The Valuation of Petroleum Lease Contracts as Real Options

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Authenticity Statement

I hereby declare that this dissertation is my own work and confirm its authenticity.

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Symbol Glossary

Uppercase Roman Letters

A, B	Constants of integration in solutions of differential equation
C	Operating cost of a project
F	Value of the opportunity to invest; value of a Futures Contract
G	Value of an option on an option
I	Capital cost of investment
M, U	Kummer Functions
$M_{t,t+1}$	Minus, marginal utility of consumption
$N(x, y)$	A normal distribution with mean x and variance y
P	Output price, crude oil
$\overline{P}$	Mean reversion value of P
Q	Quantity of resources or reserves
R	Revenue from a project
S	Scale function
T	Terminal time
V	Value of asset in place
X_t	Value of uncertain asset at time t

Lowercase Roman Letters

a, b	Coefficients of diffusion process
d	Infinitesimal increment prefix (e.g. the differential dt)
dZ	Increment of standard Wiener (Brownian motion) process
q	The output rate of a project
r	Risk-free interest rate
t	Current time

Adapted From Dixit &Pindyck (1994)

Uppercase Greek Letters

Δ	Prefix for finite but small increment (e.g., Δt)
Γ	Gamma function
φ	Payoff function

Lowercase Greek Letters

α	Drift parameter of simple Brownian motion, or proportional growth rate parameter of geometric Brownian motion
α, β, γ	Constants (Chapter 6)
θ	Variable in fundamental quadratic. Its positive and negative roots are denoted by θ^{+} and θ^{-}, respectively
ε	Random variable distributed N(0,1)
δ	Rate of return shortfall or convenience yield
η	Mean-reversion rate
μ	Risk-adjusted (CAPM) discount rate
π	Profit flow or stationary density (Chapter 3)
ϕ	Market price of risk
ρ	Correlation coefficient
σ	Variance parameter in Brownian motion
τ	One minus tax rate
ω	The rate of decline in the output rate of a project

Moment Operators

$\mathrm{E}(\ )$	Expectation operator
$Cov(\ )$	Covariance operator
$Sd(\ )$	Standard deviation operator
$Var(\ )$	Variance operator

Chapter 1

Introducing the Petroleum Lease Valuation Problem

1.0 Petroleum Leases and the need for their Valuation

Petroleum companies acquire the right to produce petroleum by means of several types of contractual arrangements with the owners of petroleum rights. A common type of these contracts is referred to as a petroleum lease. Pursuant to the terms of a petroleum lease, an owner of petroleum rights grants a lessee the exclusive rights within a defined volume of the earth's subsurface to:

(1) explore for petroleum during an initial period of time, referred to as the primary term; and
(2) develop and extract petroleum during subsequent periods of time, referred to as renewal terms.

The tenure of the primary term is, typically, five years for onshore leases and eight to ten years for offshore leases. If economic quantities of petroleum resources have been discovered, the lessee may elect successive renewal terms to develop and extract the petroleum. The consideration paid by the lessee to the owner of the petroleum rights for a lease comprises: an initial cash payment; annual rental payments; and a constant, or variable, percentage of the petroleum extracted from the leased volume of the subsurface, referred to as a royalty payment. Normally, the rental amount and the royalty percentage are selected by the owner of the petroleum rights prior to offering the lease to prospective purchasers. Competitive negotiations or tenders of sealed bids in a competitive auction determine the magnitude of the initial payment. In either case, petroleum companies require a methodology to determine the fair value of a given petroleum lease. The valuation of petroleum leases using certain mathematical finance techniques is the subject of this thesis.

This chapter opens with an overview of the production cycle followed by petroleum companies to explore for, develop and extract petroleum. A definition of a real option and a framework for casting investment decisions by petroleum companies as real option valuations follows. A list of the mathematical finance techniques which will be developed in the subsequent chapters to value petroleum leases as real options closes the chapter.

1.1 The Production Cycle of Petroleum Companies

Petroleum companies expend quantities of capital to produce petroleum, via a sequential production process comprised of five phases. The activities comprising each phase follow, in chronological order.

Prospect Generation Phase: The production cycle begins with geoscientists prospecting specific geographic areas where the subsurface may contain accumulations of petroleum substances. Acquiring the exclusive right to explore these prospects for petroleum, by means of petroleum leases, or other contracts, completes the first phase.

Exploration Phase: Prospects are explored for petroleum resources in the second phase by conducting geophysical studies, drilling exploration wells, logging the resultant well bores and testing for the presence of hydrocarbons. The collective cost of the exploration activities, referred to as exploration cost, tend to be large. Typical exploration costs for onshore and offshore prospects would be on the order of magnitude of $1 to $10 million and $100 million, respectively. If the activities in the exploration phase fail to discover resources, the case more often than not, a petroleum lease can be relinquished to the owner of the petroleum rights and the prospect abandoned. Alternatively, if a petroleum reservoir containing sufficient resources has been discovered, then a petroleum company owns the right, without the obligation, to develop the discovered resources into reserves.

Development Phase: The development phase comprises: the drilling, completing, and equipping of development wells; the construction of gathering pipelines; and the fabrication and installation of processing plants. The cost of development is the sum of the costs of the development activities and may be an order of magnitude greater than the cost of exploration. A petroleum company holding developed reserves has the right to elect to initiate extraction, pursuant to the terms of its petroleum lease.

Extraction Phase: Extracting petroleum from the subsurface involves: lifting; gathering and processing; marketing; and paying the requisite royalties and taxes. Generally, the collective costs of extraction can be allocated as fixed or variable, on the basis of 80% and 20%, respectively. As the extraction phase proceeds, the withdrawal of volumes of petroleum from the subsurface reservoir will be accompanied, in most cases, by a decline in reservoir pressure causing the rate at which the petroleum is produced to decline. The onset and rate of decline can, to some extent, be mitigated by the initiation of reservoir pressure maintenance and enhanced recovery schemes. Inevitably though, the rate of petroleum production will fall below that necessary to recover both fixed and variable extraction costs.

Abandonment Phase: When the economic limit of the extraction phase is reached, abandonment and reclamation occur. Net of salvage, expenditures in the abandonment phase have, historically, been small relative to those in the preceding phases.

The length of time required to plan and implement each phase runs to years for onshore, and decades for offshore petroleum projects. Cumulatively, the time required to complete the production cycle can be a full generation in human terms. Throughout the production cycle, uncertainties from three principal sources will challenge petroleum companies. Geologic risks, including:

(1) the absence of a reservoir; and
(2) an insufficient quantity of petroleum, if any, recoverable from the reservoir,

are significant in the exploration and development phases. The productivity of the reservoir will be definitively revealed, for the most part, in the extraction phase. Technical risks because of logistical problems on the surface and unanticipated conditions in the subsurface, in terms of rock hardness or pressure levels, can cause the costs of exploration and development operations to exceed budgeted levels. As a petroleum company explores - and perhaps subsequently develops a prospect - geological and technical information is revealed that reduces the levels of geologic and technical risk. The volatility of petroleum prices, in contrast, contributes a high level of uncertainty that pervades all phases of the production cycle.

In making capital budgeting decisions, petroleum companies' primary valuation metric has been the net present value ("NPV") rule. Pursuant to the NPV rule, an affirmative decision to undertake a project is made if and only if ("IFF") the present value of the expected future net cash flows from the project, discounted at a rate that reflects the systematic risk of the project, is greater than the capital cost of the project. Implicit in the NPV rule are the assumptions that:

(1) all cash flows will happen exactly at the times and in the amounts prescribed by the cash flow forecasts; and
(2) all investments are irreversible.

The first assumption precludes managers from using new information, as it arrives, to revise their strategy. The sequential, multiphase composition of the petroleum companies' production cycle means there are many decision points where managers can utilize the information learned in the previous and current phases to select an optimum course of action for the next phase. The assumptions of the NPV rule and the sequential, multiphase structure of the production cycle are not consistent. In the later phases of the petroleum production cycle, where most of the information that can arrive, has arrived, managerial flexibility is less valuable. During the exploration and development phases, when relatively little information regarding geologic and technical risks is available, flexibility is more valuable. The proper valuation of this flexibility may provide a petroleum company with the advantage it needs to succeed in the highly competitive market for petroleum leases. Is there another valuation approach whose formulation and underlying assumptions are more aligned with the multiphase order of the petroleum production cycle? An alternative to the NPV rule is to value a project as a real option and accept the project IFF its value is greater than its cost.

1.2 A Definition of a Real Option and its Application to Valuing Petroleum Leases

Seppi (2002) defines a real option by first considering how to define a commodity. In his construction, a commodity is formed from three attributes: the nature of a good, *G*; the time when it is present, *t*; and the location where it is available, *L*, denoted by $\{G, t, L\}$. This definition enables commodities to be a flow during a time period or a stock at a specific point in time. Consider the example of the commodity known as West Texas Intermediate ("WTI") crude oil, which is defined as a good having a sulfur content of less than 0.42% and a gravity of approximately 38 degrees API and is priced for delivery at Cushing OK. A real option, then according to Seppi, is defined as "a technology to physically convert one or more input commodities $\{G, t, L\}$ into an output commodity $\{G', t', L'\}$". Many real options can be identified in the petroleum industry by the application of Seppi's definition, including the following. The Syncrude Project, which upgrades bitumen to WTI crude oil, is a goods conversion option. A pipeline that transports crude oil from the field gate to a refinery represents an option to change locations. A crude oil storage terminal is an option to change the time crude oil is available. Most important of all, the phases of petroleum cycle represent goods conversion options, as is shown below.

Extraction Option: Extraction is the means to convert reserves into above ground barrels of crude oil. A holder of petroleum reserves owns three options, subject to the terms of its lease. The first is the extraction option. When petroleum revenues - the product of the price of crude oil and the volume extracted - exceed the cost, extraction will occur. In this state, the holder of the reserves receives the cash flow and retains a right to suspend production. Conversely, if petroleum revenues fall below lifting costs, the holder of the reserves can suspend extraction, and retains the right to wait for higher crude oil prices in the future. In effect, the holder of shut-in reserves has an out-of-the-money call on the price of crude oil. If reservoir conditions permit, at some high enough price of crude oil the holder may wish to exercise its second option: to enhance the recovery of the reserves. At some very low crude oil price, a petroleum company may exercise the third option, when the expected future net cash outflows will be greater than the cost of abandonment, by abandoning the reserves and reclaiming the surface site.

The value of the option to extract petroleum, denoted by e, is a function of three stochastic state variables with their associated risk factors: the price of crude oil, *P(t)* and the volatility of crude oil prices, given the information available during the extraction phase, $\sigma_p(e)$; the quantity of recoverable reserves, *Q(t)* and the volatility of geological risk, given the information available in the extraction phase, $\sigma_q(e)$; the cost of extraction, $C_e(t)$ and the volatility of extraction cost, $\sigma_c(e)$. The motions of the three variables are specified by the stochastic differential equations ("SDE")

$$dP(t)=\alpha_p(e)dt+\sigma_p(e)dZ(t) \qquad dQ(t)=\alpha_q(e)dt+\sigma_q(e)dZ(t)$$

$$dC_e(t)=\alpha_c(e)dt+\sigma_c(e)dZ(t).$$

The length of time extraction proceeds is T_e and the time value of money is r. Symbolically, $e=e[P(t),Q(t),C_e(t);\ \sigma_p(e),\sigma_q(e),\sigma_c(e),T_e,r]$. The quantity of information revealed by exploration and development regarding *Q(t)* and $C_e(t)$ will be large, meaning $\sigma_q(e)$ and $\sigma_c(e)$ will be small. The uncertain path followed by crude oil prices will be the dominant stochastic variable in the valuation of developed reserves.

Development Option: The technology used to convert resources into reserves is termed development. The right to develop resources into reserves, without the obligation to do so, is a real option, the value of which accrues to a holder of a resource. The holder of a resource will develop it when the value of a producing reserve, given by e, exceeds the cost of development, $C_d(t)$. The development option will have the boundary condition $Max[e(t^*)-C_d(t^*),0]$ where t^* is the optimal time to develop the resource. The value of the development option, denoted by d, will be given by the function: $d=d[e,C_d(t);\ \sigma_e(d),\sigma_{cd}(d),T_d,r]$ where T_d is the renewal term of the lease. But e is a function of P,Q and C_e, so: $d=d[P,Q,C_d,C_e;\ \sigma_p(d),\sigma_q(d),\sigma_{cd}(d),\sigma_{ce}(d),T_d,T_e,r]$ where $\sigma_q(d)$ is the risk associated with Q based on the information available prior to development, $\sigma_{cd}(d)$ is the degree of uncertainty in the cost of development. Note that the risk factors for the quantity of reserves, $\sigma_q(d)>\sigma_q(e)$, and the cost of extraction, $\sigma_{ce}(d)>\sigma_{ce}(e)$, are greater in the development phase than the extraction phase. The volatility of petroleum prices will continue to affect the motion of the value of the underlying reserves.

Exploration Option: Exploration is the technology which converts prospects into petroleum resources. It follows that a petroleum lease, which affords the holder the exclusive right, without obligation, to drill an exploration well at any time during the primary term, can be valued as an American call option that will be referred to as the exploration option. The exploration option will be exercised when the value of the development option, given by *d*, exceeds the exploration costs, denoted by $C_x(t)$. The boundary condition for the exploration option will be $Max[d(t^*) - C_x(t^*)\ ,\ 0]$ where t^* is the optimal time to explore. The value of the exploration option is a function of the state variables d and C_x with associated risk parameters $\sigma_d(x)$ and $\sigma_{C_x}(x)$, $x = x[d, C_x;\ \sigma_d(x), \sigma_{Cx}(x), T_x, r]$ where the tenure of the primary term of the lease is T_x. However, *d* is a function of *e* and C_d. Likewise, *e* is a function of *P*, *Q* and C_e. It follows that *x* is a function of these variables and parameters as well. So

$$x = x[P, Q, C_x, C_d, C_e;\ \sigma_P(x), \sigma_q(x), \sigma_{Cx}(x), \sigma_{Cd}(x), \sigma_{Ce}(x), T_x,\ T_d, T_e, r\ ].$$

Geologic risk, represented by $\sigma_q(x)$, is significant in the exploration phase, as more exploration wells are abandoned than completed. The quantity of technical risk, $\sigma_{C_x}(x)$, will also be large due the uncertain cost of drilling rock not previously penetrated, in a remote location. As the production cycle proceeds, the geologic and technical information revealed in the exploration and development phases will significantly reduce the quantity of uncertainty associated with *Q* and *C* , symbolically, $\sigma_q(x) > \sigma_q(d) > \sigma_q(q)$ and $\sigma_{Cx} > \sigma_{Cd} > \sigma_{Ce}$. The option of deferring exploration or development until some of the uncertainty can be resolved, say by competitors drilling in same the area, will be valuable.

The foregoing discussion has established the sequential arrangement of the real options to explore, develop and extract petroleum. Consequently, a petroleum lease can be valued as a three stage compound option, subject to three sources of uncertainty. The valuation of a petroleum lease is further complicated since the two variables *Q* and *C*, representing the state of the geology and technical cost, are not traded and priced in capital markets. To simplify the valuation of a petroleum lease it is assumed that: (1) the variables *Q* and *C* are deterministic; (2) the price of crude oil, which contributes a significant quantity of uncertainty in all phases of the production cycle, can be modeled by a single-factor stochastic process; and (3) consistent with the long time frames required to complete exploration, development and extraction programs, the options comprising the production cycle have infinite lives.

1.3 The Course of this Investigation

To value a petroleum lease as a compound real option, a number of mathematical techniques are needed, including the following.

(1) A method of deriving the Black-Scholes-Merton partial differential equation ("BSM PDE") is developed in Chapter 2, consistent with the fact that while barrels of crude oil

on the surface are traded in a continuous commodity market, barrels of resources and reserves in the subsurface are not.

(2) A stochastic process, the attributes of which enable it to model crude oil prices, is selected in Chapter 3, after consideration of six single-factor candidate processes.

(3) Methods to estimate the parameters of the stochastic process used to model crude oil prices are developed in Chapter 4 by employing two approaches. In the first, a regression equation specific to the stochastic process selected in Chapter 3 is derived and applied to a time-series of spot crude oil prices. The second approach consists of deriving the value of a futures contract for crude oil and using it to calibrate the parameters of the selected stochastic process to the futures market.

(4) The valuations of certain perpetuities, where the cash flow is a function of the price of crude oil modeled by the stochastic processes, are derived in Chapter 5. The perpetuity valuations are used to demonstrate an important difference between the stochastic behavior of barrels of crude oil on the surface and in the subsurface and as particular solutions to the ordinary differential equations ("ODE") derived in Chapter 6.

(5) Derivations of the BSM PDE using the selected single-factor process are developed in Chapter 6 which when solved as ODE's by repeatedly applying the boundary conditions known as value-matching and smooth-pasting value the compound real options inherent in the petroleum production cycle.

Finally, in Chapter 7, certain conclusions are presented.

Chapter 2

A Derivation of the BSM PDE to Value Real Options

2.0 Introduction

This chapter contains a derivation of the BSM PDE appropriate to value a real option. The valuation of a real option differs fundamentally from that of a financial option. Certain of the assumptions that underpin the derivation of the BSM PDE to value financial options cannot be relied upon when valuing real options. The assumption that the underlying asset is continuously traded and can be used to hedge the real option, is not consistent with the valuation problem described in Section 1.2. To identify an appropriate derivation of the BSM PDE, five alternate derivations were reviewed, considering both the number of assumptions required and the consistency of the assumptions with respect to the valuation problem. The assumptions necessary to obtain a derivation of the BSM PDE can be allocated into two broad classes: those common to all the derivations reviewed; and the incremental assumptions specific to each derivation.

Let the contingent claim to be valued, representing a firm or project, be a function of the uncertain output price received, $P(t)$, and time, t, and denoted by $F(P,t)$. All of the derivation methods considered herein are based on three common assumptions:

(1) The underlying asset, $P(t)$, follows a diffusion process specified by the SDE, $dP(t) = \alpha(P,t)dt + \sigma(P,t)dZ(t)$ (2.1)

where: (a) $\alpha(P,t)$ is a deterministic function that specifies the expected instantaneous growth rate of P, in dollars per unit-time;

(b) $\sigma(P,t)$ is a deterministic function that specifies the annualized standard deviation of the returns from P; and

(c) $dZ(t)$ is the increment of a Wiener process with zero drift and unit variance per unit of time.

(2) $F(P,t)$ is at least twice differentiable, so that Itô's Lemma can validly be applied to F.

(3) The risk-free interest rate, r, is known and constant across time.

What incremental assumptions are necessary to complete the derivation of the BSM PDE? The answer depends on the route followed to derive the BSM PDE. The principal incremental assumptions of four of the five methods: Delta Hedging, Replication, Spanning Assets, and Dynamic Programming, were found to be inconsistent with the valuation of real options. This discussion is contained in Appendix I. The fifth method, the equilibrium method due to Sick (1995), based on the Consumption Capital Asset Pricing Model ("CAPM"), is parsimonious in terms of the number of incremental assumptions required and their consistency with real option valuations. Sick's derivation of the BSM PDE is exposited below.

