Extended Binomial Tree Valuation when the Underlying Asset Distribution is Shifted Lognormal with Higher Moments

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Abstract

This paper describes a real options valuation method for situations where the underlying asset may have negative values and the underlying project present value distribution is something of the shape between normal and lognormal distribution causing skewness to the rate of return distribution. The underlying project value is assumed to follow a dynamic path having up and down movements with properties of both additive and multiplicative processes. This is described as a shifted lognormal process. A binomial tree solution, which is an extension to the common binomial tree models, is presented with an illustrative case example. The underlying assumptions about applying the valuation model follow the lines of consolidated volatility approach and marketed asset disclaimer.

JEL Classification: G31, G13, D81

Keywords: Real options, binomial tree, shifted lognormal asset dynamics, sequential investments, value under uncertainty

Introduction

Real options analysis (ROA) is a fairly new approach for valuing managerial flexibility in investment decisions. These decisions can be related for example to deferring, extending or abandoning the project. According to real options thinking, investments are characterized by sequential and irreversible investments made under conditions of uncertainty (Dixit and Pindyck 1995). The expression real options stems from utilizing and adapting the mathematics commonly used in valuing financial options related to some uncertain real investments. According to Trigeorgis and Mason (1987) and Brealey and Myers (2003), real options analysis can be also illustrated as an improved, economically corrected version of decision-tree analysis that has adopted the market perspective allowing determination of expected values using risk-neutral probabilities and risk-free discounting rate. On the other hand, real options are also often used more as a metaphor for management philosophy of uncertainty and flexibility behind the investments (Lander and Pinches 1998, Thurner 2003). However, there are also deficiencies related to the real options analysis. Mathematical complexity, parameter estimations, lack of managerial skills to do the corrective actions during the project when needed, and the problems derived from the fact that models suiting well for financial options do not necessarily fit well to real investments valuation (Lander and Pinches 1998, Busby and Pitts 1997, Miller and Waller 2003).

This paper develops a method for valuing options that may have negative values in underlying asset outcome distribution with skewness. The here suggested solution for such cases is a shifted lognormal binomial tree. The structure of the paper is as follows. Firstly, most common underlying asset dynamics applied with real options are discussed. Secondly, volatility estimation with consolidated approach is described. Thirdly, a case illustrating the utilization of the model in an uncertain multi-staged investment project is presented, and the results of the valuation method are compared with solutions provided by some other ordinary methods. Finally, the concluding section clarifies and summarizes the contribution of this paper.

Models of underlying dynamics

Basic financial options models assume that the dynamic underlying asset value process is geometric, i.e. multiplicative. Therefore, the underlying asset (stock price or project value) distribution is lognormal. The properties of lognormal distribution are that the value can not fall below zero, but it may increase to infinity without upper limit. The distribution is positively skewed, having most of the values closer to the lower limit, and the natural logarithm of the distribution yields to normal distribution. Therefore, Black and Scholes (1973) and many other option valuation methods make the assumption of lognormal underlying asset value with the normally distributed rate or return distribution.

While some assets can randomly fluctuate according to the Brownian motion, prices of raw materials and commodities tend to fluctuate randomly around certain normal value of the asset, that is, the marginal cost of producing the commodity. Such processes are mean-reverting. The

Ornstein-Uhlenbeck process is the most well-known of these. The disadvantage of the mean-reverting models is that they can not be often solved analytically, although numerical methods are nowadays sufficiently effective for solving the values. However, numerical techniques are otherwise tedious to model. Also parameterizing the models is often sufficiently hard even with historical data available. Another shortcoming of some models is the possibility of negative underlying asset values.

Another alternative for estimating the price changes over time is to assume that the price changes are additive, meaning that the price between times t and t+1 will either rise or fall by a predefined amount with different risk-neutral probabilities. Additive models are simple to use, but they lack realism if used for valuation of financial options. The stock price changes are rather geometric than arithmetic. Secondly, stock prices can never be negative, whereas additive models, if not otherwise restricted, make it possible. However, additive models are still useful for localized analyses, over short periods of time, and they are a good building block for other models. Cox and Ross (1976) suggested the constant elasticity of variance for describing financial assets dynamics for such cases where the volatility decreases as the underlying asset value increases. The rationale for the model is that all firms have fixed costs to be met regardless of the firm’s operating performance, and therefore if operating performance weakens, the volatility increases. Market observed implied volatilities also support this theory and model. Therefore, constant elasticity model price movements may be considered to have qualities of both additive and multiplicative model.

