Investment timing and dynamic operation in monopoly franchising contracts for transportation infrastructures

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Abstract

In this paper, we characterize the optimal contract for investment in transportation infrastructure and provision of the associated service in the presence of stochastic preferences. The relevant decision variables are the timing of investment and the supply path (in the limit of the installed capacity). Both variables depend on market dynamics. The need to balance the budget of the investor in expectation (the second best environment) trades-off the social efficiency against longer delay of investment and, in case the demand exhibits variable price elasticity, higher consumer rationing. The best moment to invest remains unchanged whenever the government expects to subsidize the firm if the market falls significantly and cost recovery becomes impossible. The supply path instead is downward distorted, as compared to the second best, the distortion being proportional to the level of uncertainty.

Keywords: Stochastic demand; investment timing; public-private partnership

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1 Introduction

1.1 Scope of analysis

Essential facilities, such as infrastructures for long-distance transportation services, involve significant investment costs. Public transfers, ultimately financed by taxpayers, can be avoided if the government awards exclusive franchising contracts to private consortia that are willing to invest. Examples are concession contracts for highways and the Eurotunnel project.

Most of the time, the conditions on the downstream market, from which infrastructure costs are to be recovered, are uncertain. In this sense, the benefit of involving private firms can only materialize if the financial risk of the project is transferred from the public budget to the private sector. However, firms tend to renegotiate the contracts in case bad market conditions realize after the investment process. This has been the case for many concession contracts in Latin America (see, for instance, Nombela and Rus, 2003, Guasch *et Al.,* 2003, Engels *et Al.*, 1997). The transfer of the financial risk to the firm is thus limited. Guasch *et Al.* (2003) argue that the reliance on *ex post* inefficient public transfers reduces if the initial contract internalizes the risk of renegotiation. Engels *et Al.* (1997) argue that renegotiation is avoided if the firm is perfectly insured and suggest signing flexible term contracts that allow the firms to obtain a certain amount of market returns.

In the studies previously mentioned, uncertainty is introduced in a two-period setup. For the purpose of characterizing the optimal monopoly franchising contracts, it is opportune to generalize and extend the presence of uncertainty to a longer time horizon, which better stylizes the uncertain dynamics the demand follows in real-world situations. Indeed, even though the conditions on the transportation market itself may remain unchanged over time, it is possible that exogenous shocks, which are dynamic and uncertain, affect consumer tastes and so the demand for the service. For instance, such shocks are changes in GDP across periods.

An additional decisional dimension proves to be relevant in infrastructure projects, when such a generalization is performed, namely the timing of investment. Indeed, the achievable social efficiency is the one which emerges from the trade-off between the loss which is incurred by delaying investment in any given decisional period and the benefit associated to having more optimistic forecast in the future, when the risk of market decline becomes lower. Moreover, for a given project, the financial risk exceeds the social one. Hence, if the budget balance must be satisfied, there exists a trade-off between social efficiency and financial viability of the project, while deciding when to invest.

While the real options literature has largely investigated the optimal timing for investment in different market structures (see Dixit and Pindyck, 1994 Part IV, Grenadier, 2002) the aforementioned trade-off has not been yet examined, despite the importance it acquires in the current political tendency to involve private firms in projects for essential facilities through public-private partnerships. The present paper examines this issue as for a market of transportation infrastructure in two contexts, namely no risk insurance and full risk insurance of investors, which avoids renegotiation.

In what follows we describe the methodology and the relevant decision variables in the environment just outlined.

1.2 Methodology and main insight: investment timing and endogenous congestion

Consider the recent project for building a railway to link France and Italy between Lyon and Turin. The forecasts of the demand for the perspective service are based on the current traffic in the trans-alpin area, which leads to pessimistic results. This is why it has been decided that the investment is to be postponed until the demand for other transportation modes induces more optimistic forecasts. The current demand might increase, for instance, if GDP increases during the following years. Investment timing is thus relevant because forecasted demand is dynamic.

The scale of operation is another decision to be made at the project evaluation stage, as it is strictly linked to that of timing. In order to understand this link we first notice that the capacity to be installed is not continuous in demand changes. The decision maker cannot really choose the number of kilometers to be built for covering the distance between any two regions. Divisible capacity units can be built for satisfying local demand, not the long distance demand, such as the inter-regional one in the trans-alpin area between France and Italy. Moreover, even if capacity is divisible, the dynamics of consumer tastes for the service translate into consumers exhibiting over time different sensitivity for the same congestion level, associated to any given capacity. From this perspective, the congestion level of the capacity becomes *endogenous* to the problem of choosing the best moment to invest.

Keeping in mind these circumstances, we assume that the project required one capacity unit to be installed by means of a one-shot investment, the size of which is not a choice variable. The relevant decisions are the timing of investment in this unit and the congestion level (operation scale) at which it is optimal to operate. We find that, as the two decisions are bundled, they both depend on the level of uncertainty. We subsequently study the contractual form, provided that the financial risk cannot be transferred from the public budget.

1.3 Outline

The paper is organized as follows. In Section 2 the model and the optimal bundling of investment timing and operation scale as depending on the level of uncertainty are presented. In Section 3, we move to the second-best symmetric information environment, where the private investor's budget balance is secured by relying on public subventions and market returns. In Section 4, we characterize the trade-off that arises between social efficiency and financial viability when the firm must be insured her reservation utility. Section 5 concludes.

1.4 Related literature

This paper combines works about flexible period of investment (as based on the standard real options approach) and monopoly franchising contracts.

Methodologically speaking, our paper is mainly related to Sodal (2001). The latter shows that, when a one-shot investment is to be made, a double option is available to the investor, namely the time when to invest and the scale at which to operate on the supply market. Our work differs from Sodal (2001)'s because, instead of choosing capacity, the decision-maker selects the level of congestion at which (the exogenous) capacity is to be operated.

Furthermore, this study differs from similar works about monopoly franchising contracts in that the uncertainty about the demand is dynamic. As compared to those works, the framework is here enriched in two ways. Firstly, we take the timing of investment to be a choice variable in contracting for investment and operation of the essential facilities. Secondly, we make congestion endogenous to the problem of investment timing.

2 The model

We consider a project which requires that the investor bears a significant sunk cost I for building an essential infrastructure. The social planner offers the firm a contract for investing in a long distance transport infrastructure. The firm subsequently recovers her cost from market revenues. For the sake of simplicity, both the social planner and the operator are assumed to be risk neutral. The crucial decisions to be made are the timing of investment, depending on the stochastic demand shifts, and the scale of service operation.

2.1 Market demand and value of the project

We assume that, in the market under scrutiny, the demand for transportation service changes unpredictably as random shocks occur. Time is continuous and indexed by $t \in [0, \infty)$. The inverse demand at any time $t \geq 0$ is given by

$$p_t = p(D_t) y_t. \tag{1}$$

As (1) shows, p_t is composed of two parts. The first, $p(D_t)$, is a mapping $D_t : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ describing the non-stochastic component of the inverse demand function. The second, y_t, shifts the aggregate willingness to pay between any two periods and for any quantity of the good, according to a random shocks affecting the industry.

