OPTIMALITY IN ASSET RENEWALS

By

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Abstract

Using the framework of real options, we develop a model and derive the optimal solution for the case of asset renewal. In contrast to capital replacement of physical assets, the applicable contexts for asset renewals are those in the service sector such as hotels, commercial web-sites and human resources, where the decision to renew depends on both the revenue the asset generates and the operating and maintenance cost it incurs. An analytical solution is derived for the model involving the two distinct but stochastically dependent sources of uncertainty without recourse to homogeneity of degree one to reduce the model's dimensionality. We find that under plausible conditions the value of the existing asset plus its renewal option is an increasing function of the underlying volatilities while the trigger level for revenue signalling renewal is a decreasing function. In the presence of increasing uncertainty, patience has to be exercised before making the renewal decision. Further, the capital outlay required for renewal, the discount rate and the change rate for cost have a negative effect on the value of the existing asset plus its renewal option and on the trigger level for revenue while the starting revenue following renewal and the change rate for revenue have a positive effect. Finally, we determine the conditions under which homogeneity of degree one can be justified and show that these conditions are not upheld for the present analysis.

Keywords: Replacement options, asset renewal, two stochastic variables

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1. Introduction

Investing in a productive asset creates a set of embedded real options: to renew the asset which deteriorates in quality (and possibly loses cost efficiency) over time; or offers possible expansion, contraction or abandonment alternatives. We examine the first set of renewal or revitalization or redevelopment alternatives, where other authors (such as Sick, 1989) require the assumption of homogeneity of degree one for quality and cost in order to obtain a closed-form solution.

Renewal options have their origin in the various sources of uncertainty associated with deploying the asset and arise from variability in revenues and cost amongst others, coupled with the opportunity the firm's management has to change the underlying state of the asset. At any instant of time, the management can assess the economic viability of the asset's continued deployment through conducting a real options analysis and can decide whether the time is appropriate to change the state of the asset by renewing it in some way or by replacing it with a new version. Replacement and renewal decisions typically entail determining the conditions signalling when it is economically acceptable to discontinue the use of the asset in its current state and to make an investment that will restore its former potential. The real options methodology has revealed a significant contrast between capital investment decisions formulated under the assumption of certainty with those formulated under uncertainty due to the time value of waiting. McDonald and Siegel (1986) and Dixit and Pindyck (1994) have demonstrated that in the presence of uncertainty a firm should wait before making an investment until the value of an investment opportunity exceeds the cost of that investment by an amount equalling the value of option to defer the investment. Further since the value of this waiting option increases with the underlying volatility, firms should in the presence of uncertainty act with greater patience and shun investing prematurely.

In the absence of uncertainty, the standard method for evaluating the optimal time between asset replacements is to conceive replacement as an infinite chain, to treat the present value of future cash flows as an annuity and then to minimise the equivalent annual amount. This formulation is founded on the net present value technique and presupposes that the investment cost of the replacement, the revenues it yields and the costs it incurs are all known with certainty before the original investment is made. An alternative approach that yields exactly the same solution is to reformulate the capital replacement problem using a dynamic programming model. This has the advantage that the model can be updated with new information as it becomes available. Since the dynamic programming principle is invoked that the optimal decision identified at any point in time is based on the optimality of all future subsequent decisions, the model can be re-evaluated at any time during the lifetime of the existing asset in order to incorporate new information on the exogenous parameters as and when it is revealed and a revised optimal decision can then be identified. However, both the net present value model based on equivalent annual amount and the dynamic programming formulation are not robust in the presence of uncertainty. An additional limitation is the discrete nature of these formulations that rules out the possibility of reaching any analytical solutions from which it would be possible to derive general results on the opportune conditions signalling replacement and on how the various exogenous factors influence those conditions. A real options analysis overcomes both of these limitations.

This paper applies the real options methodology to derive an analytical solution for the case of optimal asset renewal in the presence of two dependent sources of uncertainty, revenues and costs. Three similar papers, McLaughlin and Taggart (1992), Mauer and Ott (1995), Dobbs (2004), assume that the capital asset under consideration is physical equipment that provides a constant revenue over time but whose functionality degrades with accumulative usage so that its operating and maintenance costs increase with time. When the physical equipment deteriorates to the extent that its operating and maintenance costs have attained a high bar level, the asset is replaced by a new version at a known outlay cost and its costs revert to the original favourable level. Outside of this world of physical equipment exists a world of assets whose functionality similarly degrade with usage so that their operating and maintenance costs increase over time, but whose revenue stream is not constant. Hotels at the time of their establishment and newly constructed blocks of apartment flats or commercial property can all command premium prices relative to their respective incumbent rivals because of their newness, novel features or ease of use. Over time, the asset deteriorates and the differentiating factors that distinguished it from the competition fade. Consequently the operating and maintenance costs increase and revenues wane as the asset attractiveness declines. As the difference between revenues and costs narrows, a point will be reached that signals the replacement, renewal or refurbishment of the asset to bring it back to its original condition. Following the capital outlay required to restore the asset to its former state, the revenues and costs will revert to their more favourable levels. The process of asset usage accompanied by escalating costs and flagging revenues will then re-commence and continue until the conditions signalling re-investment are attained, when the entire process is repeated again. This regenerative process of eroding revenues and escalating costs followed by renewal investment seems to be a credible representation for many phenomenons other than rented real estate. It seems to be applicable for elements of the travel industry such as cruise liners and vehicle rentals, for elements of the media business such as commercial web-sites and media production, for elements of the entertainment business such as theatres, stadium and theme parks, and for elements of professional clubs such as football teams. Indeed most human resources, and especially those that are specialised like professionals, require revitalization through periodic re-education, so perceived standards are upgraded and cost efficiency restored. All these assets develop increasing costs but falling revenues from increasing accumulated usage until the asset condition is restored through an injection of capital spending that brings the revenue and operating and maintenance cost back to their former favourable levels. In this way, the asset progresses endlessly though consecutive stages of usage and renewal. Whereas physical assets seem to experience deterioration through accumulative usage that leads to escalating costs, assets deployed by service industries seem to experience both falling revenues and increasing costs in line with accumulative usage.

The presence of two or more distinct sources of uncertainty that can influence the value of an option on a real asset introduces a significantly higher level of complexity into both the formulation of the model and particularly the derivation of an analytical solution. Real option analyses involving two or more sources of uncertainty have typically adopted one of two methods to overcome the difficulty imposed by multidimensionality. The first method reduces the dimensionality of the resulting partial differential equation. Dixit and Pindyck (1994) treat the option value as a function of homogeneity of degree one. Homogeneity of degree one implies that the partial differential equation for the option value under continuance can be expressed in terms of the ratio of two variables and the resulting equation can be solved using the normal methods. An equivalent approach is to conceive the model in terms of an exchange option formulation and to use the results of Margrabe (1978) to derive the solution. Their equivalence stems from the assumption of homogeneity of degree one that Margrabe (1978) uses to derive his results. The limitation of this first method pivots on the validity of the assumption of homogeneity of degree one. Even if the underlying asset value is a function with homogeneity of degree one, the inference that the option value is similarly a function with homogeneity of degree one (as in Paxson and Pinto, 2005) is not a proof. The second method relies on numerical procedures to solve the multidimensional partial differential equation, as in Boyle (1988), Brennan and Schwartz (1978), Cortazar (2001), Geske and Shastri (1985), and Schwartz and Moon (2000). However, numerical methods do not have the elegance of an analytical solution. In our real options analysis of asset renewal under two distinct sources of uncertainty, we identify an analytical solution that satisfies the partial differential equation for the option under continuance and then proceed to derive an analytical solution. Further, we demonstrate that the analytical solution does not require an assumption of homogeneity of degree one and in fact refutes it.

Sick (1989) and Williams (1991) applied the exchange option analogy to capital budgeting ("now, if we double both P and K, the option value must double"). McLaughlin and Taggart (1992) developed an early application of real options analysis involving capital replacement. Their focus is to determine the opportunity cost of excess capacity through evaluating the change in the option value before and after diverting the existing capacity to produce an alternative product. Their formulation, however, assumes a known lifetime for the equipment and uses a discretised binomial lattice representation to determine the option value. Mauer and Ott (1995) applied a continuous time real options framework to capital equipment replacement under uncertainty. In their model, operating and maintenance cost is expressed as geometric Brownian motion with a drift parameter reflecting asset deterioration and revenues are treated as known and constant. The authors incorporate a depreciation tax shield in the cash flow, subtract the after tax salvage price gain on disposal from capital replacement outlay and postulate that depreciation and salvage price are both functions of the operating and maintenance cost. In line with the common finding between option value and waiting time, the authors demonstrate that the expected time between capital replacements is an increasing function of the underlying volatility. They also show that the expected time between capital replacements is an increasing function of the replacement outlay but a decreasing function of the salvage price. They extend their analysis to incorporate uncertainty concerning the arrival of new more efficient technologies and tax changes. Dobbs (2004) makes similar assumptions concerning the nature of cost and revenue but produces a simplified model variant by excluding the effects of depreciation and the salvage price gains on disposal. He similarly demonstrates that in the presence of uncertainty, the level of operating and maintenance cost signalling the replacement of the capital equipment is an increasing function of the underlying volatility and extends the model to reveal the relationship between the option value and the salvage price.

There exists a parallel thread of investigations emanating from the management science literature on analytical techniques for dealing with uncertainty and the capital replacement decision for real assets. Simulation techniques used in capital budgeting have been explored by Hertz (1964) and Hull (1980). Apeland and Scarf (2003) develop a Bayesian approach to updating the capital replacement decision on the release of new information. Analytical methods more aligned to the analysis of this paper include Massé (1962), Jorgenson, McCall and Radner (1967), Kamien and Schwartz (1971), Feldstein and Rothschild (1974), Rust (1987) and Ye (1990). This paper extends their analyses by determining an analytical solution to the capital replacement option function under uncertainty in the presence of two sources of stochastic variation.

The rest of the paper is organised in the following way. The next section investigates the continuous model of asset renewal under certainty by treating the variables, revenue and cost, as deterministic functions of time in order to use the results as a benchmark against which we can compare the results from the stochastic model. Section 3 is devoted to the formulation and solution of the model under uncertainty. The two uncertain variables, revenue and cost, are inserted into the fundamental partial differential equation expressing the option's value under continuance through Ito's lemma. It is demonstrated that the option's value is a function whose degree of homogeneity is never one. Although we have an analytical function of the option's value, we need numerical methods to derive the levels of the two variables signalling renewal. The numerical methods we use are discussed in section 4. This is followed by the sensitivity analysis of the solution that explores the way the various parametric values influence the solution. Since the condition of homogeneity of degree one is not fulfilled for the current formulation, section 6 investigates the circumstances that have to be obeyed in order to adopt homogeneity of degree one to safely reduce the dimensionality of the partial differential equation. The final section is the conclusion.

2. Renewal Decision under Certainty

A firm owns a capital asset that has a significant impact on its business performance and is seeking to identify the economic conditions triggering its renewal to restore its economic potential. The investment in the asset is treated as irreversible in the sense that the firm is only able to recover an insignificant proportion of the total capital outlay if the project is divested. The net cash flow (or profit) associated with this asset at any time is the differences between the revenue the asset generates and the operating and maintenance cost it incurs. The output revenue is denoted by P and the associated cost by C, so the net revenue at any time is $(P - C)$. The revenue at the origination of the asset is P_0 and P is assumed to decline at the annualised continuous rate of $\alpha \leq 0$ because of the asset's fading attractiveness to generate revenue. The operating and maintenance cost at the origination is C_0 and C is assumed to grow at the annualised continuous rate of $\theta \geq 0$ since the asset's efficiency deteriorates with usage. The initial cost of the capital investment is denoted by K. It is assumed for convenience that the residual salvage value of the asset is zero at the time of its renewal; this has no material bearing on the results since it can be absorbed in K as long as the residual value is constant or at least not stochastic. The present value of the project V for a specified lifetime T is derived from discounting future net revenue cash flows at the annualised continuous risk-adjusted rate of μ:

$$V = \int_0^T \left(P_0 e^{\alpha t} - C_0 e^{\theta t}\right) e^{-\mu t}\, dt. \tag{1}$$

We will assume that the asset is financially viable for some definite lifetime so that $\alpha < \mu$ and $\theta < \mu$. To determine the optimal lifetime for the asset, we consider the conceptual infinite project having value W, which is defined as the sequential chain of identical projects replaced at equal intervals of time T:

$$W = V + (W - K)e^{-\mu T}. \tag{2}$$

Equation (2) assumes that when the existing asset is renewed by a brand new variant so that the output revenue and cost levels revert to their original values of P_0 and C_0 respectively instead of their prevailing values. The optimal lifetime is found from differentiating equation (2):

$$\frac{dW}{dT} = \left(P_0 e^{\alpha T} - C_0 e^{\theta T}\right)e^{-\mu T} + \frac{dW}{dT}e^{-\mu T} - \mu(W - K)e^{-\mu T}. \tag{3}$$

Using the suffix * to denote the optimal value for the variables in the deterministic formulation, the first order condition for a maximum is:

$$(P * -C *)e^{-\mu T*} = \mu(W * -K)e^{-\mu T*} = \mu e^{-\mu T*}\frac{V * -K}{1 - e^{-\mu T*}},$$

so,

$$(P * - C *)\frac{\left(1 - e^{-\mu T*}\right)}{\mu} = V * -K. \tag{4}$$

The right hand side of equation (4) represents the value generated from re-investing in the asset with an optimal lifetime. In contrast, the left hand side of the equation represents an annuity with a lifetime $T*$ where the annuity amount equals the optimal net revenue. When the left hand side value is greater than $V * -K$, then it is more valuable to continue with the existing project until the equality specified by equation (4) is realised. If on the other hand, the left hand side is less than $V * -K$, then it is more valuable to discontinue the existing project and to renew it. The switching point as specified by equation (4) defines the optimal lifetime to occur when the net value generated from renewing the existing asset is equal to an annuity with the annuity amount equalling the difference between the optimal revenue and the cost and with an annuity duration equalling the project's optimal lifetime.

Since:

$$V^* = \int_0^{T^*} \left(P_0 e^{\alpha t} - C_0 e^{\theta t}\right) e^{-\mu t}$$
$$= \frac{P_0}{\mu - \alpha}\left(1 - e^{-(\mu-\alpha)T^*}\right) - \frac{C_0}{\mu - \theta}\left(1 - e^{-(\mu-\theta)T^*}\right)$$

then equation (4) becomes:

$$P^*\left(\frac{1}{\mu} + \frac{\alpha}{\mu} \times \frac{e^{-\mu T^*}}{\mu - \alpha}\right) - C^*\left(\frac{1}{\mu} + \frac{\theta}{\mu} \times \frac{e^{-\mu T^*}}{\mu - \theta}\right) = \frac{P_0}{\mu - \alpha} - \frac{C_0}{\mu - \theta} - K. \quad (5)$$

Since $P^* = P_0 e^{\alpha T^*}$ and $C^* = C_0 e^{\theta T^*}$, the optimal value of T can be found by solving equation (5); because of its non-linearity, the solution has to be evaluated through using a numerical solving procedure such as goal seek provided by Excel®.

Differentiating equation (4) again and evaluating at $\frac{dW}{dT} = 0$ yields:

$$\frac{d^2W}{dT^2} = \frac{e^{-\mu T^*}}{1 - e^{-\mu T^*}}\left(\alpha P^* - \theta C^*\right).$$

The optimal solution is a maximum whenever the incremental change in net revenues with respect to time is negative around the turning point. A negative change in net revenues heralds the trigger point for replacing the asset. Although there are a range of alternative values forcing $\left(\alpha P^* - \theta C^*\right) < 0$, a strong condition for a point of maximum occurs when simultaneously, revenues are declining proportionately, $\alpha < 0$, and costs are increasing proportionately, $\theta > 0$. We can surmise that it is only when the distance between revenues and costs sufficiently narrows that the renewal of the asset becomes economically tenable. If in contrast, the output price is an increasing function of time and unit costs is a decreasing function, then there is no apparent economic reason for renewing the existing asset at all.