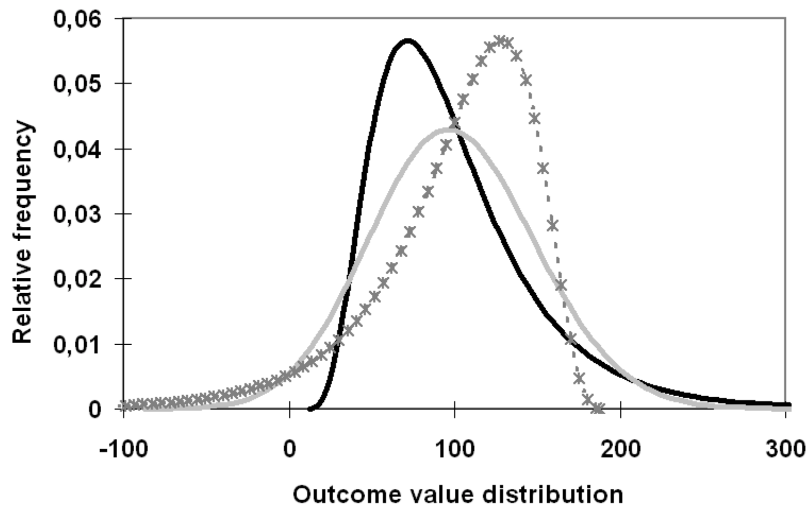
The earlier mentioned multiplicative, mean-reverting, additive, and constant elasticity models assume sufficiently continuous fluctuation and stream of information available. However, in the context of R&D projects, new knowledge may only arrive at discrete points in time, causing sudden up or down jump movements and other times the changes will probably be quite low. Therefore, Lint and Pennings (2000) and Willmer (1995) suggest using pure jump diffusion model to be used in pricing of real options. Schwartz and Moon (2001) developed their model based on Pindyck (1993) for multi-staged investments with properties of both jump process and continuous geometric Brownian motion. The jump process in various real options models is often modeled with Poisson distributions. Yet the models are not mathematically impossible to understand, it may sometimes be challenging for practitioners to find or guess the right parameterization for them. As stated by Paxson (2003), parameterizing many R&D real option models on empirical basis is often quite difficult.

The observed distributions of estimated and simulated NPV calculations do not often resemble lognormal distribution but are rather something between normal and lognormal. As a result, a better approximation for the underlying value distribution would be something between the normal and the lognormal distribution, indicating skewness and kurtosis in rate of return distribution. Some of the real options approach appliers state that it does not matter if the underlying asset distribution or the timeliness behavior of the dynamic price process does not match the often assumed multiplicative assumption, because the financial markets are not too strict about that. Presumably, these statements are based on an assumption of the. possibility to construct a dynamic tracking portfolio. With the option value mimicked exactly by the portfolio, the form of uncertainty doesn't affect the option value (Amram and Kulatilaka 1998). However, applying real options valuation without such tracking and relying on standard deviation as a measure of volatility may misguide decision making. Results of Boyarchenko and Levendorskii (2000) show that the values and investment thresholds may change significantly if Gaussian distribution is replaced with a non-Gaussian distribution having the same variance.

Moment matching is no new phenomena in options pricing. The first approach presented, taking into account the first four moments of the distribution, was presented by Jarrow and Rudd (1982). Their model, based on Edgeworth expansion, is appropriate for the valuation of European options, because such options only depend on the asset price distribution at one time point. Rubinstein (1998) expanded this idea for valuing American options with numerical binomial lattice approximation. For valuing American options, one needs to know all the price paths from time of valuation to the maturity of the option. The basic idea in Rubinstein (1998) valuation procedure is to first approximate the risk-neutral asset price distribution at the maturity, and then to deduct the earlier node values for the binomial tree by applying risk-neutral valuation with normal backward recursion. With large number of time steps, this approximation should provide internally consistent binomial tree to describe the asset price evolution with high accuracy. The parameters needed for valuation are (in addition to the basic CRR (Cox, Ross and Rubinstein 1979) parameters) skewness and kurtosis. In comparison with normal CRR and RB (Rendleman and Bartter 1979) binomial trees, the Edgeworth tree doesn't need to have a constant move size or probability. As with most other binomial models, this approximation also allows for several sequential options to exist at the same valuation model. The Edgeworth binomial tree has been applied to real options by Arnold and Crack (2003).