Let us explain first the deterministic demand. As usual in real options theory, $p(D_t)$ follows from the market conditions. For instance, Grenadier (2002), assumes that firms act on a competitive market and characterizes the Nash equilibrium in investment decisions. Boyer et *Al.* (2005) describe $p(D_t)$ as characterized by competition in a Cournot duopoly. As we refer to an "infrastructure" activity we assume that the investor is the only firm that operates the service and benefits from the demand flow throughout the duration of the contract. Indeed, firms in the transportation industry obtain monopoly franchising contracts and so competition is not feasible before the firm recovers the investment cost. We denote by Q_t the amount of service supplied at any t. Because $Q_t = D_t$, the deterministic part of inverse demand can be written as $p(Q_t)$. The literature generally assumes that the demand function exhibits constant price elasticity. In contrast, in this study, we allow for a variable elasticity. We will show that, as far as the capacity size is exogenous to investor, relaxing the assumption of constant elasticity significantly affects the optimal decisions.

Uncertainty is represented by a complete probability space $(\Omega, \mathcal{F}, P)$. The shock y_t follows a geometric Brownian motion with drift $\alpha > 0$ and volatility $\sigma \geq 0$, so that

$$dy_t = \alpha y_t dt + \sigma y_t dz_t. \tag{2}$$

y_0 is the value it takes at the current date $t_0 = 0$, while z_t is a simple Brownian motion defined on $(\Omega, \mathcal{F}, P)$. The flow of information on which decisions are based is expressed by the filtration $(\mathcal{F}_t)_{t \geq 0} = (\sigma\{y_t | v \leq t\})_{t \geq 0}$ generated by $(y_t)_{t \geq 0}$.

In what follows, we make two assumptions, which are important in the subsequent analysis. Neither of them is particularly strong, rather they are similar to hypotheses usually adopted by the literature.

- *Assumption 1*: $p(Q_t)$ is strictly decreasing, continuously differentiable and integrable on $\mathbb{R}_+$ and independent of the current shock y_t. Moreover, the mapping $Q_t \rightarrow Q_t p(Q_t)$ is strictly concave and $p(0)$ satisfies the inequalities $0 < p(0) < \infty$.

- *Assumption 2*: We denote by r, such that $0 < r < 1$ the discount rate of any riskless asset in the economy. The drift α, which we introduced in (2), and r satisfy the inequality $\alpha < r$, which means that the unit shift of the demand is lower than the unit discount rate.

Assumption 1 ensures that, excluding operational costs, an optimal scale of operation from both a social and a financial perspective exists. Moreover, Q_t at any given value y_t, is unique and finite. Assumption 2 states the convergence condition, which is standard in the real options literature and guarantees that the optimal waiting period is also finite.

2.2 Firm's technology

As previously illustrated, the project requires an irreversible fixed cost I; once the latter is borne, a total capacity $\overline{Q}$ is made available for operation. In our framework, capacity means the maximum traffic that the infrastructure to be built can technologically support during a specific period. We assume that the agent that makes the investment recovers it from market revenues. This agent either provides the service himself or he subcontracts the service. In any case, the service is provided in quantity $Q_t \leq \overline{Q}$, which imposes on the investor instantaneous costs $c\left(Q_t\right) = cQ_t$[1]. Consumers benefit from the new service immediately after the lump-sum investment is made.

2.3 Investment timing and options to scale at social optimum

The decision maker chooses the optimal threshold y that defines the stochastic period at which the investment should be realized. She also selects the optimal scale Q_t at which the firm should operate the service. In what follows, we analyze the choice of timing and scale at social optimum. At a later stage, we shall explore the more realistic cases where budget balance must be satisfied in expectation, while the firm bears the risk of market decline. Subsequently, we will treat the decisional process when the firm must be insured her reservation utility with public funds.

Assuming that the period of operation is infinite, the net social value of the project is given by the expectation of the discounted flow of net surplus. Evaluated at period τ, defined as $\tau \equiv \inf\left\{t \geq 0 \; s.t. \; y_t = y\right\}$, the net value of the project is written

$$E^y\left(\int_\tau^\infty S\left(y_t, Q_t\right) e^{-r(t-\tau)} dt - \int_0^\infty C\left(I, Q_t\right) e^{-r(t-\tau)} dt\right), \tag{3}$$

[1]Strictly speaking, an operational cost function as in the text exhibits constant returns to scale. However, the proposed formulation is adopted because it accomodates well for the presence of capacity constraints. More precisely, the marginal cost of production is assumed to be constant for $Q \leq \overline{Q}$ and to become infinite at $\overline{Q}$. Hence, up to capacity, one has constant returns in operation, but overall increasing returns to scale due to the presence of I (see Tirole, 1988).

where $C(I, Q_t) = c(Q_t) + rI$ is the instantaneous cost of operation plus the rate of investment cost.

Whenever the installed capacity is not fully used for operation, the optimal supply that derives from (3) is characterized by

$$S(y_t, Q_t) = C(I, Q_t)\frac{\varepsilon_{CQ_t}}{\varepsilon_{SQ_t}}, \tag{4}$$

$\varepsilon_{SQ_t} \equiv S_{Q_t}Q_t/S$ and $\varepsilon_{CQ_t} \equiv C_{Q_t}Q_t/C$ are the elasticities of instantaneous flows of surplus and cost with respect to Q_t at any time t. (4) says that the ratio of surplus over cost in instantaneous and absolute values must equal at social optimal supply the inverse ratio of their elasticities. This expression is similar to the one of Sodal (2001). The difference is that Q_t here means the level of supply, rather than the level of capacity, as in his analysis.

We next add two more assumptions to be relied upon in the subsequent discussion. We first define $R(y_t, Q_t) \equiv Q_t p(Q_t) y_t$ the instantaneous revenues from the service provision at any t. Moreover, we denote the elasticity of revenues to quantity by ε_{RQ_t}, so that $\varepsilon_{RQ_t} \equiv R_{Q_t}Q_t/R$. We are then ready to state:

- *Assumption 3*: $\frac{\partial}{\partial Q_t}\left(\frac{S}{C}\right) < 0$ at any $Q \geq 0$. This condition holds if and only if $\varepsilon_{CQ_t} > \varepsilon_{SQ_t}$.
- *Assumption 4:* $\frac{\partial}{\partial Q_t}\left(\frac{S}{R}\right) \geq 0$. Equivalently, $\varepsilon_{SQ_t} \geq \varepsilon_{RQ_t}$.

Assumption 3 is quite a standard one. It establishes the condition under which (4) has economic sense. Together with Assumption 3, Assumption 4 ensures that $\varepsilon_{CQ_t}/\varepsilon_{SQ_t}$ is an increasing function of Q_t. Notice that Assumption 4 is not restrictive. For instance, it is satisfied with equality if the demand exhibits constant price elasticity and it is strictly satisfied by a linear demand function.