2.1 Sick's Derivation of the BSM PDE for Real Options

According to Sick (1995), the discrete time version of the Consumption CAPM asserts that the value at time t of an uncertain asset $\widetilde{X}_t$ at time t + 1, denoted by $V_t\left[\widetilde{X}_{t+1}\right]$, is a function of: the risk-free rate, r ; the time, t, expected value of the asset at time t + 1; and the covariance of the asset price with minus the time t marginal utility of consumption at t + 1, denoted by $\widetilde{M}_{t,t+1}$. This can be represented as

$$V_t\left[\widetilde{X}_{t+1}\right](1+r) = E_t\left[\widetilde{X}_{t+1}\right] - Cov_t\left[\widetilde{X}_{t+1},\, \widetilde{M}_{t,t+1}\right] \tag{2.2}$$

In (2.2) let: $t+1 = t+\Delta t$, $\widetilde{X}_{t+\Delta t} = \widetilde{P}_{t+\Delta t} + \delta(P,t)\Delta t$,

$V_t\left[\widetilde{X}_{t+\Delta t}\right] = P_t$, and $\Delta \widetilde{P}_{t+\Delta t} = \widetilde{P}_{t+\Delta t} - P_t$,

where $\delta(P,t)$ is a deterministic function that specifies the rate, in dollars per unit time, of the income paid to the holder of the underlying asset, P. Replace r with $r\Delta t$ and $\widetilde{M}_{t,t+1}$ with $\Delta\widetilde{M}$, to allow Δt to approach zero, then

$$P_t(1+r\Delta t) = E_t\left[\widetilde{P}_{t+\Delta t} + \delta(P,t)\Delta t\right] - Cov_t\left[\widetilde{P}_{t+\Delta t} + \delta(P,t)\Delta t,\, \Delta\widetilde{M}\right]. \tag{2.3}$$

Since $\delta(P,t)$ and P_t are known at time t they can be treated as constants. For random variables X and Y any constant b, $Cov(X+b,Y) = Cov(X,Y)$. Then $\delta(P,t)$ can be removed and P_t subtracted from inside the covariance in (2.3) without effect giving (2.4),

$$E\left[\widetilde{P}_{t+\Delta t} - P_t + \delta(P,t)\Delta t\right] = rP\Delta t + Cov_t\left[\widetilde{P}_{t+\Delta t} - P_t,\, \Delta\widetilde{M}\right]. \tag{2.4}$$

Taking the limit as $\Delta t \to dt$ of (2.4) yields

$$E\left[dP + \delta(P,t)dt\right] = rPdt + Cov_t\left[dP,\, dM\right]. \tag{2.5}$$

The motions of P and M are diffusions. Assume M follows the diffusion process

$$dM = \alpha_M(M,t)dt + \sigma_M(M,t)dZ_M\,. \tag{2.6}$$

Substituting (2.1) and (2.6) in (2.5) and using $Cov(aX+b, cY+d) = ac\; Cov(X,Y)$, then

$$E\left[dP + \delta(P,t)dt\right] = rPdt\; + \sigma(P,t)\sigma_M(M,t)Cov(dz, dz_M).$$

The standard deviation of both the Brownian Motions dZ and dZ_M is $\sqrt{dt}$. Since $Cov(X,Y) = \sigma_X\;\sigma_Y\rho_{X,Y}$ then $Cov(dZ, dZ_M) = \rho_{Z,Z_M}\; dt$, where ρ_{Z,Z_M} is the correlation coefficient between dZ and dZ_M, or between returns on M and P. So

$$E\left[dP + \delta(P,t)dt\right] = rPdt + \sigma(P,t)\sigma_M(M,t)\rho_{Z,Z_M}\,dt\,. \tag{2.7}$$

The holder of P expects to receive capital gains at the rate $\alpha(P,t)$ and income at the rate $\delta(P,t)$ so over time dt,

$$E[dP + \delta(P,t)dt] = \alpha(P,t)dt + \delta(P,t)dt \,. \tag{2.8}$$

Equating (2.7) and (2.8) and eliminating dt yields

$$\alpha(P,t) + \delta(P,t) = rP + \sigma(P,t)\sigma_M(M,t)\rho_{Z,Z_M} \,. \tag{2.9}$$

This, (2.9), is the total return equation. It says rate of capital gain plus income equals the risk free return plus a risk premium. Repeating the foregoing steps, (2.2) to (2.8), leads to,

$$E[dF + \pi(P,t)dt] = rFdt + Cov_t[dF, dM], \tag{2.10}$$

an expression for the expected capital gain, dF, plus income, denoted by the deterministic function $\pi(P,t)$, from $F(P,t)$. $F(P,t)$ was assumed to be at least twice differentiable, in its first argument and differentiable in its second argument, allowing an expansion of dF by Itô's Lemma as follows,

$$dF = \left[F_t + \frac{1}{2}\sigma^2(P,t)F_{PP} + \alpha(P,t)F_P \right] dt + \sigma(P,t)F_P \, dZ \,. \tag{2.11}$$

Substitute (2.11) on both the RHS and LHS of (2.10), and the diffusion assumption for dM, eliminate dt from both sides, and collect like terms to arrive at

$$F_t + \frac{1}{2}\sigma^2(P,t)F_{PP} + [\alpha(P,t) - \sigma(P,t)\sigma_M(P,t)\rho_{P,M}]F_P - rF + \pi(P,t) = 0 \,. \tag{2.12}$$

In (2.9) it was shown: $\alpha(P,t) - \sigma(P,t)\sigma_M(P,t)\rho_{P,M} = rP - \delta(P,t)$, so

$$F_t + \frac{1}{2}\sigma^2(P,t)F_{PP} + [rP - \delta(P,t)]F_P - rF + \pi(P,t) = 0 \,. \tag{2.13}$$

If the risk factor for the marginal utility of consumption is

$$\sigma_M(M,t) = \frac{\alpha_M - r}{\sigma_M} = \frac{r_M - r}{\sigma_M} = \phi$$

which is the market price of risk, then (2.12) can be written as

$$F_t + \frac{1}{2}\sigma^2(P,t)F_{PP} + [\alpha(P,t) - \sigma(P,t)\phi\rho_{P,M}]F_P - rF + \pi(P,t) = 0 \,. \tag{2.14}$$

This general form of the BSM PDE was obtained with no reliance on assumptions of continuous trading and hedging of the underlying asset. Furthermore, (2.14) is in a general enough form to allow the use of the most appropriate diffusion process to model the behavior of P. The consideration of certain diffusion processes with a view to selecting one to model crude oil prices is the subject of the next chapter.

Chapter 3

A Stochastic Process to Model Crude Oil Prices

3.0 Introduction

In the previous chapter the deterministic functions that define the motion of the underlying asset, or output price, via the diffusion process (2.1), were left undefined. The task in this chapter is to define functions for the drift and diffusion terms in (2.1) appropriate to model the price of crude oil. The chapter opens with a discussion of the attributes of an appropriate stochastic process followed by an examination of the behavior of a process that meets the necessary criteria. A comparison of six stochastic processes and the selection of an appropriate process close the chapter.

3.1 Attributes of an Appropriate Stochastic Process

Robel (2001) posits that to model commodity prices usefully, a stochastic process, $\{P(t)\}_{t\geq 0}$, must exhibit three characteristics:

(1) generates positive values, $P(t) > 0$, for all $t \geq 0$;
(2) reverts to a mean value, $\overline{P}$, over time; and
(3) if the process involves more than one unit of the commodity, reverts to the number of units of the commodity times the reversion price of one unit.

To Robel's list a fourth characteristic is added: the property that as $t \to \infty$, $P(t)$ is not attracted to either of the boundaries $P = 0$, or $P = \infty$. The economic necessity and mathematical definition of each of these characteristics are discussed below.

Positive Prices: Commodity prices must be greater than zero, otherwise suppliers would have no motivation to vend. To demonstrate that a particular process will generate only positive values, two steps were utilized. First, a solution of the SDE that defines the process was found. The solution of the general linear single-factor SDE is given in Klebaner (1998) p. 121-123. Second, the range of the solution function, *P(t),* was shown to be positive over the function's domain.

Reversion to a Mean: Mean reversion is the tendency of a random variable if it is above (below) some normal level to drift down (up) over time, towards the normal level. There are economic arguments that commodity prices should exhibit mean reversion. In a competitive market, with no barriers to entry or exit, if a commodity price rises above an equilibrium price, new supply will come forward and demand will fall, drawing the market price back to the normal level. Conversely, supply will decrease and demand increase if the price of a commodity falls below an equilibrium level. In the long run, a commodity's price should revert to its supply cost because of competition. In a market where "pure competition" does not exist, the case for the reversion of prices is stronger. For the commodity crude oil, a cartel of suppliers, known as the Organization of Petroleum Exporting Countries ("OPEC"), has stated that it will "adjust" the supply of crude oil to obtain a price objective between $22 and $28 per barrel ("BBL"), for the "OPEC basket" marker crude. The stated intentions of OPEC augment the case for mean reverting behavior in the price path of crude oil, but in no way guarantee it. In both of the Persian Gulf Wars, 1991 and 2003, the price of crude oil spiked up, to over $40 per BBL. The opposite occurred in 1998 when over production by certain members of OPEC caused prices to fall to $10 per BBL, briefly.

To prove a given stochastic process reverts to a mean value, as $t \to \infty$, two approaches are employed. In the first, an expression for the first moment of the process is found, if it exists and the limit taken as $t \to \infty$. The second route requires the derivation of: the process' stationary density, its probability density at $t = \infty$, and the first moment of the stationary density. Application of both approaches may be necessary, because not all processes have both a first moment and a stationary density. The stationary density is defined in Klebaner (1998) p157-158.

Homogeneity Condition: This characteristic will be required if the underlying asset is the product of price and quantity, so that the price of *n* BBL's will revert to $n\overline{P}$. Robel (2001) defines the homogeneity condition for an SDE of the form

$$dP(t) = \alpha\left[P(t), \overline{P}\right] dt + \sigma\left[P(t), \overline{P}\right] dZ(t).$$

Then for any $a > 0$, the motion of the process $Y(t) = aP(t)$ should be determined by

$$dY(t) = \alpha\left[Y(t), \overline{Y}\right] dt + \sigma\left[Y(t), \overline{Y}\right] dZ(t),$$

where $\overline{Y} = a\overline{P}$. The homogeneity condition will be satisfied if the drift and volatility functions are homogeneous functions of degree one of the pair $\{P(t), \overline{P}\}$.

Boundary Behavior: In the long run, commodity prices tend neither to zero, nor to infinity. This implies that to realistically model commodity prices, a stochastic process cannot be attracted to either of the boundaries $P = 0$, or $P = \infty$. The behavior of a stochastic process at a boundary, whether it is attracted or reflected, is examined by determining the convergence or divergence, respectively, of the scale function at the boundary. The integral that defines the scale function can be found in Klebaner (1998) p 150.

3.2 The Attributes of the Inhomogeneous Geometric Brownian Motion ("IGBM") Process

The IGBM process is defined by the SDE, (3.1), with $P(0) = p_0$,

$$dP(t) = \eta\left[\overline{P} - P(t)\right] dt + \sigma P(t) dZ(t). \tag{3.1}$$

In (3.1) η is the speed, or strength of reversion to $\overline{P}$ and σ is the volatility of the diffusion. The IGBM process is studied to see if it posses the four characteristics necessary to model commodity prices, below.

Positive Prices: The direct application of the general solution for linear SDE's provides the solution of (3.1)

$$P(t) = \exp\left[-\left(\eta + \frac{\sigma^2}{2}\right)t + \sigma Z(t)\right]\left\{p_0 + \eta\overline{P}\int_0^t \exp\left[\left(\eta + \frac{\sigma^2}{2}\right)s + \sigma Z(s)\right]ds\right\}. \qquad (3.2)$$

Since $\exp(x) \geq 0$ for all x, then $P(t) > 0$ for all $t \geq 0$.

Reversion to a Mean: The first moment is found by taking the expectation at $t = 0$ on both sides of (3.1) and applying Fubini's Theorem to the LHS of (3.1)

$$\frac{d}{dt}E[P(t)] + \eta E[P(t)] = \eta\overline{P}. \qquad (3.3)$$

Equation (3.3) is a first order, non-homogeneous ODE in $E[P(t)]$ that can be solved with the integrating factor $\exp(\eta t)$ as follows,

$$\int_0^t d\left\{e^{\eta s}E[P(s)]\right\} = \eta\overline{P}\int_0^t e^{\eta s}ds. \qquad (3.4)$$

Completing the integration and substituting $E[P(0)] = p_0$ in (3.4), results in the first moment,

$$E[P(t)] = \overline{P} + \left(p_0 - \overline{P}\right)e^{-\eta t}. \qquad (3.5)$$

Taking the limit of (3.5) as $t \to \infty$ proves $P(t)$ reverts to $\overline{P}$. A second method to show that $P(t)$ reverts to $\overline{P}$ involves finding the stationary density of $P(t)$, denoted by $\pi(p)$, and deriving its first moment. The stationary density of the diffusion (2.5) and the IGBM is

$$\pi(p) = \frac{C}{\sigma(p)^2}\exp\left\{\int^P \frac{2\alpha(y)}{\sigma(y)^2}dy\right\} = \frac{C}{\sigma^2}p^{-2\left(\frac{\eta}{\sigma^2}+1\right)}\exp\left[-\frac{2\eta\overline{P}}{\sigma^2 p}\right] \qquad (3.6)$$

since for the IGBM process $\alpha(y) = \eta(\overline{P} - y)$ and $\sigma(y)^2 = \sigma^2 y^2$. The constant C is determined by setting the integral of (3.6) from 0 to ∞ equal to one. The IGBM's stationary density is

$$\pi(p) = \left(\frac{2\eta\overline{P}}{\sigma^2}\right)^{\left(\frac{2\eta}{\sigma^2}+1\right)} p^{-2\left(\frac{\eta}{\sigma^2}+1\right)} e^{-\left(\frac{2\eta\overline{P}}{\sigma^2 p}\right)} \Big/ \Gamma\left(\frac{2\eta}{\sigma^2}+1\right). \qquad (3.7)$$

In (3.7) $\Gamma(X)$ denotes the gamma function. The first moment of $\pi(p)$ is defined as

$$E(p) = \int_0^\infty p\pi(p)dp = \left(\frac{2\eta\overline{P}}{\sigma 2}\right)^{\left(\frac{2\eta}{\sigma 2}+1\right)} \int_0^\infty p^{-\left(\frac{2\eta}{\sigma 2}+1\right)} e^{-\left(\frac{2\eta\overline{P}}{\sigma 2\, P}\right)} dp \ / \ \Gamma\left(\frac{2\eta}{\sigma 2}+1\right) \quad (3.8)$$

$$E(p) = \left(\frac{2\eta\overline{P}}{\sigma^2}\right) \Gamma\left(\frac{2\eta}{\sigma^2}\right) \ / \ \Gamma\left(\frac{2\eta}{\sigma^2}+1\right) = \overline{P}.$$

Again, the IGBM process is shown to revert to $\overline{P}$ as $t \to \infty$.

Homogeneity Condition: Let $Y(t) = aP(t)$ for $a > 0$, then from (3.1)

$\alpha(Y(t), \overline{Y}) = \eta(\overline{Y} - Y(t)) = \eta(a\overline{P} + aP(t)) = a\alpha(P(t), \overline{P})$ and

$\sigma(Y(t), \overline{Y}) = \sigma Y(t) = a\sigma P(t) = a\sigma(P(t), \overline{P})$.

The IGBM process is homogeneous of the pair $\{P(t), \overline{P}\}$ satisfying the homogeneity condition.

Boundary Behavior: The scale density for the diffusion process (2.5) and the IGBM is

$$S'(p) = \exp\left\{-\int^P \frac{2\alpha(y)}{\sigma 2(y)} dy\right\} = p^{\frac{2\eta}{\sigma^2}} \exp\left\{\frac{2\eta}{\sigma^2 p}\right\}. \quad (3.9)$$

The integral of (3.9) yields the scale function for the IGBM

$$S(p) = \int^P S'(y)dy = \int^P y^{\frac{2\eta}{\sigma 2}} \exp\left\{\frac{2\eta}{\sigma 2} y^{-1}\right\} dy. \quad (3.10)$$

At the boundary $P=0$ the term $\exp[2\eta/\sigma 2P]$ will dominate the integral and the limit of $\exp[2\eta/\sigma 2P]$ as $P \to 0$ will equal ∞ if $2\eta/\sigma^2 > 0$, so the integral, $S(0)$ will not converge and the IGBM diffusion will not reach $P=0$. Similarly, at the boundary $P=\infty$, the term $P^{2\eta/\sigma^2}$ will dominate the integral and the $P\to\infty$ limit $P^{2\eta/\sigma^2} = \infty$, if $2\eta / \sigma^2 > 0$. The integral $S(\infty)$ will diverge and the IGBM diffusion will not reach $P=\infty$.

While possessed of the preceding four positive attributes, the IGBM process does have one detrimental feature. There are restrictions on the values the IGBM's parameters, η and σ, can take on, lest the variance of the process become infinite! The determination of the parameter restrictions begins by deriving an expression for the second moment of the stationary density.

$$E(p^2) = \int p^2\pi(p)dp = \left(\frac{2\eta\overline{P}}{\sigma 2}\right)^{\left(\frac{2\eta+1}{\sigma 2}\right)} \int_0^\infty p^{-\left(\frac{2\eta}{\sigma 2}\right)} e^{-\left(\frac{2\eta\overline{P}}{\sigma 2 p}\right)} dp \ / \ \Gamma\left(\frac{2\eta}{\sigma 2}+1\right)$$

$$E\left(p^2\right) = \left(\frac{2\eta\overline{P}}{\sigma^2}\right)^2 \quad \Gamma\left(\frac{2\eta}{\sigma^2}-1\right) \ / \ \Gamma\left(\frac{2\eta}{\sigma^2}+1\right) = \overline{P}^2\left(\frac{2\eta}{2\eta-\sigma^2}\right)$$

$$\text{Var}\ (p) = \left(\overline{P}\sigma\right)^2 \ / \ \left(2\eta-\sigma^2\right) \tag{3.11}$$

If $2\eta = \sigma^2$ then the denominator in (3.11) will be zero and the variance infinite. The variance must be positive, implying $\sqrt{2\eta} > \sigma$, for $\eta > 0$. The restrictions on the parameters of the IGBM process are explored further by deriving its second moment. Applying Itô's Lemma to $P(t)^2$ shows

$$d\left[P(t)^2\right] = \left[2\eta\overline{P}\,P(t) - 2\eta P(t)^2 + \sigma^2 P(t)^2\right]dt + 2\sigma P(t)^2\,dZ\ . \tag{3.12}$$

Again, on both sides of (3.12) the expectation is taken at t = 0 and Fubini's Theorem applied on the LHS, yielding

$$\frac{d}{dt}E\left[P(t)^2\right] + \left(2\eta-\sigma^2\right)E\left[P(t)^2\right] = 2\eta\,\overline{P}\,E\left[P(t)\right]\ . \tag{3.13}$$

$E\left[P(t)\right]$ is known from (3.5). As a result (3.13) is a first order, non-homogeneous ODE that can be solved using the integrating factor $\exp\left[\left(2\eta-\sigma^2\right)t\right]$,

$$\int_0^t d\left\{e^{\left(2\eta-\sigma^2\right)s}\,E\left[P(s)^2\right]\right\} = 2\eta\,\overline{P}\left\{\overline{P}\int_0^t e^{\left(2\eta-\sigma^2\right)s}ds + \left(p_0-\overline{P}\right)\int_0^t e^{\left(\eta-\sigma^2\right)s}ds\right\}. \tag{3.14}$$

The integrals on both sides of (3.14) depend on the terms in the exponential functions. Variations in the values of the exponents results in three different cases for the second moment.

Case 1: If $2\eta-\sigma^2=0$, then (3.14) becomes

$$E\left[P(t)^2\right] - p_0^2 = 2\eta\overline{P}\left\{\overline{P}\int_0^t ds + \left(p_0-\overline{P}\right)\int_0^t e^{-\eta s}ds\right\}$$

$$E\left[P(t)^2\right] = p_0^2 + 2\eta\,\overline{P}t + 2\overline{P}\left(\overline{P}-p_0\right)\left(e^{-\eta t}-1\right). \tag{3.15}$$

The limit as $t\to\infty$ of (3.15) will be ∞ because of the t in the second term on the RHS.

Case 2: If $\eta-\sigma^2 = 0$, then (3.14) becomes

$$e^{\sigma^2 t}E\left[P(t)^2\right] - p_0^2 = 2\sigma^2\overline{P}\left\{\overline{P}\int_0^t e^{\sigma^2 s}ds + \left(p_0-\overline{P}\right)\int_0^t ds\right\}$$

$$E\left[P(t)^2\right] = p_0^2 e^{-\sigma^2 t} + 2\overline{P}^2\left(1-e^{-\sigma^2 t}\right) + 2\sigma^2\overline{P}\left(p_0-\overline{P}\right)te^{-\sigma^2 t}\ . \tag{3.16}$$

The limit as $t \to \infty$ of (3.16) is $2\overline{P}^2$. While finite, the variance of the IGBM process in Case 2 may be orders of magnitude larger than σ^2, for the values associated with most commodity prices,. $Var[P(\infty)] = \overline{P}^2 >> \sigma^2$.

