strongly rectangular sets in which K_t is independent of time t.

Definition 4.2 *The uncertainty characterized by Θ^K is* IID uncertainty *(see, Epstein and Schneider (2003)) if there exists a compact subset K of $\mathbb{R}$ such that $0 \in K_t$ and*

$$\Theta^K = \{(\theta_t) \in \mathcal{L}^2 | \theta_t(\omega) \in K \quad m \otimes \mathcal{Q} - a.s.\}.$$

Definition 4.3 *The special case of the IID uncertainty Θ^K, where the set K is the symmetric interval $K = [-\kappa, \kappa]$, is called κ* -ignorance.

The real number $\kappa > 0$ represents the degree of Knightian uncertainty.

Lemma 4.1 *For any $s \geq t$ and for any $\theta \in \Theta^{K_t}$ it holds that*

$$\mathbb{E}^{\mathcal{Q}^\theta}\left[\exp\left(-\int_t^s \sigma\theta_u du + \sigma(B_s^\theta - B_t^\theta)\right)\middle|\mathcal{F}_t\right] \geq \mathbb{E}^{\mathcal{Q}^{\theta^*}}\left[\exp\left(-\int_t^s \sigma\theta_u^* du + \sigma(B_s^{\theta^*} - B_t^{\theta^*})\right)\middle|\mathcal{F}_t\right]$$

where $\theta_t^* \equiv \arg\max\{\sigma x | x \in K_t\} = \max K_t$ is degenerated measurable process.

For the proof of this lemma the reader is referred to Nishimura and Ozaki (2003), Appendix A. The degeneracy follows from the theory of support functions (cf. Chen and Epstain (2002) or Asano (2005, pp. 11-12)).

Regarding $\mathcal{Q}^\theta$ as the true measure the solution to profit equation (7) admits the following recursive form

$$\pi_s = \pi_t \exp\Big(\int_t^s (\mu - \frac{1}{2}\sigma^2) du - \sigma \int_t^s \theta_u du + \int_t^s \sigma dB_u^\theta \Big). \tag{8}$$

for $0 \le t \le s$ and where $\pi_0 > 0$ is given.

Derivation of $W(\pi_t, t)$

$$\begin{aligned}
W(\pi_t,t) =& \inf_{\theta\in\Theta} \mathbb{E}^{\mathcal{Q}^\theta}\Big[\int_t^{t+\tau} e^{-r(s-t)}\pi_s ds \Big| \mathcal{F}_t \Big] \\
=& \inf_{\theta\in\Theta} \int_t^{t+\tau} \mathbb{E}^{\mathcal{Q}^\theta}\big[e^{-r(s-t)}\pi_s \big| \mathcal{F}_t\big] ds = \\
=& \inf_{\theta\in\Theta} \int_t^{t+\tau} \pi_t e^{-r(s-t)} \exp\Big(\int_t^s (\mu - \frac{1}{2}\sigma^2) du\Big) \mathbb{E}^{\mathcal{Q}^\theta}\Big[\exp\big(\int_t^s \sigma dB_u^\theta - \int_t^s \sigma\theta_u du\big) \Big| \mathcal{F}_t\Big] ds = \\
=& \pi_t \int_t^{t+\tau} \exp\big((\mu - r - \frac{\sigma^2}{2})(s-t)\big) \mathbb{E}^{\mathcal{Q}^{\theta^*}}\Big[\exp\big(\sigma(B_s^{\theta^*} - B_t^{\theta^*}) - \int_t^s \sigma\theta_u^* du\big) \Big| \mathcal{F}_t \Big] ds = \\
=& \pi_t \int_t^{t+\tau} \exp\big((\mu - r - \frac{\sigma^2}{2})(s-t) - \int_t^s \sigma\theta_u^* du\big) \exp\big(\frac{\sigma^2(s-t)}{2}\big) ds = \\
=& \pi_t \int_t^{t+\tau} \exp\big((\mu - r)(s-t) - \int_t^s \sigma\theta_u^* du\big) ds
\end{aligned}$$

The sequence of equalities is validated by Fubini's theorem, equation (8), lemma 4.1, degeneracy of θ^* and easy algebra, respectively.