We denote $Q(y_t)$ the quantity that solves (4). Full capacity is reached whenever the hitting value $\overline{y}$ is realized, as derived from the equality $Q(\overline{y}) = \overline{Q}$. The corresponding price of the service, in the region $y_t > \overline{y}$, is given by $p\left(\overline{Q}\right) y_t$ and is such that $p\left(\overline{Q}\right) y_t > c$. In presence of a capacity constraint, the optimal supply at any $t > 0$ is thus given by

$$\begin{cases} Q(y_t), \text{ if } y_t < \overline{y} \\ \overline{Q}, \text{ if } y_t \geq \overline{y} \end{cases}$$

We thus distinguish two different regions of values y_t that define the optimal supply. They are graphically represented in Figure 1 below.

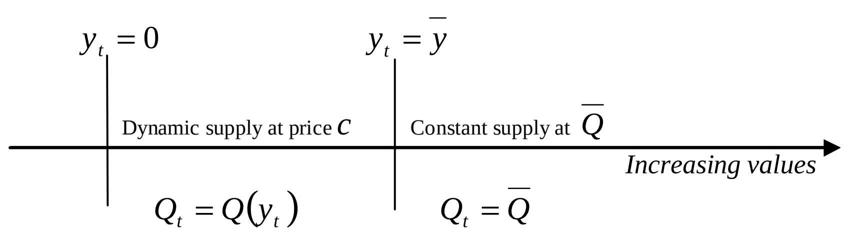


Figure 1: The path of the supply at social optimum

The value of the project differs between the two regions. For notational convenience, we denote by $\overline{\tau}$ the stochastic period when the value $\overline{y}$ realizes for the first time. Indeed, $\overline{\tau}$ is defined as $\overline{\tau} \equiv \inf \{t \geq 0 \ s.t. \ y_t = \overline{y}\}$. The derivation of the social value of the project is standard. Evaluated at τ, such that $\tau \leq \overline{\tau}$ (or $y < \overline{y}$), the general form of the social value is written as (see the Appendix for computational details)

$$
\begin{aligned}
V(y) &= E^y \int_\tau^\infty \left(S\left(y_t, Q\left(y_t\right)\right) - C\left(I, Q\left(y_t\right)\right)\right) e^{-r(t-\tau)} dt + \\
&+ \frac{1-\beta_2}{\beta_1-\beta_2}\left(\frac{y}{\overline{y}}\right)^{\beta_1}\left(S\left(\overline{y}, \overline{Q}\right)\frac{1}{r-\alpha} - E^{\overline{y}}\int_{\overline{\tau}}^\infty S\left(y_t, Q\left(y_t\right)\right) e^{-r(t-\overline{\tau})} dt\right) + \\
&+ \frac{\beta_2}{\beta_1-\beta_2}\left(\frac{y}{\overline{y}}\right)^{\beta_1}\left(\frac{C\left(I, \overline{Q}\right)}{r} - E^{\overline{y}}\int_{\overline{\tau}}^\infty C\left(I, Q\left(y_t\right)\right) e^{-r(t-\overline{\tau})} dt\right).
\end{aligned} \tag{5}
$$

The first line in the expression of $V(y)$ represents the value the project would have if, at every moment in time, marginal cost pricing (equation (4)) were to be applied and the quantity $Q(y_t)$ was not capacity constrained. Furthermore, the terms in the second and the third line express the change in discounted social value induced by the restriction of dynamic operation to values $Q(y_t) \leq \overline{Q}$. These terms are negative; $\beta_1 > 1$ and $\beta_2 < 0$ are the two roots of the quadratic equation $\beta(\beta-1)\frac{\sigma^2}{2} + \alpha\beta = r$.

Let us now move to the decision about the timing of the investment for a project of value $V(y)$. In closed form, the discounted value of the project at y_0, such that $y_0 \leq y$, is written

$$
\left(\frac{y_0}{y}\right)^{\beta_1} E^y \int_\tau^\infty \left(S\left(y_t, Q\left(y_t\right)\right) - C\left(I, Q\left(y_t\right)\right) e^{-r(t-\tau)} dt\right) + \left(\frac{y_0}{\overline{y}}\right)^{\beta_1} \Psi(S). \tag{6}
$$

In (6) we have used the notation $\Psi(S)$,

$$
\begin{aligned}
\Psi(S) &= \frac{1-\beta_2}{\beta_1-\beta_2}\left(S\left(\overline{y}, \overline{Q}\right)\frac{1}{r-\alpha} - E^{\overline{y}}\int_{\overline{\tau}}^\infty S\left(y_t, Q\left(y_t\right)\right) e^{-r(t-\overline{\tau})} dt\right) + \\
&+ \frac{\beta_2}{\beta_1-\beta_2}\left(\frac{C\left(I, \overline{Q}\right)}{r} - E^{\overline{y}}\int_{\overline{\tau}}^\infty C\left(I, Q\left(y_t\right)\right) e^{-r(t-\overline{\tau})}\right).
\end{aligned}
$$

as this expression plays no role in the investment timing decision.

Remember that $Q(y_t)$ is optimal in the social objective function (6). By the envelope theorem, the first-order condition in (6) is written as

$$E^y \int_0^\infty S(y_t, Q(y_t)) e^{-r(t-\tau)} dt = \frac{\beta_1}{\beta_1 - 1} E^y \int_0^\infty C(I, Q(y_t)) e^{-r(t-\tau)} dt \qquad (7)$$

Let us now relate the quantity decision to the timing of investment decision. At this aim, we evaluate (4) at $Q(y_t)$ and then integrate, which yields

$$E^y \int_\tau^\infty S(y_t, Q(y_t)) e^{-r(t-\tau)} dt = E^y \int_\tau^\infty C(I, Q(y_t)) \frac{\varepsilon_{CQ_t}}{\varepsilon_{SQ_t}} e^{-r(t-\tau)} dt \qquad (8)$$

Thus, if the activity is initiated at $Q(y) < \overline{Q}$, (7) and (8) must both be satisfied so that the following equality is established

$$E^y \int_\tau^\infty C(I, Q(y_t)) \frac{\varepsilon_{CQ_t}}{\varepsilon_{SQ_t}} e^{-r(t-\tau)} dt = E^y \int_\tau^\infty C(I, Q(y_t)) \frac{\beta_1}{\beta_1 - 1} e^{-r(t-\tau)} dt. \qquad (9)$$

(7) and (9) are the conditions which characterize the optimal bundle of investment timing and supply path within the first region of Figure 1.

(7) is a standard condition and suggests that, if Q_t is at $Q(y_t)$ at any t during the operation interval, then the expected flow of gross surplus exceeds the total cost of investment and operation by an amount $\beta_1/(\beta_1 - 1) > 1$, direct proportional with α and σ, the shift and volatility parameters of dy_t. There exists a trade-off between the benefit of investing immediately and the gain accrued from delaying the investment to a moment when market conditions are more favorable. Therefore, $\beta_1/(\beta_1 - 1)$ "measures" the market dynamics that determine the aforementioned trade-off.