Case 3: If neither $2\eta - \sigma^2 = 0$ nor $\eta - \sigma^2 = 0$, then (3.14) will be

$$e^{\left(2\eta-\sigma^2\right)t} E\left[P(t)^2\right] - p_0^2 = 2\eta\overline{P}\left\{ \overline{P}\int_0^t e^{\left(2\eta-\sigma^2\right)s} ds + \left(p_0 - \overline{P}\right)\int_0^t e^{\left(\eta-\sigma^2\right)s} ds \right\}$$

$$E\left[P(t)^2\right] = P_0^2 e^{-\left(2\eta-\sigma^2\right)t} + \frac{2\eta\overline{P}^2}{\left(2\eta-\sigma^2\right)}\left[1 - e^{-\left(2\eta-\sigma^2\right)t}\right] + \frac{2\eta\overline{P}}{\eta-\sigma^2}\left(p_0 - \overline{P}\right)\left[e^{-\eta t} - e^{-\left(2\eta-\sigma^2\right)t}\right]. \quad (3.17)$$

The limit of (3.17) as $t \to \infty$ is $2\eta\overline{P}^2 / \left(2\eta - \sigma^2\right)$. It follows that $Var\left[P(\infty)\right] = \left(\sigma\overline{P}\right)^2 / \left(2\eta - \sigma^2\right)$. Note this expression for the variance of the IGBM process is the same as the variance of the stationary density, (3.11), since both are taken at $t = \infty$. To summarize, the foregoing has established that in order that the variance of the IGBM process be finite and positive the parameters of the process are subject to the restrictions $\eta \neq \sigma^2$, $2\eta \neq \sigma^2$ and $2\eta > \sigma^2$.

3.3 Selection of an Appropriate Stochastic Process

The mathematical techniques employed above to discern the characteristics of the IGBM process were utilized to review the attributes of five other single-factor stochastic processes:

(1) Geometric Brownian Motion ("GBM"), $dP = \alpha P dt + \sigma P dz$;
(2) Ornstein-Uhlenbeck ("OU"), $dP = \eta\left(\overline{P} - P\right)dt + \sigma dz$;
(3) Exponential OU, $dP = \eta\left[\ln\left(\overline{P}\right) + \sigma^2 / 2\eta - \ln\left(P\right)\right] dt + \sigma P dz$;
(4) Cox-Ingersoll-Ross ("CIR"), $dP = \eta\left(\overline{P} - P\right) dt + \sigma P^{1/2} dz$; and
(5) Stochastic Logistic – Verhulst ("SLV"), $dP = \eta\left(\overline{P} - P\right)P dt + \sigma P dz$.

The results of the review are discussed and summarized in Appendix II. Of the six, single-factor stochastic processes reviewed, only the IGBM process has all the attributes listed in Section 3.1 necessary to model commodity prices. The IGBM process:

(1) generates strictly positive values;
(2) reverts to $\overline{P}$;
(3) notwithstanding its name, is homogeneous; and
(4) will be reflected at either of the boundaries $P = 0$ or $P = \infty$, if $2\eta / \sigma^2 > 0$.

For these reasons the IGBM process is selected to model crude oil prices. Whether, or not, the parameter restrictions found in Section 3.2 will prevent the IGBM process from modeling the price of crude oil is addressed in the next chapter, wherein values of its parameters are estimated.

Chapter 4

Estimation of Parameters

4.0 Introduction

In this chapter two approaches to estimate the parameters necessary for the IGBM process to model crude oil prices are implemented. In the first approach a regression equation is derived and then applied to an historic time-series of spot crude oil prices. The second approach involves deriving formulas for futures contracts and call options on futures contracts and calibrating these formulas to the prices prevailing in the markets for each of these contracts. The estimates obtained from the two approaches are compared and a selection of appropriate parameters is made in the closing section.

4.1 Time-Series Approach

Derivation of Regression Equation: A continuous time solution of the IGBM SDE is derived and then restated in discrete time to obtain a recursive form, below.

$$dP(t) = \eta \left[\overline{P} - P(t) \right] dt + \sigma P(t)\, dZ(t) \tag{4.1}$$

To solve (4.1), let $Y(t) = \eta \left[\overline{P} - P(t) \right]$ and apply Itô's Lemma to obtain

$$dY(t) + \eta Y(t) dt = -\eta\sigma P(t)\, dZ(t). \tag{4.2}$$

The resulting SDE can be solved with the integrating factor, $\exp(\eta t)$, which gives

$$\int_0^t d\left[e^{\eta s}\, Y(s) \right] = -\eta \int_0^t e^{\eta s}\, \sigma P(s) dZ(s). \tag{4.3}$$

Integrate (4.3) and substitute for $Y(t)$ to find

$$P(t) = \overline{P}\left(1 - e^{-\eta t}\right) + P(0) e^{-\eta t} + \int_0^t e^{-\eta (t-s)} \sigma P(s)\, dZ(s). \tag{4.4}$$

Evenly subdivide the interval $[0,T]$ into N subintervals. Let $t_i = iT/N$ for $i = 1,....,N$ and denote each time step as $\Delta t = t_i - t_{i-1}$. Then (4.4) recast in discrete time is

$$P(t_i) = \overline{P}(1 - e^{-\eta \Delta t}) + P(t_{i-1}) e^{-\eta \Delta t} + \int_{t_{i-1}}^{t_i} e^{-\eta (t_i - s)} \sigma P(t_i)\, dZ(s). \tag{4.5}$$

By the Itô Isometry Theorem the integral on the RHS of (4.5) will be a $N(0, \sigma^2)$ random variable. So

$$P(t_i) = \overline{P}\,(1-e^{-\eta\,\Delta t}) + P(t_{i-1})\; e^{-\eta\,\Delta t} + \sigma_\varepsilon \varepsilon \quad , \tag{4.6}$$

where ε is a random variable distributed $N(0,1)$. Let $\varepsilon(t_i) = \int_{t_{i-1}}^{t_i} e^{-\eta(t_i - s)} \sigma\, P(t_i)^\gamma\, dZ(s)$. To obtain $E\left[\varepsilon(t_i)^2\right]$ the multiplication rule, $dZ(s)^2 = ds$, and Fubini's theorem are utilized, so

$$E_{t_{i-1}}\left[\varepsilon(t_i)^2\right] = \int_{t_{i-1}}^{t_i} e^{-2\eta(t_i - s)} E_{t_{i-1}}\left[\sigma^2 P(t_i)^2\right] ds \;=\; \frac{\sigma^2}{2\eta} P(t_{i-1})^2 \left[1 - e^{-2\eta\,\Delta t}\right]. \tag{4.7}$$

For small Δt, $E_{t_{i-1}}\left[\sigma^2 P(t_i)^2\right]$ is approximated by $\sigma^2 P(t_{i-1})^2$ in (4.7). Since $E_{t_{t-1}}\left[\varepsilon(t_i)\right] = 0$, then the variance of $\varepsilon(t_i)$, σ_ε^2, for the IGBM will be given by the last term in (4.7). Then from (4.6)

$$\frac{P(t_i) - P(t_{i-1})}{P(t_{i-1})} = \left(e^{-\eta\,\Delta t} - 1\right) + \overline{P}\left(1 - e^{-\eta\,\Delta t}\right)\left(\frac{1}{P(t_{i-1})}\right) + \sigma\sqrt{\frac{1 - e^{-2\eta\,\Delta t}}{2\eta}}\;\varepsilon\,. \tag{4.8}$$

Equation (4.8) suggests the linear regression approach

$$\frac{P(t_i) - P(t_{i-1})}{P(t_{i-1})} = \hat{a} + \hat{b}\left[\frac{1}{P(t_{i-1})}\right] + \hat{\varepsilon}\,, \tag{4.9}$$

where $\hat{a}$ is the intercept, $\hat{b}$ is the slope and σ_ε^2 is the standard error of residuals. The parameters of the IGBM process are obtained from the above statistics by

$$\eta = -\frac{1}{\Delta t} \ln\ (\hat{a} + 1), \qquad \overline{P} = \hat{b}/(-\hat{a}), \qquad \text{and} \qquad \sigma = \sigma_\varepsilon \sqrt{\frac{2\eta}{1 - e^{-2\eta\,\Delta t}}}\,.$$

Application to a Time-Series: A monthly time-series comprised of spot WTI crude oil prices for the period from May 1983 to August 2003 was obtained. Some basic statistics for the time-series follow in Appendix III. The regression scheme specified by (4.9) was applied to the whole of the time-series and a subset from August 1998 to August 2003. The subset of the time-series was selected to see whether the parameter estimates varied with the sample period. The estimates contained in Table 4.1 were computed.

Table 4.1 Estimates of IGBM Parameters, Time-Series Approach

Parameter	May '83 – Aug. '03	Aug. '98 – Aug. '03
Speed of reversion (η)	0.419	0.393
Mean Reversion Value $(\overline{P})$	$21.827	$29.633
Annual Volatility (σ)	34.232%	38.790%

While the speed of reversion and the volatility have remained somewhat constant over the sample period of twenty years, the mean reversion value, $\overline{P}$, appears to have increased. The observed upward movement in $\overline{P}$ is consistent with certain supply policies announced by

OPEC. The upward shift in $\overline{P}$ is likely the reason for the deviations from a straight line, between \$22 and \$30 per BBL, in the percentile versus percentile plot in Figure 4.1.

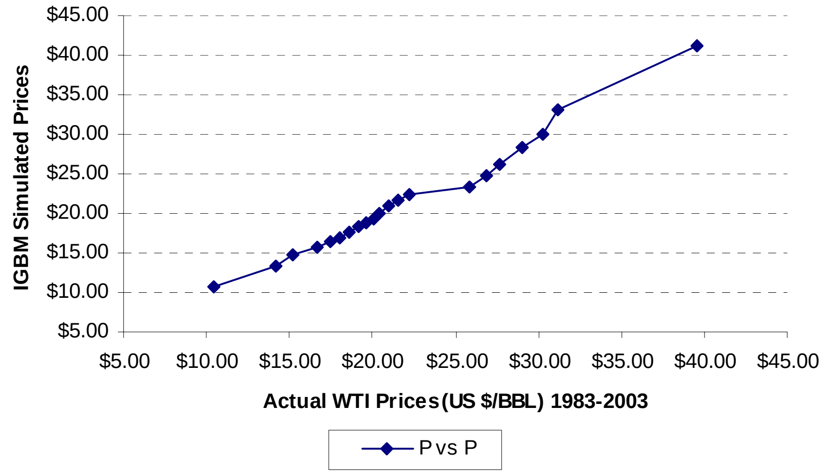

Figure 4.1 Percentile vs Percentile (Monthly Prices)

In Figure 4.1 the percentiles for the monthly time-series of both historic and simulated crude oil prices are compared. The simulated prices were generated using the Milstein Scheme for the IGBM given by (4.10) using the estimates of η, $\overline{P}$, and σ determined by the regression for the May 1983 to August 2003 time-series. The Milstein Scheme in (4.10) was derived by the direct application of (4.9) and (4.10), P. 33 in Jackel (2002) to (3.1).

$$P(t_{i+1}) = \eta\,\overline{P}\Delta t + P(t_i)\left\{1-\eta\,\Delta t + \sigma\,\varepsilon\sqrt{\Delta t} + \frac{1}{2}\sigma^2\left[\varepsilon^2-1\right]\Delta t\right\} \qquad (4.10)$$

Ten time-series of crude oil prices were generated with (4.10) and the percentiles for each calculated. The median percentile for the sample of ten time-series is plotted on the vertical axis on Figure 4.1. The points in Figure 4.1 form an approximately straight line through the origin, consistent with the notion the IGBM can model crude oil prices. Note there is little deviation at both ends of the distribution.

Time-series data is also used to estimate the risk premium for crude oil, $\rho_{PM}\,\sigma_P\,\phi$, as defined in Chapter 2. The correlation between the returns from holding crude and the equity market is denoted by ρ_{PM}. Monthly returns for each of crude oil and common equity were computed using WTI spot prices and the Standard & Poors' 500 ("S&P 500") total return index, respectively. The relationship between the monthly returns for the S&P 500 and WTI crude oil is illustrated on the previous page. The correlation coefficient between the two return time-series, for 20 years of data, is -0.142. While the magnitude of ρ is small, its sign has a significant implication: the risk premium $\rho_{PM}\,\sigma_P\,\phi$ is negative.

Time-series data was used in Weir (2002) to estimate ranges for the historical equity risk premium, $\mu_M - r$, and equity volatility of 3.8% to 5.1% and 13% to 23%, respectively. Single point estimates of 4.5% and 17% for the equity risk premium and equity market volatility, respectively, were selected. This implies a market price of risk of ϕ = 0.265, since $\phi = (\mu_M - r)/\sigma_M$.

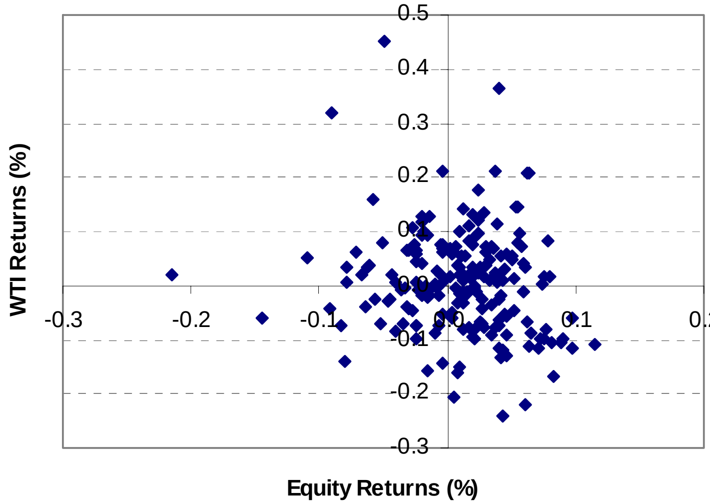

Figure 4.2 Monthly Returns from WTI vs Equity

4.2 Calibration Approach

The calibration approach to estimating parameters entails two steps. First, a valuation model of a futures contract is derived, assuming the spot price evolves through time as an IGBM. Second, a search is conducted for those values of the parameters that will calibrate a sequence of futures valuations to a sequence of prices of traded futures contracts. The sequence of traded futures contracts, ordered according to the increasing maturity of the contracts, is referred to as the term structure, or futures curve.

Valuation of Futures Contracts: The value of a futures contract at time t that settles at a later time T, denoted by $F_t(T)$, according to Seppi (2002) is equal to the time t risk-neutral expectation of the spot price at time T,

$$F_t(T) = \hat{\mathrm{E}}_t\left[P(T)\right] = \mathrm{E}_t\left[\hat{P}(T)\right]. \tag{4.11}$$

The risk neutral motion of P, denoted by $\hat{P}$, is given by an IGBM process with a risk-neutral drift, $\hat{\alpha}(P,t)$. Subtracting the risk premium, $\rho_{PM}\,\sigma(P)\phi$, from the real world drift results in $\hat{\alpha}(P,t)$. The volatility of an IGBM process is, $\sigma(P) = \sigma P$, so

$$\hat{\alpha}(P,t) = \eta\left[\overline{P} - P(t)\right] - \rho\sigma(P)\phi = (\eta + \rho\sigma\phi)\left[\frac{\eta\,\overline{P}}{\eta + \rho\sigma\phi} - P(t)\right].$$

Let $\hat{\eta} = \eta + \rho\sigma\phi$ and $\hat{\overline{P}} = \eta\,\overline{P}/(\eta + \rho\sigma\phi)$ so that the risk-neutral process for the spot price is

$$d\hat{P}(t) = \hat{\eta}\left[\hat{\overline{P}} - \hat{P}(t)\right]dt + \sigma\,\hat{P}(t)\,dZ(t). \tag{4.12}$$

Since the expected value at time t of a stochastic variable following an IGBM is (3.5), then, from (4.11), (3.5) and (4.12) the value of a futures contract is

$$F_t(T) = \mathrm{E}_t\left[\hat{P}(T)\right] = \hat{\bar{P}} + \left[P(t) - \hat{\bar{P}}\right] e^{-\hat{\eta}(T-t)}$$

$$F_t(T) = \frac{\eta\,\bar{P}}{\eta + \rho\sigma\phi} + \left[\,P(t) - \frac{\eta\,\bar{P}}{\eta + \rho\sigma\phi}\right] e^{-(\eta + \rho\,\sigma\,\phi)(T-t)}. \tag{4.13}$$

The above relationship, (4.13), has also been derived as a solution of the PDE for futures prices by Bos, Ware, and Pavlov (2003). The value of a futures contract, given by (4.13), can be interpreted as having two components. The first term is a long run mean. The second term is a revision in expectations. The long run mean can be obtained by taking the limit of (4.13) as $T \to \infty$.

Consider the ability of (4.13) to model the observed term structure of crude oil futures prices. When the spot price, *P(t)*, is greater (less) than the long run mean, then the second term will be positive (negative) and will decline monotonically with the maturity of a contract, since $dF_t(T)/dT$ is negative. Futures curves having a negatively, or positively, sloped term structure are said to be in "backwardation" or "contango", respectively. Both types of futures curves can be modeled by (4.13). The expression (4.13) is, however, not capable of representing a term structure that has both a positive and a negative slope, at different maturities. Such term structures arise when the mid-maturity contracts are higher, or lower, than both the spot price and long dated contracts. The monotonic expression, exp $[-(\eta + \rho\sigma\phi)(T-t)]$ inhibits (4.13) from modeling term structures that are convex, or concave.

Estimates by Calibration to the Futures Curve: Baker, Mayfield and Parsons (1998) and Bessembinder, Coughenour, Sequin, and Smolder (1998) have observed that when spot crude oil prices are subject to a shock up, or downwards, the subsequent term structure tends to be negatively or positively sloped, respectively. This observation is consistent with the view that crude oil prices are mean reverting and suggests that the futures markets' view of the long run mean reverting price is the price of the longest dated future. For crude oil, this implies $\bar{P} = F_t(6)$, since crude oil futures contracts are traded having maturities of up to six years. Using the assumption that $\bar{P} = F_t(6)$ and the futures curve $F_t(0)$ to $F_t(6)$, the SOLVER tool in Excel is used to search for values of the parameters η and $\rho\sigma\phi$ that calibrate (4.13) to the observed term structure of crude oil futures, with the minimum absolute error. The estimates of parameters obtained by calibrating (4.13) to the futures curve at two arbitrary dates are shown in Table 4.2.

Table 4.2 Estimates of IGBM Parameters, Calibration Approach

Parameter	October 13, 2003	October 30, 2003
η	1.369	2.462
$\rho\sigma\phi$	3.05%	7.89%
$\bar{P}$	\$26.09	\$26.69

The calibration procedure is illustrated in Figure 4.3 for the crude oil futures curve on October 30, 2003.

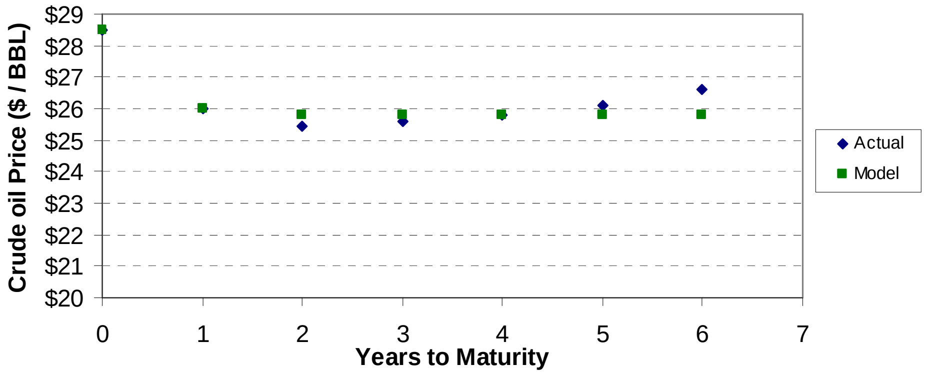

Figure 4.3 Actual vs Model Futures Prices

The above graph illustrates the difficulty of replicating a convex futures curve with (4.13). The parameters estimated by the search algorithm place the model in the midst of the actual term structure. The relative error in applying the model to the October 30,2003 curve is approximately 1%. Nonetheless, the monotonic property of (4.13) means it cannot fully capture the shape of the term structure of futures contracts in all possible cases.