Another common problematic assumption is that the underlying asset has only positive values, which is always true for stocks. Unfortunately, this doesn't hold for all real investments, because the operating cash flows – also after the initial staged investment outlays - may still be negative. This is not very common to happen, but it is still possible, and then negative underlying values can cause problems in volatility estimation and option valuation. In such cases, additive process can be used, as well as Rhys and Tippett (2003) analytical solution for generalized Student distributions allowing also negative underlying values.

The models that build on the assumption of geometric Brownian motion (gBm) may give erroneous results if applied to such cases were the actual outcome distribution and the process causing it do not follow gBm properties. Negative underlying asset values, skewness, and kurtosis, for example, may change the valuation significantly. Therefore, mean and standard deviation of the underlying asset outcome distribution or the mean and volatility of the rate of return distribution are not always sufficient for describing a sufficiently smooth dynamic process without jumps or mean-reverting behavior. The following Graph 1 with different project outcome distributions illustrates this.



Graph 1: Project outcome distributions with the mean of 100 and standard deviation of 50. Applying methods assuming gBm to other than dynamic process leading into lognormal underlying asset outcome value distribution (the black line in picture) may cause erroneous results

Volatility estimation with consolidated approach

In real options analysis, the standard deviation of the expected price change over a year is used to measure the uncertainty and volatility of the underlying asset. Volatility may be also regarded as the second moment of the value distribution. It is probably the most difficult input parameter to estimate in real options analysis (Mun 2002), which is also the case with financial options. However, there exists historical data for financial markets that can be used for choosing and comparing different alternative stochastic models and their parameterizations to find the appropriate volatility measurement. Also futures market information can be applied. However, in case of other investments, especially if related to R&D, there does not necessarily exist such information (Newton and Pearson 1994). Volatility is not equal to the volatility of the company's equity, because usually the companies depend on multiple interacting and diversified different projects (Mun 2002). Therefore, the volatility of the pool of projects is smaller than the volatility of the single project (Borissiouk and Peli 2002). Nor is volatility often equal to the volatility of single input variable such as volatility of the commodity, even though the commodity volatility affects the valuation often significantly. Depending of the case, Mun (2002) presents as the real options volatility estimation methods logarithmic cash flow returns approach, logarithmic present value approach, garch approach, management assumption approach, and market proxy assumption.

The approach of consolidated volatility, the logarithmic present value approach in terms of Mun, was first introduced by Copeland and Antikarov (2001). The method relies on marketed asset disclaimer and Samuelson's poof (1965) of correctly estimated rate of return of any asset to follow random walk regardless of the pattern of the cash flows. The approach is based on the idea that an investment with real options should be valued as if it was a traded asset in markets even though it would not be publicly listed. According to Copeland and Antikarov, the present value of the cash flows of the project without flexibility is the best unbiased estimate of the market value of the project were it a traded asset. They call this assumption *marketed asset disclaimer*. Therefore, simulation of cash flows should provide a reliable estimate of the investment's volatility. The method is also applied elsewhere, e.g. in Mun (2002, 2003), Herath and Park (2002), and Cobb and Charnes (2004).

The method of consolidated approach is can be justified especially when no historical data is available for volatility estimation. Then a Monte Carlo simulation on project's present value is used to develop a hypothetical distribution of one period returns. On each simulation trial run, the value of the future cash flows is estimated at two time periods, one for the first time period and another for the present time. The cash flows are discounted and summed to the time 0 and 1, and the following logarithmic ratio is calculated:

$$z = \ln\left(\frac{PV_1 + FCF_1}{PV_0}\right) \qquad (1)$$

where PV_1 means present value at time t=1, FCF_1 means free cash flow at time 1, and PV_0 project's present value at the beginning of the project at time t=0. Present value at each moment x can be calculated according to the following equation:

$$PV_x = \sum_{t=x+1}^{n} \frac{FCF_t}{(1+WACC)^{t-1}} \qquad (2)$$

The model simulated is often a conventional present value calculation where uncertainties related to parameters are presented as subjective distributions (Copeland and Antikarov 2001). After the simulation, the mean and the standard deviation of the rate of return distribution are calculated. The consolidated volatility approach is analogous to stock price simulations where the theoretical stock price is the sum of all future dividend cash payments, and with real options, these cash payments are the free cash flows. The sum of free cash flows present valued to time zero is the current stock price (asset value), and at time one, the stock price in the future. The natural logarithm of the ratio of these sums is analogous to the logarithmic returns of stock prices. As stock price at time zero is known while the future stock price is uncertain, only the uncertain future stock price is simulated (Mun 2003).