Furthermore, (9) reveals that the ratio $\varepsilon_{CQ_t}/\varepsilon_{SQ_t}$ is increasing in $\beta_1/(\beta_1 - 1)$. From Assumptions 3 and 4 it follows that $Q(y_t)$ is also increasing in $\beta_1/(\beta_1 - 1)$. Indeed, whenever the investment is delayed because uncertainty is high, optimal operation shifts to a higher trajectory. If the inequality $\frac{\varepsilon_{C\overline{Q}}}{\varepsilon_{S\overline{Q}}} < \frac{\beta_1}{\beta_1 - 1}$ holds, then the operation starts at full capacity. Throughout the rest of our analysis, without loss of generality, we assume that $\frac{\varepsilon_{C\overline{Q}}}{\varepsilon_{S\overline{Q}}} > \frac{\beta_1}{\beta_1 - 1}$ so that the operation starts optimally at $Q(y) < \overline{Q}$. Once the hitting value y and the quantities $Q(y_t)$, where $y_t \geq y$ are determined, the stochastic period $\overline{\tau}$ when the capacity is expected to be entirely used for the first time, is characterized as well.

The interesting aspect of the solution above is the following. It suggests that it may be optimal to invest at a stage when the demand is not very large and the available capacity is not entirely used in operation. Even though the installed capacity $\overline{Q}$ can be used for no purpose other than the provision of the transportation service, the social benefit of early investment, when some capacity remains idle (at least for some periods) maybe high enough, when compared to the cost of investment and operation, that any additional delay is wasteful. This result is not *per se* surprising but it leads to an important insight for the transportation industry. Two characteristics of this market are essential for the conclusion we have drawn: firstly, the investment in a new railway or highway is indivisible, whereas the traffic (hence, the extent to which such capacity is utilized) may well fluctuate across periods. Indivisibility makes the level of capacity congestion (at investment period) to be endogenous to the problem of investment timing. On the other hand, optimal congestion is dynamic and follows the evolution of consumers preferences during the period of operation.

Provided that timing and scaling are bundled decisions, it becomes important to understand how they are jointly set when the price of the transportation service must be selected so that it satisfies the investor's budget constraint. This issue is examined below. We will show that, in the presence of a profit constraint, investment associated with partial use of capacity is even more easily justified. We first argue and show analytically that the social planner relies on sources other than public money for investment, because this strategy avoids costly public funds. We assume that a private investor, who recovers the cost from market revenues, is available for contracting. We then investigate the optimal timing-scale bundle that ensures the investor's participation. We finally examine the situation when the firm must be insured her reservation utility against market decline during the contract interval; we explore the associated implications that follow in terms of timing-scale combination.

3 Second-best: budget balance and no insurance

3.1 Subventions, market revenues and motivation for private participation

It can be easily shown that the solution $(y, Q(y_t))$ that solve (7) and (9) is not financially viable. However, a public firm must be insured budget balance *ex post* investment while a private firm would not participate if the expected payoff were negative. Moreover, if the social planner provides subventions, they come at a marginal social cost λ.

Suppose that the consumer surplus and the profit of the firm are attributed the same weight in social welfare. The programme is written as

$$\begin{cases} \underset{y,Q_t}{Max} \left(\frac{y_0}{y}\right)^{\beta} \left[E^y \int_{\tau}^{\infty} \left(S\left(y_t, Q_t\right) - C\left(I, Q_t\right)\right) e^{-r(t-\tau)} dt - \lambda x\right] \\ s.t.\ E^y \int_{\tau}^{\infty} \left(R\left(y_t, Q_t\right) - C\left(I, Q_t\right)\right) e^{-r(t-\tau)} dt + x \geq 0 \end{cases} \quad (10)$$

The objective function is that of the previous Section less the social cost of the public transfer x. The constraint suggests that a combination of market revenues and direct subventions satisfy the budget balance of the firm. We denote by μ the marginal cost of the profit constraint. The Lagrangian of (10) is written

$$\left(\frac{y_0}{y}\right)^{\beta}\left[E^y\int_{\tau}^{\infty}\left(S\left(y_t,Q_t\right)-C\left(I,Q_t\right)\right)e^{-r(t-\tau)}dt+\left(\mu-\lambda\right)x+\right.\tag{11}$$
$$\left.\mu\left(E^y\int_{\tau}^{\infty}\left(R\left(y_t,Q_t\right)-C\left(I,Q_t\right)\right)e^{-r(t-\tau)}dt+x\right)\right]$$

In case $\mu\leq\lambda$ the marginal loss attached to the taxation system is higher than the welfare loss from consumers' rationing, when the price is such that the budget constraint is satisfied. Otherwise, state aid should complete market returns. However, in both cases, outside investment recovered from market returns is optimal (at least partially).

The planner can rely either on capital market or on a private firm that is offered a monopoly franchising contract. We assume in our study that the second solution is adopted. Subsequently, the constraint in (10) is satisfied with equality at social optimum, which means that the private investor bears all the risk of market decline during the period of operation. In the next Section, we will relax this assumption and show how the social decisions change when the firm must be insured her reservation utility in operation.

3.2 Solution

When the constraint (10) is satisfied with equality, the function (11) to be maximized is rewritten as

$$\left(\frac{y_0}{y}\right)^{\beta_1}\left(E^y\int_{\tau}^{\infty}\left(S\left(y_t,Q_t\right)-C\left(I,Q_t\right)\right)e^{-r(t-\tau)}dt+\right.\tag{12}$$
$$\left.\lambda\left(E^y\int_{\tau}^{\infty}\left(R\left(y_t,Q_t\right)-C\left(I,Q_t\right)\right)e^{-r(t-\tau)}dt\right)\right).$$

The optimal quantity $Q\left(y_t\right)$ at any t such that the capacity is not entirely used solves the standard Ramsey-Boiteaux formula

$$S\left(y_t,Q_t\right)=C\left(I,Q_t\right)\frac{\varepsilon_{CQ_t}}{\varepsilon_{SQ_t}}+\frac{\lambda}{1+\lambda}\left(S\left(y_t,Q_t\right)-\frac{\varepsilon_{SQ_t}}{\varepsilon_{RQ_t}}R\left(y_t,Q_t\right)\right).\tag{13}$$

Obviously, the need to satisfy the budget balance induces consumers rationing with respect to the social optimum. Indeed, comparing (13) with (4), we notice that by Assumption 3, $Q\left(y_t\right)$ is lower in (13) than the one at social optimum. However, in order to evaluate the second best supply we need to find the values y_t at which the equation (13) is evaluated.

Similarly to the benchmark case in the previous Section, the discounted value of the project in closed form and optimal supply is written

$$\left(\frac{y_0}{y}\right)^{\beta_1} E^y \int_\tau^\infty \left(S\left(y_t, Q\left(y_t\right)\right) + \lambda R\left(y_t, Q\left(y_t\right)\right) - (1+\lambda) C\left(I, Q\left(y_t\right)\right) e^{-r(t-\tau)} dt\right) +$$
$$+ \left(\frac{y_0}{\overline{y}}\right)^{\beta_1} \Psi\left(S + \lambda R\right)$$

The functional form above is found by replacing the instantaneous flow $S(.) - C(.)$ from social optimum of previous Section by $S(.) + \lambda R(.) - (1+\lambda) C(.)$.