Estimates by Calibration to Options on Futures: A direct estimate of the volatility of crude oil is not available by calibrating (4.13) to the futures curve. Instead, valuation models of call options exercisable for crude oil futures contracts are calibrated to market prices to extract an estimate of the volatility of crude oil futures. Applying the futures call option model in Black (1976) to a sequence of at-the-money calls resulted in the following estimates of implied volatility of the underlying futures contracts.

Table 4.3 Implied Volatility of Crude Oil Futures At November 3, 2003

Call Expiry	Feb. 2004	June 2004	Dec. 2004	Dec. 2005
Volatility	31.7%	30.7%	25.9%	25.7%

A trend of declining implied volatility with increasing maturity of the futures contracts is apparent in the Table 4.3. What does this say about the volatility of spot prices? The answer depends on the model of futures prices used in the call valuation.

Black's 1976 call valuation model utilizes the "cash and carry" model of futures prices, $F(P,T) = P\exp\left[(r-\delta)(T-t)\right]$, where delta is the convenience yield net of carrying costs. Applying Itô's Lemma to find dF and computing returns by dividing dF by F, shows that the volatility of the futures prices, σ_F, equals that of crude oil prices, σ_P. Which of the above list of implied volatilities should we select as being an estimate of σ_P? Since Black's 1976 model is based on the assumption that the underlying is following a GBM, it cannot model the attenuation of volatility with increasing maturity of the futures contracts.

Clewlow and Strickland (2000) propose modeling the attenuation of volatility with the increasing maturity of futures contracts explicitly, using the SDE

$$dF_t(T) = \sigma e^{-\eta(T-t)} F_t(T)\, dZ(t). \qquad (4.14)$$

There is no drift term in (4.14). Clewlow and Strickland state that in a risk-neutral world a futures contract that had no cost can offer an investor no return. Utilizing (4.14) to model the underlying forward contract that matures at time *s*, Clewlow and Strickland (2000) derive the value, denoted by C, at time t of a European call option with strike price K that expires at time T, as follows

$$C\left[t, F(t,s);\, K, T, s\right] = P(t,T)\left[\; F(t,s)N(h) - KN(h-\sqrt{\omega})\right], \quad \text{where}$$

$$h = \frac{\log\left(F\left(t,s\right) / \, K\right) + \frac{1}{2}\omega}{\sqrt{\omega}} \qquad \omega = \frac{\sigma^2}{2\eta}\left(e^{-2\eta\left(s-T\right)} - e^{-2\eta\left(s-t\right)}\right). \tag{4.15}$$

The valuations of call options exercisable for futures given by (4.15) are used to calibrate the market prices of a set of at-the-money futures options to a corresponding term structure of prices of crude oil futures contracts. Calibration proceeds by searching for the magnitudes of σ and η that minimize the difference between the model values and market prices of the futures call options. The search is conducted using the SOLVER Tool in the Excel spreadsheet located in Appendix IV. The estimates for σ and η found by the search, using the market prices prevailing on November 4, 2003, are 35% and 0.624, respectively.

4.3 Comparison of Estimates and Selection of Parameters

To implement valuation models, where the price of crude oil is assumed to follow the IGBM process, estimates of four parameters are required: the mean-reverting price, $\overline{P}$; the speed of reversion, η; the volatility of prices, σ_P; and the risk premium, $\rho_{PM}\,\sigma_P\,\phi$. The estimates from the time-series and calibration approaches are compared and an appropriate selection for each parameter is made, below.

Parameter $\overline{P}$: The time-series approach estimates for $\overline{P}$ are \$21.83 and \$29.63 per BBL for samples comprising the last twenty years and five years, respectively. The assumption that $\overline{P}$ has remained constant for the entire twenty-year sample period is not consistent with the \$8 per BBL difference between the two estimates. The dichotomy between the quantity and the relevance of the time-series data reduces the weight that can be given to the time-series estimates of $\overline{P}$. The futures curve estimate, based on the longest dated futures price of \$26.04 to \$26.69 per barrel, is the market price today for delivery in six years. As such it is the market price that will balance supply and demand in the future, bearing in mind the OPEC range of \$24 to \$30 per BBL. Placing more weight on the estimate drawn from the longest dated future, results in \$27 per BBL being selected as an estimate of $\overline{P}$.

Parameter η: For the speed of reversion, η, three estimates were acquired, based on: the time-series regression approach, of 0.393 to 0.419; the calibration to the futures curve approach, of 1.369 to 2.462; and the calibration to call options on futures approach, of 0.624. The calibration to futures estimates are an order of magnitude larger than those obtained from the other two approaches. On the two days these estimates were computed, the crude oil futures curve had a steep negative slope. The calibration to futures estimate of η on these days had to have sufficient magnitude to bend the model futures curve to fit the price of both the first and last months' futures. While there should be a reluctance to select parameters other than those consistent with the futures curve, more weight is placed on the time-series estimate, which is a direct measure of the speed of reversion of spot prices and the calibration of futures call options estimate, which directly addresses the attenuation of futures volatility. An appropriate speed of reversion parameter for spot crude oil prices lies midway between the estimates of 0.4 and 0.6, say 0.5.

Parameter σ : The time-series and the calibration to call options on futures approaches delivered estimates of the volatility of spot crude oil prices of 34.2% to 38.8% and 35%, respectively. Selecting 35% as the magnitude of spot crude oil price volatility is consistent with both historic and market-derived information.

Parameter $\rho\sigma\phi$: The time-series approach found estimates of ρ_{PM} and ϕ of -0.142 and 0.265, respectively. With σ = 35% the implied time-series estimate of $\rho\sigma\phi$ is -1.32%. In contrast, the calibration to the futures curve approach produced estimates of 3.1% to 7.9%. The later estimates of the risk premium are positive and consistent with the levels for risk classes such as equities or "high-yield" debt. An argument could be made that higher returns from crude oil mean lower profits in the rest of the economy and ergo lower stock returns. Furthermore, there is some empirical evidence, as we shall see in Chapter 5, that the correlation coefficient between returns from equities and reserves of crude oil, held in the subsurface, is also negative. More weight is assigned the time-series estimate and the range –1.0% < $\rho\sigma\phi$ < 1.0% is selected for the parameter.

In selecting the parameters above, weight was given to both the time-series and the calibration approaches. The later estimates are based on market prices on or about October 30, 2003. The parameters selected are not necessarily appropriate for some other date.

Chapter 5

The Valuation of Petroleum Reserves as Perpetuities

5.0 Introduction

In this chapter valuations of producing petroleum reserves are derived by discounting perpetual, risk-neutral cash flow streams at the risk-free rate. The risk-neutral cash flow streams are functions of forward prices. Let the rate at which the risk-neutral cash flow stream is paid to the holder of the perpetuity be denoted by $\hat{\pi}(t)$. Then, during the period of time, *dt*, the payment is $\hat{\pi}(t)dt = \pi[F_0(t)]dt$, where $F_0(t)$ is the value of a forward contract at time zero that matures at time *t*. A forward contract is a risk-neutral contract that can be discounted at the risk-free rate, *r*. It follows that the value of a perpetuity at *t*=0, denoted by $V[t_S, \infty;\ \pi(F_0(t))]$ that commences payments at time t_s, is given by

$$V\Big[t_s, \infty;\ \pi\Big(F_0(t)\Big)\Big] = \int_{t_s}^{\infty} \pi\Big[F_0(t)\Big] e^{-rt} dt \,. \tag{5.1}$$

The valuation of a forward, given by (4.11), enables (5.1) to be written as the expectation

$$V\Big[\ t_s, \infty;\ \pi(P_0; \hat{P})\Big] = \int_{t_s}^{\infty} \pi \left\{ E_0 \Big[\hat{P}(t) \mid P(0) = P_0 \Big] \right\} e^{-rt} dt \,. \tag{5.2}$$

While the perpetuity valuations determined by (5.2) are used in subsequent chapters as boundary conditions, here they are utilized to demonstrate an important difference between the stochastic behaviors of surface and subsurface barrels of crude oil.

5.1 Valuation of Producing Petroleum Reserves – IGBM Prices

Level Extraction Rates: Consider a perpetuity that pays $\pi(t)dt = \tau\ [P(t) - C_O]\ dt$, where $P(t)$ follows an IGBM given by (3.1). The risk-neutral motion of (3.1) is given by (4.12) and the first moment of an IGBM by (3.5). Then the perpetuity's value will be, from (5.2),

$$\begin{aligned} V\Big[\ t_s, \infty;\ \pi(P_0; \hat{P})\Big] &= \int_{t_s}^{\infty} \tau\, \mathrm{E}_0 \Big[\ \hat{P}(t) - C_O\ \Big]\ e^{-rt} dt \\ &= \tau \int_{t_s}^{\infty} \mathrm{E}\Big[\hat{P}(t) \mid P(0) = p_0\Big]\ e^{-rt} dt - C_O \int_{t_s}^{\infty} e^{-rt} dt \\ &= \tau\, (\hat{\bar{P}} - C_O)\ \int_{t_s}^{\infty} e^{-rt} dt + \tau\, (p_0 - \hat{\bar{P}})\ \int_{t_s}^{\infty} e^{-(\hat{\eta} + r)t} dt \end{aligned}$$

$$V(p_0,t_s) = \tau\left(\frac{\hat{\bar{P}} - C_O}{r}\right)e^{-r t_s} + \frac{\tau(p_0 - \hat{\bar{P}})\, e^{-(\hat{\eta}+r)\, t_s}}{(\hat{\eta}+r)}.$$

If the payments begin immediately so that t_s=0, then

$$V(p_0) = \tau\left[\frac{\hat{\bar{P}} - C_O}{r} + \frac{p_0 - \hat{\bar{P}}}{\hat{\eta}+r}\right]. \tag{5.3}$$

The value of the perpetuity (5.3) is equal to τ times the sum of: risk-neutral mean-reverting price, $\hat{\bar{P}}$, less operating costs, C_O, capitalized at the risk-free rate; plus the difference between the initially observed price and the risk-neutral mean-reverting price capitalized at rate $(r+\hat{\eta})$. Note that the denominator in the second term will be negative unless the restriction, $r+\hat{\eta} = r+\eta+\rho\sigma\phi > 0$, is imposed. This is a subtle but important point, the significance of which will be examined in the next chapter. In real world parameters (5.3) is

$$V(p_0) = \tau\left\{\frac{\eta\bar{P}}{r(\eta+\rho\sigma\phi)} - \frac{C_O}{r} + \frac{p_0 - \dfrac{\eta\bar{P}}{\eta+\rho\sigma\phi}}{r+\eta+\rho\sigma\phi}\right\}. \tag{5.4}$$

Declining Extraction Rates: Consider a perpetuity with declining revenues, $R(t) = q(t)P(t)dt$, where $P(t)$ follows an IGBM and $q(t)$ is deterministic and declines exponentially. An SDE for the motion of $R(t)$ and its expected value are required. The motions of $P(t)$ and $q(t)$ are given by the SDE for the IGBM process (3.1) and $dq(t) = -\omega\, q(t)dt$, respectively. So by the product rule, $d[R(t)] = d[qP]$ is

$$dR(t) = \left[\eta\bar{P}q(t) - (\eta+\omega)R(t)\right]dt + \sigma\, R(t)\, dZ. \tag{5.5}$$

Let $\eta' = (\eta+\omega)$ and $\bar{R}(t) = \eta\bar{P}q(t)/(\eta+\omega)$ in (5.5), then

$$dR(t) = \eta'\left[\bar{R}(t) - R(t)\right]dt + \sigma\, R(t)\, dZ. \tag{5.6}$$

To find the first moment of $R(t)$ the expectation at t=0 is taken on both sides of (5.6) and Fuibini's Theorem is applied on the LHS yielding

$$\frac{d}{dt}E\left[R(t)\right] + \eta' E\left[R(t)\right] = \eta'\bar{R}(t). \tag{5.7}$$

The ODE (5.7), is solved with the help of the integrating factor $\exp(\eta' t)$ so

$$\int_0^t d\left[e^{\eta' s}R(s)\right] = \int_0^t \eta' e^{\eta' s}\bar{R}(s)\, ds.$$

Since $dq(t) = -\omega\, q(t)\, dt$, then $q(t) = q(0)\exp(-\omega t)$ and $\bar{R}(s) = \eta\bar{P}q(0)\exp(-\omega s)/\eta'$, so

$$e^{\eta' t} E\big[R(t)\big] - E\big[R(t)\big] = \eta\, \overline{P} q(t) \int_0^t e^{\eta s} ds$$

$$E\big[R(t)\big] = e^{-\eta' t}\left\{ P(0)\, q(0) + \overline{P} q(0) \left[e^{\eta t} - 1 \right] \right\}$$

$$\mathrm{E}\,\big[R(t)\big] = q(0)\, e^{-\omega t} \left\{ \overline{P} + \left[P(0) - \overline{P} \right] e^{-\eta t} \right\} = q(t)\, E\big[P(t)\big]. \tag{5.8}$$

According to (5.8) the expected revenue is equal to the product of the deterministic production rate and the expected stochastic price. Then a perpetuity that pays $\pi(t)dt = \left[\tau\, R(t) - C_O\right] dt$ will have a value given by (5.2) and (5.8),

$$V\left[t_s, \infty;\ \pi(P_0, q_0; \hat{R}) \right] = \int_{t_s}^{\infty} \mathrm{E}\left[\hat{\pi}(t) \mid R(0) = p_0\, q_0 \right] e^{-rt} dt = \int_{t_s}^{\infty} \left[\tau\, q(t)\, E\left[\hat{P}(t) \right] - C_0 \right] e^{-rt} dt$$

$$= \int_{t_s}^{\infty} \left\{ \tau\, q_0\, e^{-\omega t} \left[\hat{\overline{P}} + (p_0 - \hat{\overline{P}})\, e^{-\hat{\eta} t} \right] - C_O \right\} e^{-rt} dt$$

$$= \tau\, q_0 \hat{\overline{P}} \int_{t_s}^{\infty} e^{-(\omega + r)t} dt + \tau\, q_0 \left(p_0 - \hat{\overline{P}} \right) \int_{t_s}^{\infty} e^{-(\omega + \hat{\eta} + r)t} dt - C_O \int_{t_s}^{\infty} e^{-rt} dt$$

$$V(t_s;\ p_0, q_0) = \frac{\tau\, q_0\, \hat{\overline{P}}\, e^{-(\omega + r)\, t_s}}{\omega + r} + \frac{\tau\, q_0 (p_0 - \hat{\overline{P}})\, e^{-(\omega + \hat{\eta} + r)\, t_s}}{\omega + \hat{\eta} + r} - \frac{C_O\, e^{-r t_s}}{r}.$$

If the payments begin immediately so that t_s=0, then

$$V\left(p_0, q_0\right) = \frac{q_0\, \tau \hat{\overline{P}}}{\omega + r} + \frac{q_0 \tau \left(p_0 - \hat{\overline{P}}\right)}{\omega + \hat{\eta} + r} - \frac{C_O}{r}. \tag{5.9}$$

The value of the perpetuity (5.9) can be interpreted as the sum of the net risk-neutral mean-reverting revenues, $q_0\, \tau \hat{\overline{P}}$, capitalized at the risk-free rate, plus the rate of decline, $r + \omega$; plus the capitalized net incremental revenues, positive or negative, due to the initial difference between the observed price and the risk-neutral mean-reverting price; minus the capitalized fixed costs, C_0 / r. In real world variables (5.9) is

$$V\left(P, q_0\right) = \frac{\tau\, q_0\, \eta\, \overline{P}}{(\omega + r)(\eta + \rho\,\sigma\,\phi)} + \frac{\tau\, q_0 \left(P - \dfrac{\eta\, \overline{P}}{\eta + \rho\,\sigma\,\phi} \right)}{(\omega + \eta + \rho\,\sigma\,\phi + r)} - \frac{C_O}{r}. \tag{5.10}$$

If (5.9) is re-written in the form $V(P) = b + mP$, where b is the term representing the capitalized mean-reverting revenues less capitalized fixed costs and m is $q_0 \tau / \left(r + \omega + \hat{\eta}\right)$, then Itô's Lemma can be used to show

$$Var(dV/V) = [\sigma/(1 + b/mP)]^2\, dt. \tag{5.11}$$

Since P follows an IGBM it is greater than zero. The imposed restriction, $r + \hat{\eta} > 0$, ensures m>0. Since the petroleum reserve is on production, net revenues must exceed costs, so b>0. It follows that the denominator in (5.11) is greater than zero and volatility of holding a reserve of crude oil, $Sd(dV/V)$, is less than that of crude oil on the surface, σ. Is there any evidence that in-ground barrels are less volatile than barrels of crude on the surface?

5.2 Estimates of the Volatility of Returns from Holding Reserves

Estimating the volatility of returns from holding in-ground barrels of crude oil is more difficult than for above-ground barrels. There is no organized marketplace for reserves where homogeneous volumes trade hands pursuant to standardized contracts on a regular schedule. Rather, from time-to-time, buyers and sellers of reserves negotiate unique transactions. The reserves sold in successive transactions differ from each other in terms of location, quality, fiscal burdens, extraction costs and decline rate. Reserves of natural gas and crude oil are commingled in many transactions. Nonetheless, several sources of estimates of the volatility of returns from holding petroleum reserves can be cited.

Adelman and Watkins (2003) studied a database of 6,000 reserve transactions in the U.S. during the period from 1982 to 2002. They focused on the approximately 28% of the transactions where information regarding both the total purchase price and the volumes of reserves traded was available. By regressing the realized purchase price against the volumes of crude oil and natural gas acquired in each transaction, Adelman and Watkins (2003) obtained estimates of the annual average transaction price per BBL of crude oil and per MCF of natural gas. Annual holding period returns were then computed by expressing the change in the capital value of a BBL of petroleum, or MCF of natural gas, as a percentage of the prior year's value. The volatility of the time series of returns for reserves of crude oil and natural gas were estimated to be 35.8% and 42.9%, respectively. It should be noted that Adelman and Watkins' (2003) holding period returns do not include any of the cash flow that would be received by the owner of reserves.

Chen and Antonacci (2003) estimate holding period returns for reserves, including the annual cash flow, by considering notional, rather than real, reserves. Declining production streams for the notional reserves of crude oil and natural gas were estimated based on U.S. averages. The income portion of the return was calculated as the product of the annual forecast volume times the spot field price net of costs, again based on U.S. averages. At each year end the discounted expected future cash flow from the reserves was computed using the forecast production decline curves and the forward price curves for crude oil and natural gas to obtain a capital value. The annual income and capital gains were computed to derive a time series of total returns for the period 1982 to 2002. The volatility of the time series of total returns found by Chen and Antonacci was 21.64%. This is much less than Adelman and Watkins' (2003) estimates. The difference may be, in part, due to Chen and Antonacci including the annual income in their estimate of total return. However, the arithmetic average of total returns estimated by Chen and Antonacci (2003), of 5.53% for reserves of petroleum and natural gas, in equal parts, is not that different from the changes in the capital values for petroleum and natural gas of 4.5% and 11.5%, respectively, estimated by Adelman and Watkins (2003). Lastly, Chen and Antonacci (2003) estimate the correlation coefficient between returns on large capitalization stocks and reserves of crude oil and natural gas during the period 1982 to 2002 was –0.20.

The foregoing estimates of the volatility of the returns from holding reserves may not be sufficient to confirm the conjecture: that the volatility of holding reserves is less than, or equal to, the volatility of holding crude oil on the surface. The estimates are, however, consistent with the conjecture. Furthermore, the volatility estimates contradict the assertions of GBM based valuation models, see Appendix V, that the volatility of holding in-ground barrels is greater than, or equal to, holding above-ground barrels. In any case, the observed stochastic behaviors of barrels of crude oil on the surface and in the ground are different.

Chapter 6

The Valuation of Petroleum Leases

6.0 Introduction

The previous chapters contain certain seemingly disparate mathematical techniques that, in this chapter, will be assembled to value petroleum leases as perpetual options. This chapter opens with the derivation and solution of a valuation equation for a claim on a mean reverting asset. The general form of the BSM PDE, found in Chapter 2, is configured with the IGBM diffusion process, selected in Chapter 3, and solved for the perpetual case. Valuations of perpetual calls and puts, along with their sensitivity to certain parameters, are obtained in sections three and four, respectively. The call and put valuations, together with the valuation of a level perpetuity, found in Chapter 5, are used to value an extraction and development options in section five. The value of a call option exercisable for a declining, perpetual cash flow stream is developed in section seven. The parameters selected to model crude oil prices in Chapter 4 are used to value examples of both tar-sands and conventional petroleum leases in sections six and eight, respectively.