This procedure works sufficiently well if the underlying asset values are always positive in simulation (Mun 2003, Razgaitis 2003), and if the existing underlying asset value distribution shape is similar to the lognormal distribution shape. Unfortunately, these conditions do not hold very often in practical cases. As a reminder, it should be remembered that according to the common geometric Brownian motion assumption, the *underlying value distribution* as a result of the underlying process, is lognormal by nature. This distribution can only have non-negative values from zero to infinity. Therefore, the rate of return distribution, a natural logarithm of the underlying value distribution, is a normal distribution. If the underlying value distribution has negative values, the rate of return distribution calculation is erroneous, because natural logarithm for negative values is not defined. If the method is incorrectly used with negative underlying values, the consolidated volatility parameter estimation with Copeland and Antikarov (2001) gives values that are higher than the true volatility of the rate of return.

The problem of the negative cash flow can be solved simply by shifting the negative cash flows to positive values, because arithmetic translation of adding values to all cash flows and making them positive will only shift the expected returns, but keep constant the second moment or volatility (Mun 2003). This doesn't however solve the problem of having the wrong distribution shape, and the negative cash flow correction procedure by arithmetic addition worsens the distribution shape problem related to higher moments. Despite of this, the advantages of having better estimated volatility is such that the arithmetic transition is justified.

For taking into account the higher moments in valuation, Rubinstein (1998) presented an Edgeworth expansion for binomial tree, where Edgeworth expansion is used to transform a standard binomial density into a unimodal standardized discrete density, evaluated at equally spaced points, with approximately prespecified skewness and kurtosis. Also Gram-Charlier series using the cumulants of moments can be used in valuation. However, both of these methods are polynomials approximations, and they yield erroneous negative probabilities for many skewness-kurtosis value pairs, if the rate of return distribution is not close enough to the normal distribution having skewness of 0 and kurtosis of 3. In the following, a shifted lognormal binomial valuation method for allowing negative cash flows in the underlying asset outcome distribution considering also skewness and kurtosis in rate of return distribution is presented.

Shifted lognormal binomial tree

One solution to the problem discussed in the previous section is to use a shifted lognormal distribution in underlying asset process modeling if the shape of the underlying distribution is lognormal, or something between normal and lognormal, and even has negative values. Approach of applying the shifted lognormal terminal value distribution for valuing basket options was presented by Brigo et al. (2004) to take into account the skewness of value distribution. However, the shifted lognormal distribution approach with appropriate parameterization can also be applied for valuing options with negative underlying asset values. Many simulation softwares allow the user to examine the qualities of the underlying asset value distribution, and they also provide opportunity to fit the simulated results into certain well-known distributions. For example, in Palisade's BestFit software (included in @Risk 4.54 Industrial edition), underlying value distribution can be fitted into normal, lognormal and shifted lognormal distributions. These distributions are sufficiently easy to use in the modeling of the dynamic underlying price movements.

Shifted lognormal distribution has three parameters: *shifting pseudo mean*, *standard deviation*, and *shifting parameter*. Together with these three parameters, fitted distribution is able to very accurately mimic most of the present value distributions having a shape between normal and a lognormal distribution. Proper parameterization ensures that mean, variance, and also the shape of the distribution curve are correct, whereas commonly used lognormal distribution only handles the mean and standard deviation. This gives the possibility to model dynamic asset price movements which are not purely additive or multiplicative by nature but something in between having qualities of these both, while also having negative underlying values with skewness.

The logic of the distribution fitting is the following. Firstly, *standard deviation* is calculated as usually, and it defines how spread the distribution is. Secondly, the mean of the distribution is calculated. Thirdly, it is also known that the mean of the distribution is the sum of the first parameter, *shifting pseudo mean,* and the *shifting parameter* (the latter being often negative by value). Now, because the mean and the standard deviation are known, software can be used to change the shifting parameter value until it finds the best measure of fit to the distribution. In this process, the first and the second moment of the distribution (mean and standard deviation) do not change. Actually, ordinary lognormal distribution (parameterized as μ, σ) can be thought as a special case of this distribution having the shifting parameter of zero.