By the envelope theorem, the optimality condition for the value of y that triggers investment is

$$E^y \int_\tau^\infty S\left(y_t, Q\left(y_t\right)\right) e^{-r(t-\tau)} dt = \frac{\beta}{\beta - 1} E^y \int_\tau^\infty C\left(I, Q\left(y_t\right)\right) e^{-r(t-\tau)} dt + \frac{\lambda}{1+\lambda} E^y \int_\tau^\infty \left(S\left(y_t, Q\left(y_t\right)\right) - R\left(y_t, Q\left(y_t\right)\right)\right) e^{-r(t-\tau)} dt \tag{14}$$

We rewrite (13) at optimal supply $Q(y_t)$ and integrate it, so that we obtain

$$E^y \int_\tau^\infty S\left(y_t, Q\left(y_t\right)\right) e^{-r(t-\tau)} dt = E^y \int_\tau^\infty C\left(I, Q\left(y_t\right)\right) \frac{\varepsilon_{CQ_t}}{\varepsilon_{SQ_t}} e^{-r(t-\tau)} dt + + \frac{\lambda}{1+\lambda} E^y \int_\tau^\infty \left(S\left(y_t, Q\left(y_t\right)\right) - \frac{\varepsilon_{RQ}}{\varepsilon_{SQ}} R\left(y_t, Q\left(y_t\right)\right)\right) e^{-r(t-\tau)} dt \tag{15}$$

(14) and (15) hold simultaneously when their right hand sides are equal so that

$$E^y \int_\tau^\infty C\left(I, Q\left(y_t\right)\right) \frac{\varepsilon_{CQ_t}}{\varepsilon_{SQ_t}} e^{-r(t-\tau)} dt = = E^y \int_\tau^\infty \left(C\left(I, Q\left(y_t\right)\right) \frac{\beta}{\beta - 1} - \frac{\lambda}{1+\lambda} \left(1 - \frac{\varepsilon_{RQ}}{\varepsilon_{SQ}}\right) R\left(y_t, Q\left(y_t\right)\right) e^{-r(t-\tau)} dt\right. \tag{16}$$

Like in benchmark case, (14) and (16) characterize the optimal bundle of investment timing and operation scale in the constrained problem. Equations (14) and (15) show that $Q(y_t)$ and y must be such that the discounted gross surplus is higher than the one at social optimum, due to the internalization of the cost of the constraint. Either downward distorting the supply or delaying the investment until the market is higher than at social optimum allow the firm's budget constraint be satisfied. Furthermore, any supplementary delay induces higher trajectory of optimal supply, as shown in the previous Section. Henceforth, the path of the output that is bundled to the trigger decision derives from two countervailing effects: downward "Ramsey-Boitteaux" distortion versus upward scaling when the investment decision is delayed.

Once the presence of the two effects is understood, we can observe the resulting second best supply path in (16). Remember that by Assumption 4, $\varepsilon_{RQ} \leq \varepsilon_{SQ}$.

Consider first the case where the price elasticity along the demand curve is constant. $\varepsilon_{RQ} = \varepsilon_{SQ}$ in (16) and the path of the supply turns to be the social optimal one, expressed in (9). Waiting for higher demand and scaling upwards the operation perfectly compensates the Ramsey-Boitteaux distortion. This result is determined by the fact that the price $p(Q_t)\, y_t$ in (13) is constant at any $Q(y_t) < \overline{Q}$, whenever the elasticity of the demand is constant. Thus, a unit change of y_t causes a constant change of $p(Q_t)$ at any t.

In contrast, the case where $\varepsilon_{RQ} < \varepsilon_{SQ}$ corresponds to a decreasing price elasticity of the demand along the demand curve. The optimal price that satisfies the Ramsey-Boitteaux formula differs at different realizations of y_t. Consequently, quantity reduction dominates the flow of options to scale up the operation through additional delay. Decreasing elasticity at higher output makes the quantity decision a valuable instrument for solving the trade-off between the social efficiency and the financial viability of the social project.

However, the fact that the market is dynamic reduces the cost of the constraint in consumers utilities. Indeed, the distortion of any quantity in (13) is evidently lower when the supply (or pricing) decision is bundled with the investment timing decision in a dynamic market, as in (16).

3.3 Examples

Figures 2.1 and 2.2 illustrate the findings of the first and the second-best analyses, for a linear demand function (LD) and a constant elasticity demand function (CED) respectively.

In Figure 2.1., the inverse demand function is assumed to be $p(Q_t) = 30 - 0,3Q_t$. In Figure 2.2., the deterministic inverse demand is $p(Q_t) = 60Q_t^{-\frac{1}{1,3}}$. The exogenous capacity is assumed to be $\overline{Q} = 96$ and its installation costs $I = 6000$. The unit cost of operation is $c = 9$. The other default parameter values are $r = 0.05$, $\alpha = 0.03$, $\sigma = 0.175$ and $y_0 = 0.5$.

We illustrate a specific path of y_t and the corresponding quantities $Q(y_t)$. We use both vertical axes to define the relevant scales. Thus, the left hand vertical axis in each figure presents the scale of y_t while the right hand one presents the scale of quantities $Q(y_t)$. The trajectory of y_t is the same in both graphs, allowing for the values of $Q(y_t)$ to be easily compared between the two cases.

First, we notice that the path of $Q(y_t)$ of the CED function is below the one of the LD function, but that the values $Q(y_t)$ in the former case increase faster. Indeed, in the solution (9) of first-best analysis, ε_{SQ_t} is constant in the CED case while it decreases with Q_t in the LD case since the price elasticity of LD is decreasing along the demand curve. As a consequence, the first-best operation starts in Figure 2.1. at higher values ($Q(y) = 67$) than in Figure 2.2. ($Q(y) = 16$). Subsequently, the effect of any upward change of y_t in $Q(y_t)$, once the operation starts, is lower in Figure 2.1. than in Figure 2.2. Moreover, the investment is made earlier in the LD case (period 1-2) than in the CED case (period 4-5). Indeed, the benefit of delaying the investment reduces if the demand becomes more inelastic at higher supply, in contrast to the case where it remains constant.

At second-best, the initial supply is downward distorted from the first-best solution for LD ($Q(y) = 58$) while it is equal to the first-best solution for CED ($Q(y) = 16$). The investment is delayed in both Figures from first-best: period 4-5 in Figure 2.1. and period 7-8 in Figure 2.2. Thus, the initial demand is rationed in Figure 2.1., as suggested by the solution (16) for the case when the demand function has decreasing elasticity along the demand curve. In contrast, for CED, (16) shows that scaling up the operation while delaying investment perfectly compensates the quantity rationing imposed by the constraint, as in Figure 2.2.