3. Renewal Decision under Uncertainty

We will develop the stochastic model by allowing the output revenue P and operating and maintenance cost C to follow distinct but dependent stochastic processes, instead of behaving as deterministic functions of time. In line with the previous real options analysis on capital replacement by Mauer and Ott (1995) and Dobbs (2004), we will adopt the simplest stochastic process that is the basis for many continuous time real options analyses, see Dixit and Pindyck (1994), Trigeorgis (1996), by assuming that each of the two variables follows the geometric Brownian motion process with drift. The stochastic process for revenue P is described by:

$$dP = \alpha P dt + \sigma_P P\, dz_P . \tag{6}$$

The change in the value of P over an interval t to $t + dt$ is explained by a proportional deterministic drift component α and an incremental Wiener term comprising the stochastic component. The Wiener process is defined by $dz_P = u_{t,P}\sqrt{dt}$ where $u_{t,P}$ follows a standard Normal distribution with zero mean and unit variance, implying that $E\left[dP\right] = \alpha P dt$ and $Var\left[dP\right] = \sigma_P^2 P^2\, dt$. In the same way, the stochastic process for C is described by:

$$dC = \theta C dt + \sigma_C C\, dz_C , \tag{7}$$

with similar definitions. We will treat the stochastic evolutions of revenue and cost to be dependent by allowing the exogenous shocks affecting the variables P and C to be correlated. This potential dependence is described by the covariance of the two shock variables, and so:

$$Cov\left[dP, dC\right] = \rho \sigma_P \sigma_C P C dt ,$$

where the correlation coefficient ρ is constrained by $|\rho| \leq 1$.

The decision facing management is whether the existing asset should be renewed immediately by a new variant or whether it should continue to be deployed until less favourable values of P and C are revealed. The decision that management makes ought to be based on the prevailing values of P and C. The solution for the deterministic model tells us that the capital asset will be renewed only when the levels of the revenue and cost have sufficiently narrowed and that the optimal values of P and C signalling renewal occurs simultaneously at a single point in time. Let the levels triggering the renewal of the asset under uncertainty be $\tilde{P}$ and $\tilde{C}$ for P and C respectively. Treating P and C as independent, we can envisage a trigger revenue level $\tilde{P}$ where $\tilde{P} \leq P_0$ that will prompt the renewal of the asset, and a trigger cost level $\tilde{C}$ where $\tilde{C} \geq C_0$ that will prompt the asset's renewal. This is in line with the strong sufficient condition for a maximum for the deterministic model. However, in contrast to the deterministic model, we are not necessarily searching for single pair of trigger points $\{\tilde{P}, \tilde{C}\}$ when the variables are stochastic. The time when P attains its trigger level may not coincide exactly with the time when C attains its trigger level. Moreover, it is as rational to replace the asset when relatively P is moderate when C is low as when P is high when C is moderate. The conditions signalling the asset's renewal can only be described by a possibly infinite set of pairs $\{\tilde{P}, \tilde{C}\}$, which can be represented by the locus function $G(\tilde{P}, \tilde{C}) = 0$ such that the asset is renewed whenever $G(\tilde{P}, \tilde{C}) \leq 0$ and is not renewed when otherwise. At every instant of time, management has to compute the value of the function $G(P, C)$ based on the prevailing values of P and C, and then make a decision based on the functions' value on whether the existing asset should continue to be deployed or whether it should be renewed.

We introduce the function F, which is defined as the value of the existing asset plus the option to renewed it at the exercise price K. The function F will depend on the prevailing values of both revenue and cost, so $F = F(P, C)$. Our analysis is to identify the function F and to reveal its properties. We first proceed to identify the relevant boundary conditions for the function F, which are expressed as its asymptotic values and the economic requirements governing the renewal of the existing asset at the cost of K. Secondly, we determine the function F under the condition for the continuance of the existing asset, and then finally use the boundary conditions to identify the pairs of trigger points $\tilde{P}$ and $\tilde{C}$.

The function F, which has to be non-negative otherwise there would be no investment in the asset, can be separated in principle between the value of the existing asset $F_V(P, C)$ and the renewal option value $F_O(P, V)$, so $F = F_V + F_O$. Since the option value is always non-negative, $F \geq F_V$, Trigeorgis (1996). Assuming an infinite lifetime,

$$F_V(t \to \infty) = \frac{P}{\mu - \alpha} - \frac{C}{\mu - \theta} \tag{8}$$

When P becomes very large, there is little or no economic justification for replacing the asset so the renewal option value tends to zero and $F_V(t \to \infty)$ dominates the value of F. In contrast, a value of P close to zero makes the renewal of the asset almost inevitable and this is reflected in a significantly large value for the renewal option so the value of F_O dominates the value of F_V. Similarly, we can argue that there is little or no economic justification for renewing the asset when the costs become very small and consequently, the value of the renewal option will be relatively small and dominated by the value of F_V. In contrast, there is strong economic justification for renewing the asset when C becomes significantly large. This implies that the value of the renewal option becomes relatively large and dominated the value of F_V. We can express all these conditions as:

$$F_o(\infty, C) = 0, F_o(0, C) = \infty, F_o(P, 0) = 0, F_o(P, \infty) = \infty. \tag{9}$$

We can extend the boundary condition developed by Dobbs (2004) to the current formulation. The boundary condition defining the switching region at $P = \tilde{P}$ and $C = \tilde{C}$ is derived by replacing W by F in equation (2) and temporarily treating F as a function of T:

$$F(T) = \int_{T}^{T*} (P - C) e^{-\mu t} dt + (F(0) - K) e^{-\mu(T* - T)}. \tag{10}$$

Allowing T to equal $T*$ and rewriting equation (10) in terms of the variables P and C:

$$F(\tilde{P}, \tilde{C}) = F(P_0, C_0) - K. \tag{11}$$

This is the value matching condition, Dixit and Pindyck(1994). It defines the point values for variables $P = \tilde{P}$ and $C = \tilde{C}$ when management is indifferent between the continuance of the existing asset and its renewal and it represents the condition defining the trigger for switching between the existing asset and its renewal. Associated with this is the smooth pasting condition that demands that the two functions representing the continuance of the existing asset and its renewal, expressed respectively as the left and right hand sides of equation (11), have identical slopes at the trigger point. This requires that:

$$\left.\frac{\partial F}{\partial P}\right|_{P=\tilde{P},C=\tilde{C}} = 0\,, \tag{12}$$

and

$$\left.\frac{\partial F}{\partial C}\right|_{P=\tilde{P},C=\tilde{C}} = 0\,, \tag{13}$$

since the right hand side of equation (11) is a constant.

Applying the general result developed by Dixit and Pindyck(1994), the function F assuming continuance of the existing asset is derived from the dynamic programming principle:

$$\mu F dt = (P - C)\,dt + E\left[dF\right], \tag{14}$$

where E denotes the expectation operator. Equation (14) states that over the time interval t to $t + dt$, the expected return from owning the asset F is equal to the profit flow $(P - C)\,dt$ that the asset generates and the expected capital appreciation of F. As in the deterministic case, we assume that $\mu - \alpha > 0$ and $\mu - \theta > 0$. As a function of both P and C, dF can be found using Ito's lemma, Trigeorgis(1996):

$$\begin{aligned} dF = & \left(\tfrac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 F}{\partial P^2} + \tfrac{1}{2}\sigma_C^2 C^2 \frac{\partial^2 F}{\partial C^2} + \rho\sigma_P\sigma_C PC \frac{\partial^2 F}{\partial P \partial C}\right) dt \\ & + \left(\alpha P \frac{\partial F}{\partial P} + \theta C \frac{\partial F}{\partial C} + \frac{\partial F}{\partial t}\right) dt + \sigma_P P \frac{\partial F}{\partial P} dz_P + \sigma_C C \frac{\partial F}{\partial C} dz_C \end{aligned} \tag{15}$$

When taking the expectation of dF, the terms involving dz_P and dz_C in equation (15) fall to zero. Also, we can set $\frac{\partial F}{\partial t} = 0$ since the conceptual time horizon is infinity and the parameters α and θ are treated as independent of time, Dixit and Pindyck (1994). It follows that equation (14) after dividing by dt can be rewritten as:

$$\begin{aligned} & \tfrac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 F}{\partial P^2} + \tfrac{1}{2}\sigma_C^2 C^2 \frac{\partial^2 F}{\partial C^2} + \rho\sigma_P\sigma_C PC \frac{\partial^2 F}{\partial P \partial C} \\ & + \alpha P \frac{\partial F}{\partial P} + \theta C \frac{\partial F}{\partial C} - \mu F + (P - C) = 0 \end{aligned} \tag{16}$$

The solution to this partial differential equation is made up of the particular solution and the homogenous solution. The particular solution is:

$$F = \frac{P}{\mu - \alpha} - \frac{C}{\mu - \theta}, \tag{17}$$

which takes the same form as (8). This implies that the homogenous solution represents the renewal option value minus a quantity reflecting the extent that the existing asset only has a finite lifetime.

The simplest kind of function satisfying the homogenous part of the partial differential equation takes the form:

$$F = AP^{\beta}C^{\eta} \tag{18}$$

where A is a parameter to be determined. This generic functional form can be justified on two counts. In their analysis of cost and price uncertainty, Dixit and Pindyck (1994) reach the same functional form as (18) with the additional requirement that the exponents comply with homogeneity of degree one, $\beta + \eta = 1$. In our formulation, we make no assumptions on the degree of homogeneity of F. Secondly, the functional form (18) satisfies the partial differential equation (16); substituting (18) in (16) yields:

$$\left(\tfrac{1}{2}\sigma_P^2\beta\left(\beta - 1\right) + \tfrac{1}{2}\sigma_C^2\eta\left(\eta - 1\right) + \rho\sigma_P\sigma_C\beta\eta + \alpha\beta + \theta\eta - \mu\right)F = 0.$$

It follows that the parameters β and η must satisfy the relationship:

$$Q\left(\beta, \eta\right) = \tfrac{1}{2}\sigma_P^2\beta\left(\beta - 1\right) + \tfrac{1}{2}\sigma_C^2\eta\left(\eta - 1\right) + \rho\sigma_P\sigma_C\beta\eta + \alpha\beta + \theta\eta - \mu = 0. \tag{19}$$

The function $Q\left(\beta, \eta\right) = 0$ defines an ellipse since the square of the coefficient for the term $\beta\eta$ is less than four times the product of the coefficients for the terms β^2 and η^2, Kaplan and Lewis (1971), that is:

$$\left(\rho\sigma_P\sigma_C\right)^2 < 4\left(\tfrac{1}{2}\sigma_P^2\,\tfrac{1}{2}\sigma_C^2\right).$$

The function $Q\left(\beta, \eta\right) = 0$ crosses the η axis for the pair of β values:

$$\beta_{+,-} = \left(\tfrac{1}{2} - \frac{\alpha}{\sigma_P^2}\right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{\alpha}{\sigma_P^2}\right)^2 + \frac{2\mu}{\sigma_P^2}}, \tag{20}$$

Dixit and Pindyck (1994) establish that $\beta_+ \geq 0$ and $\beta_- \leq 0$. Similarly, the function $Q\left(\beta, \eta\right) = 0$ crosses the β axis for the pair of η values:

$$\eta_{+,-} = \left(\tfrac{1}{2} - \frac{\theta}{\sigma_C^2}\right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{\theta}{\sigma_C^2}\right)^2 + \frac{2\mu}{\sigma_C^2}}, \tag{21}$$

where $\eta_+ \geq 0$ and $\eta_- < 0$.

The ellipsoidal shape $Q(\beta, \eta) = 0$ is illustrated in figure (1a, 1b and 1c), which exhibits the function Q for the following set of parametric values that define the base case:

α	θ	μ	σ_P	σ_C
-5%	10%	25%	25%	25%

and for ρ equal to 0.5, 0.0 and -0.5 respectively. Because of (20) and (21), the function $Q(\beta, \eta) = 0$ intersects the two axes at identical points irrespective of the value of ρ: for the base case these are:

	Positive Root	Negative Root
β	4.413	-1.813
η	1.935	-4.135

Figure (1) reveals that the parameter ρ influences the tilt of the ellipsoid. Denoting by φ the angle required to rotate the axes in a clockwise direction in order to eliminate the term in $\beta\eta$, then:

$$\varphi = \tfrac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_P\sigma_C}{\sigma_C^2 - \sigma_P^2}\right),$$

except when $\sigma_P = \sigma_C$, in which case $\varphi = \pm\frac{\pi}{4}$ whose sign depends on the sign of ρ, Kaplan and Lewis (1971).

Although constrained by $Q(\beta, \eta) = 0$, the possible values of β and η can belong to any of the four quadrants, that is:

$$\begin{aligned}
&\{\beta_1, \eta_1\}\ \beta_1 \geq 0, \eta_1 \geq 0;\\
&\{\beta_1, \eta_2\}\ \beta_1 \geq 0, \eta_2 \leq 0;\\
&\{\beta_2, \eta_1\}\ \beta_2 \leq 0, \eta_1 \geq 0;\\
&\{\beta_2, \eta_2\}\ \beta_2 \leq 0, \eta_2 \leq 0.
\end{aligned}$$

This implies that the homogenous solution has the general form:

$$F = A_1 P^{\beta_1} C^{\eta_1} + A_2 P^{\beta_1} C^{\eta_2} + A_3 P^{\beta_2} C^{\eta_1} + A_4 P^{\beta_2} C^{\eta_2}. \tag{22}$$

The values of the coefficients for equation (22) can be assessed in line with the conditions expressed in equation (9). Since $F_0(0, C) = \infty$ then $\beta < 0$, and since $F_0(P, \infty) = \infty$ then $\eta > 0$, which together imply that $A_3 \neq 0$. In contrast, since $F_0(\infty, C) = 0$ then A_1 and A_2 are both zero, and since $F_0(P, 0) = 0$ then $A_4 = 0$. Collectively, equation (22) reduces to:

$$F = A_3 P^{\beta_2} C^{\eta_1}. \tag{23}$$

Stitching together the homogenous solution (23) with the particular solution (17) yields the overall solution:

$$F = A_3 P^{\beta_2} C^{\eta_1} + \frac{P}{\mu - \alpha} - \frac{C}{\mu - \theta}. \tag{24}$$

Substituting the value matching condition (11) in (24) yields:

$$A_3 \tilde{P}^{\beta_2} \tilde{C}^{\eta_1} + \frac{\tilde{P}}{\mu - \alpha} - \frac{\tilde{C}}{\mu - \theta} = A_3 P_0^{\beta_2} C_0^{\eta_1} + \frac{P_0}{\mu - \alpha} - \frac{C_0}{\mu - \theta} - K. \tag{25}$$

The two smooth pasting conditions, equations (12) and (13) respectively, imply that:

$$A_3 \beta_2 \tilde{P}^{\beta_2 - 1} \tilde{C}^{\eta_1} + \frac{1}{\mu - \alpha} = 0, \tag{26}$$

$$A_3 \eta_1 \tilde{P}^{\beta_2} \tilde{C}^{\eta_1 - 1} - \frac{1}{\mu - \theta} = 0. \tag{27}$$

Collectively, the four equations (19), (25), (26) and (27) contain five unknowns; these are β_2, η_1, A_3, $\tilde{P}$ and $\tilde{C}$. Since it is not possible to determine unique values for all five unknowns, the model seems to be indeterminate and on the surface flawed.