For modeling of the dynamic asset movements, the simulated and fitted present value model has to be forced to follow some stochastic diffusion process over the time. If the underlying asset value is assumed to behave according to the lognormal distribution, the following equations for expected value (3) and standard deviation (4) in time can be calculated:

$$E(S_T) = S_0 e^{\mu T} \tag{3}$$

$$\sigma = S_0 e^{\mu T} \sqrt{e^{\sigma^2 T} - 1} \tag{4}$$

In case of shifted lognormal binomial tree, the underlying asset value increases each time period according to the equation (3). This is the same assumption that is made in Copeland and Antikarov (2001). However, the variance of the process is the same as suggested by the equation (4) only in the beginning and at the maturity. Similarly to Rubinstein's (1998) Edgeworth binomial tree, and contrary to the ordinary Cox, Ross and Rubinstein (1979) binomial lattice, shifted lognormal distribution valuation tree doesn't have a constant move size or variance in each node and time period. The values for the inner nodes in the binomial lattice valuation cone are calculated by dynamic back-rolling from the leaves down to tree root using risk-neutral probabilities, assuming this to describe dynamic underlying price process between the end nodes and the tree root better than any other model. This is similar to the idea of Rubinstein's (1998) Edgeworth binomial tree inner node calculations.

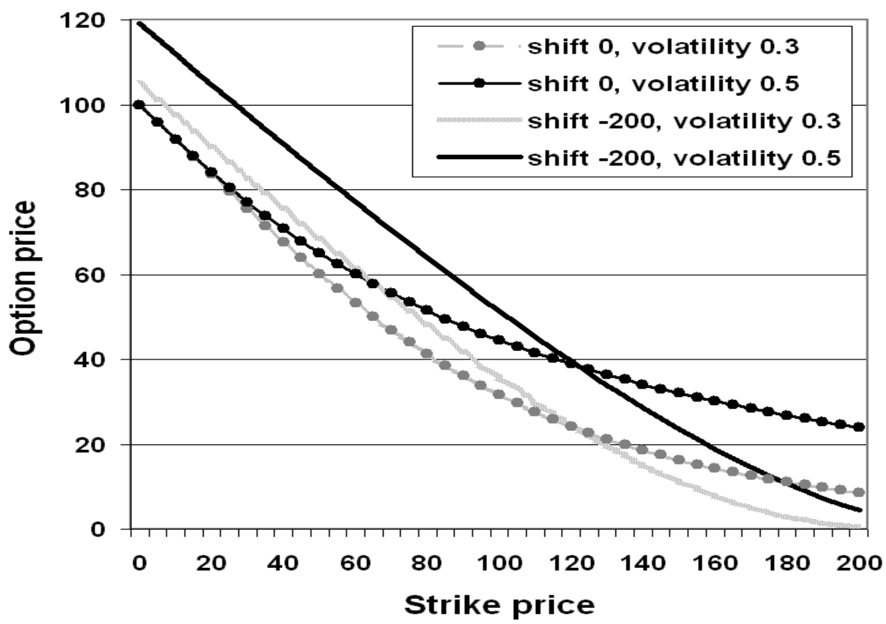
The event tree building of the shifted lognormal binomial tree is similar to ordinary Cox, Ross and Rubinstein (1979) or Rendleman and Bartter (1979) binomial lattice construction. The modelling starts by using the first *shifted pseudo mean* parameter as the underlying asset price. The common volatility parameter used at this point may be set to any arbitrary value. After the construction of the ordinary CRR lattice, the *shifting* parameter multiplied by e^{rt} is added to the values at the end nodes (the leaves of the binomial tree cone).

After calculating the values at the end nodes, the node values between the end nodes and the tree root are computed using risk-neutral probabilities with back-rolling method starting from the end nodes. Now, the mean of the stochastic process increases according to the risk-free interest rate between each period. After that, standard deviation of the distribution is calculated with the end node values of the lattice, and the result is set to equal the standard deviation provided by the simulation and distribution fitting. This can be done with Excel's solver function or by manually changing the volatility value until the value of the lattice standard deviation matches that of the distribution fitting standard deviation parameter multiplied by the risk-free time value of the money.

Finally, the valuation lattice is used similarly as the ordinary binomial lattice of Cox, Ross and Rubinstein (1979) or Rendleman and Bartter (1979). However, it should be noted that even though the event tree building is done by changing the volatility until the match between standard deviation is found, the term 'volatility' should not be used anymore for this parameter, but it should rather be called a *shifting volatility*. The only time the shifting volatility parameter value and the commonly used volatility parameter value in binomial tree would be the same is when the shifting parameter would be zero.