Figure 2.1 -The case of linear demand

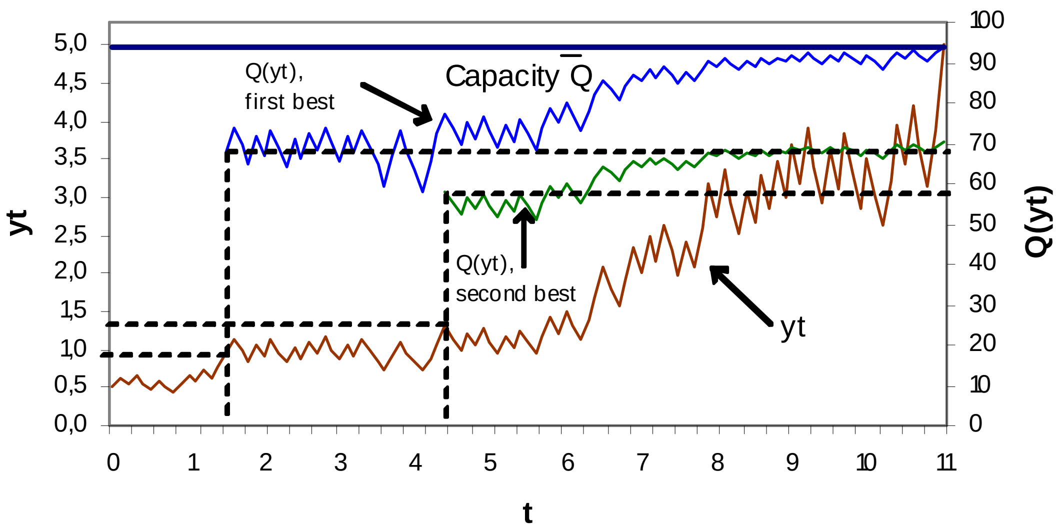
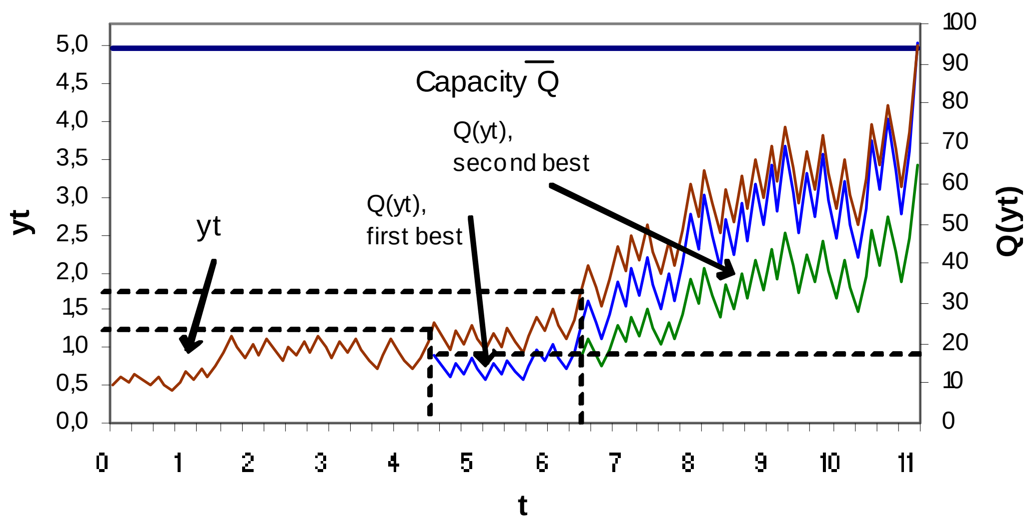


Figure 2.2 - The case of constant elasticity demand



4 Risk insurance

All along the previous analysis, we have assumed that the decision variables are optimally chosen in such a way that the investment cost is expected to be recovered over the entire contract duration (in our case, the entire life of the asset). This is the case when the risk can be transferred to the firm, the latter's budget constraint being met only in expected terms.

In reality, if market conditions fall significantly during the contract execution, then governments tend to intervene and provide subsidies to the firms so as to cover their losses. Thus, governments revise the initial contracting terms in order to limit the award of *ex post* inefficient transfers. In what follows, we develop this point.

As at the second best, budget balance must be satisfied in order to ensure firm's participation. We have

$$x + E^y \int_{\tau}^{\infty} \left(R\left(y_t, Q_t\right) - C\left(I, Q_t\right)\right) e^{-r(t-\tau)} dt \geq 0. \tag{17}$$

If the firm is to be perfectly insured, then a supplementary constraint adds to the social problem. Evaluated at the period τ of investment, the new constraint writes as

$$\left(\frac{y}{y_b}\right)\left(x' + E^{y_b} \int_{\tau_b}^{\infty} \left(R\left(y_v, Q_v\right) - C\left(I, Q_v\right)\right) e^{-r(v-\tau)} dv\right) \geq 0. \tag{18}$$

This equation states that if the demand falls at some y_b during the operation period, where $y_b < y$, an additional transfer $x\prime$ needs be offered to the firm, so that the latter is able to cover the expected losses of $E^{y_b}\left(R\left(y_v, Q_v\right) - C\left(I, Q_v\right)\right) < 0$ at any $v \geq \tau_b$.

The stochastic period when the amount $x\prime$ of public funds should be transferred to the firm is defined as $\tau_b \equiv \inf\left\{t \geq 0 \; s.t. \; y_t = y_b\right\}$. We first saturate (17) and (18) and then replace into the social problem, so that we obtain the new social function

$$\left(\frac{y_0}{y}\right)^{\beta_1}\left[E^y \int_{\tau}^{\infty} \left(S\left(y_t, Q_t\right) - C\left(I, Q_t\right)\right) e^{-r(t-\tau)} dt + \right. \tag{19}$$
$$\lambda E^y \int_{\tau}^{\infty} \left(R\left(y_t, Q_t\right) - C\left(I, Q_t\right)\right) e^{-r(t-\tau)} dt +$$
$$\left. \lambda\left(\frac{y}{y_b}\right)\left(E^{y_b} \int_{\tau_b}^{\infty} \left(R\left(y_v, Q_v\right) - C\left(I, Q_v\right)\right) e^{-r(v-\tau_b)} dv\right)\right].$$

The problem is to be solved by backward induction. Firstly, the social planner finds the optimal demand at which it is preferable to cover the firm's costs by means of public subsidies, rather than through market revenues. Secondly, he identifies the optimal time-scale bundle.

Public subsidies of expected value x' should be provided when the demand is at the value y_b that satisfies the first order condition of (19), which is written as

$$E^{y_b} \int_{\tau_b}^{\infty} \left(R\left(y_v, Q_v\right) - \frac{\beta_2}{\beta_2 - 1} C\left(I, Q_v\right) \right) e^{-r(v-\tau_b)} dv = 0. \qquad (20)$$

In some sense, the interpretation of the value y_b that solves (20) is similar to that of the "bankruptcy" trigger found by Leland (1994). The latter argues that a manager who bears the risk of a project decides to declare bankruptcy whenever the expected revenues are "marked down" by $\beta_2 / \left(\beta_2 - 1\right) < 1$ from the cost incurred by continuing the activity. Our solution shows that a public authority facing a risky project takes a "similar" decision. She identifies the optimal level of the demand at which the transfer x' should take place. Because the public transfer is costly, it is optimal to rely on public funds just when the market revenues are a fraction $\beta_2 / \left(\beta_2 - 1\right)$ of operation costs.