In their analysis of investment under two distinct sources of uncertainty, Dixit and Pindyck (1994) instead of identifying unique levels signalling investment for the two stochastic variables, determine a boundary that discriminates between continuance and investment based on the values for two variables. Their solution involves identifying the function that defines discriminating boundary as a function of the two variables rather than their unique levels. A single pair of trigger values $\{\tilde{P}, \tilde{C}\}$ signalling the investment does not make economic sense in the presence of two sources of uncertainty since we can envisage any number of distinct scenarios all indicating asset renewal. Management would equally renew the asset when C is high and P is moderate as when C is moderate and P is low. Further, since the probability of simultaneously attaining the single pair of trigger values $\{\tilde{P},\tilde{C}\}$ is remote, we could argue that the appropriate trigger region should be specified by say $\{P \le \tilde{P}, C \ge \tilde{C}\}$, which states that the renewal of the existing asset would be signalled when the prevailing revenue falls below some floor level and prevailing cost rises above some ceiling level. However, this decision rule ignores the possibility of any trade-off between the respective movements of revenue on one hand and operating and maintenance cost on the other. When the boundary discriminating between continuance and renewal is smooth, then along this boundary management would hold on to the same decision since any positive (negative) increment in C could be compensated by a positive (negative) increment in P. In the presence of a trade-off, the boundary discriminating between continuance and renewal is described by a smooth function, $G(\tilde{P},\tilde{C}) = 0$.

The function $G(\tilde{P},\tilde{C}) = 0$ is derived using the following procedure. We start by specifying the value of one of the unknowns, say $\tilde{C}$. From this starting value, it should be feasible at least in principle to determine the values of the remaining four unknowns, including the corresponding value of $\tilde{P}$. By varying the starting value of $\tilde{C}$ in a systematic way, the corresponding values of $\tilde{P}$ can be generated and we can build up the set $\{\tilde{P},\tilde{C}\}$ and map out the locus of values satisfying $G(\tilde{P},\tilde{C}) = 0$. A more detailed description of the mechanics of the procedure is discussed in the next section.

Combining (26) and (27), we obtain:

$$A_3 = -\frac{\tilde{P}}{\beta_2(\mu-\alpha)} \times \frac{1}{\tilde{P}^{\beta_2} C^{\eta_1}} = \frac{\tilde{C}}{\eta_1(\mu-\theta)} \times \frac{1}{\tilde{P}^{\beta_2} C^{\eta_1}}, \tag{28}$$

and:

$$\tilde{P} = -\frac{\beta_2}{\eta_1} \times \frac{\mu-\alpha}{\mu-\theta} \times \tilde{C}. \tag{29}$$

Substituting the values of A_3 and $\tilde{P}$ from (28) and (29) respectively into (25) yields:

$$\frac{\tilde{C}}{\eta_1(\mu-\theta)}\left(1-\beta_2-\eta_1-\frac{P_0^{\beta_2}C_0^{\eta_1}}{\tilde{C}^{\beta_2+\eta_1}}\left(\frac{-\beta_2(\mu-\alpha)}{\eta_1(\mu-\theta)}\right)^{-\beta_2}\right)=\frac{P_0}{\mu-\alpha}-\frac{C_0}{\mu-\theta}-K. \quad (30)$$

For a pre-specified value of $\tilde{C}$, the solution values of β_2 and η_1 can be derived from (19) and (30). Since the functions involved are non-linear in character and it is not feasible to develop analytical solutions for β_2 and η_1, we have to resort numerical techniques to derive their solution. This is explained in the next section. From knowing the values of $\tilde{C}$, β_2 and η_1, the corresponding value of $\tilde{P}$ can be sought from (29). By varying the pre-specified value of $\tilde{C}$ over a reasonable range the corresponding value of $\tilde{P}$ can be evaluated and the resulting pairs of values then satisfy the function $G(\tilde{P},\tilde{C})=0$..

Since both $\tilde{P}$ and $\tilde{C}$ are intrinsically non-negative and $\mu-\alpha$ and $\mu-\theta$ are assumed to be non-negative, then (29) implies that these conditions are upheld only when either $\beta_2<0$ and $\eta_1>0$ or $\beta_2>0$ and $\eta_1<0$ apply. The second of these conditions means that the renewal of the asset occurs for very large values of P and very small values of C and since this defies economic logic, we can safely ignore this possibility. Equation (29) suggests that $\tilde{P}$ is an increasing function $\tilde{C}$ but this has to be tested since the parameters β_2 and η_1 are variable rather than fixed constants. Further, since $\beta_2<0$ and $\eta_1>0$ then (28) implies that $A_3>0$.

Since the quantity:

$$F(P_0,C_o)-K>0,$$

otherwise we would not invest in the asset in the first place, then combining (25), (28) and (29) yields:

$$F(\tilde{P},\tilde{C})=\frac{\tilde{P}}{\beta_2(\mu-\alpha)}(\beta_2+\eta_1-1)=\frac{\tilde{C}}{\eta_1(\mu-\theta)}(1-\beta_2-\eta_1)>0. \quad (31)$$

From (31), it follows that $\eta_1<1-\beta_2$. The function F is not homogenous of degree one but less than one, so that a doubling of P and C will not produce a doubling of the value of F, but rather a value slightly less than a doubling.

The stochastic model for asset renewal should revert to its deterministic variant when the underlying volatilities are set equal to zero. Specifying in (16), $\sigma_P = 0$ and $\sigma_C = 0$:

$$\alpha P \frac{\partial F}{\partial P} + \theta C \frac{\partial F}{\partial C} - \mu F + \left(P - C\right) = 0.$$

The solution to this partial differential equation takes the form:

$$F = AP^{\beta}C^{\eta} + \frac{P}{\mu - \alpha} - \frac{C}{\mu - \theta}$$

Focusing on the homogenous solution, then:

$$Q\left(\beta, \eta\right) = \alpha\beta + \theta\eta - \mu = 0. \tag{32}$$

To demonstrate the equivalence of the solutions for the deterministic and stochastic variants of the model when $\sigma_P = 0$ and $\sigma_C = 0$ requires establishing from the value matching condition (25) that the quantity:

$$A\tilde{P}^{\beta}\tilde{C}^{\eta} - AP_0^{\beta}C_0^{\eta} + \frac{\tilde{P}}{\mu - \alpha} - \frac{\tilde{C}}{\mu - \theta} \tag{33}$$

equals the left hand side of equation (5). Combining (29) and (32) to eliminate η yields:

$$\frac{1}{\beta} = \frac{\alpha}{\mu} - \frac{\theta}{\mu}\left(\frac{\tilde{C}}{\mu - \theta}\right)\left(\frac{\mu - \alpha}{\tilde{P}}\right). \tag{34}$$

We note that:

$$\left(\frac{P_0}{\tilde{P}}\right)^{\beta}\left(\frac{C_0}{\tilde{C}}\right)^{\eta} = \exp\left(-\left(\alpha\beta + \theta\eta\right)\tilde{T}\right) = \exp\left(-\mu\tilde{T}\right). \tag{35}$$

Using (29), (34), and (35) sequentially, then after some simplification (33) becomes:

$$\begin{aligned}
&A\tilde{P}^{\beta}\tilde{C}^{\eta} - AP_0^{\beta}C_0^{\eta} + \frac{\tilde{P}}{\mu - \alpha} - \frac{\tilde{C}}{\mu - \theta} \\
&= -\frac{\tilde{P}}{\beta\left(\mu - \alpha\right)} + \frac{\tilde{P}}{\beta\left(\mu - \alpha\right)} \times \frac{P_0^{\beta}C_0^{\eta}}{\tilde{P}^{\beta}\tilde{C}^{\eta}} + \frac{\tilde{P}}{\mu - \alpha} - \frac{\tilde{C}}{\mu - \theta} \\
&= \tilde{P}\left(\frac{1}{\mu} + \frac{\alpha}{\mu} \times \frac{e^{-\mu\tilde{T}}}{\mu - \alpha}\right) - \tilde{C}\left(\frac{1}{\mu} + \frac{\theta}{\mu} \times \frac{e^{-\mu\tilde{T}}}{\mu - \theta}\right),
\end{aligned}$$

which is the required result. The formulation and the solution for the stochastic variant of the model based on two sources of uncertainty produce identical results as the deterministic variant when the underlying volatilities for the two variables are set equal to zero. The derivation did not require any statement on the feasible signs of the various parameters since it is implicit. Equations (26) and (27) imply that A and η share the same sign, but the opposite sign to β. Rewriting equation (30) as:

$$AP_0^{\beta}C_0^{\eta}\left(\exp\left(\mu\tilde{T}\right)-1\right)=\left(\frac{P_0-\tilde{P}}{\mu-\alpha}\right)-\left(\frac{C_0-\tilde{C}}{\mu-\theta}\right)-K, \tag{36}$$

and since to be economically meaningful, the optimal value of the asset, which is:

$$V\left(\tilde{T}\right)=\left(\frac{P_0-\tilde{P}}{\mu-\alpha}\right)-\left(\frac{C_0-\tilde{C}}{\mu-\theta}\right),$$

has to be greater than the cost of the capital investment K, it follows that the right hand side of equation (36) has to be positive, and so $A>0$. By implication, $\beta<0$ and $\eta>0$.

4. Applied Numerical Methods for the Stochastic Model

Since (19) and (30) involve complicated non-linear relationships between the unknown quantities and no analytical solution exists, this section discusses the numerical procedures and techniques used to evaluate the solution for the stochastic model and the numerical sensitivity analysis that is performed in the next section. We define the function $H\left(\beta_2,\eta_1\right)=0$ from (30):

$$\begin{aligned}H\left(\beta_2,\eta_1\right)=\frac{\tilde{C}}{\eta_1\left(\mu-\theta\right)}\left(1-\beta_2-\eta_1-\frac{P_0^{\beta_2}C_0^{\eta_1}}{\tilde{C}^{\beta_2+\eta_1}}\left(\frac{-\beta_2\left(\mu-\alpha\right)}{\eta_1\left(\mu-\theta\right)}\right)^{-\beta_2}\right)\\-\frac{P_0}{\mu-\alpha}+\frac{C_0}{\mu-\theta}+K=0\end{aligned} \tag{37}$$

which is explored numerically by extending the base case to:

P_0	C_0	K	$\tilde{C}$	α	θ	μ	σ_P	σ_C	ρ
80	20	100	40	-5%	10%	25%	25%	25%	50%

The function $H\left(\beta_2,\eta_1\right)=0$ is displayed in figure (2) for the quadrant values where $\eta_1>0$ and $\beta_2<0$. Although the function is continuous over the quadrant, the exhibit reveals a discontinuity at the apex, which is due to imprecision arising from the numerical method used. It can be demonstrated that by re-writing equation (37) as:

$$\frac{\tilde{C}}{\left(\mu-\theta\right)}\left(1-\beta-\eta-\frac{P_0^{\beta}C_0^{\eta}}{\tilde{C}^{\beta+\eta}}\left(\frac{\eta\left(\mu-\theta\right)}{-\beta\left(\mu-\alpha\right)}\right)^{\beta}\right)-\eta\left(\frac{P_0}{\mu-\alpha}+\frac{C_0}{\mu-\theta}+K\right)=0$$

that $\beta_2 \to 0-$ and $\eta_1 \to 0+$ is a solution to $H(\beta_2, \eta_1) = 0$ and that the function exhibits a cusp and local maximum for β_2 at these values. This revealed numerical discontinuity has no significant effect on evaluating the pair of values, η_1 and β_2. The function $Q(\beta_2, \eta_1) = 0$, (19), can only pass through the origin for η_1 and β_2 both zero when the rate of return $\mu = 0$, which would entail that either or both of the assumptions that $\mu - \alpha > 0$ and $\mu - \theta > 0$ were violated. We can discount the possibility that the solution values for η_1 and β_2 being infinitesimally close to zero to be irrelevant in practice. In its continuous form, the function takes on a hair-pin shape with its apex at the origin $(\eta_1 = 0, \beta_2 = 0)$ and with two splayed legs emanating from the apex, both having negative slope.

The solution values of η_1 and β_2, which occur at the intersection of the two functions $Q(\beta_2, \eta_1) = 0$ and $H(\beta_2, \eta_1) = 0$, can be found from figure (3). Since the function $H(\beta_2, \eta_1) = 0$ has two splayed legs, there exists in principle two points of intersection and it becomes necessary to identify which of the two solution points is relevant. For the extended base case, the values of η_1 and β_2 at the two points of intersection with the corresponding value of $\tilde{P}$ are as follows:

	Intersection Point	
	I	II
η_1	1.586	0.824
β_2	-0.994	-1.622
$\tilde{P}$	50.12	157.51

One of these two points of intersection has to be excluded for the solution to provide a unique answer. In the presence of a single source of uncertainty, it is known that the separate conditions on the two parameters are that $\beta_2 < 0$ when revenue is the stochastic variable, Adkins (2005) and $\eta_1 > 1$ when cost is the stochastic variable, Dobbs (2004). Although the condition on β_2 is satisfied for either point of intersection, the condition on η_1 is only satisfied for intersection point I. We would also expect the point triggering a capital renewal for the variable $\tilde{P}$ to lie in the region

$\tilde{P} < P_0$ when the cost variable takes on the value $\tilde{C} = 40$, which again eliminates point II. We can safely conclude that the intersection point of interest occurs when $\eta_1 > 1$ and possibly when $\tilde{P} < P_0$, and our analysis will focus on point I solutions only. The solution space is therefore defined by the region bounded by $\eta_1 > 1$, $\beta_2 < 0$ and $\eta_1 < 1 - \beta_2$. This solution space can be narrowed for β_2 even further by leveraging the information contained in the function $Q(\beta_2, \eta_1) = 0$. When $\eta_1 = 1$, the negative root for β_2 is:

$$\beta_L = \left(\tfrac{1}{2} - \frac{\alpha}{\sigma_P^2} - \frac{\rho\sigma_C}{\sigma_P}\right) - \sqrt{\left(\tfrac{1}{2} - \frac{\alpha}{\sigma_P^2} - \frac{\rho\sigma_C}{\sigma_P}\right)^2 + \frac{2(\mu - \theta)}{\sigma_P^2}}\,.$$

When $\eta = 1 - \beta$, the negative root for β_2 is:

$$\beta_U = \left(\tfrac{1}{2} - \frac{(\alpha - \theta)}{\sigma^2}\right) - \sqrt{\left(\tfrac{1}{2} - \frac{(\alpha - \theta)}{\sigma^2}\right)^2 + \frac{2(\mu - \theta)}{\sigma^2}}\,,$$

where $\sigma^2 = \sigma_P^2 + \sigma_C^2 - 2\rho\sigma_P\sigma_C$. The solution space for the two parameters is specified by their lower and upper bounds:

$$\beta_L \le \beta_2 \le \beta_U\,,$$
$$1 \le \eta_1 \le 1 - \beta_U\,.$$

Solving for η_1 and β_2 from the two equations $Q(\beta_2, \eta_1) = 0$ and $H(\beta_2, \eta_1) = 0$ is not straightforward since both functions are non-linear and identifying the relevant point of intersection requires an effective numerical solution method. The usual technique for solving for n unknown variables given n independent non-linear equations is to use Newton's method of successive approximations. This begins the process of deriving the solution values by starting with realistic initial guesses of the n unknown variables and then uses the analytical first derivatives (Jacobian matrix) to generate the successive approximations. In theory, Newton's method converges to the correct solution provided that the functions are continuously differentiable, that the Jacobian matrix is not close to singularity and that the initial guesses are sufficiently close to the roots. A further complication arises when one or more of the functions in the set of n equations is so mathematically complicated that determining the derivative analytically and then numerically is prone to error. In their work on the computational analysis of economic and finance problems, Miranda and Fackler (2002) advocate the use of Broyden's method. Like Newton's method, this alternative also uses realistic guesses of the n unknown variables to start the process of evaluating the roots and generates the successive approximations based on the first derivatives which are numerically evaluated instead of being derived from the analytical Jacobian matrix. Determining the Jacobian matrix numerically by calculating the slopes over very small intervals around the focal values rather than analytically can be more efficient since the algorithm can be applied to any set of n equations and it avoids any pitfalls of having to derive the derivatives analytically and then numerically without error.