Graph 2 illustrates how the call option value changes as a function of strike price for different distributions according to the shifting parameter values (0 and -200) and different volatilities

(0.3 and 0.5). Underlying asset present value is 100, risk-free interest rate 5 %, and time to maturity four years. Shifting parameter of zero means that the option is a common plain vanilla option following geometric Brownian motion and having lognormal underlying asset distribution. Volatilities in case of shifted distributions mean that the variance of the shifted process is at maturity the same with the lognormal distribution resulted from the common gBm diffusion process. It can be seen that the shifting of the underlying parameter significantly? affects the value of the option. For example, with strike price of 50, call option value with volatility of 0.5 and shift of 0 is smaller than the option value with volatility of 0.3 and shifting of -200. On the other hand, with the strike price of 190, option with volatility of 0.5 and shifting of -200 has smaller value than an option with volatility of 0.3 and shift of 0.



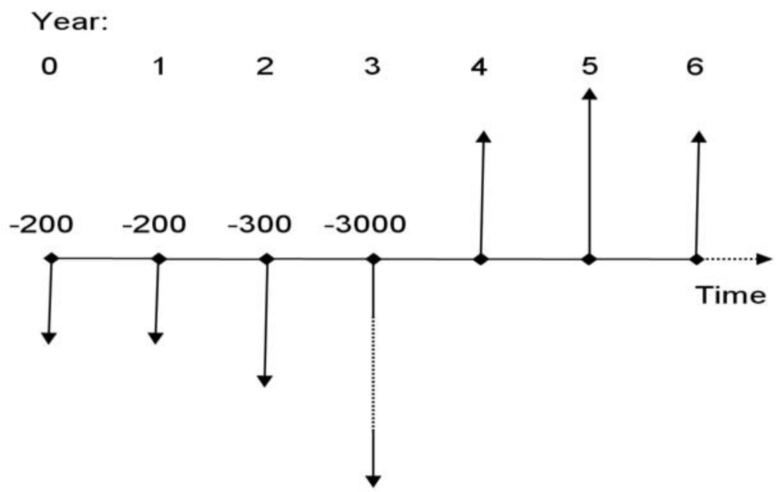
Graph 2: Option values as a function of strike price for different lognormal distributions according to the shifting parameter values (0 and -200) and volatilities (0.3 and 0.5).

The valuation approach presented here suffers from the same problem as the consolidated approach of Copeland and Antikarov (2001). As Brandao and Dyer (2003) noted, the use of the marketed asset disclaimer and consolidated volatility as the basis for creating a complete market for an asset that is not traded, may lead to significant errors, since the valuation is based on assumptions regarding the project value that cannot be tested in the market place. Shifted lognormal binomial tree valuation used is also vulnerable to the same errors.

Case example

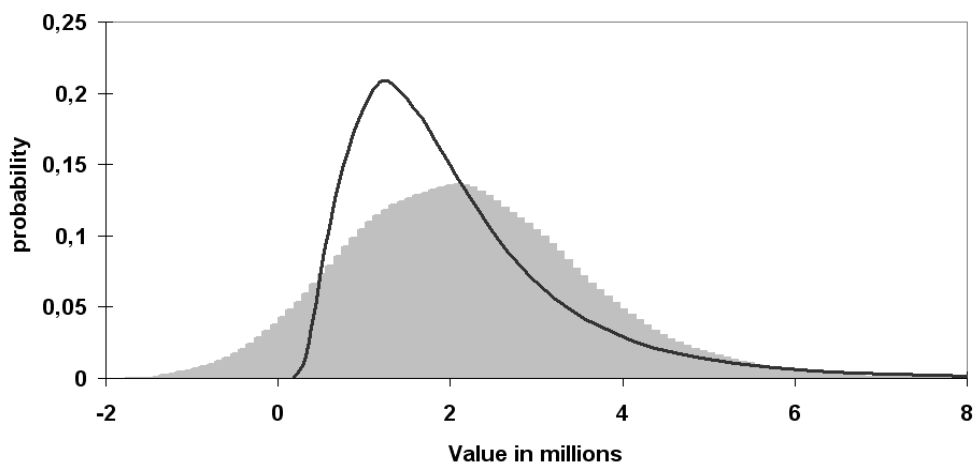
In the following a simplified case example is presented to illustrate the shifted lognormal binomial tree valuation in practice. No technical risk is assumed to exist and hence the only reason for the project to be abandoned is due to market related risks, which mostly concern demand, material price, and selling price uncertainties. All the market risk is assumed to be resolved during time to maturity. The valuation process is done according to the four stage process of Copeland and Antikarov (2001). The first step in the model is the calculation of the net present value of the project less the investment costs using weighted average cost of capital or some other justified discounting rate. This is assumed to represent project value without flexibility. In the second stage, an event tree is built to model the uncertainty and dynamic behavior of the underlying value describing how much the value is likely to move up and down during the investment period. This is usually done with CRR binomial tree, but this paper uses shifted lognormal binomial tree instead. The third step is to identify and incorporate managerial flexibility into the event tree. In fourth step, the real options analysis is conducted using simple algebraic (binomial) methodology with back-rolling and risk-neutral valuation.