6.1 Solutions of the BSM Equation for an IGBM Motion

The general form of BSM PDE derived in Chapter 2 is

$$V_t + \frac{1}{2}\sigma^2(P,t)V_{PP} + \left[\alpha(P,t) - \rho\sigma(P,t)\phi\right]V_P - rV + \pi(P) = 0\,.$$

If an asset, P , follows a motion determined by the IGBM SDE (3.1),

$$dP(t) = \eta[\overline{P} - P(t)]\,dt + \sigma\,P(t)\,dZ(t) \qquad \text{with} \quad P(0) = P_0\,.$$

Then, in the BSM PDE the coefficient of the diffusion term will be, $\sigma^2(P,t) = \sigma^2 P^2$ and the coefficient of the convection term will be, $\left[\alpha(P,t) - \rho\sigma(P,t)\phi\right]$ = $\left[\eta\,\overline{P} - (\eta + \rho\sigma\phi)P\right]$. For a perpetual American claim on P , such that $V_t = 0$, while it is optimal to hold V, its value will be given by the ODE (6.1)

$$\frac{1}{2}\sigma^2 P^2 V_{PP} + \left[\eta\,\overline{P} - (\eta + \rho\sigma\phi)P\right]V_P - rV + \pi(P) = 0\,. \qquad (6.1)$$

The homogeneous portion of (6.1) after dividing through by $\frac{1}{2}\sigma^2$ is

$$P^2 V_{PP} + \left[\frac{2\eta\overline{P}}{\sigma^2} - \frac{2(\eta + \rho\sigma\phi)}{\sigma^2} P \right] V_P - \frac{2r}{\sigma^2} V = 0 .$$

Let $\alpha = -2(\eta + \rho\sigma\phi)/\sigma^2$, $\quad \beta = 2\eta\overline{P}/\sigma^2 \quad$ *and* $\quad \gamma = 2r/\sigma^2$, then

$$P^2 V_{PP} + [\beta + \alpha P] V_P - \gamma V = 0 . \tag{6.2}$$

A general approach to solving ODE's with polynomial coefficients is given in Bateman (1953) and a specific substitution for the independent variable in (6.2) is in Robel (2001). It follows that to solve (6.2) the changes of the dependent variable, $V(P) = \xi^{\theta} H(\xi)$, and the independent variable, $\xi = \beta P^{-1}$, are appropriate.

$$V(P) = \xi(P)^{\theta} H[\xi(P)] \tag{6.3}$$

$$V_P = \frac{-1}{\beta}\left[\theta\xi^{\theta+1} H(\xi) + \xi^{\theta+2} \frac{dH}{d\xi} \right] \tag{6.4}$$

$$V_{PP} = \frac{1}{\beta^2}\left[\theta(\theta+1)\xi^{\theta+2} H(\xi) + 2(\theta+1)\xi^{\theta+3} \frac{dH}{d\xi} + \xi^{\theta+4} \frac{d^2H}{d\xi^2} \right] \tag{6.5}$$

Substitute (6.3), (6.4), and (6.5) in (6.2) and collect like terms in $\xi^{\theta+1}$ and $\xi^{\theta} H(\xi)$.

$$\xi^{\theta+1}\left\{ \xi \frac{d^2H}{d\xi^2} + [2(\theta+1) - \alpha - \xi] \frac{dH}{d\xi} - \theta H \right\} + \left[\theta^2 + (1-\alpha)\theta - \gamma\right] \xi^{\theta} H = 0 \tag{6.6}$$

Since neither $\xi^{\theta+1}$ nor $\xi^{\theta} H$ are equal to zero, then (6.6) will be equal to zero IFF

$$\xi \frac{d^2H}{d\xi^2} + [2\theta + 2 - \alpha - \xi] \frac{dH}{d\xi} - \theta H = 0 \tag{6.7}$$

and

$$\theta^2 + (1-\alpha)\theta - \gamma = 0 . \tag{6.8}$$

The ODE (6.7) is Kummer's equation, whose solution is

$$H(\xi) = A_1 M(\theta, 2\theta + 2 - \alpha; \xi) + A_2 U(\theta, 2\theta + 2 - \alpha; \xi),$$

where $M(a, b; x)$ and $U(a, b; x)$ are Kummer's functions as defined in Abramowitz and Stegun (1964) ("A&S") 13.1.2 and Spanier and Oldham (1987) ("S&O") 48:3:1, respectively, and A_1 and A_2 are arbitrary constants. Kummer's U function is also referred to as Tricomi's confluent hypergeometric function. In the original variable, P, the homogeneous solution of (6.2) is

$$ {}_{H}V(P) = \left(\frac{\beta}{P}\right)^{\theta} \left\{ A_1\, M\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right) + A_2\, U\left(\theta, 2\theta + 2 - \alpha\,;\, \frac{\beta}{P}\right)\right\}. \tag{6.9} $$

To confirm that (6.9) represents two independent solutions of (6.2), substitute each of the solution terms and their respective derivatives into (6.2). For the solution containing Kummer's *M* function these are,

$$ {}_{H}V(P) = \left(\frac{\beta}{P}\right)^{\theta} M\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right) \tag{6.10} $$

$$ {}_{H}V'(P) = \frac{-\theta}{P}\left(\frac{\beta}{P}\right)^{\theta} M\left(\theta + 1,\ 2\theta + 2 - \alpha; \frac{\beta}{P}\right) \tag{6.11} $$

$$ {}_{H}V''(P) = \frac{\theta}{P^3}\left(\frac{\beta}{P}\right)^{\theta} \left\{ (\alpha P + \beta) M\left(\theta + 1, 2\theta + 2 - \alpha; \frac{\beta}{P}\right) + (\theta + 1 - \alpha) P M\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right)\right\}. \tag{6.12} $$

Substitution of (6.10), (6.11) and (6.12) in (6.2) leads to

$$ \left(\frac{\beta}{P}\right)^{\theta} M\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right) \left(\theta^2 + (1 - \alpha)\theta - \gamma\right) = 0\,. \tag{6.13} $$

Similarly, the solution containing Kummer's *U* function and its derivatives are,

$$ {}_{H}V(P) = \left(\frac{\beta}{P}\right)^{\theta} U\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right) \tag{6.14} $$

$$ {}_{H}V'(P) = \frac{\theta(1 + \theta - \alpha)}{P}\left(\frac{\beta}{P}\right)^{\theta} U\left(\theta + 1, 2\theta + 2 - \alpha; \frac{\beta}{P}\right) \tag{6.15} $$

$$ {}_{H}V''(P) = \frac{\theta(1 + \theta - \alpha)}{P^3}\left(\frac{\beta}{P}\right)^{\theta} \left\{ P U\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right) - (\alpha P + \beta) U\left(\theta + 1, 2\theta + 2 - \alpha; \frac{\beta}{P}\right)\right\}. \tag{6.16} $$

Substitution of (6.14), (6.15) and (6.16) in (6.2) leads to

$$ \left(\frac{\beta}{P}\right)^{\theta} U\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right) \left(\theta^2 + (1 - \alpha)\theta - \gamma\right) = 0\,. \tag{6.17} $$

The quadratic (6.8) that determines θ has two solutions, denoted by θ^+ and θ^- for the positive and negative roots, respectively. The magnitudes of θ^+ and θ^- are given by

$$\theta^{+/-} = \left[(\alpha - 1) \pm \sqrt{(1-\alpha)^2 + 4\gamma} \right] / 2. \tag{6.18}$$

Since $r > 0$ and $\sigma^2 > 0$, then it is the case that $\gamma > 0$. It follows that $(1-\alpha)^2 < (1-\alpha)^2 + 4\gamma$ which implies $\theta^- < 0 < \theta^+$. With two possible values for θ and two solution terms in (6.9) then there are four "solutions" that will satisfy (6.2). Which two of these solutions will be useful in determining the value of a call, or a put option on *P*? Consider the values of θ and the specifications of the free boundary problems for put and call options. Robel (2001) has discussed the requirements for a perpetual American call option. The free boundary conditions for put and call options are in Table 6.1.

Table 6.1 Free Boundary Conditions for Perpetual Puts and Calls on an IGBM Asset P

Option	Put	Call	
Boundary Behaviour	$\lim_{p \to \infty} V(P) = 0$	$\lim_{P \to 0} V(P) =$ bounded	(6.19)
First Derivative	$V'(P) < 0$	$V'(P) > 0$	(6.20)
Second Derivative	$V''(P) > 0$	$V''(P) > 0$	(6.21)

Robel (2001) has concluded that for a call option $A_1 = 0$, the valuation being provided by Tricomi's function with $\theta = \theta^+$. Robel (2001) showed that ${}_H V(P)$ is bounded utilizing the asymptotic expansion of Kummer's $U(a,c;x)$ function (see A&S 13.5.2 or S&O 48:6:1),

$$U\ (a,c;x) \approx (x)^{-a} \left\{ 1 - \frac{ab}{x} + \frac{a(a+1)b(b+1)}{2x^2} - \ldots\ldots \right\}, \tag{6.22}$$

where $b = 1 + a - c$. Then, if Kummer's *U* function in (6.14) is expanded using (6.22), then

$$\lim_{P \to 0}\ {}_H V(P) = \lim_{P \to 0} \left\{ 1 - \theta(\alpha - \theta - 1)\left(\frac{P}{\beta}\right) + \frac{\theta(\theta+1)(\alpha-\theta-1)(\alpha-\theta)}{2}\left(\frac{P}{\beta}\right)^2 - \ldots\ldots \right\} = 1.$$

To show ${}_H V'(P) > 0$ for $P > 0$, examine (6.15). By manipulation, $\theta(1 + \theta - \alpha) = \gamma = 2r/\sigma^2 > 0$, for either θ^- or θ^+. For $P > 0$ the only term in the first derivative of $V(P)$ given by (6.15) that can be negative is $U(\theta + 1, 2\theta + 2 - \alpha; \beta / P)$. According to S&O, $U(a,b;x)$ will be positive and defined for $a > 0, b > 0; x > 0$. Since $P > 0$, ensuring $V'(P) > 0$ comes down to selecting the first parameter greater than zero. Since $\theta^- < 0 < \theta^+$, then $\theta^- + 1 < 1 < \theta^+ + 1$, so selecting $\theta = \theta^+$ will ensure the first parameter is positive. The second parameter, $2\theta + 2 - \alpha$, is equal to $1 - \sqrt{(1-\alpha)^2 + 4\gamma}$ and $1 + \sqrt{(1-\alpha)^2 + 4\gamma}$ for θ^- and θ^+, respectively. Again, the choice of $\theta = \theta^+$ will make the second parameter positive as well.

The condition necessary to protect the positive value of the second derivative of *V*(*P*), given by (6.12), was determined by Robel (2001) to be $\alpha < \gamma$. Recall that in Chapter 5 the valuations of perpetuities, whose cash flow followed an IGBM, had discount factors in their denominators of the form, $r+\eta+\rho\sigma\phi$. To have positive perpetuity values, then $r+\eta+\rho\sigma\emptyset$ must be greater than zero or $\rho\sigma\phi > -(r+\eta)$ or $-(\eta+\rho\sigma\phi) < r$. Dividing both sides of the last inequality by $\sigma^2/2$, yields α and γ, respectively. From the restriction, $\alpha < \gamma$, it follows that

$$2\alpha < -2\alpha + 4\gamma, \qquad \alpha^2 + 2\alpha + 1 = \alpha^2 - 2\alpha + 1 + 4\gamma \quad \text{and} \quad \frac{\alpha+1}{2} < \frac{\sqrt{(1-\alpha)^2 + 4\gamma}}{2}$$

hence $\alpha < \theta^+$ or $\theta^+ + 2 < 2\theta^+ + 2 - \alpha$. This is an inequality Robel (2001) uses to show $V''(P) > 0$. Robel (2004) has observed, "that the same assumption $[\alpha < \gamma]$ which is needed to ensure the underlying asset has a finite value is exactly the same assumption which also ensures that the problem of valuing the [contingent] asset is well-posed". It can be added that the ability for the risk premium to be negative, albeit not too negative, is useful when working with certain commodities such as crude oil where the correlation of holding period returns with the market maybe negative.

For the valuation of a put option it must be shown that the limit as $P \to \infty$ of (6.10),

$$\lim_{P\to\infty} \left(\frac{\beta}{P}\right)^{\theta} M\left(\theta,\ 2\theta+2-\alpha;\ \frac{\beta}{P}\right) = \lim_{P\to\infty} \left(\frac{\beta}{P}\right)^{\theta} \lim_{P\to\infty} M\left(\theta,\ 2\theta+2-\alpha; \beta/P\right) \quad (6.23)$$

is zero. Consider the first limit. If $\theta = \theta^+ > 0$ then the limit will be equal to zero. The second limit can be obtained from the definition of the Kummer M function,

$$\lim_{P\to\infty} M\left(\theta,\ 2\theta+2-\alpha;\ \frac{\beta}{P}\right) = \lim_{P\to\infty} \left\{1+\theta(2\theta+2-\alpha)\left(\frac{\beta}{P}\right)........\right\}$$

which is one. Hence, the limit as $P \to \infty$ of (6.23) is the product of zero and one equal to zero, if $\theta = \theta^+$.

The put option valuation formula must have a negative first derivative. The first derivative is given by (6.11) which has a negative sign. Since $\theta = \theta^+ > 0$, $\beta > 0$ and $P > 0$ then for (6.11) to be negative we must select the parameters of $M(\theta+1,\ 2\theta+2-\alpha;\ \beta/P)$ such that it is positive. S&O show that $M(a,b;z)$ will be strictly positive when $a > 0,\ b > 0,\ and\ z > 0$. Since $\theta = \theta^+$, $\theta^+ + 1 > 1$ and $2\theta^+ + 2 - \alpha > 0$ then $M(\theta+1,\ 2\theta+2-\alpha;\ \beta/P) > 0$ and (6.11) will be negative. Similarly, the second derivative will be positive as well.

6.2 Valuation of a Perpetual American Call

The value of a perpetual American call on one unit of an asset following an IGBM process with a price of P is given by (6.9) with $A_1 = 0$. To find the constant A_2, a value of P, denoted by P^*, that maximizes the value of $V(P)$ for all $P < P^*$ must be found. Imposing the value-matching and smooth-pasting boundary conditions accomplishes this. For a call option, having an exercise price of I and the payoff function $\varphi(P) = Max[0,\ P-I]$, then the value-matching and smooth-pasting boundary conditions are

$$V(P^*) = \varphi(P^*) = P^* - I \quad \textit{and} \quad V^{'}(P^*) = \varphi'(P^*) = 1, \quad \text{respectively.}$$

The first derivative of $V(P)$ is (6.15), which at P^* must equal 1, so

$$A_2 = P^* / \gamma \left(\beta / P^*\right)^{\theta} U\left(\theta + 1,\ 2\theta + 2 - \alpha;\ \beta / P^*\right). \tag{6.24}$$

Since $P^* - V\left(P^*\right) = I$, then

$$P^* - A_2\left(\beta / P^*\right)^{\theta} U\left(\theta,\ 2\theta + 2 - \alpha; \beta / P^*\right) = I\,. \tag{6.25}$$

By substituting (6.24) for A_2 into (6.25) an expression for P^* is found,

$$P^* - \frac{P^* U\left(\theta, 2\theta + 2 - \alpha; \beta / P^*\right)}{\gamma U\left(\theta + 1, 2\theta + 2 - \alpha; \beta / P^*\right)} = I\,. \tag{6.26}$$

A numerical solution of (6.26) will yield P^* which can be substituted into (6.24) to find A_2 and thus fully parameterize the value function (6.9). The value of a perpetual call is

$$V(P) = A_2 \left(\frac{\beta}{P}\right)^{\theta} U\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right) \qquad P \le P^*$$

$$= P - I \qquad P \ge P^*.$$

As a demonstration of the foregoing, consider the valuation of a perpetual, American call option on spot crude oil with the parameters: $r = 0.05$, $\rho\sigma\phi = -0.01$, $\sigma = 0.35$, $\overline{P} = \$27$ and $\eta = 0.5$, estimated in Chapter 4. The variation of the optimal exercise price, P^*, with the mean reversion price, $\overline{P}$, is illustrated by the valuations of a call option exercisable at a cost of $I =$ \$27.00 for an asset following an IGBM in the graph on the LHS of Figure 6.1.

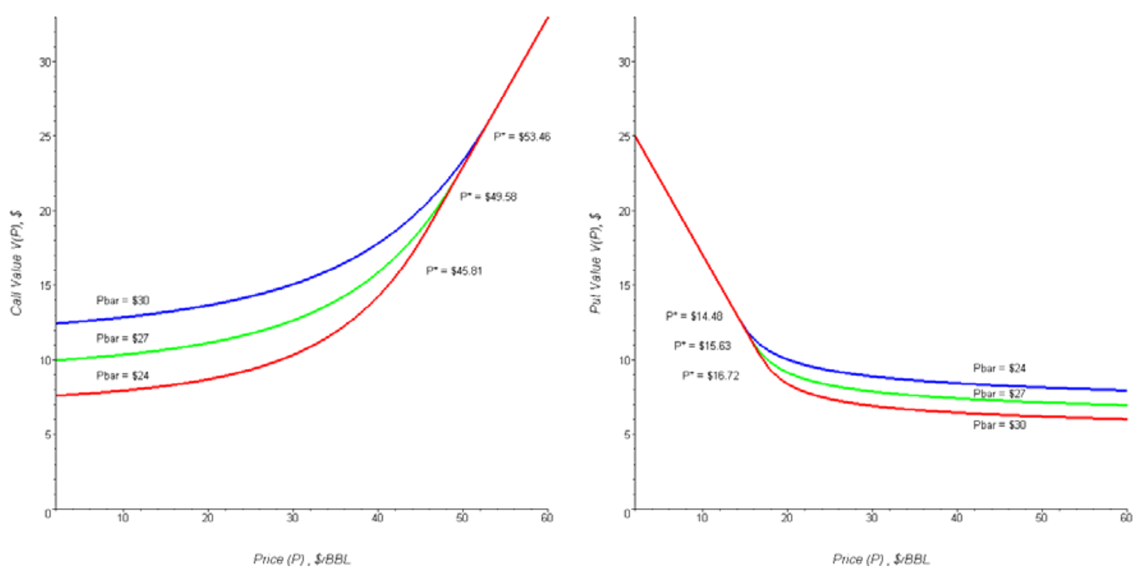

Figure 6.1 Comparison of Call and Put Valuations

6.3 Valuation of a Perpetual American Put

Analogously to the valuation of the call, the put option exercisable for one unit of an asset following an IGBM process with a price of P is given by (6.9) with $A_2 = 0$. To find the constant A_1 , a value of P , again denoted by P^* , that maximizes the value of $V(P)$ for all P greater than P^* must be found. Again, imposing the value-matching and smooth-pasting boundary conditions will accomplish this. For a put option, with an exercise price of I and a payoff function of $\varphi(P) = Max[0, I - P]$, then the value-matching and smooth-pasting boundary conditions are

$$V(P^*) = \varphi(P^*) = I - P^* \quad and \qquad V'(P^*) = \varphi'(P^*) = -1, \text{ respectively.}$$

The first derivative of $V(P)$ is (6.11), which at P^* must equal -1, so

$$A_1 = P^*/\theta \; (\beta / P^*)^{\theta} M(\theta + 1, 2\theta + 2 - \alpha; \beta / P^*). \tag{6.27}$$

And since $V(P^*) + P^* = I$, then

$$A_1 (\beta / P^*)^{\theta} M(\theta, 2\theta + 2 - \alpha; \beta / P^*) + P^* = I \, . \tag{6.28}$$

By substituting (6.27) for A_1 , in (6.28) an expression for P^* is found

$$P^* + \frac{P^* M(\theta, 2\theta + 2 - \alpha; \beta / P^*)}{\theta M(\theta + 1, 2\theta + 2 - \alpha; \beta / P^*)} = I \, . \tag{6.29}$$

A numerical solution of (6.29) will yield P^* and A_1 will follow from (6.27). The value of a perpetual put is

$$\begin{aligned} V(P) &= I - P & P \le P^* \\ &= A_1 \left(\frac{\beta}{P} \right)^{\theta} M\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P} \right) & P > P^* . \end{aligned}$$

Valuations of the put options, as a function of price, with the critical exercise prices varying with $\overline{P}$ are compared on the RHS of Figure 6.1. Note that in the graph for the put valuations, the vertical order of the curves is reversed from that in the call valuations.