Like Newton's method, the effectiveness of Broyden to yield robustly precise solutions for the roots depends on the good behaviour of the first derivatives of the functions and the closeness of the initial guesses of the roots. Figure (3) reveals that the choice of the initial guesses is instrumental in identifying the relevant solution from the two possibilities and concentrating the search process to an area of the graph where the Jacobian matrix is well defined. Since a poor choice for the initial guesses may entail the unacceptable root to emerge or the first derivative of $H(\beta_2, \eta_1) = 0$ to become irregular, then for a given value of $\tilde{C}$, the initial guesses for the roots were identified visually from figure (3). Broyden's method was then applied to yield the solution values of η_1 and β_2 satisfying the two equations $Q(\beta_2, \eta_1) = 0$ and $H(\beta_2, \eta_1) = 0$. It was observed that the convergence towards the solution was rapid, due to the distinctive slopes of the two functions. The corresponding value of $\tilde{P}$ was then evaluated from equation (29). A small change was made to the variable $\tilde{C}$, its value was inserted into equation (37) and Broyden's method was re-applied using the solution values of η_1 and β_2 from the previous process as the initial guesses. After evaluating the solution values for η_1 and β_2 for this value of $\tilde{C}$, the corresponding value of $\tilde{P}$ was calculated. In this way, it is possible to build up two vectors of values for $\tilde{P}$ and $\tilde{C}$ that satisfy the relationship $G(\tilde{P}, \tilde{C}) = 0$. These are displayed in figure (4).

This figure reveals that the relationship between $\tilde{P}$ and $\tilde{C}$, the pairs of values signifying the trigger point for asset renewal, is positive and that the strategy of continuing with the existing asset is sustainable when an increase in costs is supported by a sufficiently high level of or significant increase in revenue. Figure (4) also shows that for the majority of occasions the value $\tilde{P}$ that triggers the renewal exceeds the corresponding value $\tilde{C}$ and that profit (the difference between revenue and cost) has to drop below a positive floor level in order to signal the switch between the existing asset and its brand new version. Although the relationship between $\tilde{P}$ and $\tilde{C}$ is monotonically increasing, it increases at a decreasing rate so that for large values of $\tilde{P}$ and $\tilde{C}$ the absolute increase in the value of $\tilde{P}$ is less than that for $\tilde{C}$ so that inevitably the value of $\tilde{C}$ exceeds $\tilde{P}$. In principle the decision to replace the existing asset with a new version may have to wait until the profit becomes negative. The cut-off value for $\tilde{C}$ where the decision to switch is based on a zero profit floor level is derived by eliminating β_2 from (19) through using (29) when $\tilde{P} = \tilde{C}$ and solving the quadratic equation for single variable η_1. The positive solution value of η_1 is then used to derive β_2, and these values are fed into equation (30) to identify the value of $\tilde{C}$. The values specifying a zero profit floor level are:

η	β	$\tilde{C}$	$\tilde{P}$
1.6805	-0.8403	103.3849	103.3849

Based on these values, we can discount the possibility that the trigger point occurs for zero profit or less for our base case since it is highly unlikely that the calculated values of $\tilde{P}$ and $\tilde{C}$ will ever be attained in practice.

The solution based on the stochastic framework involving the two sources of uncertainty can be compared with its deterministic variant. For the base case conditions assuming no stochastic variability $\sigma_P = 0$ and $\sigma_C = 0$, we use (5) to determine the optimal time for replacing the asset T^*. Using the goal seek facility, the value of T^* is 7.2566 years and the corresponding optimal values of P and C are $P^* = 55.6565$ and $C^* = 41.3217$. For the base case conditions with stochastic variability and letting the pre-specified value of $\tilde{C}$ be $\tilde{C} = 41.3217$, the corresponding optimal value of P is $\tilde{P} = 51.2498$. This suggests that for a given value of $\tilde{C}$, the optimal renewal rule for P based on the deterministic formulation of the model exceeds that for the stochastic variant and that management should exercise patience in the presence of uncertainty before renewing the existing asset. Before renewing the existing asset with its net operating revenues, management has to be reassured that the existing asset is genuinely performing unsatisfactorily and that it is economically prudent to renew the asset at the investment cost that entails. Dixit and Pindyck (1994) refer to this waiting to make sure that the revenue has hit some unsustainable floor level rather than being the result of a statistical aberration as hysteresis and this reflects that the process of decision making is path dependent.

5. Sensitivity Analysis

This section explores the sensitivity analysis performed on the solution in order to establish that the results engendered by our analysis are intuitively sound and that the underlying analytical methods are credible. A combination of numerical and analytical methods is used to identify the effects of parametric changes on the value of the existing asset plus the renewal option and the trigger levels signalling the asset's renewal. The impact of the various parameters on the values of the η_1 and β_2, and consequently on the values of $\tilde{P}$ and $\tilde{F} = F\left(\tilde{P}, \tilde{C}\right)$, is not straightforward due to their complicated behaviour as expressed by the functions $Q\left(\beta_2, \eta_1\right) = 0$ and $H\left(\beta_2, \eta_1\right) = 0$. We relegate to the Appendix most of the analysis specifying the values and signs of the key derivatives that form the building blocks for determining the effect that the exogenous parameters exert on endogenous quantities.

Initially, we will explore the impact of the volatilities, σ_P and σ_C, on the values of the parameters η_1 and β_2, and through them on the shape and behaviour of the functions $\tilde{P}$ and $\tilde{F}$, while keeping $\tilde{C}$ a constant. We establish that the value of the parameter ρ exerts significant influence on the shape and behaviour of these functions and on how the remaining exogenous factors affect the endogenous quantities.

The effect of changes in σ_P (σ_C) on $\tilde{P}$ and $\tilde{F}$ are transmitted through β_2 and η_1 along the functions Q and H. In the Appendix, we establish that the signs of $\frac{\partial \beta_2}{\partial \sigma_P}$ and $\frac{\partial \eta_1}{\partial \sigma_P}$ depend on the signs of $\frac{\partial Q}{\partial \sigma_P}$ and $-\frac{\partial Q}{\partial \sigma_P}$ respectively. Now, from (19):

$$\frac{\partial Q}{\partial \sigma_P} = \sigma_P \beta_2 (\beta_2 - 1) + \rho \sigma_C \beta_2 \eta_1 \geq 0. \tag{38}$$

so $\frac{\partial \beta_2}{\partial \sigma_P} \geq 0$ provided that $\rho \leq 0$. However, when $\rho > 0$, the second term on the right hand side of (38) can dominate the first term, which happens when σ_P is relatively small, and then $\frac{\partial \beta_2}{\partial \sigma_P} < 0$. The quantity $\frac{\partial \beta_2}{\partial \sigma_P} < 0$ when $\sigma_P^2 < \frac{\eta_1}{1 - \beta_2} \times \rho \sigma_P \sigma_C$; that is when the variance for the stochastic variations in revenue is less than the covariance multiplied by the factor $\frac{\eta_1}{1 - \beta_2}$. Also, the value of Q attains a minimum when:

$$\sigma_P = \frac{\eta_1}{1 - \beta_2} \times \rho \sigma_C.$$

A similar argument can be used to establish that $\frac{\partial \eta_1}{\partial \sigma_P} < 0$ and that η_1 is a decreasing function of σ_P when $\sigma_P^2 > \frac{\eta_1}{1 - \beta_2} \times \rho \sigma_P \sigma_C$, which is always true for $\rho \leq 0$.

The possible convex and concave nature of β_2 and η_1 are exhibited respectively in figures (5a and b). These two graphs represent the relationships between σ_P with β_2 and η_1 respectively for ρ equalling -1.0, -0.5, 0.0, 0.5 and 1.0. All five curves for the two respective relationships radiate from the same points where $\sigma_P = 0$. From there on, the five curves can be distinguished by whether or not they possess a turning point. For $\rho \leq 0$, β_2 and η_1 are respectively increasing and decreasing functions of σ_P and do not display a turning point over its recorded range. In contrast, when

$\rho = 0.5$ or $\rho = 1.0$, the relationships between σ_P with β_2 and η_1 respectively reveal a point of minimum and maximum respectively, after which the curve follows a similar shape as for the cases when $\rho \le 0$. The turning point can be derived from (38) except that both β_2 and η_1 vary with σ_P and so it is more convenient to identify the turning points for β_2 and η_1 numerically. These are exhibited in table (1) with the corresponding value of σ_P.

Table 1: Turning point values for β_2, η_1, $\tilde{P}$ and $F\left(\tilde{P},\tilde{C}\right)$, and corresponding values of σ_P for the specified values of ρ

ρ	$\text{Min}\{\beta\}$	$\text{Max}\{\eta\}$	$\text{Max}\{\tilde{P}\}$	$\text{Min}\{F(\tilde{P},\tilde{C})\}$	σ_P
-1.0	-1.0495	1.6422	51.0620	65.7225	0.0000
-0.5	-1.0495	1.6422	51.0620	65.7225	0.0000
0.0	-1.0495	1.6422	51.0620	65.7225	0.0000
0.5	-1.1011	1.6975	51.8931	63.4082	0.1010
1.0	-1.3368	1.9355	55.2554	55.2968	0.2071

Provided that $\sigma_P^2 > \frac{\eta_1}{1-\beta_2} \times \rho\sigma_P\sigma_C$, implying that $\frac{\partial\beta_2}{\partial\sigma_P} > 0$ and $\frac{\partial\eta_1}{\partial\sigma_P} < 0$, $\tilde{P}$ is a monotonic decreasing function and $\tilde{F}$ is a monotonic increasing function of σ_P: see the Appendix and the figures (6a and b) respectively that depict the profiles for various specified values of ρ. However, since $\sigma_P^2 < \frac{\eta_1}{1-\beta_2} \times \rho\sigma_P\sigma_C$ can arise for $\rho > 0$ and certain small values of σ_P, the profiles for $\tilde{P}$ and $\tilde{F}$ can exhibit a point of maximum and minimum respectively. When a turning point does exist, it occurs at the value of σ_P that yields turning points for β_2 and η_1. The corresponding maximal values for $\tilde{P}$ and minimal values for $\tilde{F}$ are displayed in table 1 for the various specified values of ρ.

We can apply the same line of argument for determining the effect of variations in σ_C on the endogenous quantities. In the Appendix, we establish that the signs of $\frac{\partial \beta_2}{\partial \sigma_C}$ and $\frac{\partial \eta_1}{\partial \sigma_C}$ depend on the signs of $\frac{\partial Q}{\partial \sigma_C}$ and $-\frac{\partial Q}{\partial \sigma_C}$ respectively. Again, from (19):

$$\frac{\partial Q}{\partial \sigma_C} = \sigma_C \eta_1 (\eta_1 - 1) + \rho \sigma_P \beta_2 \eta_1 \geq 0,$$

when $\sigma_C^2 \geq \frac{-\beta_2}{\eta_1 - 1} \times \rho \sigma_P \sigma_C$, which is always true provided $\rho \leq 0$. The possible convex and concave nature of β_2 and η_1 are exhibited respectively in figures (7a and b). The turning points exhibited in figures 5 and 7 are not identical because of the differences in the underlying conditions for each case. Figures (8a and b) exhibit the profiles for $\tilde{P}$ and $\tilde{F}$ versus σ_C for various specified values of ρ. These show that $\tilde{P}$ and $\tilde{F}$ are respectively monotonically decreasing and increasing functions of σ_C for all values of $\sigma_C \geq \frac{-\beta_2}{\eta_1 - 1} \times \rho \sigma_P$.

A common result of options analysis is that the option value is a monotonically increasing function of the underlying volatility, Dixit and Pindyck (1994). Our findings from this analysis agree with this common result provided that $\rho \leq 0$. When this condition is not met, the function $\tilde{F}$ is possibly convex and exhibits a point of minimum. The convexity of the function $\tilde{F}$ is due to the convexity of β_2, or equivalently the concavity of η_1, and this arises when the function $Q(\beta_2, \eta_1)$ exhibits a turning point with respect to one of σ_P or σ_C. At each of these turning points, the function $Q(\beta_2, \eta_1)$ minimised. Since the quantity:

$$Q_1(\beta_2, \eta_1) = \tfrac{1}{2}\sigma_P^2 \beta_2 (\beta_2 - 1) + \tfrac{1}{2}\sigma_C^2 \eta_1 (\eta_1 - 1) + \rho \sigma_P \sigma_C \beta_2 \eta_1$$

is a measure of composite volatility, $\tilde{F}$ is a convex function when the measure of composite volatility displays a minimum. The function $\tilde{F}$ can decrease for increasing values of an underlying volatility when the composite volatility measure is decreasing. Any convexity displayed by $\tilde{F}$ is reflected in the concavity of $\tilde{P}$, but when the condition stated above is met then $\tilde{P}$ is a monotonically decreasing function of the underlying volatility. Each of the previous real option analyses of capital replacement, Mauer and Ott (1995), Dobbs (2004) and Adkins(2005), demonstrates that a rise in the underlying volatility causes an increase in the trigger level for the operating and maintenance cost signalling asset replacement or a decrease for the revenue, and that management has to exercise greater patience before making the replacement decision. The current model involving two sources of uncertainty supplies similar results as the single dimensional variants.

The behaviour of the function representing the value of the existing asset and its renewal option and of the trigger level signalling renewal conform to previous studies and intuition. Any convexity in $\tilde{F}$ or concavity in $\tilde{P}$ can be explained by a decrease in the composite volatility measure even though one element of the volatility is increasing. However, since the convexity in $\tilde{F}$ or concavity in $\tilde{P}$ is eliminated when the correlation coefficient ρ is equal to zero or less, it is reasonable to enquire whether realistic values for ρ are confined to the region $-1 \le \rho \le 0$. The context for the model is an asset that deteriorates and fades through usage so that the operating and maintenance cost it incurs tend to increase over time and the revenue it generates tends to decrease over time. In this context, an exogenous shock leading to a positive (negative) change in the operating and maintenance cost is more likely to be correlated with one that leads to a negative (positive) change in revenue. Because of this, it is reasonable to expect that the stochastic evolutions for revenue and cost are negatively correlated so that we can treat the correlation coefficient as falling in the region $-1 \le \rho \le 0$. This implies that the functions $\tilde{F}$ and $\tilde{P}$ are respectively monotonically increasing and decreasing with the underlying volatility. In the remainder of this section, we will investigate the effect of other changes in the exogenous factors on the behaviour of $\tilde{F}$ and $\tilde{P}$ by adopting a neutral position with $\rho = 0$.

The effect of variations in the investment outlay K required to renew the asset on the value of the existing asset and its renewal option $\tilde{F}$ and the trigger level for revenue $\tilde{P}$ signalling a renewal is exhibited in figure 9. Both of the profiles are downward sloping. The effect of increasing the investment outlay is to reduce the net value generated by the asset and its renewal option. The investment outlay enters the value matching condition as a negative quantity and its effect is to depress $\tilde{F}$. The increasing decline in the net value generated by the asset will inevitably reach the point $\tilde{F} < 0$ when it will be no longer viable to replace the existing asset. Since the way to compensate for increases in the investment outlay is to retain the asset for a longer period, the trigger level for revenue $\tilde{P}$ signalling asset renewal will have to be increased to maintain the asset's viability.

The profiles of the value of the asset and its renewal option $\tilde{F}$ and the trigger level for revenue $\tilde{P}$ signalling a renewal due to variations in the revenue immediately following renewal P_0 are exhibited in figure 10. Both of the profiles are upward sloping. Since the value of any capital budgeting project is improved through increases in the positive cash flows, we would expect P_0 to have a positive effect on $\tilde{F}$. Any increase in the revenue level immediately following renewal is reflected in the value of asset and its renewal option through the value matching condition. Further, any improvement in $\tilde{F}$ implies that the asset can be held for a longer time before it becomes opportune to renew it. It follows that any increase in P_0 will be manifested in an increase of $\tilde{P}$.