Lemma 4.2 *Let $0 \le t_1 \le t_2 \le T$ and let X be an $\mathcal{F}_T$-measurable random variable. If Θ is a strongly rectangular then under the assumption that the minima exist,*

$$\min_{\theta\in\Theta} \mathbb{E}^{\mathcal{Q}^\theta}\Big[\mathbb{E}^{\mathcal{Q}^\theta}[X|\mathcal{F}_{t_2}] \Big| \mathcal{F}_{t_1}\Big] = \min_{\theta\in\Theta} \mathbb{E}^{\mathcal{Q}^\theta}\Big[\min_{\theta'\in\Theta} \mathbb{E}^{\mathcal{Q}^{\theta'}}[X|\mathcal{F}_{t_2}] \Big| \mathcal{F}_{t_1}\Big] \tag{9}$$

Proof: see Nishimura and Ozaki (2003), Appendix B.

$$\begin{aligned}
F_t =& \max_{t'\ge t} \min_{\mathcal{Q}\in\mathcal{P}^\Theta} \mathbb{E}^{\mathcal{Q}}\Big[\int_{t'}^{t'+\tau} e^{-r(s-t)}\pi_s ds - e^{-r(t'-t)} I \Big| \mathcal{F}_t\Big] = \\
=& \max\Big\{ \min_{\mathcal{Q}\in\mathcal{P}^\Theta} \mathbb{E}^{\mathcal{Q}}\Big[\int_t^{t+\tau} e^{-r(s-t)}\pi_s ds \Big| \mathcal{F}_t\Big] - I, \max_{t'\ge t+dt} \min_{\mathcal{Q}\in\mathcal{P}^\Theta} \mathbb{E}^{\mathcal{Q}}\Big[\int_{t'}^{t'+\tau} e^{-r(s-t)}\pi_s ds - e^{-r(t'-t)} I \Big| \mathcal{F}_t\Big]\Big\}
\end{aligned}$$

Plugging W_t and applying the tower property, yield

$$\begin{aligned}
F_t =& \max\Big\{ W_t - I, e^{-rdt} \max_{t'\ge t+dt} \min_{\mathcal{Q}\in\mathcal{P}^\Theta} \mathbb{E}^{\mathcal{Q}}\Big[\int_{t'}^{t'+\tau} e^{-r(s-t-dt)}\pi_s ds - e^{-r(t'-t-dt)} I \Big| \mathcal{F}_t\Big]\Big\} = \\
=& \max\Big\{ W_t - I, e^{-rdt} \max_{t'\ge t+dt} \min_{\theta\in\Theta} \mathbb{E}^{\mathcal{Q}^\theta}\Big[\mathbb{E}^{\mathcal{Q}^\theta}\big[\int_{t'}^{t'+\tau} e^{-r(s-t-dt)}\pi_s ds - e^{-r(t'-t-dt)} I | \mathcal{F}_{t+dt}\big] \Big| \mathcal{F}_t\Big]\Big\} =
\end{aligned}$$

By lemma 4.2 we get

$$\max\left\{W_t - I, e^{-rdt}\max_{t'\geq t+dt}\min_{\theta\in\Theta}\mathbb{E}^{\mathcal{Q}^\theta}\left[\min_{\theta'\in\Theta}\mathbb{E}^{\mathcal{Q}^{\theta'}}\left[\int_{t'}^{t'+\tau} e^{-r(s-t-dt)}\pi_s ds - e^{-r(t'-t-dt)}I|\mathcal{F}_{t+dt}\right]\middle|\mathcal{F}_t\right]\right\} =$$

$$= \max\left\{W_t - I, e^{-rdt}\min_{\theta\in\Theta}\mathbb{E}^{\mathcal{Q}^\theta}\left[\max_{t'\geq t+dt}\min_{\theta'\in\Theta}\mathbb{E}^{\mathcal{Q}^{\theta'}}\left[\int_{t'}^{t'+\tau} e^{-r(s-t-dt)}\pi_s ds - e^{-r(t'-t-dt)}I|\mathcal{F}_{t+dt}\right]\middle|\mathcal{F}_t\right]\right\} =$$

$$= \max\left\{W_t - I, e^{-rdt}\min_{\theta\in\Theta}\mathbb{E}^{\mathcal{Q}}\left[dF_{t+dt}\middle|\mathcal{F}_t\right]\right\}$$