6.4 Valuation of a Plant with a Level Extraction Rate and Options on Same

Extraction Option: Consider a plant that extracts one unit of output per unit of time that:

(1) receives at the plant gate a stochastic price, P, where P follows an IGBM;
(2) incurs costs per unit time equal to C; and
(3) pays taxes on its net profits $P - C$ at a rate of one minus τ .

The plant's instantaneous profit will equal $\tau(P-C)dt$ and its value, $V(P)$, is given by the perpetuity valuation (5.3)

$$V(P) = \tau\left[\frac{\hat{\bar{P}} - C}{r} + \frac{P - \hat{\bar{P}}}{r+\hat{\eta}}\right] = \tau\left[\frac{\hat{\eta}\,\hat{\bar{P}} - (r+\hat{\eta})C}{r(r+\hat{\eta})} + \frac{P}{r+\hat{\eta}}\right], \tag{6.30}$$

where $\hat{\eta} = (\eta + \rho\sigma\phi)$ and $\hat{\bar{P}} = \eta\,\bar{P}/(\eta + \rho\sigma\phi)$. The slope of (6.30) is $V'(P) = \tau/(r + \eta + \rho\sigma\phi)$, which by restriction is greater than zero. For an initial P less than $C + \hat{\eta}\left(C - \hat{\bar{P}}\right)/r$, the value of the plant will be negative, assuming the manager of the plant does not have the flexibility to suspend the operation of the plant.

Now, let the manager of the plant be empowered with the authority to shut-in production when $P < C$ and restart extraction if $P > C$, incurring no cost to take either course of action. The value of a unit flow of extraction, again denoted by $V(P)$, will be given by the ODE (6.1) where $\pi(P) = MAX\left[0, \tau(P-C)\right]$. The homogeneous solution of ODE (6.1) follows from (6.9) and the particular solution from (6.30). Application of the value-matching and smooth-pasting boundary conditions at $P = C$ will determine the constants in the solution. Now consider the value of the plant when $P < C$ and $P > C$, denoted by ${}_{P<C}V(P)$ and ${}_{P>C}V(P)$, respectively.

When $P < C$, then $\pi(P) = 0$. In the region $0 \le P < C$ the situation of an owner of a non-producing plant is analogous to that of the holder of an out-of-the-money call option. If, in the future, the price rises above the cost of production, C, the operator of the plant can recommence production. This and the boundary condition $V(P) \to 0$ as $P \to 0$ implies $A_1 = 0$ in (6.9). So,

$${}_{P<C}V(P) = A_2\left(\frac{\beta}{P}\right)^{\theta} U\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right). \tag{6.31}$$

When $P > C$, then $\pi(P) = \tau(P - C)$. The solution of (6.1) will then comprise the homogeneous solution (6.9) plus a particular solution. The particular solution, ${}_{P}V(P)$, of (6.1) is given by the risk-neutral value of the perpetuity (6.30). Differentiation and substitution of (6.30) into (6.1) shows it is indeed a particular solution. Hence, the solution of (6.1) is

$${}_{P>C}V(P) = {}_{H}V(P) + {}_{P}V(P) \tag{6.32}$$

$$_{P>C}V(P)=B_1\left(\frac{\beta}{P}\right)^{\theta} M\left(\theta,2\theta+2-\alpha;\frac{\beta}{P}\right)+B_2\left(\frac{\beta}{P}\right)^{\theta} U\left(\theta,2\theta+2-\alpha;\frac{\beta}{P}\right)+\tau\left[\frac{\hat{\bar{P}}-C}{r}+\frac{P-\hat{\bar{P}}}{r+\hat{\eta}}\right].$$

At some "high enough" price, $V(P)$ will just be equal to the capitalized value of the cash flow stream given by the perpetuity (6.30). Since $(\beta/P)^{\theta}U(\theta,2\theta+2-\alpha;\,\beta/P)$ goes to infinity as P does, then B_2 in (6.32) must equal zero. This is equivalent to saying that when $P > C$, the value of a producing property is the sum of the capitalized cash flow, from (6.30), plus the right to suspend production, being a put, given by the first term in (6.32). So,

$$_{P>C}V(P)=B_1\left(\frac{\beta}{P}\right)^{\theta} M\left(\theta,2\theta+2-\alpha;\frac{\beta}{P}\right)+\tau\left[\frac{\hat{\bar{P}}-C}{r}+\frac{P-\hat{\bar{P}}}{r+\hat{\eta}}\right]. \qquad (6.33)$$

To find the constants, A_2 and B_1, the further boundary conditions that both $V(P)$ and $V'(P)$ must be continuous at $P = C$ are imposed. The requirements for continuity provide two linear equations in two unknowns, A_2 and B_1. Since the second constant and the independent variable of both Kummer functions, M and U, are the same, for notational simplicity let $M(\theta,2\theta+2-\alpha;\,\beta/P)$ be denoted by $M(\theta)$, $M(\theta+1,2\theta+2-\alpha;\,\beta/P)$ be denoted by $M(\theta+1)$ and the analogous terms in U by $U(\theta)$ and $U(\theta+1)$. Thus, at $P=C$,

$$_{P<C}V(C)={}_{P>C}V(C) \qquad A_2\,U(\theta)-B_1\,M(\theta)=\frac{\hat{\eta}}{r}\left(\frac{C}{\beta}\right)^{\theta}\frac{\left(\hat{\bar{P}}-C\right)}{r+\hat{\eta}} \qquad (6.34)$$

$$_{P<C}V'(C)={}_{P>C}V'(C) \qquad A_2\theta(\gamma)U(\theta+1)+B_1\,\theta M(\theta+1)=\left(\frac{C}{\beta}\right)^{\theta}\frac{\tau C}{r+\hat{\eta}}. \qquad (6.35)$$

Solving (6.34) and (6.35) for A_2 and B_1 yields

$$A_2=\frac{\left(\frac{C}{\beta}\right)^{\theta}\frac{\tau\theta}{r+\hat{\eta}}\left[\frac{\hat{\eta}}{r}\left(\hat{\bar{P}}-C\right)M(\theta+1)+C\,M(\theta)\right]}{\theta U(\theta)M(\theta+1)+\gamma U(\theta+1)M(\theta)} \qquad (6.36)$$

and

$$B_1=\frac{\left(\frac{C}{\beta}\right)^{\theta}\frac{\tau}{r+\hat{\eta}}\left[C\,U(\theta)-\frac{\hat{\eta}}{r}\left(\hat{\bar{P}}-C\right)(\gamma)\,U(\theta+1)\right]}{\theta U(\theta)M(\theta+1)+\gamma U(\theta+1)M(\theta)}. \qquad (6.37)$$

Development Option: Consider the value of the option, $F(P)$, to develop a plant with the capacity of extracting one unit of output per unit of time, worth $V(P)$. The value of the call option will be determined by the ODE (6.1) with $\pi(P)=0$, since the holder of the option will not receive any cash flow from nonexistent plant. Along with the imposition of the value-matching and the smooth-pasting boundary conditions,

$$F\left(P_D^{*}\right)=V\left(P_D^{*}\right)-\tau I_D \qquad \textit{and} \qquad F'\left(P_D^{*}\right)=V'\left(P_D^{*}\right),$$

where P_D^* is the optimal price of crude oil at which to exercise the development option *F(P)* and construct the additional capacity, *V(P)* is the value of one BBL per annum of capacity, including the value of the right to suspend production, and τ is the after-tax capital cost of constructing the incremental capacity. *F(P)* will be given by (6.9) with $A_I = 0$

$$F(P) = K_2 \left(\frac{\beta}{P}\right)^{\theta} U\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right).$$

$V(P)$ will be given by (6.33). The only case considered is the case for $V(P)$ where $P > C$, since it would not be rational to build a plant just to shut it. Lastly, I_D is reduced by τ, since the capital costs are deductible for computing the resource owner's net profit interest and the cash inflows were adjusted down by τ.

At P_D^* the value matching condition, $F\left(P_D^*\right) = V\left(P_D^*\right) - \tau I_D$, leads to

$$K_2 = B_1 \frac{M(\theta)}{U(\theta)} + \left(\frac{\beta}{P_D^*}\right)^{-\theta} \frac{\tau}{U(\theta)} \left\{ \frac{\hat{\bar{P}} - C}{r} + \frac{P_D^* - \hat{\bar{P}}}{r + \hat{\eta}} - I_D \right\}. \qquad (6.38)$$

The smooth-pasting condition $F'\left(P_D^*\right) = V'\left(P_D^*\right)$ leads to

$$K_2 \frac{\gamma}{P_D^*} \left(\frac{\beta}{P_D^*}\right)^{\theta} U(\theta + 1) = B_1 \left(\frac{-\theta}{P_D^*}\right) \left(\frac{\beta}{P_D^*}\right)^{\theta} M(\theta + 1) + \frac{\tau}{r + \hat{\eta}}. \qquad (6.39)$$

Then substitution of (6.38) in (6.39) results in an expression for P^*

$$B_1 \left(\frac{\beta}{P_D^*}\right)^{\theta} \left\{ \frac{\gamma U(\theta + 1) M(\theta)}{U(\theta)} + \theta M(\theta + 1) \right\} + \ldots.. \qquad (6.40)$$

$$+ \frac{\tau P_D^*}{r + \hat{\eta}} \left\{ \frac{\gamma U(\theta + 1)}{U(\theta)} - 1 \right\} - \frac{\tau \gamma U(\theta + 1)}{U(\theta)} \left[\frac{C}{r} + I_D - \frac{\hat{\eta} \hat{\bar{P}}}{r(r + \hat{\eta})} \right] = 0.$$

Since B_1 is known from (6.37), then (6.40) can be solved numerically to find P_D^*. Substitution of P_D^* in (6.38) will determine K_2. The value of a call option exercisable for one BBL per annum of production capacity is

$$F(P) = K_2 \left(\frac{\beta}{P}\right)^{\theta} U\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right) \qquad P < P_D^* \qquad (6.41)$$

$$= B_1 \left(\frac{\beta}{P}\right)^{\theta} M\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right) + \tau \left[\frac{\hat{\bar{P}} - C}{r} + \frac{P - \hat{\bar{P}}}{r + \hat{\eta}} \right] - \tau I_D \qquad P \geq P_D^*. \qquad (6.42)$$

Exploration Option: Consider the value of the option, $G(P)$, to explore for the opportunity to develop a plant, given by $F(P)$, to extract one unit of output per unit of time, worth $V(P)$. The value of this call option will, again, be given by the ODE (6.1) with $\pi(P) = 0$, since the holder of $G(P)$ will not receive any cash flow from the unexplored and undeveloped lease

$$G(P) = L_2\left(\frac{\beta}{P}\right)^{\theta} U\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right). \tag{6.43}$$

If P_X^* and I_X are the optimal exercise price and the cost of exercise, respectively, for the exploration option, then the value-matching and smooth-pasting boundary conditions are

$$G\left(P_X^*\right) = F\left(P_X^*\right) - \tau I_X \qquad \textit{and} \qquad G'\left(P_X^*\right) = F'\left(P_X^*\right). \tag{6.44}$$

A choice between (6.41) and (6.42) to represent *F(P)* must be made to apply the boundary conditions (6.44) based on whether $P_X^* > P_D^*$ or $P_X^* < P_D^*$. Dixit and Pindyck (1994) observe that "intuitively we would expect $P_X^* > P_D^*$". Suppose $P_X^* < P_D^*$ so that $F\left(P_X^*\right)$ is given by (6.41) and since $G'\left(P_X^*\right) = F'\left(P_X^*\right)$ implies that $K_2 = L_2$, but this is contradicted by the boundary condition $G\left(P_X^*\right) = F\left(P_X^*\right) - I_X$. It follows that $P_X^* > P_D^*$ and $F\left(P_X^*\right)$ is given by (6.42). Then the value-matching condition says

$$L_2 = B_1 \frac{M(\theta)}{U(\theta)} + \left(\frac{\beta}{P_X^*}\right)^{-\theta} \frac{\tau}{U(\theta)} \left\{\frac{\hat{\bar{P}} - C}{r} + \frac{P - \hat{\bar{P}}}{r + \hat{\eta}} - \left(I_D + I_X\right)\right\}. \tag{6.45}$$

The smooth-pasting boundary condition leads to

$$L_2 \frac{\gamma}{P_X^*}\left(\frac{\beta}{P_X^*}\right)^{\theta} U(\theta + 1) = B_1\left(\frac{-\theta}{P_X^*}\right)\left(\frac{\beta}{P_X^*}\right)^{\theta} M(\theta + 1) + \frac{\tau}{r + \hat{\eta}}. \tag{6.46}$$

Then substitution of (6.45) in (6.46) and collecting terms in power of P_X^* results in a familiar expression

$$B_1\left(\frac{\beta}{P_X^*}\right)^{\theta} \left\{\frac{\gamma U(\theta+1) M(\theta)}{U(\theta)} + \theta M(\theta+1)\right\} + \ldots..$$

$$+ \frac{\tau P_X^*}{r + \hat{\eta}}\left\{\frac{\gamma U(\theta+1)}{U(\theta)} - 1\right\} - \frac{\tau \gamma U(\theta+1)}{U(\theta)}\left[\frac{C}{r} + \left(I_X + I_D\right) - \frac{\hat{\bar{P}}\hat{\eta}}{r(r + \hat{\eta})}\right] = 0. \tag{6.47}$$

Again, since B_1 is known from (6.37), then (6.47) can be solved numerically to find P_X^*. Substitution of P_X^* in (6.45) determines L_2. The value of the exploration option is

$$G(P) = L_2\left(\frac{\beta}{P}\right)^{\theta} U\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right) \qquad P < P_X^{*} \quad (6.48)$$

$$= B_1\left(\frac{\beta}{P}\right)^{\theta} M\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P}\right) + \tau\left[\frac{\hat{\bar{P}} - C}{r} + \frac{P - \hat{\bar{P}}}{r + \hat{\eta}}\right] - \tau\left(I_X + I_D\right) \quad P \geq P_X^{*}. \quad (6.49)$$

6.5 Valuation of Syncrude Extraction, Development, and Exploration Options

An example of a petroleum project with a non-declining rate of extraction is the Syncrude Project. Located near Fort McMurray in the Athabasca region of northeast Alberta, Syncrude mines bituminous oil sands, extracts crude bitumen and upgrades the bitumen to synthetic crude oil. Syncrude holds three long-term oil sands leases comprising nearly 200,000 acres of Crown rights. The magnitude of the resource underlying the three leases has been estimated by the Alberta Energy and Utilities Board (the “AEUB”) at 9.0 billion barrels of crude bitumen. Of this resource, the AEUB estimates that 6.0 billion BBL are recoverable by surface mining, representing the Project’s initial established reserves. Cumulative bitumen extraction since 1978 is approximately 1.7 billion BBL, leaving remaining reserves of 4.3 billion BBL. With mining and extraction capacity of 302,000 BBL per day of bitumen, Syncrude is currently operating at an average rate of 261,000 BBL per day of bitumen, equivalent to 86% of capacity. A capacity expansion program, known as the Stage 3 Expansion, is underway which when completed, in 2006, will increase Syncrude’s bitumen capacity to 500,000 BBL per day. At the current or expanded capacity rates of extraction the Project’s reserves of bitumen will be depleted in 45 and 24 years, respectively. Syncrude’s exiting reserves together with even a small portion of the reserves held by third parties without an existing mine and upgrading plant located in the Athabasca region, estimated by the AEUB at 24.2 billion BBL, should be sufficient to enable the Project to continue operating for many decades and be valued as a perpetuity.

The bitumen stream, at the inlet to Syncrude’s upgrading plant, is very viscous, short of hydrogen and high in impurities, including sulphur and heavy metals. Upgrading removes the excess carbon, via a coking process and the impurities. The Project’s yield of synthetic crude oil output is equal to approximately 86% of the volume of bitumen input. Currently, Syncrude’s capacity and output of crude oil are 260,000 BBL per day and 224,700 BBL per day, respectively. The Stage 3 Expansion is planned to add 110,000 BBL per day of incremental crude oil output capacity.

Extraction Option: The value of the option to extract crude oil from the oil sands, *V(P)*, is given by (6.31) and (6.33) with the constants A_2 and B_1 determined by (6.36) and (6.37), respectively. To implement the extraction model parameters for P, C and τ are required. At the plant gate, the crude oil price Syncrude’s output receives approximates the WTI reference price. The parameters estimated for WTI crude oil in Chapter 4 were: $\eta = 0.5$; $\bar{P} =$ \$27 with the OPEC range =\$24 to \$30; $\sigma = 0.35$; and $\rho\sigma\phi = -0.01$. Operating costs, following the completion of the Stage 3 Expansion, are forecast at \$16 per BBL, including expensed and capitalized items. The fiscal arrangement between Syncrude and the owner of the resource, the Province of Alberta, is a one percent royalty on sales, prior to recovery of capital costs, then a 25% share of net profits, represented by 1-τ. The foregoing can be utilized to value one BBL per annum of Syncrude production.

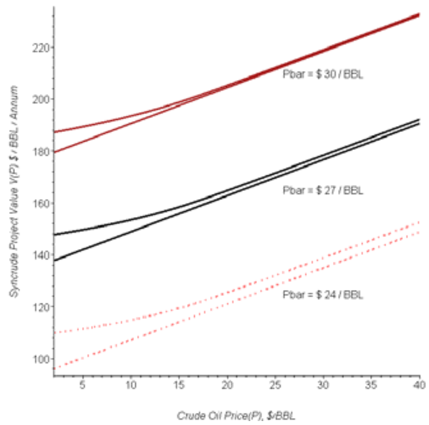

Figure 6.2 Comparison of Syncrude Production Option and Perpetuity Valuations

Figure 6.2 illustrates how the value of the Syncrude project, *V(P),* varies as the mean-reverting price of crude oil changes. Intuitively, *V(P)* should shift higher, or lower, with increased, or decreased, levels of $\overline{P}$. For each pair of curves, the trajectory of the upper curve is given by (6.31) and (6.33) and that of the lower curve by the perpetuity (6.30). The value of flexibility is equal to the distance between the curves comprising each pair. If $\overline{P}$ = \$30, the curves representing the project's value are close together, consistent with the intuition that at "high prices" the value of the right to suspend extraction is small. This occurs because $\overline{P}$ = \$30 >> C = \$16 and the strength of reversion of P is "strong", η being 0.5. For $\overline{P}$ = \$ 27, the pair of value curves are farther apart, reflecting the greater chance that P will fall below C, increasing the value of the put term in (6.33). The value of the right to shut-in the project for $\overline{P}$ = \$27 is approximately \$2.50 per BBL per annum. When multiplied by the annual capacity of Syncrude, 135 MMBBL per annum at the completion of the Stage 3 Expansion, the value of the option to suspend production is worth approximately \$338 million to the holders of the Project. A downward shift of $\overline{P}$ to \$24 increases the probability that P will fall below C = \$16. It follows that the value of the option to suspend production of crude oil will be larger than that for $\overline{P}$ = \$27. The separation of the bottom pair of curves is approximately \$5.00 per BBL per annum, equivalent to \$675 million of project value.