The effect of variations in the rate of return μ on the value of the asset plus the renewal option $\tilde{F}$ and the trigger level for revenue $\tilde{P}$ signalling a renewal are exhibited in figure 11. Both profiles are downward sloping. Generally, the discount rate has a negative impact on the value of a capital budgeting project since it contracts the present value of future cash flows. This negative effect is exhibited by the profile of the asset value and its renewal option. While the graph has been drawn for a discount rate in the range $15\% \leq \mu \leq 30\%$, reductions in the value of μ will continuously force up the value of $\tilde{F}$ until tends to infinity. Increases in the discount rate imply that the cost of renewing the asset becomes increasingly more expensive. Because of this, management will defer and wait longer before replacing the existing asset and this will cause the trigger level for revenue to fall as a consequence.

The impact of change in the continuous rate of change for revenue on the value of the asset plus the renewal option $\tilde{F}$ and the trigger level for revenue $\tilde{P}$ signalling a renewal are exhibited in figure 12. Both of the profiles are upward sloping. Increases in the parameter α will make the asset more valuable because of the rise in the revenue cash flow and this will be reflected positively in the value of the existing asset and its renewal option. Although the value of the parameter α was specified as negative, we can observe from the figure that the model accommodates both positive and negative values for α. When α becomes increasingly more negative the economic viability of owning the asset declines until $\tilde{F}=0$ when we question the benefit from holding the asset. In contrast, increases in the value of α make owning the asset increasingly more desirable and the value of $\tilde{F}$ tends to infinity as $\alpha \to \mu$. Besides making the asset more attractive, increases in α mean that we can afford to renew the asset more readily and this is reflected in the increased trigger level for revenue $\tilde{P}$.

The effect of variations in the continuous rate of change for the operating and maintenance cost θ on the value of the asset plus the renewal option $\tilde{F}$ and the trigger level for revenue $\tilde{P}$ signalling a renewal are exhibited in figure 13. Both profiles are downward sloping. Positive changes in the value of θ lead to increases in the future operating and maintenance cost and this increase contributes negatively to the value of the asset plus its renewal option. As the value of θ increases towards μ, the value of $\tilde{F}$ becomes infinitely negative. Since increases in the value of θ reduces the value of $\tilde{F}$, there exists a value of $\theta < \mu$ for which $\tilde{F}=0$ when we are indifferent between holding the asset or not. Increasing the value of θ also depresses the trigger level for revenue $\tilde{P}$ signalling asset renewal. An increase in the operating and maintenance cost due to a rise in θ means we have to keep the asset for a longer period in order to maintain the same return from the asset and this implies that the trigger level for revenue will decline as a consequence.

6. Homogeneity and Model Indeterminacy

In their real options analysis of price and cost uncertainty, Dixit and Pindyck (1994) derive their solution from a partial differential equation similar to (16), a value matching condition and two smooth pasting conditions. From these four equations, the authors have to derive solutions for the five unknowns. They proceed to reduce the dimensionality of the model and more particularly of the partial differential equation by assuming that the function representing the option value is homogenous of degree one. They argue that when revenues and costs are doubled, this will double the value of the project, and then assume that the value of the underlying option will consequently double. Homogeneity of degree one implies that the optimal decision will depend only on the ratio of the two variables and their solution for the trigger values, which is expressed in terms of this ratio, is represented by a ray passing through the origin. This ray separates the values of the two focal variables into two distinct regions that distinguish the type of decision to be pursued. When the prevailing values of the two focal variables belong to one side of the ray, the optimal decision is continuance whilst the optimal decision is to exchange the option for the asset if otherwise. The optimal decision rule is founded on a set of optimal pairs of values for the two focal variables. Model indeterminacy of the type experienced in our model should not therefore be interpreted as a flaw in the model's construction. It is the recognition that the instance when the option is exercised and the existing asset is renewed cannot be captured by a single pair of points but by a set of infinite pairs of points.

An alternative procedure for reducing the dimensionality of the partial differential equation, which is applied by Errais and Sadowsky (2005), relies on the work by Margrabe (1978) on exchange options. It conceives the investment opportunity as an option to exchange one stochastic asset for another stochastic asset. In this way, the authors show that the resulting partial differential equation describing the exchange of stochastic assets can be represented by an equation involving a single variable instead of two variables. The solution that they derive is identical in form with the one presented by Dixit and Pindyck (1994). It is interesting to note that in his analysis, Margrabe (1978) uses the properties of homogenous functions of degree one in order to derive his results.

The condition of homogeneity of degree one can be perceived as a ploy for reducing the dimensionality of the model or as a desirable assumption that facilitates the analytical solution of the model. Although the assumption of homogeneity of degree one can make the model more tractable and simplify the analysis, the underlying validity for adopting this assumption lies not in making a mental jump from inferring that since the underlying assets possess this property so the option must also possess this property, but within the nature of the value matching and smooth pasting conditions of the model in hand. Generally, the value matching condition can be expressed as the equality at the time of exercising the option between the function $f_1\left(\underline{X},\underline{Y}\right)$, which represents the value of the option to continue, and the two functions $f_2\left(\underline{X}\right)$ and $f_3\left(\underline{Y}\right)$, whose difference represents the net value generated from exercising the option, where the vectors $\underline{X}$ and $\underline{Y}$, of size n and m respectively are defined by $\underline{X}=\left\{X_1,X_2,\ldots,X_n\right\}$ and $\underline{Y}=\left\{Y_1,Y_2,\ldots,Y_m\right\}$. The value matching condition is:

$$f_1\left(\underline{X},\underline{Y}\right)=f_2\left(\underline{X}\right)-f_3\left(\underline{Y}\right)$$

Now we know from the properties of homogenous functions of degree one that if both f_2 and f_3 are homogenous functions of degree one then f_1 is also a homogenous function of degree one. This result can also be demonstrated from the smooth pasting conditions:

$$\frac{\partial f_1}{\partial X_i}=\frac{\partial f_2}{\partial X_i}\ \forall i,$$
$$\frac{\partial f_1}{\partial Y_j}=-\frac{\partial f_3}{\partial Y_j}\ \forall j.$$

This implies that:

$$\sum_{i=1}^{n}X_i\frac{\partial f_1}{\partial X_i}+\sum_{j=1}^{m}Y_j\frac{\partial f_1}{\partial Y_j}=\sum_{i=1}^{n}X_i\frac{\partial f_2}{\partial X_i}-\sum_{j=1}^{m}Y_j\frac{\partial f_3}{\partial Y_j}.$$

Then by Euler's equation:

$$\sum_{i=1}^{n}X_i\frac{\partial f_1}{\partial X_i}+\sum_{j=1}^{m}Y_j\frac{\partial f_1}{\partial Y_j}=f_2-f_3=f_1,$$

which implies that f_1 is a homogenous function of degree one. The assumption of homogeneity of degree one can be tested for the model in hand from the property of its value matching condition. It can be easily verified from the value matching condition for our model, (25), that the function depicting the option value is not homogenous of degree one so we would have been mistaken to make this assumption.

Equally, our analysis has demonstrated that the function is not homogenous of degree one.

7. Conclusion

The context of previous studies investigating the capital renewal decision using continuous time real options models has been mainly confined to physical assets, plant and equipment, where there is a single focal stochastic variable, operating and maintenance costs. This paper has considered an alternative contextual canvas and has extended the analysis to two focal stochastic variables, revenue and cost. This contextual canvas is characterised by assets whose deterioration through usage leads to revenue decline due to the asset's erosion in market attractiveness and an escalation in operating and maintenance cost due to the loss in equipment's efficiency. Because of their deteriorating value, these assets are after a period of sustained use renewed through a capital injection that restores their worth through bringing its earning potential and cost structure back to more favourable levels. Typical assets falling into this category are those used in the service sector, including amongst others real estate, theatres, commercial web-sites and human resources. Through usage, these assets experience variations in both the revenue they generate and the cost they incur and then they have to be renewed whenever their revenue and cost hit critical levels.

The real options model of asset renewal involving two distinct but stochastically dependent variables is constructed from the concept that the overall return from an asset equals its periodic cash flow plus its expected capital gain over the period. The two sources of stochastic variability are introduced through Ito's lemma and the resulting partial differential equation is solved in conjunction with the value matching and smooth pasting conditions to yield an analytical functional form for the value of the existing asset and its renewal option. In contrast to previous studies involving two stochastic variables, no resort is made to homogeneity of degree one or a similar ploy to reduce the problem's dimensionality. In fact, the solution demonstrates that the resulting analytical form does not comply with homogeneity of degree one, and so its adoption would have produced a flawed solution. When the number of stochastic variables included in the model increases above one, then in line with other studies involving two sources of uncertainty we set out to identify the boundary discriminating between continuance and renewal, which is defined by a certain function. Unfortunately, an analytical solution cannot be derived for the function defining the boundary discriminating between the decisions of continuance and renewal owing to the complexity of functional forms. Instead, we map out the function by using numerical methods. These methods, which have the advantages of being adequately robust and easily transferable to other models, enables us to map out the boundary function that is monotonically increasing, which demonstrates that the decision between continuance and renewal implies a trade-off between revenue and cost.

The credibility of the formulation and the results it produces is tested by examining whether the consequences derived from the model are sound and intuitively appealing. Our analysis starts by identifying the solution to the deterministic variant of the model, which is constructed on the replacement chain concept and uses non-random functions to describe the evolutions of the two variables. We demonstrate that the solution to the stochastic variant of the model always reduces to the deterministic solution when the underlying volatilities are set equal to zero. Both the deterministic and stochastic variants produce identical solutions in the presence of certainty. Secondly, we consider the behaviour of the value of the existing asset and its renewal option and the trigger level for revenue that signals asset renewal. A common result from the analysis of options is that the option value varies increasingly with increases in the underlying volatility. The results from the present analysis echo this conclusion with one insignificant limitation. Our analysis demonstrates that the value of the existing asset and its renewal option is a monotonically increasing function of the underlying volatility from either of the two sources provided that the correlation coefficient for the two sources of uncertainty is not positive. This limitation is not significant for two reasons. Since the context for the model is an asset whose value deteriorates through usage owing to declining revenue and escalating cost, we would expect any exogenous stochastic shift to impact on the two variables quite differently. An exogenous shift that is adverse is likely to produce an increase in cost and at the same time a decrease in revenue. For our context, we can treat $\rho \leq 0$. Secondly, we suggest that the case of a fall in the value of the existing asset plus its renewal option due to an increase in volatility arises when the composite variance measure decreases owing to the nature of the correlation between the two variables. We can safely assert that our model supports common result that the option value varies increasingly with increases in the underlying volatility. Within this limitation, the analysis also demonstrates that the trigger level for revenue that signal asset renewal is a monotonic decreasing function of the underlying volatility so that management has to act with greater patience in the presence of increasing uncertainty. Finally, we examined the behaviour of two endogenous quantities in the light of separate changes in the other parameters including the capital outlay required to renew the asset, the starting revenues following renewal, the discount rate, and the rates of change for revenue and cost. In each case, the results conform to intuition.

The method of involving homogeneity of degree one in order to reduce the model's dimensionality down to one was not adopted in the present analysis since it did not apply. This raises the question of when homogeneity of degree one, or similar ploys, is a valid assumption to make. We demonstrate that the validity of homogeneity of degree one emanates fundamentally from the nature of the value matching condition. If the expressions on the right hand side of the value matching condition, representing the value of the asset after the change in state, are individually homogenous functions of degree one and if they are linearly combined to produce the left hand side, then the expression on the left hand of the value matching condition, representing the option under continuance, is also a homogenous function of degree one. This is established either from the properties of homogenous functions of degree one or equivalently from the smooth pasting conditions. Before homogeneity of degree one is invoked to reduce the model's dimensionality, we can test the validity of the assumption from the value matching condition. Whenever the value matching condition fails to conform to homogeneity of degree one, as in the present analysis, then adopting this assumption will clearly lead to error.

The present analysis represents an important extension to the previous studies on capital replacement by expanding the context to consider asset renewal and by including two distinct sources of uncertainty through Ito's lemma. It also extends previous analytically based studies on real options involving two sources of uncertainty by not having recourse to homogeneity of degree one or other ploys to reduce the model's dimensionality. Although we apply the dynamic programming principle to derive the formulation and the solution, the model can be recast in a contingent claims framework provided that twinning market traded assets exists for the two sources of uncertainty. The methods adopted in this paper can be applied to consider alternative sources of uncertainty including the salvage price on disposal, the amount of the capital injection required for renewal, or other exogenous factors that are naturally stochastic. The focus of the analysis has been asset renewal. There exists a family of options other than renewal, including the option for the original investment opportunity, for the suspension of activities, expansion or divestment. These can be gainfully incorporated into the analysis if they are relevant for the particular context under consideration.

Appendix: Derivation of the Derivatives

The derivatives of the various functions developed in this appendix assume that $\tilde{C}$ is a constant unless specifically qualified. The subscripts for β and η have been omitted without loss of meaning.

From (19) and (37) respectively, we have:

$$Q(\beta,\eta) = \tfrac{1}{2}\sigma_P^2\beta(\beta-1) + \tfrac{1}{2}\sigma_C^2\eta(\eta-1) + \rho\sigma_P\sigma_C\beta\eta + \alpha\beta + \theta\eta - \mu = 0,$$

$$H(\beta,\eta) = \frac{\tilde{C}}{\eta(\mu-\theta)}\left(1-\beta-\eta-\frac{P_0^{\beta}C_0^{\eta}}{\tilde{C}^{\beta+\eta}}\left(\frac{-\beta(\mu-\alpha)}{\eta(\mu-\theta)}\right)^{-\beta}\right) - \frac{P_0}{\mu-\alpha} + \frac{C_0}{\mu-\theta} + K = 0.$$

We note that:

$$\frac{\partial H}{\partial \sigma_P} = \frac{\partial H}{\partial \sigma_C} = 0.$$

From equations (29) and (31), we define the functions G_1 and G_2 respectively:

$$G_1\left(\beta,\eta,\tilde{P}\right) = \tilde{P} + \frac{\beta}{\eta}\times\frac{\mu-\alpha}{\mu-\theta}\times\tilde{C} = 0, \tag{39}$$

$$G_2\left(\beta,\eta,\tilde{F}\right) = \tilde{F} - \frac{\tilde{C}}{\eta(\mu-\theta)}(1-\beta-\eta) = 0. \tag{40}$$

Then:

$$\frac{\partial G_1}{\partial \beta} = \frac{\tilde{C}}{\eta}\frac{\mu-\alpha}{\mu-\theta} > 0, \tag{41}$$

$$\frac{\partial G_1}{\partial \eta} = -\frac{\beta\tilde{C}}{\eta^2}\frac{\mu-\alpha}{\mu-\theta} > 0, \tag{42}$$

$$\frac{\partial G_2}{\partial \beta} = \frac{\tilde{C}}{\eta} \frac{1}{\mu - \theta} > 0, \tag{43}$$

$$\frac{\partial G_2}{\partial \eta} = \frac{\tilde{C}}{\eta^2} \frac{1 - \beta}{\mu - \theta} > 0. \tag{44}$$

Variations in $\tilde{P}$ due to changes in σ_P are delivered through β and η along the functions Q, H and G_1. Then from Kaplan and Lewis (1971):

$$\frac{\partial \tilde{P}}{\partial \sigma_P} = -\frac{\dfrac{\partial(Q, H, G_1)}{\partial(\beta, \eta, \sigma_P)}}{\dfrac{\partial(Q, H, G_1)}{\partial(\beta, \eta, \tilde{P})}} \tag{45}$$

where:

$$\frac{\partial(Q, H, G_1)}{\partial(\beta, \eta, \sigma_P)} = \begin{vmatrix} \dfrac{\partial Q}{\partial \beta} & \dfrac{\partial Q}{\partial \eta} & \dfrac{\partial Q}{\partial \sigma_P} \\ \dfrac{\partial H}{\partial \beta} & \dfrac{\partial H}{\partial \eta} & \dfrac{\partial H}{\partial \sigma_P} \\ \dfrac{\partial G_1}{\partial \beta} & \dfrac{\partial G_1}{\partial \eta} & \dfrac{\partial G_1}{\partial \sigma_P} \end{vmatrix} \tag{46}$$

and:

$$\frac{\partial(Q, H, G_1)}{\partial(\beta, \eta, \tilde{P})} = \begin{vmatrix} \dfrac{\partial Q}{\partial \beta} & \dfrac{\partial Q}{\partial \eta} & \dfrac{\partial Q}{\partial \tilde{P}} \\ \dfrac{\partial H}{\partial \beta} & \dfrac{\partial H}{\partial \eta} & \dfrac{\partial H}{\partial \tilde{P}} \\ \dfrac{\partial G_1}{\partial \beta} & \dfrac{\partial G_1}{\partial \eta} & \dfrac{\partial G_1}{\partial \tilde{P}} \end{vmatrix} \tag{47}$$