The project contains three years of R&D with annual costs of 200, 200 and 300 for each year. The company considers whether the project has positive expectation at the end of each year and whether the it should be continued or not. After the R&D, company decides whether to invest 3000 in the production unit. A year after that and beyond the product is supposed to bring positive operating cash flows worth present value of 2077. Risk free interest rate is assumed as 4 % and the company's weighted average cost of capital is 10 %. Graph 3 presents the simplified cash flow structure of the project. Hence, the net present value using standard NPV calculation gives a result of -772, and according to this calculation, the project should be abandoned.



Graph 3: Estimated cash flows of the project

Firstly, present value less investment costs using discounted cash flow valuation model was calculated to discover the project's value without flexibility and investments. Graph 4 presents the distribution of the project present value less costs with filled grey area. This is the underlying project value with the mean of 2077 and standard deviation of 2358. The black line in Graph 4 presents the NPV distribution with the same mean and variance if the distribution would be lognormal. As the graph illustrates, the underlying asset value distribution is not very close to the lognormal shape. However, many models and cases presented in both managerial and scientific articles make this assumption or simplification.



Graph 4: Distribution of the underlying project value. The black line depicts the underlying value if it was assumed to be lognormal, which is a common assumption with many real options models. Both distributions have the same mean (2077) and standard deviation (2358).

Firstly, the model of Copeland and Antikarov (2001) was used to estimate the volatility according to the equations 1 and 2. With their method applied without checking whether the underlying behaves according to gBm assumptions and applying the calculation method erroneously, the volatility of the project was calculated to be 88 %. This result is wrong because of the negative underlying asset values. The problem was solved by applying equation 4 and setting the lognormal distribution to have the same mean and variance as the simulated distribution. After that, the annual volatility was found to be 53 %.

According to the normal binomial tree for sequential investments, the project was valued with binomial CRR tree with quarterly steps, totaling 12 steps for three years. Number of time steps was kept significantly small to be able to present the calculations and the results in appendix 1. In practice, a larger number of time steps should be used. However, the calculations done with a larger number of time steps did not change the results significantly in any case. The project value was found to be 123. If the wrong volatility measurement of 88 % would have been used, the project value would have been 589, which is significantly more than the result calculated with corrected volatility. However, neither of these results would be correct, because the distribution shape differs so significantly from lognormal.

Finally, the project was valued with shifted lognormal distribution. The value for the pseudo mean parameter was 11347, standard deviation 2358, and the shifting parameter 9270. Together, these resulted into the annual shifted volatility of 12%. With the shifted lognormal method, the project was valued to be worth 178, which is 55 more than the result given by the CRR binomial tree. The CRR tree would have given the same result if its volatility would have been 57 %. Table 1 summarizes the results of the different valuation procedures presented. Event trees and valuation trees for correctly parameterized CRR and shifted lognormal binomial tree are presented in the appendix I.

Table 1: Summary of the different valuation method results

Method	Value
Net present value	-772
CRR Binomial tree approach (wrong volatility, wrong shape of distribution)	589
CRR Binomial tree approach (corrected volatility, wrong shape of distribution)	123
Shifted lognormal binomial tree approach	178

As the results in Table 1 indicate, the different methods of net present value, binomial tree and the shifted lognormal binomial tree, result in significantly different project values. As assumed, net present value method provides clearly lower estimation for the project value than the other applied methods do. The differences between the results of binomial tree valuation clearly indicate that a great care should be taken when applying methods originally developed under gBm assumptions in situations where these assumptions do not hold. In such cases, other valuation methods should be selected, and also underlying asset diffusion process should be investigated and modeled carefully. Shifted log-normal binomial tree suggested in this paper is illustrated to provide a more accurate value of the alternatives, because it follows the underlying value dynamics more accurately than the other alternatives because of its ability to handle negative underlying values and third and fourth moment of the distribution.