The positive intercepts of the vertical axis and modest slopes of the value curves indicate an important attribute of the valuations of Syncrude by (6.30), (6.31) and (6.33): they are not very sensitive to changes in crude oil prices, given the selected strength of mean reversion of P. At P = 0, the cash flow from the Project, that will be capitalized in (6.30), can be written as $\eta(\overline{P}-C)-(r+\rho\sigma\phi)C$. This cash flow will be positive for $\eta > [(r+\rho\sigma\phi)C/(\overline{P}-C)]$. For $\overline{P}$ = \$27, r = 0.05, and $\rho\sigma\phi$ = -0.01, then for any η > 0.058 the value of the perpetuity, (6.30), will be greater than zero. The shallow slope of the mean-reverting value functions contrasts with steep assent, with increasing P, of the value function for a flexible project where the output price, P, follows a GBM. The value, $V(P)$, is given in Dixit and Pindyck (1994), p. 188-189, by

$$V(P) = K_2 P^{\alpha^+} \qquad P < C$$
$$= B_1 P^{\alpha^-} + P/\delta - C/r \qquad P \geq C.$$

The value functions for the Syncrude project assuming P follows either a GBM or an IGBM are contrasted in Figure 6.3.

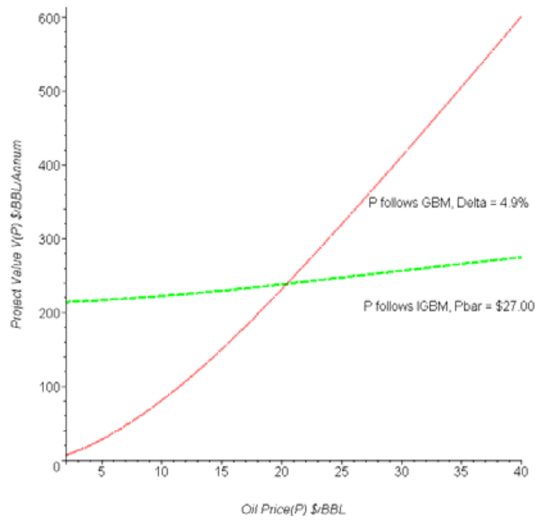

Figure 6.3 Comparison of Valuations of the Syncrude Project, Tau=1.0

Modeling P as a GBM, with σ=0.35 and δ=4.9% calibrated to the October 30, 2003 futures curve, results in a value function with a steep, asymptotic slope approximated by the reciprocal of the discount factor in the perpetuity, $1/\delta$ = 20.4. The value function for the project where P follows an IGBM has an asymptotic slope, again approximated by the reciprocal of the perpetuity's discount factor, see (6.30), equal to 1 / $(r+\eta+\rho\sigma\phi)$ = 1.85 (ignoring τ = 0.75). If crude oil did not exhibit the speed of revision found in Chapter 4, η= 0.5, then the slope of $V(P)$ would be much greater. For example, if the half-life of reversion was five years, not 1.4 years, then η would equal 0.14 and the slope of $V(P)$ would be 5.6. If the price of crude oil is as mean-reverting as both the historical data and the futures curve imply, then the value of the Syncrude project with C = $16 will not vary a large amount with changes in P.

Development Option: The value of a call option, denoted by $F(P)$, to build one BBL per annum of incremental Syncrude capacity is given by (6.41) and (6.42) with the optimal exercise price, P^*, determined by the numerical solution of (6.40). The exercise price of the call, I_D, can be estimated from the cost of adding an incremental BBL per annum of capacity in the Stage 3 Expansion. The management of the Syncrude Project now expects to complete the Stage 3 Expansion in late 2006, with the course of construction having run nearly five years. The originally budgeted cost for the Stage 3 Expansion of Cdn. $4.1 billion, in July 2001, was later revised to Cdn. $5.7 billion and is currently estimated at Cdn. $7.8 billion. At this cost level the incremental capacity to be added, say an effective rate of 100,000 BBL per day, has a capital cost equivalent to $160.28 per BBL per annum, before recoveries from Alberta's net profit interest. The value of the option to develop one BBL per annum of Syncrude capacity with a capital cost of $160 per BBL per annum is shown in Figure 6.4.

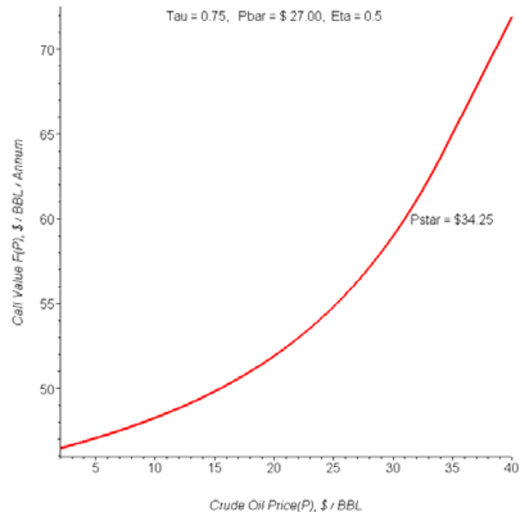

Figure 6.4 Call Value for One BBL per Annum of Syncrude Project

Figure 6.4 shows that the value of a call on incremental capacity at Syncrude will remain valuable, between \$46-\$65 per BBL per annum, over a wide range of crude oil prices up to \$35 per BBL. At current prices, \$35 per BBL, investment in additional capacity, even at a cost of \$160 per BBL per annum, is optimum since P^* = \$34.25. Before taking such a decision, though, a decision-maker may wish to expand the scope of his thinking to include the effect on P^* of OPEC's actions with respect to its targeted price band of \$24 to \$30 per BBL; and cost under-runs or over-runs of, say, 20% in the capital cost. Then using (6.41) and (6.42), P^* can be calculated reflecting the changes in $\overline{P}$ and I_D.

Table 6.2 P^* As a Function of $\overline{P}$ and I_D

$\overline{P}$	I_D = \$128	I_D = \$160	I_D = \$192
\$30	27.77	31.18	35.04
\$27	29.87	34.25	39.72
\$24	33.90	41.02	51.34

Based on $\overline{P}$ = \$27 per BBL capacity additions costing \$160.28 per BBL per annum are optimum at a crude oil price of \$35 per BBL. However, if Syncrude were to experience capital cost overruns, a spot price of nearly \$40 per BBL will be required to make the addition economic. Lastly, OPEC's role in providing its competitors with a "price umbrella" can be seen in Table 6.5.1. If OPEC targets the high end of its range, \$30 per BBL of WTI, the Syncrude Expansion is economic at \$35 per BBL, even with cost overruns. Only by targeting the lower end of its range, \$24 per BBL of WTI, will OPEC rule out the expansion of a project like the Syncrude Expansion, unless a 20% decrease in the capital costs per unit of capacity can be attained.

Beyond the completion of the Stage 3 Expansion, the Syncrude planners are considering two further expansions, known as the Stage 4 and Stage 5 Expansions, which could add incremental crude oil output capacities of 70,000 BBL per day and 120,000 BBL per day, respectively. If approved, construction of Stage 4 may commence in 2007 and continue till 2010. Syncrude has estimated the cost of the Stage 4 Expansion at Cdn. \$2.3 billion, equivalent to \$90 per BBL per annum. The capital cost estimate will be revised upon the completion of a pre-engineering study, which is underway and a definitive engineering study and construction budget. For now, perhaps the best estimate available of the capital cost of adding capacity is Syncrude's experienced costs with the Stage 3 Expansion.

Exploration Option: The value of a call option, denoted by $G(P)$, to acquire the option to build one BBL per annum of incremental Syncrude capacity is given by (6.48) and (6.49) with the optimal exercise price, P_X^*, determined by the numerical solution of (6.47). Comparisons of (6.41) and (6.42) to (6.48) and (6.49) and (6.40) to (6.47) show that the value of $G(P)$ is given by $F(P)$ where the cost of exploration, I_X, has been added to the cost of development, I_D. In Table 6.2 it was demonstrated that increasing I_D caused P^* to increase. It follows that a valuation of the option to explore the Syncrude leases could be calculated by adding the exploration costs to the development costs and utilizing (6.41) and (6.42). The oil sands leases held by Syncrude are fully explored at this time so no additional costs are added in our calculations.

Syncrude Lease Valuation: Combining the preceding calculations facilitates the valuation of Syncrude's holding of oil sands leases. Assuming that the capacity utilization rate in the Syncrude plant will reach 90%, then the output rate of crude oil for the Project at the completion of the Stage 3 Expansion will approximate 333,000 BBL per day. Similarly, the Stage 4 and Stage 5 Expansions may add output rates of crude oil of 63,000 BBL per day and 108,000 BBL per day, respectively. Based on the parameters estimated in Chapter 4 with $\overline{P} =$ \$27 per BBL and a spot price of \$30 per BBL the valuation in Table 6.3 results.

Table 6.3 Pre-Income Tax Valuation of Syncrude Leases at Oct. 31, 2003

	\$/BBL/Annum	\$ MM Value
Extraction Option	176	21,392
Less: Cost to Complete Stage 3		(2,000)
		19,392
Development Options		
Stage 4	59	1,357
Stage 5	59	2,326
		3,683
Total Value		23,075

6.6 Valuation of an Option on a Project with a Declining Extraction Rate

Extraction and Development Option: Consider a project that extracts $q(t)$ units of output per unit of time, declining exponential at a fixed rate ω and receives a price for its output, $P(t)$, where P follows an IGBM, then its gross revenues will be $R(t)=q(t)P(t)$. The instantaneous cash flow that will accrue to the project will be $\pi(t)=\tau[R(t)-C_0]dt$ and the value of the project, with no flexibility, is given by (5.9) which can be written in the form

$$V(q_0;P) = b + mP \qquad (6.50)$$

where $b = \dfrac{q_0\,\tau\,\hat{\eta}\,\hat{\overline{P}}}{(r+\omega)(r+\omega+\hat{\eta})} - \dfrac{C_0}{r}$ *and* $m = \dfrac{q_0\tau}{(r+\omega+\hat{\eta})}$.

The value of the perpetual call option to develop a project with a declining revenue stream will be given by the ODE (6.1) with $\pi(P)=0$ along with the value-matching and smooth-pasting boundary conditions,

$$F\left(P_D^*\right)=V\left(P_D^*\right)-I_D \qquad and \qquad F'\left(P_D^*\right)=V'\left(P_D^*\right),$$

where P_D^* is optimal price of crude oil to exercise $F(P)$ and develop the project. $V(P)$ is specified by (6.50). For $P<P_D^*$, then $F(P)$ will be given by (6.9) with $A_1=0$. For $P\geq P_D^*$, application of the value-matching condition results in

$$b+mP_D^*-K_2\left(\frac{\beta}{P_D^*}\right)^{\theta}U(\theta)=I_D. \tag{6.51}$$

And the smooth-pasting condition leads to

$$K_2=\left(\frac{mP_D^*}{\gamma U(\theta+1)}\right)\left(\frac{\beta}{P_D^*}\right)^{-\theta}. \tag{6.52}$$

Substitution of (6.52) in (6.51) results in an expression for P^*

$$b+mP_D^*\left[1-\frac{U(\theta)}{\gamma U(\theta+1)}\right]=I_D \tag{6.53}$$

which is solved numerically to find P_D^* and then K_2 follows from (6.52). The value of F is

$$F(P)=K_2\left(\frac{\beta}{P}\right)^{\theta}U\left(\theta,2\theta+2-\alpha;\frac{\beta}{P}\right) \qquad P<P_D^* \tag{6.54}$$

$$=b+mP-I_D \qquad P\geq P_D^*. \tag{6.55}$$

Exploration Option: Consider the value of the option, $G(P)$, to explore for the opportunity to develop the project, given by $F(P)$. Again, the value of the call option will be given by the ODE (6.1) with $\pi(P)=0$. If P_X^* and I_X are the optimal exercise price and cost of exercise, respectively, for $G(P)$, its solution can be found by applying the value-matching and smooth-pasting boundary conditions to (6.9) with $A_1 = 0$,

$$G\left(P_X^*\right)=F\left(P_X^*\right)-I_X \qquad and \qquad G'\left(P_X^*\right)=F'\left(P_X^*\right)-I_X.$$

As in Section 6.4 the restriction that $P_X^*>P_D^*$ is made implying $F(P)=b+mP-I_D$. Then the value-matching condition leads to

$$b+mP_X^*-L_2\left(\frac{\beta}{P_X^*}\right)^{\theta}U(\theta)=\left(I_X+I_D\right) \tag{6.56}$$

and the smooth-pasting condition leads to

$$L_2 = \left(\frac{mP_X^*}{\gamma U(\theta+1)} \right) \left(\frac{\beta}{P_X^*} \right)^{-\theta} . \tag{6.57}$$

Substituting (6.57) in (6.56) again leads to an expression in P_X^*,

$$b + mP_X^* \left[1 - \frac{U(\theta)}{\gamma U(\theta+1)} \right] = (I_D + I_X)$$

which can be solved numerically for P_X^*. Substitution of P_X^* in (6.57) determines L_2. The value of the exploration option is

$$G(P) = L_2 \left(\frac{\beta}{P} \right)^{\theta} U\left(\theta, 2\theta + 2 - \alpha; \frac{\beta}{P} \right) \qquad P < P_X^* \tag{6.58}$$

$$= b + mP - (I_X + I_D) \qquad P \geq P_X^* . \tag{6.59}$$

6.7 Valuation of a Conventional Petroleum Lease

As an example of a small, conventional petroleum project in the Western Canadian Sedimentary Basin, consider the valuation of a lease covering, say, 640 acres that prospectively contains 300,000 BBL of crude reserves. When explored and developed the lease will have four wells capable of initially extracting crude oil at a rate of 100 BBL per day, declining at a rate of 12% per annum. Fixed extraction costs are assumed to be $72,000 per annum. Variable costs, represented by τ, are assumed to total 30% of the WTI reference price comprised of: royalties, 15%; transportation and quality differentials, 7.5%; and variable extraction costs, 7.5%. The value of the development option is given by (6.54) and (6.55). If the cost of development, I_D, is $750,000, then the value of the development option is illustrated in Figure 6.5 below.

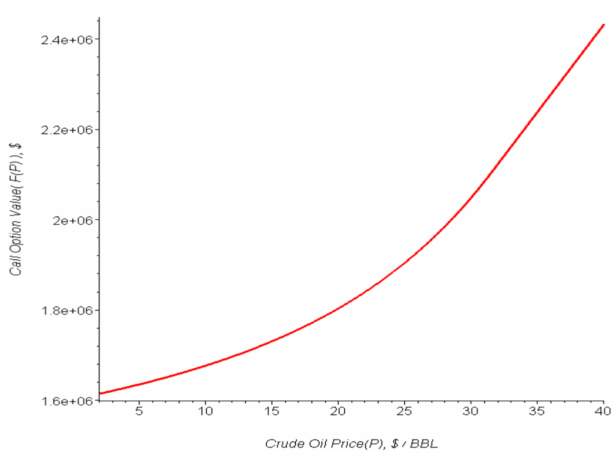

Figure 6.5 Value of Conventional Development Option

Chapter 7

Conclusions

7.0 Conclusions from This Investigation

The principal conclusions to be drawn from the foregoing investigation include the following:

Petroleum leases are real options. Petroleum lease contracts can be analyzed and valued as real options, pursuant to Seppi's definition. The valuation of the rights accruing to the holder of a petroleum lease proceeds in a reverse-recursive manner via the petroleum production cycle: from the extraction phase; through the development phase; and finally, to the exploration phase.

Real options can be valued using the BSM PDE. Notwithstanding that certain of the assumptions, such as the continuous trading and the use of the underlying asset to hedge the contingent claim being valued, that underpin the derivation of the BSM PDE to value financial options, cannot be relied upon, there exist alternate assumptions consistent with the characteristics of real options which facilitate the derivation of the BSM PDE to value real options. A derivation of the BSM PDE to value real options, supported by valid assumptions and parsimonious with respect to the number of required assumptions, is Sick's derivation, based on the Consumption CAPM. Sick's derivation not only yields (2.14), a general form of the BSM PDE suitable for any stochastic process that the underlying asset follows, but also the total rate of return equation (2.9).

Commodity prices can be modeled by the IGBM process. It was argued in Chapter 3, to model commodity prices appropriately, a stochastic process must possess four attributes:

(1) generate strictly positive values;
(2) revert to a mean value, $\overline{P}$;
(3) have drift and volatility functions that are homogeneous of the pair $\{P(t), \overline{P}\}$; and
(4) as $t \to \infty$, not be attracted to either of the boundaries $P = 0$ or $P = \infty$.

The mathematical techniques necessary to determine the foregoing attributes were applied to six, single-factor stochastic processes: only the IGBM process was found to have all four attributes, subject to the restrictions $\eta \neq \sigma^2$, $2\eta \neq \sigma^2$ and $2\eta > \sigma^2$.

The IGBM process can be parameterized to model crude oil prices. Two approaches, a time-series approach using historic crude oil prices, and a calibration approach based on the market prices of crude oil futures and options exercisable for crude oil futures, were employed to estimate the parameters of the IGBM process. The implementation of the time-series approach encountered the dichotomy between the quantity and the relevance of historic data. While more confidence can be placed on the sample comprising twenty years of data than the five year sample, the parameter, $\overline{P}$, appears to vary with time, implying that the estimate of $\overline{P}$ from the five year sample is more relevant for valuation purposes. The most relevant estimate of $\overline{P}$, however, is the one obtained from the market price of the longest dated futures contract. It is commodity market participants' forward looking estimate of the price required to balance supply and demand in the long-run. The other two parameters estimated by the time-series approach, η and σ, appear to be time invariant. The calibration approach's estimate of σ was approximately the same as the time-series approach, while its estimate of η was an order of magnitude larger. Giving weight to the estimates obtained from both approaches, the magnitudes of the selected parameters for crude oil prices, $\eta = 0.5$ and $\sigma = 0.35$, do not violate the parameter restrictions for the IGBM process, $\eta \neq \sigma^2$, $2\eta \neq \sigma^2$ and $2\eta > \sigma^2$.

Reserves of petroleum can be valued as perpetuities. The value of a petroleum reserve, where the extracted crude oil receives prices following an IGBM process, can be derived for level and declining rates of extraction: the valuations are given by (5.3) and (5.9), respectively.

The riskiness of petroleum reserves is less than crude oil on the surface. Empirical evidence, prepared by others, was presented demonstrating: the volatility of returns from holding petroleum in the ground is less than, or equal to, that from holding crude oil on the surface. The volatility of holding a petroleum reserve was derived by applying Itô's Lemma to (5.9) and is given by (5.11). Examination of the coefficients of (5.11) results in the view that the volatility of returns from holding a petroleum reserve is less than that of crude oil, consistent with the empirical evidence. In contrast, reserves valued assuming prices follow a GBM have volatilities greater than crude oil's.

Valuations of American style, perpetual options exercisable for an asset following an IGBM process can be derived. The value of a perpetual option, exercisable at any future time, for an asset, the price of which follows an IGBM, is given by the ODE (6.1). For the homogeneous portion of the ODE (6.1) it was shown that (6.9) is the solution and that there are four solution candidates. Of these solutions, the two for which for $\theta = \theta^+$ and containing Kummer's *U* and *M* functions, can be used to value perpetual, American calls and puts, respectively. The effect of varying $\overline{P}$ on the values and optimal exercise prices of perpetual call and put options is illustrated in Figure 6.1.

The value of a petroleum lease contract is found by the valuation of the compound options to extract, develop and explore. In the case of a level extraction rate: a petroleum lease valuation begins by determining the value of a plant, *V*(*P*), with the flexibility to suspend and restart extraction, given by (6.31) and (6.32) when $P < C$ and $P > C$, respectively, with constants specified by (6.36) and (6.37). Letting *V*(*P*) be the underlying asset, the value of the next option in the sequence - the development option, *F*(*P*) – is found by imposing the value-matching and smooth-pasting boundary conditions. *F*(*P*) has the value given by (6.41) and (6.42) for $P < P_D^*$, and $P \geq P_D^*$, respectively. By letting *F*(*P*) now be the underlying asset and again imposing the value-matching and smooth-pasting boundary conditions the value of the last option in the sequence – the exploration option, *G(P)* - is acquired. For $P < P_X^*$ and $P > P_X^*$, *G(P)* is given by (6.48) and (6.49), respectively, with the constant specified by (6.45).

In the case of a declining extraction rate, a petroleum lease valuation begins by deriving the value of a perpetuity, denoted by *V*(*P*) and given by (6.50). The value of the development option, *F*(*P*), is obtained by letting *V*(*P*) be the underlying asset and imposing the value-matching and smooth-pasting boundary conditions. The relationship between the value of the development option and the price of crude oil is given by (6.54) and (6.55) for $P < P_D^*$ and $P \geq P_D^*$, respectively. Making $F(P)$ be the underlying asset for the exploration option, the value of which is denoted by $G(P)$, and again imposing the value-matching and smooth-pasting boundary conditions results in (6.58) and (6.59) for $P < P_X^*$ and $P \geq P_X^*$, respectively, with the constant determined by (6.57).