Similarly, variations in F due to changes in σ_P is derived from:

$$\frac{\partial \tilde{F}}{\partial \sigma_P} = -\frac{\dfrac{\partial(Q, H, G_2)}{\partial(\beta, \eta, \sigma_P)}}{\dfrac{\partial(Q, H, G_2)}{\partial(\beta, \eta, \tilde{F})}} \tag{48}$$

where:

$$\frac{\partial\left(Q,H,G_2\right)}{\partial\left(\beta,\eta,\sigma_P\right)}=\begin{vmatrix}\frac{\partial Q}{\partial\beta} & \frac{\partial Q}{\partial\eta} & \frac{\partial Q}{\partial\sigma_P}\\ \frac{\partial H}{\partial\beta} & \frac{\partial H}{\partial\eta} & \frac{\partial H}{\partial\sigma_P}\\ \frac{\partial G_2}{\partial\beta} & \frac{\partial G_2}{\partial\eta} & \frac{\partial G_2}{\partial\sigma_P}\end{vmatrix} \tag{49}$$

and:

$$\frac{\partial\left(Q,H,G_2\right)}{\partial\left(\beta,\eta,\tilde{F}\right)}=\begin{vmatrix}\frac{\partial Q}{\partial\beta} & \frac{\partial Q}{\partial\eta} & \frac{\partial Q}{\partial\tilde{F}}\\ \frac{\partial H}{\partial\beta} & \frac{\partial H}{\partial\eta} & \frac{\partial H}{\partial\tilde{F}}\\ \frac{\partial G_2}{\partial\beta} & \frac{\partial G_2}{\partial\eta} & \frac{\partial G_2}{\partial\tilde{F}}\end{vmatrix} \tag{50}$$

Now since:

$$\frac{\partial Q}{\partial\tilde{P}}=\frac{\partial H}{\partial\tilde{P}}=\frac{\partial Q}{\partial\tilde{F}}=\frac{\partial H}{\partial\tilde{F}}=0,$$

and:

$$\frac{\partial G_1}{\partial\tilde{P}}=\frac{\partial G_2}{\partial\tilde{F}}=1$$

then it follows that from equations (47) and (50) that:

$$\frac{\partial\left(Q,H,G_1\right)}{\partial\left(\beta,\eta,\tilde{P}\right)}=\frac{\partial\left(Q,H,G_2\right)}{\partial\left(\beta,\eta,\tilde{F}\right)}=D,$$

where:

$$D=\frac{\partial Q}{\partial\beta}\times\frac{\partial H}{\partial\eta}-\frac{\partial Q}{\partial\eta}\frac{\partial H}{\partial\beta}. \tag{51}$$

We will now proceed to derive all the partial derivatives forming the expressions in (45) and (48).

Differentiating $Q\left(\beta,\eta\right)=0$, (19), with respect to β yields:

$$\frac{\partial Q}{\partial\beta}=\sigma_P^2\beta-\tfrac{1}{2}\sigma_P^2+\rho\sigma_P\sigma_C\eta+\alpha. \tag{52}$$

Since $\beta < 0$ and we expect $\alpha \le 0$, then the first, second and fourth terms on the right hand side of equation (52) are negative. Further, we can expect the absolute sum of these terms to exceed the third term and we can treat $\frac{\partial Q}{\partial \beta} \le 0$.

Differentiating function $Q(\beta, \eta) = 0$ with respect to η yields:

$$\frac{\partial Q}{\partial \eta} = \sigma_C^2 \eta - \tfrac{1}{2}\sigma_C^2 + \rho\sigma_P\sigma_C\beta + \theta. \tag{53}$$

Since $\eta > 1$ and we expect $\theta \ge 0$, the first and fourth terms on the right hand side of equation (53) are both positive and we can expect their sum to exceed the sum of the second and third terms. We can treat $\frac{\partial Q}{\partial \eta} \ge 0$.

Differentiating function $H(\beta, \eta) = 0$ with respect to β yields:

$$\begin{aligned}\frac{\partial H}{\partial \beta} = \frac{-\tilde{C}}{\eta(\mu-\theta)}&\left\{1 + \ln\left(\frac{P_0}{\tilde{C}}\right)\left(\frac{P_0}{\tilde{C}}\right)^{\beta}\left(\frac{C_0}{\tilde{C}}\right)^{\eta}\left(\frac{-\beta}{\eta}\times\frac{\mu-\alpha}{\mu-\theta}\right)^{-\beta}\right\} \\ &\frac{-\tilde{C}}{\eta(\mu-\theta)}\left\{\left(\frac{P_0}{\tilde{C}}\right)^{\beta}\left(\frac{C_0}{\tilde{C}}\right)^{\eta}\ln\left(\frac{-\beta e}{\eta}\times\frac{\mu-\alpha}{\mu-\theta}\right)\left(\frac{-\beta}{\eta}\times\frac{\mu-\alpha}{\mu-\theta}\right)^{-\beta}\right\}\end{aligned} \tag{54}$$

Substituting the relationship (29) between $\tilde{C}$ and $\tilde{P}$, (54) becomes:

$$\frac{\partial H}{\partial \beta} = \frac{-\tilde{C}}{\eta(\mu-\theta)}\left\{1 + Z\left[\ln\left(\frac{P_0}{\tilde{P}}\right) - 1\right]\right\}, \tag{55}$$

where $Z = \left(\frac{P_0}{\tilde{P}}\right)^{\beta}\left(\frac{C_0}{\tilde{C}}\right)^{\eta}$. Now we expect $\frac{P_0}{\tilde{P}} > 1$, so $\left(\frac{P_0}{\tilde{P}}\right)^{\beta} < 1$; also we expect $\frac{C_0}{\tilde{C}} < 1$, so $\left(\frac{C_0}{\tilde{C}}\right)^{\eta} < 1$; so we can treat the quantity $Z\left[\ln\left(\frac{P_0}{\tilde{P}}\right) - 1\right]$ as a small negative number which is dominated by the value one. Therefore, we can treat $\frac{\partial H}{\partial \beta} < 0$.

Differentiating function $H(\beta, \eta) = 0$ with respect to η yields:

$$\frac{\partial H}{\partial \eta} = -\frac{\tilde{C}(1-\beta)}{\eta^2(\mu-\theta)} + \frac{\tilde{C}}{\eta^2(\mu-\theta)}\left(\frac{P_0}{\tilde{C}}\right)^{\beta}\left(\frac{C_0}{\tilde{C}}\right)^{\eta}\left(\frac{-\beta}{\eta}\times\frac{\mu-\alpha}{\mu-\theta}\right)^{-\beta}$$
$$-\frac{\tilde{C}}{\eta(\mu-\theta)}\left(\frac{P_0}{\tilde{C}}\right)^{\beta}\left(\frac{C_0}{\tilde{C}}\right)^{\eta}\left(\frac{-\beta}{\eta}\times\frac{\mu-\alpha}{\mu-\theta}\right)^{-\beta}\ln\left(\frac{C_0}{\tilde{C}}\right) \tag{56}$$
$$-\frac{\tilde{C}\beta}{\eta^2(\mu-\theta)}\left(\frac{P_0}{\tilde{C}}\right)^{\beta}\left(\frac{C_0}{\tilde{C}}\right)^{\eta}\left(\frac{-\beta}{\eta}\times\frac{\mu-\alpha}{\mu-\theta}\right)^{-\beta}$$

Substituting the relationship (29) between $\tilde{C}$ and $\tilde{P}$, (56) becomes:

$$\frac{\partial H}{\partial \eta} = -\frac{\tilde{C}(1-\beta)}{\eta^2(\mu-\theta)}\left\{1 - Z\left[1-\frac{\eta}{1-\beta}\ln\left(\frac{C_0}{\tilde{C}}\right)\right]\right\}. \tag{57}$$

Using a similar train of arguments, we can treat $\frac{\partial H}{\partial \eta} < 0$.

Given the signs of the various terms comprising D, equation (51) it follows that $D > 0$.

Since variations in β arising from changes in σ_P occur along both the functions $Q(\beta,\eta) = 0$ and $H(\beta,\eta) = 0$, then $\frac{\partial \beta}{\partial \sigma_P}$ is found from:

$$\frac{\partial \beta}{\partial \sigma_P} = -\frac{\dfrac{\partial(Q,H)}{\partial(\sigma_P,\eta)}}{\dfrac{\partial(Q,H)}{\partial(\beta,\eta)}}, \tag{58}$$

where:

$$\frac{\partial(Q,H)}{\partial(\sigma_P,\eta)} = \begin{vmatrix} \dfrac{\partial Q}{\partial \sigma_P} & \dfrac{\partial Q}{\partial \eta} \\ \dfrac{\partial H}{\partial \sigma_P} & \dfrac{\partial H}{\partial \eta} \end{vmatrix} = \frac{\partial Q}{\partial \sigma_P}\times\frac{\partial H}{\partial \eta}.$$

Since:

$$\frac{\partial(Q,H)}{\partial(\beta,\eta)} = \begin{vmatrix} \dfrac{\partial Q}{\partial \beta} & \dfrac{\partial Q}{\partial \eta} \\ \dfrac{\partial H}{\partial \beta} & \dfrac{\partial H}{\partial \eta} \end{vmatrix} = D,$$

then from (58):

$$\frac{\partial \beta}{\partial \sigma_P} = -\frac{1}{D}\frac{\partial Q}{\partial \sigma_P} \times \frac{\partial H}{\partial \eta}. \tag{59}$$

From (59), it follows that the sign of $\frac{\partial \beta}{\partial \sigma_P}$ depends on the sign of $\frac{\partial Q}{\partial \sigma_P}$.

The sign of $\frac{\partial \eta}{\partial \sigma_P}$ can be determined in a similar fashion:

$$\frac{\partial \eta}{\partial \sigma_P} = -\frac{\frac{\partial(Q,H)}{\partial(\beta,\sigma_P)}}{\frac{\partial(Q,H)}{\partial(\beta,\eta)}} \tag{60}$$

where:

$$\frac{\partial(Q,H)}{\partial(\beta,\sigma_P)} = \begin{vmatrix} \frac{\partial Q}{\partial \beta} & \frac{\partial Q}{\partial \sigma_P} \\ \frac{\partial H}{\partial \beta} & \frac{\partial H}{\partial \sigma_P} \end{vmatrix} = -\frac{\partial Q}{\partial \sigma_P} \times \frac{\partial H}{\partial \beta}.$$

From equation (60), it follows that:

$$\frac{\partial \eta}{\partial \sigma_P} = \frac{1}{D}\frac{\partial Q}{\partial \sigma_P} \times \frac{\partial H}{\partial \beta}. \tag{61}$$

It follows that the sign $\frac{\partial \eta}{\partial \sigma_P}$ depends on the sign of $-\frac{\partial Q}{\partial \sigma_P}$.

.

From equation (29), differentiating $\tilde{P}$ with respect to σ_P yields:

$$\frac{d\tilde{P}}{d\sigma_P} = -\left(\frac{\mu-\alpha}{\mu-\theta}\right)\frac{\tilde{C}}{\eta}\frac{\partial \beta}{\partial \sigma_P} + \left(\frac{\mu-\alpha}{\mu-\theta}\right)\frac{\beta\tilde{C}}{\eta^2}\frac{\partial \eta}{\partial \sigma_P},$$

so:

$$\frac{d\tilde{P}}{\tilde{P}d\sigma_P} = \frac{1}{\beta}\frac{\partial \beta}{\partial \sigma_P} - \frac{1}{\eta}\frac{\partial \eta}{\partial \sigma_P}. \tag{62}$$

Substituting the results from equations (59) and (61), then we can re-write (62) as:

$$\frac{d\tilde{P}}{\tilde{P}d\sigma_P} = \frac{N_1}{D}\frac{\partial Q}{\partial \sigma_P},$$

where:

$$
\begin{aligned}
N_1 &= -\frac{1}{\beta}\frac{\partial H}{\partial \eta} - \frac{1}{\eta}\frac{\partial H}{\partial \beta} \\
&= \frac{\tilde{C}(1-\beta)}{\beta\eta^2(\mu-\theta)}\left\{1 - Z + \frac{\eta}{1-\beta} Z \ln\left(\frac{C_0}{\tilde{C}}\right)\right\} \\
&\quad + \frac{\beta\tilde{C}}{\beta\eta^2(\mu-\theta)}\left\{1 - Z + Z\ln\left(\frac{P_0}{\tilde{P}}\right)\right\} \\
&= \frac{\tilde{C}}{\beta\eta^2(\mu-\theta)}\left[1 - Z + Z\ln(Z)\right].
\end{aligned}
\tag{63}
$$

Now for reasonable values of $\tilde{P}$ and $\tilde{C}$, $N_1 < 0$. It follows that:

$$
\frac{d\tilde{P}}{\tilde{P}d\sigma_P} = \frac{N_1}{D}\frac{\partial Q}{\partial \sigma_P} < 0, \tag{64}
$$

provided that $\frac{\partial Q}{\partial \sigma_P} > 0$.

From (31):

$$
F = \frac{\tilde{C}}{\eta(\mu-\theta)}(1-\beta-\eta),
$$

differentiating with respect to σ_P yields:

$$
\frac{\partial F}{F\partial \sigma_P} = -\frac{1}{\eta(1-\beta-\eta)}\left\{(1-\beta)\frac{\partial \eta}{\partial \sigma_P} + \eta\frac{\partial \beta}{\partial \sigma_P}\right\}. \tag{65}
$$

Substituting the results from equations (59) and (61), then we can re-write equation (65) as:

$$
\frac{\partial F}{F\partial \sigma_P} = -\frac{1}{\eta(1-\beta-\eta)}\frac{\partial Q}{\partial \sigma_P}\frac{N_2}{D}
$$

where:

$$
\begin{aligned}
N_2 &= (1-\beta)\frac{\partial H}{\partial \beta} - \eta\frac{\partial H}{\partial \eta} \\
&= -\frac{\tilde{C}(1-\beta)}{\eta(\mu-\theta)}\left\{1-\left(\frac{P_0}{\tilde{P}}\right)^{\beta}\left(\frac{C_0}{\tilde{C}}\right)^{\eta}\left(1-\ln\left(\frac{P_0}{\tilde{P}}\right)\right)\right\} \\
&\quad + \frac{\tilde{C}(1-\beta)}{\eta(\mu-\theta)}\left\{1-\left(\frac{P_0}{\tilde{P}}\right)^{\beta}\left(\frac{C_0}{\tilde{C}}\right)^{\eta}\left(1-\eta\ln\left(\frac{C_0}{\tilde{C}}\right)\right)\right\} \\
&= \frac{\tilde{C}(1-\beta)}{\eta(\mu-\theta)}\left\{\left(\frac{P_0}{\tilde{P}}\right)^{\beta}\left(\frac{C_0}{\tilde{C}}\right)^{\eta}\left(\ln\left[\left(\frac{P_0}{\tilde{P}}\right)^{\beta-1}\left(\frac{C_0}{\tilde{C}}\right)^{\eta}\right]-\beta\right)+\beta\right\}.
\end{aligned}
\tag{66}
$$

Now for reasonable values of $\tilde{P}$ and $\tilde{C}$, $N_2 < 0$. It follows that:

$$
\frac{\partial F}{\partial \sigma_P} > 0, \tag{67}
$$

provided that $\dfrac{\partial Q}{\partial \sigma_P} > 0$.