It is noteworthy that, since the value y_b is endogenous to the social problem, so is the risk of having to spend public money to cover operation losses. The government trades-off the efficiency of time-scale bundle (the first-best one) against both *ex ante* budget balance (the second-best constraint) and *ex post* budget balance (the risk-insurance constraint).

Let us next evaluate the social function in (19) at the optimal value of y_b. This yields

$$\left(\frac{y_0}{y}\right)^{\beta_1} \Bigg[E^y \int_{\tau}^{\infty} \left(S\left(y_t, Q_t\right) - C\left(I, Q_t\right)\right) e^{-r(t-\tau)} dt + \qquad (21)$$
$$\lambda \left(E^y \int_{\tau}^{\infty} \left(R\left(y_t, Q_t\right) - C\left(I, Q_t\right)\right) e^{-r(t-\tau)} dt \right) +$$
$$\lambda E^{y_b} \int_{\tau_b}^{\infty} \frac{1}{\beta_2 - 1} C\left(I, Q_v\right) e^{-r(v-\tau)} dv \Bigg]$$

We can now find the optimal supply trajectory in the current context. Whenever capacity is not entirely used, the first-order condition with respect of Q_t is given by

$$S_t = \begin{cases} C_t \frac{\varepsilon_{CQ_t}}{\varepsilon_{SQ_t}} + \frac{\lambda}{1+\lambda} \left(S_t - \frac{\varepsilon_{SQ_t}}{\varepsilon_{RQ_t}} R_t \right), & \text{if } \tau \le t < \tau_b, \\ C_t \frac{\varepsilon_{CQ_t}}{\varepsilon_{SQ_t}} + \frac{\lambda}{1+\lambda} \left(S_t - \frac{\varepsilon_{SQ_t}}{\varepsilon_{RQ_t}} R_t \right) - \frac{1}{\beta_2 - 1} C_t \frac{\varepsilon_{CQ_t}}{\varepsilon_{SQ_t}}, & \text{if } t \ge \tau_b. \end{cases} \qquad (22)$$

Looking at the first line of (22), we notice that, before the demand falls significantly, the first-order condition with respect to Q_t is the same as in the second-best problem, namely (13). The first-order condition in the second line of (22), captures the risk of public transfers which arises when the market falls.

The optimal path of the demand, evaluated at optimal investment timing, solves the equation

$$\int_0^\infty C_t \frac{\varepsilon_{CQ_t}}{\varepsilon_{SQ_t}} e^{-r(t-\tau)} dt = \int_0^\infty \left(C_t \frac{\beta_1}{\beta_1 - 1} - \frac{\lambda}{1+\lambda} \left(1 - \frac{\varepsilon_{SQ_t}}{\varepsilon_{RQ_t}}\right) R_t \right) e^{-r(t-\tau)} dt - \frac{1}{1-\beta_2} \int_{\tau_b}^\infty C_t \frac{\varepsilon_{CQ_t}}{\varepsilon_{SQ_t}} e^{-r(t-\tau)} dt \quad (23)$$

We are now able to discuss how the optimal supply is determined in the presence of risk insurance. Let us compare the solution above with the one we found in the second-best environment. The difference resides in the additional term $-\frac{1}{1-\beta_2} \int_{\tau_b}^\infty C_t \frac{\varepsilon_{CQ_t}}{\varepsilon_{SQ_t}} e^{-r(t-\tau)} dt < 0$ which appears in (23). The sign of this term, together with Assumption 3, which involves that, at each t during the operation interval, $\varepsilon_{CQ_t}/\varepsilon_{SQ_t}$ is set lower than in the second-best case, reveals that the expected trajectory of the supply is downward distorted as compared to the second best.

Equation (23) completely characterizes the optimal trade-off in a project with flexible investment timing and risk insurance. The first term in the right-hand side suggests that the timing of investment is delayed and the quantity is increased when uncertainty increases. Indeed, $\beta_1/(\beta_1 - 1)$ is larger, the higher the uncertainty, so that $\varepsilon_{CQ_t}/\varepsilon_{SQ_t}$ also increases and the quantity path moves upward. This was found already in the first-best environment.

The second term, namely $-\lambda/(1+\lambda) \int_0^\infty (1 - \varepsilon_{SQ_t}/\varepsilon_{RQ_t}) R_t e^{-rt} dt$, reveals that the quantity results as the net effect between rationing required under budget balance and upward scaling induced by investment delay. As the reader should recall, this appeared already in the second-best scenario, where we also pointed that the term previously mentioned behaves differently according to the elasticity of the demand function and that the investment delay is necessary to avoid inefficient public transfers.

Lastly, the third term in the right-hand side of (23), $-\frac{1}{1-\beta_2} \int_{\tau_b}^\infty C_t \frac{\varepsilon_{CQ_t}}{\varepsilon_{SQ_t}} e^{-r(t-\tau)} dt$, is specific of the regime under scrutiny and shows that quantity rationing occurs also when the operator is risk insured. This allows to reduce the risk that subventions need to be awarded during the operation interval. In this sense, the timing-scale bundle is such that social efficiency is traded-off against ex ante budget balance and ex post budget balance (that is, risk insurance). A couple of observations are worth making, before concluding. Firstly, the quantity distortion previously mentioned is present whatever the demand elasticity. Secondly, as the volatility σ^2 of the exogenous demand path gets higher, β_2 becomes less negative and quantity rationing increases. The intuition behind this is that, the higher the uncertainty, the more important the risk of providing public transfers during operation, which negatively impacts social efficiency.

5 Conclusion

In this paper, we have considered an infrastructure project in the presence of stochastic preferences for the associated service and characterized the optimal timing of investment and scale of operation in the limit of the installed capacity. We have found that if the capacity is given to the decision maker, the optimal congestion level is endogenous to the problem of investment timing and varies during the operation period according to the changes in the preferences of the infrastructure users.

In the second-best environment, the budget constraint of the investor must be satisfied. The investment needs be postponed with respect to the socially optimal moment. Moreover, quantity is rationed in case the price elasticity of demand varies in the quantity itself. The second-best time-quantity bundle adjusts more easily, being the congestion level endogenous. In definitive, the level of congestion and the investment time are those which solve the trade-off between social efficiency and financial viability of the project.

Providing full insurance to the investor by transferring public funds further increases consumer rationing, an effect which is present indifferently of the demand elasticity.

Our results suggest that the traffic prevision that is at the basis of the evaluation of transportation projects, should clearly differentiate between the demand component which depends on the characteristics of the service and the exogenous demand component. The former is an instrument under the control of the decision maker, whereas the latter results from exogenous shocks in the economy, hence it is not manageable.

Our analysis also reveals the importance of the demand elasticity to price (whether constant or variable). Many times in transportation studies, it is assumed that the elasticity of the demand is constant. This hypothesis may prove inappropriate to an accurate evaluation of such projects.

In the framework of this study, the issue of contract duration becomes interesting. Observe that a flexible contract term, of the kind proposed by Engels et Alii (1997), would be endogenously determined in our timing-scale allocation. Duration would be infinite in our second-best environment. On the other hand, it would reduce proportionally in the level of uncertainty, should the operator be insured against market fall. However, to clearly understand how the contract duration should be endogenized in the timing-scale decision, we should explicitly model the post-contract social benefits. This is left for further research.