By the first order Taylor approximation of e^{-rdt} we obtain

$$F(W_t) = \max\left\{W_t - I, \min_{\theta\in\Theta}\mathbb{E}^{\mathcal{Q}^\theta}[dF_t|\mathcal{F}_t] + F(W_t) - rF(W_t)dt\right\}.$$

Appendix C: Comparative statics results

The impact of ambiguity aversion on π^*:

$$\frac{\partial\pi^*}{\partial\kappa} = \pi^*\Bigg(\underbrace{-\frac{1}{\beta(\beta-1)}\frac{\partial\beta}{\partial\kappa} + \frac{1}{\delta}\frac{\partial\delta}{\partial\kappa}}_{\text{perepetual project effect (+)}}\underbrace{-\frac{\tau\exp(-\delta\tau)}{1-\exp(-\delta\tau)}\frac{\partial\delta}{\partial\kappa}}_{\text{finite life-time effect (-)}}\Bigg).$$

Notice that $\left(\beta - \frac{1}{2}\right)\sigma^2 + \mu - \kappa\sigma = \sqrt{\left((\mu-\kappa\sigma) - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2}$ and thus

$$\frac{\partial\beta}{\partial\kappa} = \frac{1}{\sigma}\left(1 - \frac{\mu - \kappa\sigma - \frac{1}{2}\sigma^2}{\sqrt{\left((\mu-\kappa\sigma) - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2}}\right) = \frac{\sigma\beta}{\sigma^2\left(\beta-\frac{1}{2}\right) + \mu - \kappa\sigma}.$$

$$-\frac{1}{\beta(\beta-1)}\frac{\partial\beta}{\partial\kappa} + \frac{1}{\delta}\frac{\partial\delta}{\partial\kappa} = -\frac{1}{\beta-1}\frac{\sigma}{\left(\beta-\frac{1}{2}\right)\sigma^2 + \mu - \kappa\sigma} + \frac{\sigma}{r - \mu + \kappa\sigma} =$$

$$= \frac{\sigma}{\beta-1}\frac{-r + (\beta-1)\sigma^2(\beta-\frac{1}{2}) + \beta(\mu-\kappa\sigma)}{\left(\left(\beta-\frac{1}{2}\right)\sigma^2 + \mu - \kappa\sigma\right)\left(r - \mu + \kappa\sigma\right)} =$$

$$= \frac{\sigma}{\beta-1}\frac{\frac{\sigma^2}{2}(\beta-1)^2}{\left(\left(\beta-\frac{1}{2}\right)\sigma^2 + \mu - \kappa\sigma\right)\left(r - \mu + \kappa\sigma\right)} > 0$$

The last equality follows after substitution of the fundamental quadratic.

The sign of derivative for π^* depends on τ:

$$\frac{\partial\pi^*}{\partial\kappa} \gtreqless 0$$

$$\Leftrightarrow \quad \frac{\frac{\sigma^2}{2}(\beta-1)}{\left(\beta-\frac{1}{2}\right)\sigma^2 + \mu - \kappa\sigma} \gtreqless \frac{\tau\delta}{\exp(\tau\delta) - 1}.$$

The RHS can be made large for τ small, whereas the LHS is independent on τ.

Appendix D: Ironing the investment trigger

Proof:

$$\frac{\partial \pi^*(\kappa,\tau(\kappa))}{\partial \kappa} = \pi^*\Bigg(\frac{\partial \ln(\frac{\beta(\kappa)}{\beta(\kappa)-1} I\delta(\kappa))}{\partial \kappa} - \frac{\partial \ln(1-\exp(-\delta(\kappa)\tau(\kappa)))}{\partial \kappa}\Bigg).$$