We will now determine the derivatives defining the variations in $\tilde{P}$ and $F(\tilde{P},\tilde{C})$ due to a change in σ_C. Differentiating function $Q(\beta,\eta) = 0$ with respect to σ_C yields:

$$
\frac{\partial Q}{\partial \sigma_C} = \sigma_C\eta(\eta-1) + \rho\sigma_P\beta\eta. \tag{68}
$$

Since $\beta < 0$ and $\eta > 1$, then $\dfrac{\partial Q}{\partial \sigma_C} \geq 0$ whenever $\rho \leq 0$; when $\rho > 0$ then $\dfrac{\partial Q}{\partial \sigma_C} \geq 0$ provided $\sigma_C \geq \dfrac{-\rho\sigma_P\beta}{\eta-1}$ but if σ_C is sufficiently small, $\sigma_C < \dfrac{-\rho\sigma_P\beta}{\eta-1}$, $\dfrac{\partial Q}{\partial \sigma_C} < 0$.

The partial derivative $\dfrac{\partial \beta}{\partial \sigma_C}$ is found from:

$$
\frac{\partial \beta}{\partial \sigma_C} = -\frac{\dfrac{\partial(Q,H)}{\partial(\sigma_C,\eta)}}{\dfrac{\partial(Q,H)}{\partial(\beta,\eta)}},
$$

where:

$$\frac{\partial(Q,H)}{\partial(\sigma_C,\eta)} = \begin{vmatrix} \frac{\partial Q}{\partial \sigma_C} & \frac{\partial Q}{\partial \eta} \\ \frac{\partial H}{\partial \sigma_C} & \frac{\partial H}{\partial \eta} \end{vmatrix} = \frac{\partial Q}{\partial \sigma_C} \times \frac{\partial H}{\partial \eta}.$$

It follows that:

$$\frac{\partial \beta}{\partial \sigma_C} = -\frac{1}{D} \times \frac{\partial Q}{\partial \sigma_C} \times \frac{\partial H}{\partial \eta}. \tag{69}$$

Given the signs of the various terms in equation (69), which have been derived above, the sign of $\frac{\partial \beta}{\partial \sigma_C}$ depends on the sign of $\frac{\partial Q}{\partial \sigma_C}$.

The value of $\frac{\partial \eta}{\partial \sigma_C}$ can be found in a similar fashion:

$$\frac{\partial \eta}{\partial \sigma_C} = -\frac{\frac{\partial(Q,H)}{\partial(\beta,\sigma_C)}}{\frac{\partial(Q,H)}{\partial(\beta,\eta)}},$$

where:

$$\frac{\partial(Q,H)}{\partial(\beta,\sigma_C)} = \begin{vmatrix} \frac{\partial Q}{\partial \beta} & \frac{\partial Q}{\partial \sigma_C} \\ \frac{\partial H}{\partial \beta} & \frac{\partial H}{\partial \sigma_C} \end{vmatrix} = -\frac{\partial Q}{\partial \sigma_C} \times \frac{\partial H}{\partial \beta}.$$

It follows that:

$$\frac{\partial \eta}{\partial \sigma_C} = \frac{1}{D} \times \frac{\partial Q}{\partial \sigma_C} \times \frac{\partial H}{\partial \beta}. \tag{70}$$

Given the signs of the various terms in equation (70), which have been derived above, the sign of $\frac{\partial \eta}{\partial \sigma_C}$ depends on the sign of $-\frac{\partial Q}{\partial \sigma_C}$.

Following the same analytical arguments as for the derivations of $\frac{d\tilde{P}}{d\sigma_P}$ and $\frac{dF(\tilde{P},\tilde{C})}{d\sigma_P}$, then it can be established that:

$$\frac{d\tilde{P}}{d\sigma_C} = \frac{\tilde{P}}{D}\frac{\partial Q}{\partial \sigma_C} \times N_1 < 0,$$

and:

$$\frac{dF}{d\sigma_P} = -\frac{\tilde{C}}{\eta^2(\mu-\theta)}\frac{\partial Q}{\partial \sigma_P}\frac{N_2}{D} > 0,$$

provided that in each case $\frac{\partial Q}{\partial \sigma_C} \geq 0$; otherwise the sign is reversed.

Further derivatives are available from the first author on request.

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Figure 1a: Profile of $Q(\beta, \eta) = 0$ for $\rho = 0.5$