Conclusions

In this paper a real options valuation method was developed for situations where the underlying asset may have negative values and the underlying project present value distribution is something of the shape between normal and log-normal distribution with skewness in rate of return distribution. The presented shifted lognormal binomial tree and process can be applied with Cox, Ross and Rubinstein (1979) or Rendleman and Bartter (1979) binomial trees. The model is consistent with the risk-neutrality assumption required for pricing options. The shifted lognormal model is as flexible and easy to use as the ordinary CRR binomial lattice, and as a lattice approach, it is appropriate for valuation of uncertain multi-staged investments with several embedded real options. Utilization of the model was illustrated with a simple multi-staged investment case example, which was also used as a basis for comparison of the results provided by the three different models. The results between different valuation methods suggest that a great care should be taken when considering the underlying asset diffusion process. Also, applying ordinary valuation methods developed under geometric Brownian motion assumption may lead to erroneous results if these assumptions are violated.

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Appendix I: the binomial tree comparison between Cox, Ross and Rubinstein (1979) and the shifted lognormal binomial tree.

Binomial tree by Cox-Ross-Rubinstein (1979)

Event tree

Underlying	2077	**Time**	3	**p**	0,453	**Up**	1,303	**r**	4 %
Volatility	53 %	**n**	12	**1-p**	0,547	**Down**	0,767		

0	1	2	3	4	5	6	7	8	9	10	11	12
2077	2707	3529	4599	5995	7814	10185	13276	17304	22554	29398	38318	49945
	1593	2077	2707	3529	4599	5995	7814	10185	13276	17304	22554	29398
		1223	1593	2077	2707	3529	4599	5995	7814	10185	13276	17304
			938	1223	1593	2077	2707	3529	4599	5995	7814	10185
				720	938	1223	1593	2077	2707	3529	4599	5995
					552	720	938	1223	1593	2077	2707	3529
						424	552	720	938	1223	1593	2077
							325	424	552	720	938	1223
								249	325	424	552	720
									191	249	325	424
										147	191	249
											113	147
												86

Valuation tree

X1					**X2**				**X3**				**X4**
	-200				-200				-300				-3000
	0	1	2	3	4	5	6	7	8	9	10	11	12
	323	594	1064	1833	2979	4823	7089	10125	14121	19643	26457	35348	46945
		104	216	447	918	1877	3036	4706	7003	10364	14363	19584	26398
			13	30	66	510	952	1710	2892	4903	7244	10305	14304
				0	0	69	153	342	762	1835	3054	4844	7185
					0	0	0	0	0	443	859	1629	2995
value:						0	0	0	0	48	106	237	529
123							0	0	0	0	0	0	0
								0	0	0	0	0	0
									0	0	0	0	0
										0	0	0	0
											0	0	0
												0	0
													0

Shifted lognormal binomial tree

Event tree

Std.Dev	2358								
Underlying	11347	**Time**	3	**p**	0,569	**Up**	1,062	**r**	4 %
Volatility	12 %	**n**	12	**1-p**	0,431	**Down**	0,942	**Shift**	-9270

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
2077	2685	3336	4033	4777	5572	6421	7328	8296	9329	10431	11606	12860
	1323	1890	2496	3145	3840	4582	5375	6222	7127	8093	9124	10224
		607	1134	1699	2303	2950	3643	4383	5174	6019	6922	7886
			-75	416	941	1504	2107	2752	3442	4180	4969	5812
				-722	-267	221	744	1305	1906	2549	3237	3973
					-1339	-917	-464	22	543	1102	1701	2342
						-1927	-1536	-1116	-665	-181	338	895
							-2487	-2126	-1737	-1319	-870	-388
								-3021	-2687	-2328	-1942	-1526
									-3531	-3224	-2892	-2535
										-4018	-3735	-3431
											-4483	-4225
												-4929

Valuation tree

	X1				**X2**				**X3**				**X4**
	-200				**-200**				**-300**				**-3000**
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
	378	579	869	1268	1777	2605	3346	4186	5113	6417	7490	8636	9860
		122	211	363	627	1195	1688	2315	3061	4215	5152	6153	7224
			8	14	24	361	574	900	1386	2314	3078	3952	4886
				0	0	50	89	157	279	897	1359	1999	2812
					0	0	0	0	0	174	308	548	973
value:						0	0	0	0	0	0	0	0
178							0	0	0	0	0	0	0
								0	0	0	0	0	0
									0	0	0	0	0
										0	0	0	0
											0	0	0
												0	0
													0