In both the level and declining extraction rate cases, $G(P)$ values a petroleum lease as a function of the current price of crude oil, assuming there is no geological risk or technical risk: the extraction rate and all costs, C_e, I_D, and I_X are all known with certainty. In addition, determining the constants for the solutions of the valuation ODE's also yields the optimal prices of crude oil, P_X^* and P_D^*, at which to exercise the exploration and development options, respectively.

The Syncrude lease valuation example illustrates the appropriateness and limitations of real options as a valuation approach. The application of the real option methodology to the Syncrude Project, in Chapter 6, resulted in:

(1) the valuation of a plant with extraction flexibility as a function of the current price of crude oil, see Figure 6.2;
(2) a demonstration that a plant with extraction flexibility is worth an amount greater than or equal to a plant without it, compare the pairs of curves in Figure 6.2;
(3) the valuation of the perpetual call option to develop a plant with extraction flexibility as a function of the price of crude oil, see Figure 6.4; and
(4) the calculation of the optimal price of crude oil at which to exercise the option to develop the flexible plant, as a function of $\overline{P}$ and I_D.

The experience of the owners' of the Syncrude Project during the Stage 3 Expansion demonstrates two of the major differences between real options and financial options. First, the time required to exercise a real option, the course of construction, is not instantaneous, as in the case of financial options. Second, there can be significant uncertainty associated with the magnitude of the exercise price of a real option on a large, capital intensive project: as the 90% cost overrun during the Stage 3 Expansion illustrates.

Appendix 1

Comparison of Incremental Assumptions to Derive the BSM PDE

Delta-Hedging: The delta-hedging method, as defined in Wilmott, Howinson, & Dewynne (1995) p42-43, assumes the underlying asset is both continuously traded and can be combined with the contingent asset being valued to form hedged, or risk-free portfolios. These portfolios may be comprised of: a long position of one unit of the contingent claim and a short position of delta units of the underlying asset; or a long position in one unit of the underlying asset and a short position in delta units of the contingent claim. Neither assumption is tenable for the purpose of valuing petroleum assets. While subsurface barrels trade hands in privately negotiated transactions from time to time, there is no organized market where homogeneous barrels of reserves trade in a continuous manner. Hedged portfolios comprised of, say, a long position in the contingent claim and a short position in a barrel of subsurface crude oil cannot be formed in practice. How could someone lend a barrel of crude oil to a short seller that had not yet been discovered, developed or extracted?

Replication: A prescribed alternative derivation, equivalent to the delta-hedging method, is the replication method, as defined in Dixit & Pindyck (1994) p116-117. Assuming the claim to be valued, $F(P,t)$ is not traded, a notional portfolio that will replicate the risk and return characteristics of F is formed, comprised of: the underlying asset P, which is assumed to be traded; and risk-free bonds. While the replication method could be used to value the extraction phase, where the project's output of crude oil is traded, application to the earlier phases, where the underlying asset is not traded, becomes more problematic.

Spanning-Assets: Another method, advocated by both Dixit & Pindyck (1994) and Robel (2001), is derived by the use of spanning-assets. This method requires the assumed existence of a traded asset that can be sold short; is perfectly correlated with the underlying asset, P, so that it will replicate, or "span", the movements in P; and follows its own stochastic process. The further assumption that the expected returns of the underlying asset and the spanning-asset are equal and given by the CAPM, is required to complete the derivation. Is there a traded asset that could replicate the returns of subsurface barrels of petroleum? Publicly traded royalty trust units and master limited partnership units might fill this role for developed barrels but not for undiscovered barrels of petroleum.

Dynamic-Programming: Dixit & Pindyck (1994) also consider the dynamic-programming method of deriving the BSM PDE. Pursuant to this method, the value of the contingent claim $F(P,t)$ is expressed in terms of Bellman's recursive form, see p122, and simplified using a Taylor series expansion to order dt and Itô's Lemma. The resulting PDE is analogous to the BSM PDE, except that the convection term contains the real drift and the discount term contains the exogenous discount rate. The substitution of the risk-neutral for the real drift in the convection term and the risk-free for the exogenous discount rate in the discount term results in the BSM PDE. This recasting of the BSM PDE from the real to risk-neutral world obscures the underlying assumptions being relied upon to obtain the result. Such a transformation means the resulting PDE relies on the assumptions that characterizes the risk-neutral world of no arbitrage. The no arbitrage assumption is equivalent to assuming that both the underlying asset and the contingent claim trade continuously and can be sold short as in the delta-hedging method.

Appendix II

Comparison of Single-Factor Stochastic Processes

GBM Process: Frequently utilized, the GBM process generates positive values and satisfies the homogeneity condition. The first moment of the GBM is given by $p_0 \exp[\alpha t]$ so that as $t \to \infty$, it will not revert to a mean. For $\sigma > \sqrt{2\alpha}$, the motion of the GBM process will be attracted to the boundary $P = 0$. The boundary behavior of the GBM is of practical importance, given the high volatility of crude oil and the low level of risk-free interest rates.

OU Process: The first moment of the OU process is also given by (3.5) showing it reverts to $\overline{P}$. The solution of the OU SDE contains a normally distributed random variable and can generate negative values. While the probability of this occurring for crude oil is low, based on the parameters selected in Chapter 4, the terminal density is N (28, .35), the OU Process violates the homogeneity condition.

CIR Process: While generating positive values and reverting to $\overline{P}$, the CIR process violates the homogeneity condition. For the parameters for crude oil the CIR process should not be attached to either of the boundaries $P = 0$ or $P = \infty$.

Exponential OU Process: This is the stochastic version the Gompertz ODE. The solution of the exponential OU comprises power and exponential functions and only generates positive values. The first moment exists but when the limit as $t \to \infty$ is taken, the reversion value is found to be $\overline{P} \exp[\sigma^2 / 4\eta]$. Also, the Exponential OU does not satisfy the homogeneity condition.

SLV Process: This is the stochastic version of the Verhulst ODE. The solution to the SLV SDE, given in Kloeden and Platen (1999) and Robel (2001), is comprised of exponential functions in the numerator and denominator that will generate positive values for $t > 0$. The first moment of the SLV process is a function of the second moment, the second of the third, and so on. According to Robel (2004) the moments of the SLV process are unknown. The stationary density of the SLV does exist and its first moment is $\overline{P}\left(1 - \sigma^2 / 2\eta\overline{P}\right)$. Consequently, the SLV does not revert to $\overline{P}$. Furthermore, the SLV process does not satisfy the homogeneity condition. The boundary behavior of the SLV process is the same as the CIR process.

Table A-II-1 Comparison of Single-Factor Stochastic Processes

				Boundary Behavior			
				$P = 0$		$P = \infty$	
Process	$\{P(t) \geq 0\}$ **All** t**?**	**Satisfies Homogeneity Condition**	**Reverts To?**	**Attracted**	**Reflected**	**Attracted**	**Reflected**
GBM	Yes	Yes	N/A	$\sigma > \sqrt{2\alpha}$	$\sigma < \sqrt{2\alpha}$	$\sigma < \sqrt{2\alpha}$	$\sigma > \sqrt{2\alpha}$
CIR	Yes	No	$\overline{P}$	$\sigma > \sqrt{2\eta\overline{P}}$	$\sigma \leq \sqrt{2\eta\overline{P}}$	$\frac{2\eta\overline{P}}{\sigma^2} \leq 0$	$\frac{2\eta\overline{P}}{\sigma^2} > 0$
O-U	No	No	$\overline{P}$	$\eta > 0$	$\eta < 0$	$\eta < 0$	$\eta > 0$
IGBM	Yes	Yes	$\overline{P}$		$2\eta / \sigma^2 > 0$		$2\eta / \sigma^2 > 0$
Exp. O-U	Yes	No	$\overline{P}\exp(\sigma^2 / 4\eta)$				
SLV	Yes	No	$\overline{P}\left(1 - \frac{\sigma^2}{2\eta\overline{P}}\right)$	$\sigma > \sqrt{2\eta\overline{P}}$	$\sigma \leq \sqrt{2\eta\overline{P}}$	$\frac{2\eta\overline{P}}{\sigma^2} \leq 0$	$\frac{2\eta\overline{P}}{\sigma^2} > 0$

Appendix III

Statistics for Spot Crude Oil Time-Series Data

The time series data comprises spot WTI crude oil prices, observed monthly, for the period from May 31, 1986 until August 29, 2003. The start date was not chosen; rather it is the earliest date for which Bloomberg has data. Also considered was a subset of the last five year's data. The summary statistics are:

Table A-III-1 Summary Statistics for Spot Crude Oil Time-Series

Statistic	May '83 – Aug. '03	Aug. '89 – Aug. '03
No. of observations	244	61
Min.	\$10.40	\$11.26
Max.	39.60	36.60
Location		
Mean	21.94	25.09
Mode	30.25	N/A
Median	20.42	26.80
Shape		
Standard Deviation	\$5.63	\$6.11
Skewness	0.46	-0.65
Kurtosis	-0.51	-0.27

Comments regarding the time-series include:

1. The means of the two samples appear to be different, with the mean increasing from \$21.94 per BBL to \$25.09 per BBL in the last five years.

2. The dispersion, as measured by the standard deviation, has increased slightly.

3. The 20 year sample displays positive skewness, indicating a skew to the right and some negative, excess kurtosis, indicating a flat peak density.

The spot crude oil prices comprising both the "long" and "short" time-series were used to implement the linear regression approach specified by (4.9). The absolute value of the $\hat{t}$-statistics for the intercept and slope of the regression (4.9), $\hat{a}$ and $\hat{b}$, respectively, are shown below.

Table A-III-2 $\hat{t}$ -Statistics for Regression Parameters		
	May '83 – Aug. '03	Aug. '89 – Aug. '03
$\hat{a}$	3.11	1.60
$\hat{b}$	3.43	2.17
Degrees of Freedom	242	59

The $\hat{t}$ distribution value for a two sided test at the .005 significance level with 200 degrees of freedom, $t_{.005,200}$ is 2.838. Since $\hat{t} > t$ it follows that for the "long" time-series the null hypothesis H_o: $\hat{a} = 0$ and H_o: $\hat{b} = 0$ can be rejected at the 0.5% level, implying $\hat{a}$ and $\hat{b} = 0$ differ from zero. For the "short" time-series less confidence can be placed on the regression coefficients. For the intercept, $\hat{a}$; $t_{20,60} = 1.296$, so the null hypothesis H_o: $\hat{a} = 0$ can only be rejected at the 20% confidence level. The situation is better for the slope parameter $\hat{b}$ where $t_{.05,60} = 2.0$ so H_o: $\hat{b} = 0$ can be rejected at the 5% level.

Appendix IV
Estimates of Parameters by Calibrating Calls on Crude Oil Futures

Call Option on a Forward Contract		Feb-04	Mar-04	Apr-04	May-04	Jun-04	Jul-04	Aug-04	Sep-04	Oct-04	Nov-04	Dec-04	Mar-05	Jun-05	Dec-05
Call Option Maturity	(T-t)	0.250	0.333	0.417	0.500	0.583	0.667	0.750	0.833	0.917	1.000	1.083	1.333	1.583	1.080
Risk Free Rate	r	0.010	0.010	0.010	0.011	0.011	0.011	0.012	0.012	0.013	0.013	0.013	0.015	0.017	0.0188
Discount Factor	P(t,T)	1.002	1.003	1.004	1.005	1.006	1.008	1.009	1.010	1.012	1.013	1.015	1.020	1.027	1.021
Maturity of Futures Contract	(s-t)	0.250	0.333	0.417	0.500	0.583	0.667	0.750	0.833	0.917	1.000	1.080	1.330	1.583	1.080
Price of Futures Contract	F(t,s)	$28.47	$28.19	$27.91	$27.63	$27.35	$27.10	$26.88	$26.66	$26.48	$26.32	$26.16	25.800	25.590	$25.51
Volatility	Sigma	0.350													
Speed of Reversion	Alpha	0.624													
First Term		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.004	1.004	1.000	1.000
Second Term		0.732	0.660	0.595	0.536	0.483	0.435	0.392	0.354	0.319	0.287	0.260	0.190	0.139	0.260
	w	0.026	0.033	0.040	0.046	0.051	0.055	0.060	0.063	0.067	0.070	0.073	0.080	0.085	0.073
Option Strike Price	K	$28.50	$28.00	$28.00	$27.50	$27.50	$27.50	$26.00	$27.00	$26.50	$31.00	$26.00	$26.00	$26.00	$25.00
	h	0.075	0.128	0.084	0.129	0.088	0.056	0.258	0.076	0.126	-0.486	0.158	0.114	0.091	0.210
Cumm Norm(h)	N(h)	0.530	0.551	0.533	0.551	0.535	0.522	0.602	0.530	0.550	0.313	0.563	0.545	0.536	0.583
CummNorm(h-sqrt(w))	N(h-sqrt(w))	0.465	0.478	0.454	0.466	0.446	0.429	0.506	0.430	0.447	0.226	0.455	0.433	0.421	0.476
Call Price(Model)	P(t,T)[F(t,s)N(h) -KN(h-w)]	$1.83	$2.15	$2.19	$2.42	$2.40	$2.38	$3.06	$2.55	$2.75	$1.25	$2.93	$2.87	$2.86	$3.03
Call Price (Market)		$1.73	$2.17	$2.24	$2.51	$2.47	$2.33	$3.10	$2.43	$2.59	$0.90	$2.73	$2.73	$2.87	$3.84
		$0.10	$0.02	$0.05	$0.09	$0.07	$0.05	$0.04	$0.12	$0.16	$0.35	$0.20	$0.14	$0.01	$0.81
Abs. Error		$2.21													
Square Error		$0.01	$0.00	$0.00	$0.01	$0.00	$0.00	$0.00	$0.01	$0.02	$0.12	$0.04	$0.02	$0.00	$0.65
		$0.90													

Appendix V

Valuation of a Producing Petroleum Reserve – GBM Prices

Consider the valuation of a stream of cash flows from a petroleum reserve. The gross revenues receivable are the product of: the price, $P(t)$; the rate of crude oil production, $q(t)$, which declines by a constant percentage, ω; and the amount of time, dt, equal to $P(t)q(t)dt$. Let $R(t) = q(t)P(t)$. Since $dq(t) = -\omega\, q(t)$, then, by the product rule, the real motion of $R(t)$ will be given by the SDE (AV.1).

$$dR(t) = (\alpha_p - \omega)R(t) + \sigma_P\, R(t)dz \qquad R(0) = q_0 p_0 \qquad \text{(AV.1)}$$

The risk-neutral version of R will have a drift term of:

$$\hat{\alpha}(R,t) = (\alpha_P - \omega)R - \rho\sigma_P \emptyset R = (r - \delta_P - \omega)R$$

Where δ_P is the convenience yield of crude oil. The risk-neutral motion of $R(t)$ is (AV.2).

$$d\hat{R}(t) = [r - (\delta_P + \omega)]\, \hat{R}(t) + \sigma_P \hat{R}(t)dz \qquad \hat{R}_0 = q_0 P_0 \qquad \text{(AV.2)}$$

If the holder of the petroleum reserve must pay fixed and variable operating costs, represented by C_O and τ, respectively, then the instantaneous cash flow will be $\pi(t)dt = (\tau R(t) - C_O)dt$. The value of a perpetuity paying $\pi(t)dt$ from (AV.2) is:

$$V\left[t_S, \infty; \pi\left(R_0, \hat{R}\right)\right] = \int_{t_S}^{\infty} \mathrm{E}_0\left[\hat{\pi}(t) \mid R(0) = p_0 q_0\right] e^{-rt} dt$$

$$= \int_{t_S}^{\infty} \left(\tau\, \mathrm{E}_0\left[\hat{R}(t) \mid R(0) = p_0 q_0\right] e^{-rt} - C_O\, e^{-rt}\right) dt$$

$$= \tau\, p_0 q_0 \int_{t_S}^{\infty} e^{-(\delta_P + \omega)t} dt - C_O \int_{t_S}^{\infty} e^{-rt} dt$$

$$V\left[R_0, t_S\right] = \frac{\tau\, p_0 q_0\, e^{-(\delta_p + \omega)t_S}}{\delta_P + \omega} - \frac{C_O\, e^{-r t_S}}{r}$$

If $t_S = 0$, then:

$$V\left(R_0\right) = \frac{\tau\, p_0 q_0}{\delta_P + \omega} - \frac{C_O}{r} \qquad \text{(AV.3)}$$

The aggregate value of all the barrels comprising a petroleum reserve given by (AV.3) is equal to the capitalized value of the net revenues, discounted at the rate ($\delta_P + \omega$), less the capitalized fixed costs, discounted at the risk-free rate. Over the life of the reserve the total quantity of barrels extracted will be $Q = q_0 / \omega$. Hence, the value of a single barrel of crude oil in the reserve, $B = V / Q$, is:

$$B(P) = \frac{\tau \omega P}{\delta_P + \omega} - \frac{C_O}{rQ} \qquad \text{(AV.4)}$$

Now consider some of the attributes of the valuation model defined by (AV.4). Since $dB / d\tau$ is positive, as royalties and taxes increase, the value of a barrel will go down. Now consider how B varies with the production decline rate, ω.

$$\frac{dB}{d\omega} = \frac{\tau P \delta_p}{(\delta_p + \omega)^2}$$

So long as $\delta_p > 0$, then $dB / d\omega$ will be positive, so as the decline rate increases, pushing more production to the front end of the time line, then B will increase. But is δ_p always greater than zero? Remember, δ_p is the convenience yield on a barrel of crude oil and is estimated from the futures curve using the "cash and carry" futures model. So, when the term structure is "backwardated", $\delta_p > 0$. But if the term structure is in "contango" then $\delta_p < 0$. In this instance, where longer-dated barrels are worth more than short-dated barrels, B will decline with increases in ω. Since ω is a portion of the reserves produced, as well as the decline rate, (AV.4) says producers facing a "contango" in the futures market should "throttle back" their wells. Doing so will decrease the capitalization rate in (AV.4), and correspondingly increase the value of the revenue stream. The limit of this strategy will be reached when $\omega = \delta_p$; after this point the "capitalized revenue" term of (AV.4) will become negative. This is not consistent with reality!

Another problem with GBM-based valuation models concerns the volatility of their returns. By applying Itô's Lemma to (AV.4) it can be shown that:

$$Var\left(\frac{dB}{B}\right) = \left[\frac{\sigma}{1 - \frac{C_O(\delta_p + \omega)}{\tau \omega r P Q}}\right]^2 dt \qquad \text{(AV.5)}$$

Consider the relationship between the volatilities of each of a barrel of subsurface petroleum, $Sd(dB / B)$ and a barrel of crude oil on the surface, σ, specified by (AV.5). The magnitude of the second term in the denominator on the RHS of (AV.5), relative to one, will determine whether $Var(dB / B)$ is greater, or less than σ^2. The capitalized fixed costs per barrel, C_0 / rQ, must be significantly less than the net price per barrel, τP, or the reserve would not have been placed on production. It follows that $0 < \left(C_0 / rQ\right) / \tau P < 1$. If $-1 < \delta_P / \omega < 2$, then the remaining portion of the denominator of (AV.5), $\left(\delta_P / \omega + 1\right)$ will be less than one. In this event, the whole of the denominator will be less than one and $Var\left(dB / B\right) > \sigma^2$, otherwise the converse applies.

Pindyck (2001) estimated that the mean and standard deviation of the three-month convenience yield for crude oil are 6.94% and 9.76%, respectively, based on samples of spot and futures prices from January 1, 1984 to January 31, 2001. Experience shows that the rate of production declines in most, but not all, petroleum reserves at annual rates between 5% and 30%, with a range of 12% to 20% being "typical". Adjusting the annual decline rate to quarterly rates shows that for "typical", but not all, estimates, the ratio δ_P / ω will lie in the interval (-1, 2). It follows that for "typical" estimates of δ_P and ω, the variance of a barrel of reserves, valued by (AV.4), will be greater than that of a spot barrel. This prediction contradicts the available evidence.

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