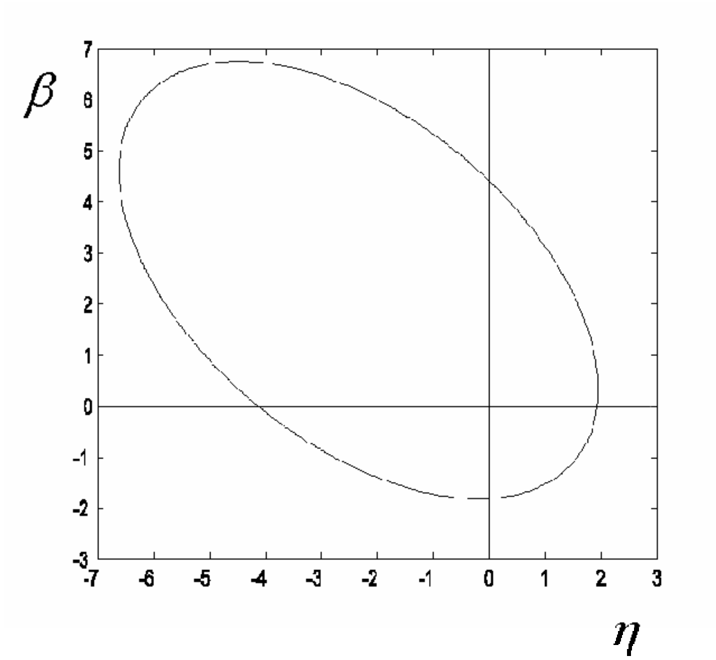

Figure 1b: Profile of $Q(\beta, \eta) = 0$ for $\rho = 0$

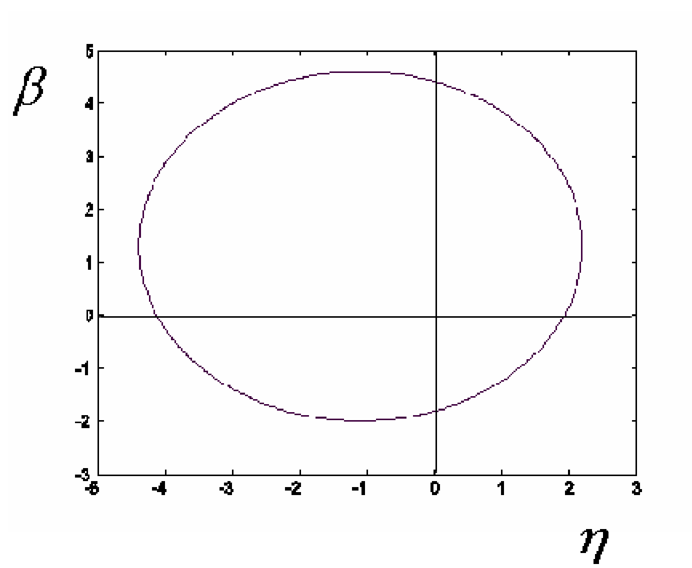

Figure 1c: Profile of $Q(\beta,\eta)=0$ for $\rho=-0.5$

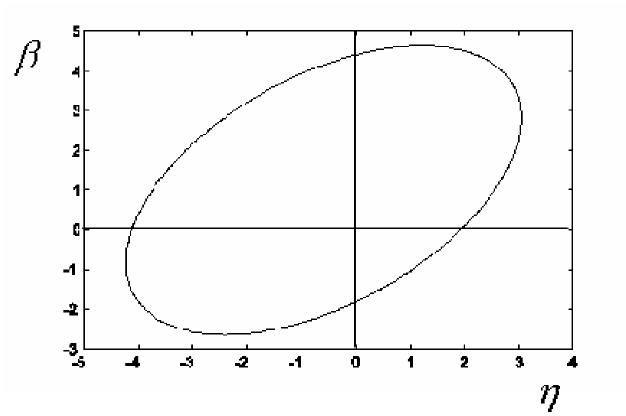

Figure 2: Profile of $H(\beta_2,\eta_1)=0$

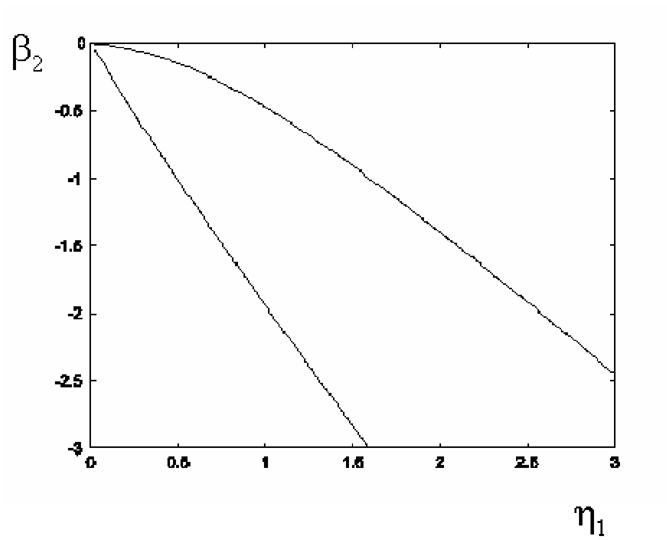

Figure 3: Profiles of $Q(\beta_2, \eta_1) = 0$ and $H(\beta_2, \eta_1) = 0$

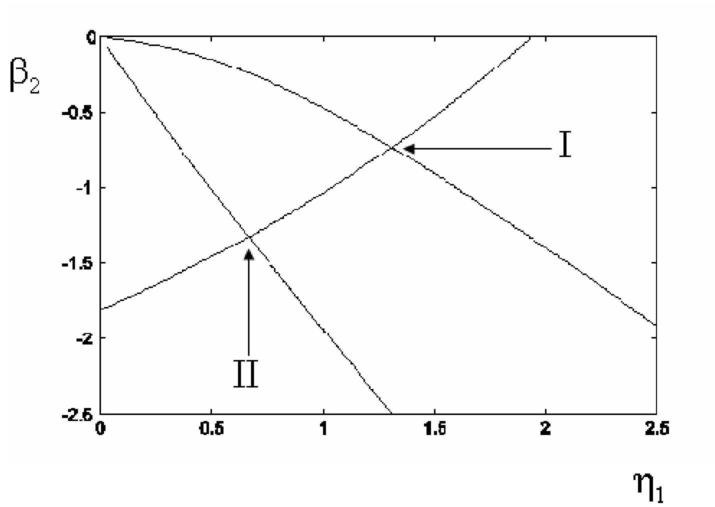

Figure 4: Profile of $G(\tilde{P}, \tilde{C}) = 0$

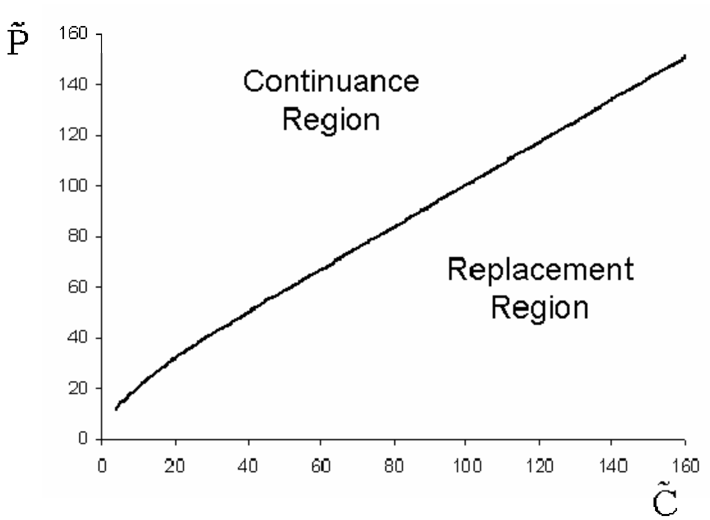

Figure 5a: Profiles of β_2 versus σ_P for Variations in ρ

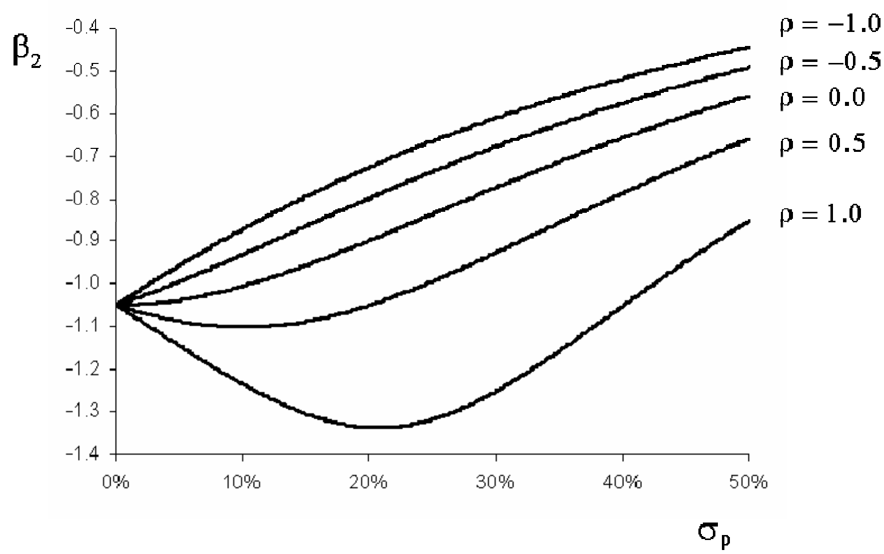

Figure 5b: Profiles of η_1 versus σ_P for Variations in ρ

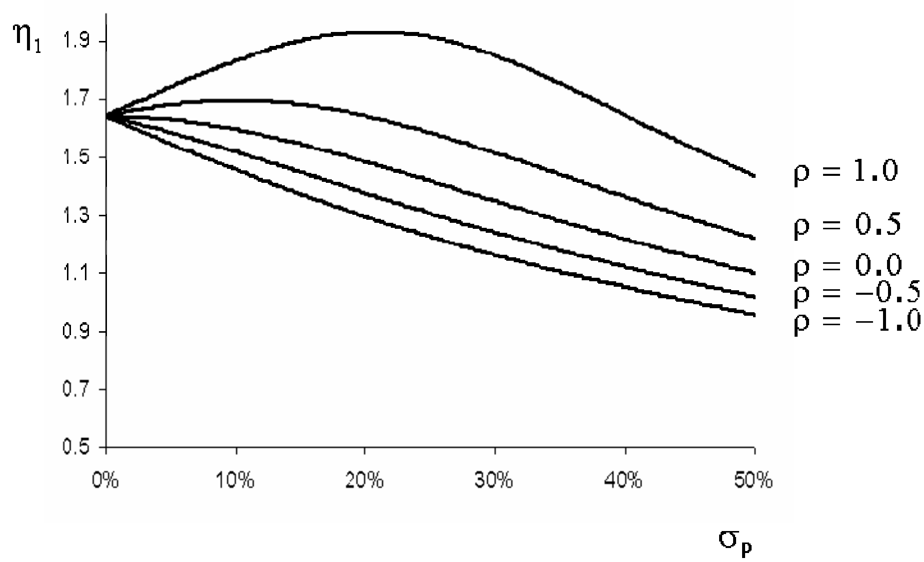

Figure 6a: Profiles of $\tilde{P}$ versus σ_P for Variations in ρ

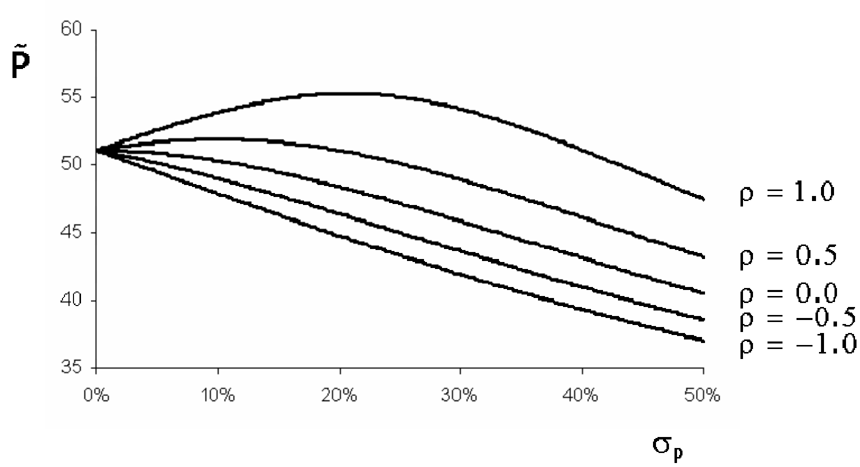

Figure 6b: Profiles of $\tilde{F}$ versus σ_P for Variations in ρ

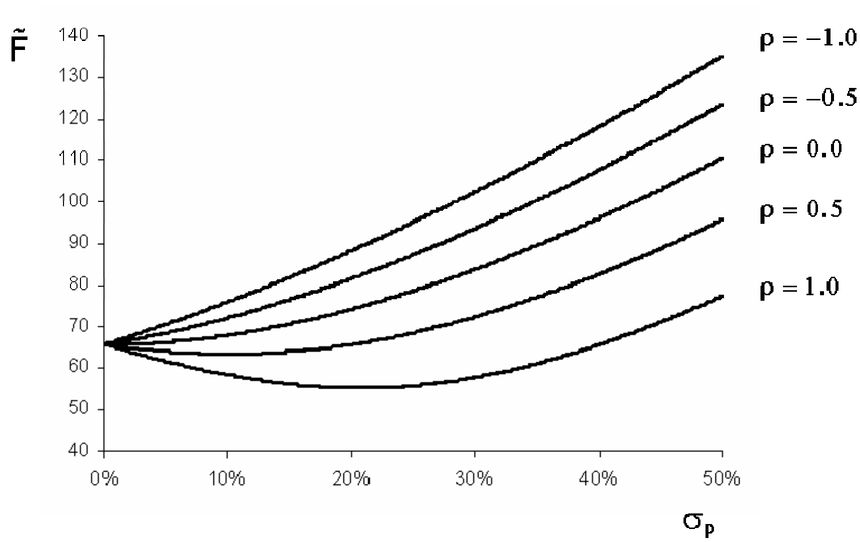

Figure 7a: Profiles of β_2 versus σ_C for Variations in ρ

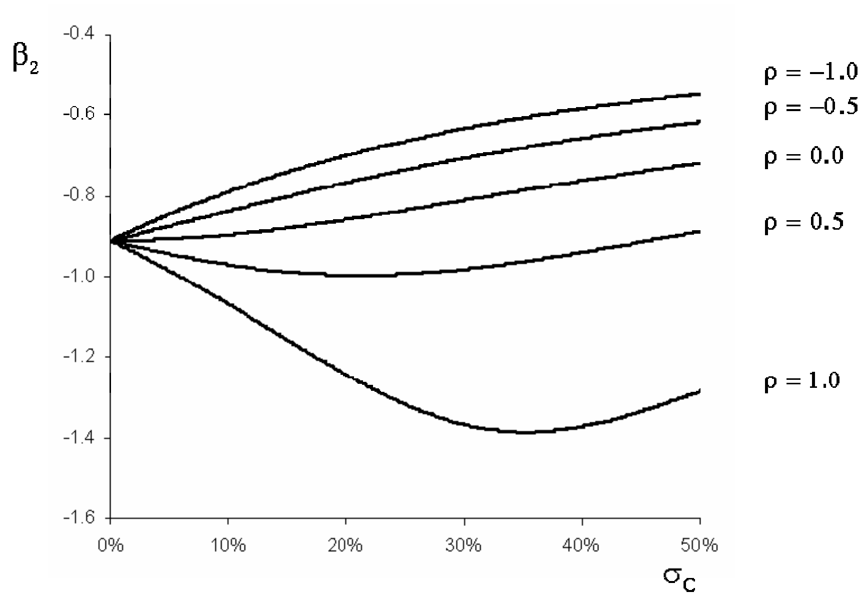

Figure 7b: Profiles of η_1 versus σ_C for Variations in ρ

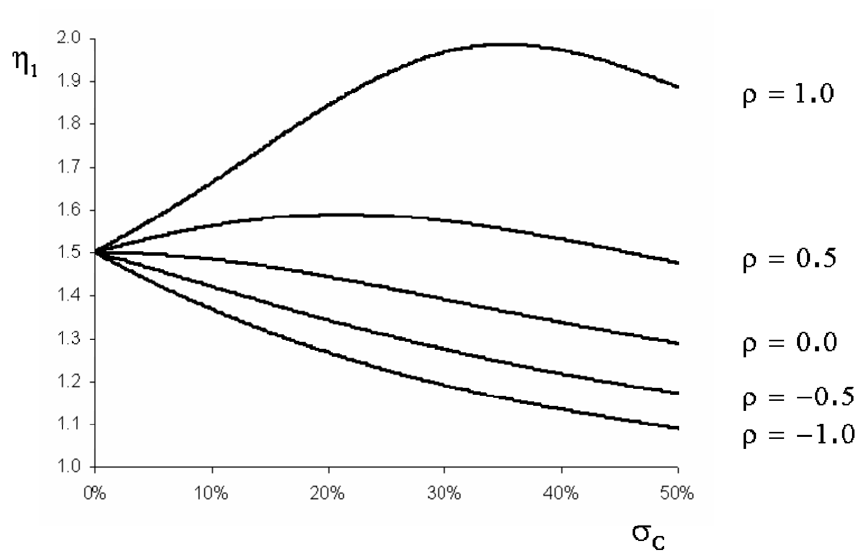

Figure 8a: Profiles of $\tilde{P}$ versus σ_C for Variations in ρ

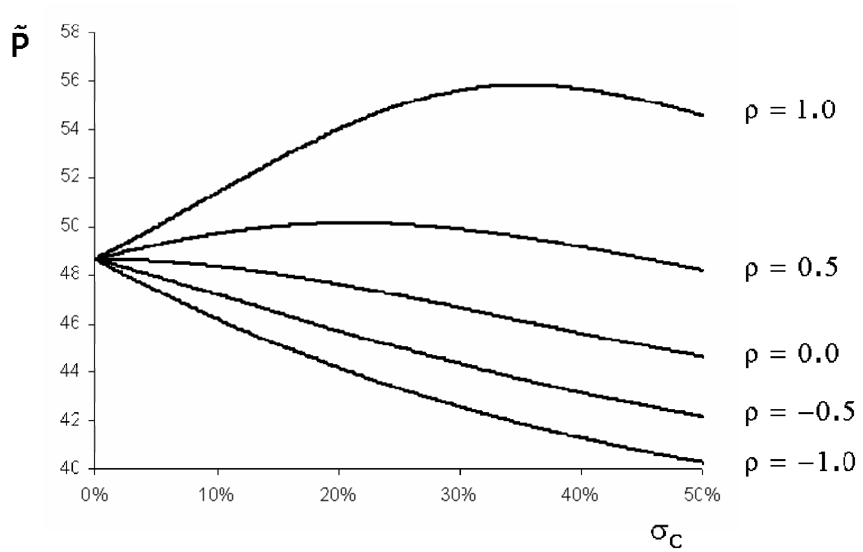

Figure 8b: Profiles of $\tilde{F}$ versus σ_C for Variations in ρ

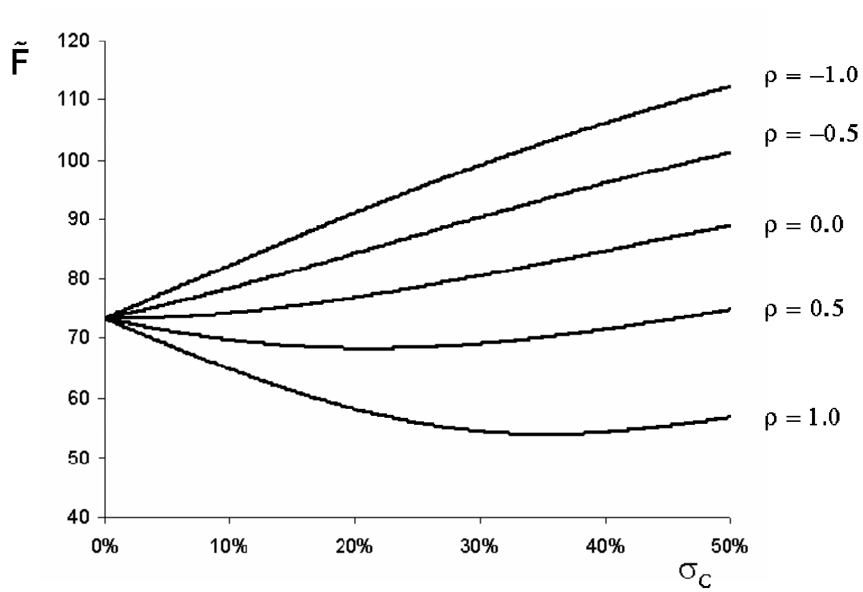

Figure 9: Profiles of $\tilde{F}$ and $\tilde{P}$ versus for variations in K

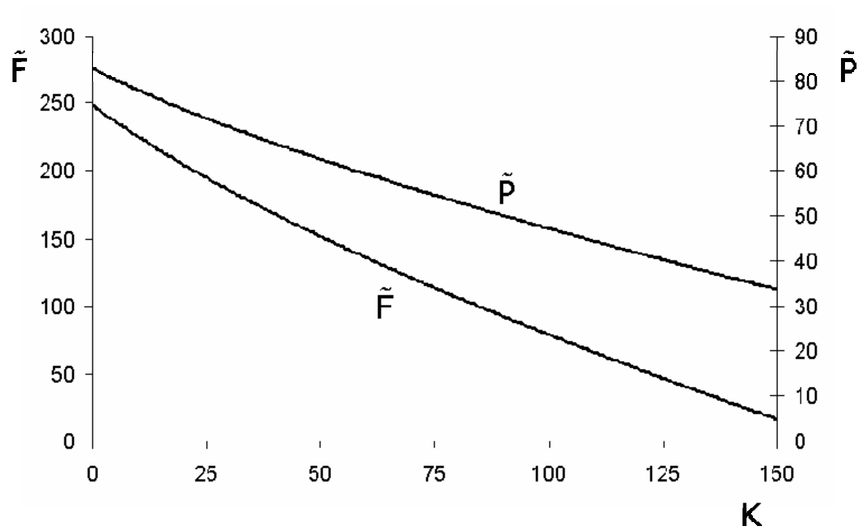

Figure 10: Profiles of $\tilde{F}$ and $\tilde{P}$ versus for variations in P_0

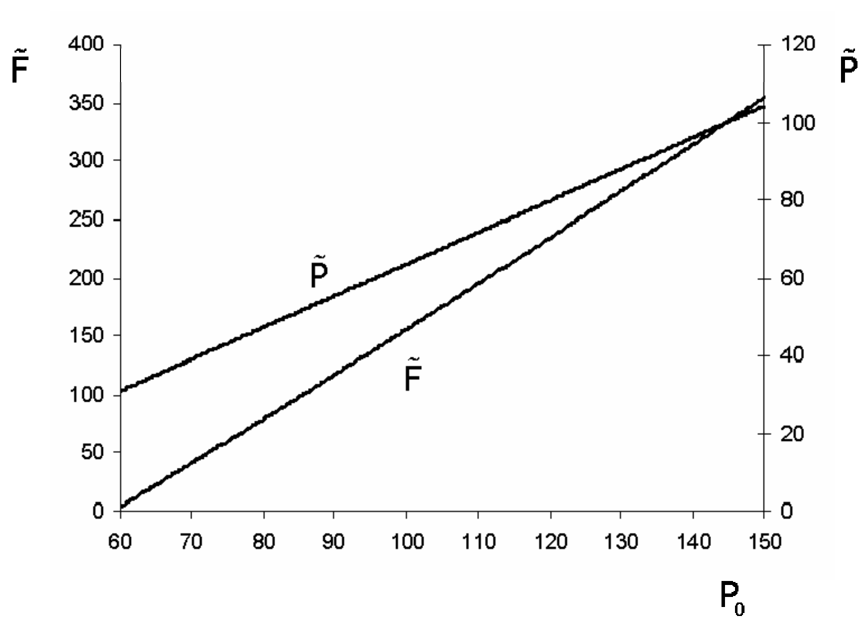

Figure 11: Profiles of $\tilde{F}$ and $\tilde{P}$ versus for variations in μ

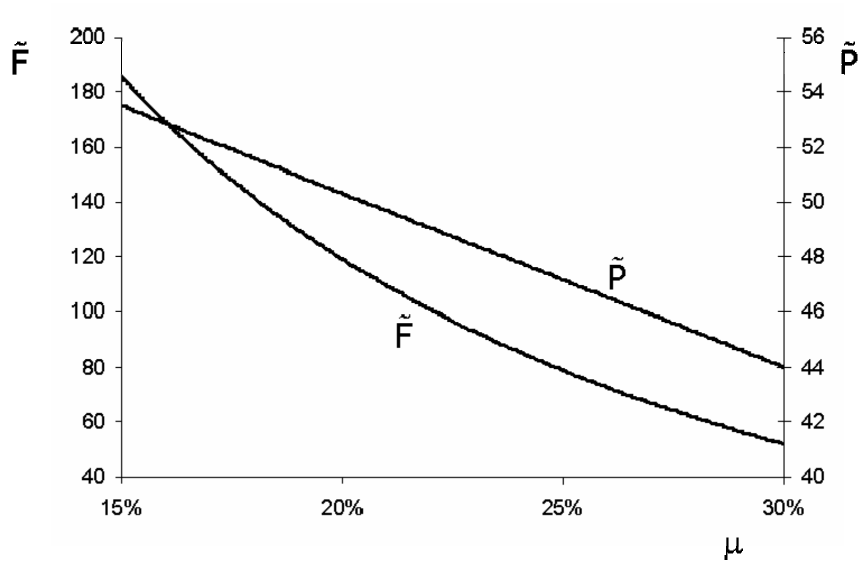

Figure 12: Profiles of $\tilde{F}$ and $\tilde{P}$ versus for variations in α

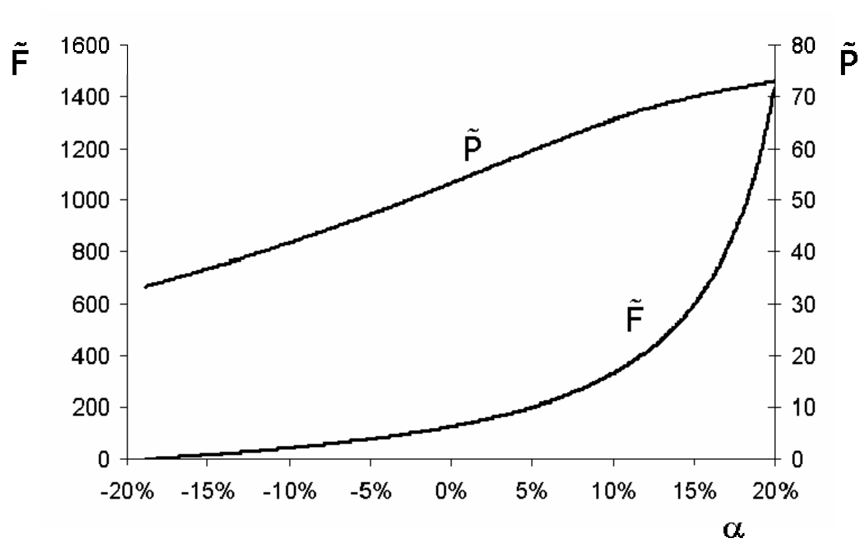

Figure 13: Profiles of $\tilde{F}$ and $\tilde{P}$ versus for variations in θ