In our work, operation frictionlessly adjusts to demand variations. Yet it is not clear that, in reality, supply can immediately adapt to any exogenous change in consumers preferences. In railways, for instance, adjustment costs weaken supply adaptability to demand. In highways, by contrast, the traffic flow fully reflects the dynamics of road user valuations and sensitivity to congestion across periods. This suggests that the various modes would require specific approaches, as they exhibit different degree of adjustment flexibility, due to their operational technologies.

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Appendix

Le us take a value $F(y_t)$ based on the stochastic process (2); any such value satisfies the standard Ito's lemma

$$\alpha y_t F_{y_t} dt + \frac{1}{2}\sigma^2 y^2 F_{y_t y_t} - rF + A_0 = 0, \tag{24}$$

that has the general solution

$$F(y_t) = A_0 + A_1 y_t^{\beta_1} + A_2 y_t^{\beta_2}, \tag{25}$$

where $\beta_1 > 1$ and $\beta_2 < 0$ are known constants (see Dixit and Pindyck, 1994), while A_0, A_1, A_2 are constants of derivation, as determined by the boundary conditions of (24).

The value of the project at any period τ when the stochastic demand takes the value $y_\tau = y$ is written in general form as

$$V(y) = E^y \int_\tau^\infty (S(y_t, Q_t) - C(I, Q_t))\, e^{-r(t-\tau)} dt + A_1 y^{\beta_1} + A_2 y^{\beta_2}. \tag{26}$$

The first term of the right hand side is the value of the project if the marginal cost pricing is held constant. Equivalently, the optimal supply is always evaluated at $Q(y_t)$. The other two terms express the fact that $Q(y_t)$ is restricted to the region $\left[0, \overline{Q}\right]$, and so the marginal cost pricing is restricted. The constant A_1 is determined at the intersection of the first and second region in Figure 1 in the main text. The term $A_2 y^{\beta_2}$ expresses the possibility of temporary suspension in the future, should the demand be very low. We exclude this case from our analysis as it is not really relevant for the transportation industry. Therefore we set $A_2 = 0$.

At any y in the future such that $y > \overline{y}$, the value of the project is written in general form as

$$\overline{V}(y) = E^y \int_\tau^\infty \left(S\left(y_t, \overline{Q}\right) - C\left(I, \overline{Q}\right)\right) e^{-r(t-\tau)} + B_1 y^{\beta_1} + B_2 y^{\beta_2}. \tag{27}$$

The first term of the right hand side is the discounted flow of net surplus, in expectation over y, from operation at full capacity all over the future. $B_1 y^{\beta_1}$ is the expected change of social flow if y_t increases in the future. This term is economically irrelevant so that we set $B_1 = 0$. The term $B_2 y^{\beta_2}$ shows how the social flow changes when y_t falls below $\overline{y}$ in the future and the social value becomes again that of (26).

For the subsequent analysis we need to express the relative values of the net flows at $Q(y_t)$ and $\overline{Q}$. By envelope theorem,

$$\begin{aligned}
& \frac{\partial}{\partial y} E^y \int_\tau^\infty \left(s\left(Q\left(y_t\right)\right) y_t - c\left(I, Q\left(y_t\right)\right)\right) e^{-r(t-\tau)} dt \\
= \; & E^y \int_\tau^\infty \left[\left(s'\left(Q_t\right) y_t - c\right) Q'\left(y_t\right) e^{-(r-\alpha)(t-\tau)} + s\left(Q\left(y_t\right)\right) e^{-(r-\alpha)(t-\tau)}\right] dt \\
= \; & E^y \int_\tau^\infty s\left(Q\left(y_t\right)\right) e^{-(r-\alpha)(t-\tau)} dt.
\end{aligned}$$

At $\overline{Q}$, the relative value is written

$$\frac{\partial}{\partial y} E^y \int_0^\infty \left(s\left(\overline{Q}\right) y_t - c\left(I, \overline{Q}\right)\right) e^{-rt} dt = s\left(\overline{Q}\right) \frac{1}{r-\alpha}.$$

The absolute and relative values of (26) and (27) are equal at $\overline{y}$.

It follows that

$$\begin{aligned}
E^{\overline{y}} \int_\tau^\infty \left(s\left(Q\left(y_t\right)\right) y_t - C\left(I, Q\left(y_t\right)\right)\right) e^{-r(t-\tau)} dt + A_1 \overline{y}^{\beta_1} = \\
= E^{\overline{y}} \int_\tau^\infty \left(S\left(y_t, \overline{Q}\right) - C\left(I, \overline{Q}\right)\right) e^{-r(t-\tau)} dt + B_2 \overline{y}^{\beta_2} \\
E^{\overline{y}} \int_\tau^\infty s\left(Q\left(y_t\right)\right) e^{-(r-\alpha)(t-\tau)} dt + \beta_1 A_1 \overline{y}^{\beta_1 - 1} = s\left(\overline{Q}\right) \frac{1}{r-\alpha} + \beta_2 B_2 \overline{y}^{\beta_2 - 1}
\end{aligned}$$

From which it follows that

$$\begin{aligned}
\beta_2 A_1 \overline{y}^{\beta_1} &= \beta_2 \left(s\left(\overline{Q}\right) \frac{y}{r-\alpha} - E^{\overline{y}} \int_\tau^\infty s\left(Q\left(y_t\right)\right) y_t e^{-r(t-\tau)} dt \right) - \\
& \quad -\beta_2 \left(\frac{C\left(I, Q\right)}{r} - \int_\tau^\infty C\left(I, Q\left(y_t\right)\right) e^{-r(t-\tau)} \right) + \beta_2 B_2 \overline{y}^{\beta_2} \\
\beta_1 A_1 \overline{y}^{\beta_1} &= s\left(\overline{Q}\right) \frac{y}{r-\alpha} - E^{\overline{y}} \int_\tau^\infty s\left(Q\left(y_t\right)\right) y e^{-(r-\alpha)(t-\tau)} dt + \beta_2 B_2 \overline{y}^{\beta_2}
\end{aligned}$$

Substracting the two equations above gives

$$\begin{aligned}
A_1 \overline{y}^{\beta_1} &= \frac{1-\beta_2}{\beta_1-\beta_2} \left(s\left(\overline{Q}\right) \frac{\overline{y}}{r-\alpha} - E^{\overline{y}} \int_\tau^\infty s\left(Q\left(y_t\right)\right) y e^{-(r-\alpha)(t-\tau)} dt \right) + \\
& \quad + \frac{\beta_2}{\beta_1-\beta_2} \left(\frac{C\left(I, Q\right)}{r} - E^{\overline{y}} \int_\tau^\infty C\left(I, Q\left(y_t\right)\right) e^{-r(t-\tau)} \right).
\end{aligned} \tag{28}$$

We replace (28) in (26) so that we obtain the value of the project $V(y)$ expressed in (5) in the main text.