Equating the derivative above to 0 and solving the ODE obtained give

$$\ln(\frac{\beta(\kappa)}{\beta(\kappa)-1} I\delta(\kappa)) + c_0 = \ln(1-\exp(-\delta(\kappa)\tau(\kappa))),$$

where c_0 is an integration constant. Thus, $C_0\frac{\beta(\kappa)}{\beta(\kappa)-1}I\delta(\kappa) = 1-\exp(-\delta(\kappa)\tau(\kappa))$ with $C_0 := \exp(c_0)$. Rearranging and employing the initial condition reflecting the case without ambiguity, yield $C_0 = \frac{1}{\pi^*(0,\tau(0))}$ and $\tau(\kappa) = \frac{1}{\delta(\kappa)}\ln\left(1-\frac{1}{\pi^*(0,\tau(0))}\frac{\beta(\kappa)}{\beta(\kappa)-1}I\delta(\kappa)\right)^{-1}$. Now, the formula in the proposition now follows by easy algebra: $\tau(\kappa) = \frac{1}{\delta(\kappa)}\ln\left(1-\frac{\pi^*(\kappa,\infty)}{\pi^*(0,\tau(0))}\right)^{-1}$. The solution is meaningful if $\tau(\kappa) > 0$, which corresponds to $\pi^*(0,\tau(0)) > \pi^*(\kappa,\infty)$.

Proof: We examine the monotonicity (in κ) of the probability of investment over a H-period horizon, given that the project's life-time is $\tau(\kappa)$. By definition $\pi^*(\kappa,\tau(\kappa)) = \pi^*(0,\tau(0))$ and thus

$$\begin{aligned}\frac{\partial \mathcal{Q}^\kappa(\sup_{0\le t\le H}\pi_t \ge \pi^*(\kappa,\tau(\kappa)))}{\partial \kappa} =& \frac{\partial\Phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial \kappa} + \frac{\partial\Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \kappa}\left(\frac{\pi^*(0,\tau(0))}{\pi_0}\right)^{\frac{2\hat{\mu}}{\sigma^2}} \\ &+\Phi(d_2)\left(\frac{\pi^*(0,\tau(0))}{\pi_0}\right)^{\frac{2\hat{\mu}}{\sigma^2}}\ln\left(\frac{\pi^*(0,\tau(0))}{\pi_0}\right)\frac{\partial(\frac{2\hat{\mu}}{\sigma^2})}{\partial\kappa}\end{aligned}$$

It is easy to see that the second term is positive, whereas the remaining ones are negative. The total sign of derivative is difficult to verify analytically.

References

[1] Asano, T., 2005. "Irreversibilities and the optimal timing of enviromnental policy under Knightian uncertainty", Discussion paper, Osaka University.

[2] Chen, Z. and L. Epstein, 2002. "Ambiguity, risk and asset returns in continuous time", Econometrica, 70, 1403-1443.

[3] Dixit, A.K., and R.S. Pindyck, 1994. *"Investment under Uncertainty"*, Princeton University Press.

[4] Dobbs, I.M., 2004. "Replacement investment: optimal economic life under uncertainty", Journal of Bussiness Finance and Accounting, 31, 729-?.

[5] Epstein, L. and M. Schneider, 2003. "IID: Independently and indistinguishable distributed", Journal of Economic Theory, 113, 32-50.

[6] Gryglewicz, S., K.J.M. Huisman, and P.M. Kort, 2005. "A non-monotonic uncertainty effect in the value of waiting to invest", working paper, Tilburg University.

[7] Harrison, M.J., 1985. *"Brownian Motion and Stochastic Flow Systems"*, Robert E. Krieger Publishing Company.

[8] Knight, F.R., 1921. *"Risk, Uncertainty and Profit"*, Houghton Mifflin, Boston.

[9] Knight, F.R., 2002. *"Risk, Uncertainty and Profit"*, Beard Books, Washington.

[10] Lund, D., 2005. "How to analyze the investment-uncertainty relationship in real options models?", Review of Financial Economics, 14, 311-322.

[11] McDonald, R. and D. Siegel, 1986. "The value of waiting to invest", Quarterly Journal of Economics, 101, 707-727.

[12] Miao, J. and N. Wang, 2006. "Risk, uncertainty and option exercise", Working paper, Boston University and Columbia University.

[13] Nishimura, K.G and H. Ozaki, 2003. "Irreversible investment and Kinghtian uncertainty", Working paper, University of Tokyo.

[14] Sarkar, S., 2003. "The effect of mean reversion on investment under uncertainty", Journal of Economics Dynamics and Control, 28, 377-396.