Real Option Analysis for Adjacent Gas Producers to Choose Optimal Operating Strategy, such as Gas Plant Size, Leasing rate, and Entry Point

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1 Introduction

In the oil and gas industry, producers often happen to own adjacent lands in which they may do production in the future. Should their lands are not next to each other, producers' decision may become very well simple. Their optimal operating strategy can be achieved by following a classical real option approach, which has been addressed by many of the precedential real option papers. However, it is worth noting that there is some difference when these producers are adjacent. The network effect resulting from the reduced toll rate charged by the pipeline company might motivate the leader(who starts investment and production first) to build a larger gas plant so that the leader can use a reasonable leasing rate to induce the follower to start production earlier without building up its own gas plant. This paper will actually discuss the the dynamics of leasing fee and the network effect and the interaction between these two factors and the optimal entry point for both the leader and the follower under some game theory consideration.

2 The assumptions of the Model

In the model, there are two gas producers A and B. They have purchased two adjacent lands for gas exploration and producing. There are two kinds of uncertainties involved. The first is the technical uncertainty, i.e. the reserve quantity in the land. We denote it as:

$$dQ_A = \mu_A(Q_A)dt + \sigma_A(Q_A)dz_A$$

$$dQ_B = \mu_B(Q_B)dt + \sigma_B(Q_B)dz_B$$

where the correlation $\rho_1 = corr(dz_A, dz_B)$.

More appropriately, we should model the Q_t as a martingale since $E_t[Q_{t+1}] = Q_t$. Thus $\mu_A(Q_A) = \mu_B(Q_B) = 0$. Once starting the production, the reserve quantity will become:

$$dQ_A = [\mu_A(Q_A) - q_A]dt + \sigma_A(Q_A)dz_A = -q_A dt + \sigma_A(Q_A)dz_A$$

$$dQ_B = [\mu_B(Q_B) - q_B]dt + \sigma_B(Q_B)dz_B = -q_B dt + \sigma_B(Q_B)dz_B$$

Model the exponentially declining production volume as

$$q_A = \alpha_A Q_A$$

$$q_B = \alpha_B Q_B$$

where the production rate is α_A, α_B. There are two constraints on the production rate. One is the technological constraint $\overline{\alpha}$, which is determined by the engineer. The other is the capacity constraint $\alpha^c = \frac{q^c}{Q}$, which is determined by the plant size. Therefore, we should restrain the production rate α as follows:

$$\alpha \leq \overline{\alpha}, \text{ if } \overline{\alpha} < \alpha^c \quad (2.1)$$

$$\alpha \leq \alpha^c, \text{ if } \alpha^c < \overline{\alpha} \quad (2.2)$$

(Notice, all the above variables α, q, Q could have subscripts A or B to stand for producer A or B.)

The second uncertainty is the economic uncertainty, i.e. the market price of gas. We denote it as

$$dP = \mu(P)dt + \sigma(P)dz_P$$

where we assume the correlation $\rho_2 = corr(dz_P, dz_A) = 0 = corr(dz_P, dz_B)$. The cost of constructing a gas plant with capacity of q_A^c and q_B^c has fixed and variable components:

$$K_A(q_A^c) = a_A + b_A q_A^c$$

$$K_B(q_B^c) = a_B + b_B q_B^c$$

where the parameters $a_A, b_A, a_B, b_B > 0$. Therefore, if only A (or B) is producing in A's (or B's) plant, $q_A \leq q_A^c$ and $q_B \leq q_B^c$.

3 The leader and follower strategy

Suppose that both producers have no gas processing facility in their land at the initial stage. If their lands are not adjacent, the problem could have been a classical real option problem. The two producers both have a call option on the gas in the land they have purchased and they will start to construct the facility at the first moment that π_t equals or exceeds the exercise value. Here, we define the cash flow as $\pi_t : \mathbb{R} \times \mathbb{R} \rightarrow \pi_t = f(P_t, Q_t)$. This will be discussed in detail in section 3. However, this paper will focus on the strategy of two producers whose lands are adjacent. In this case, one's operating decision might affect on the other's.

Basically, there are two factors need to be considered by the two producers. First, the leader could charge the follower a leasing rate if the follower wants to rent the leader's facility to process gas. Of course, the follower could choose either to rent the facility or to build his own gas plant. Second, there is a toll rate charged by the pipeline company. If the leader and the follower are both producing and transporting out gas, the pipeline company will be able to reduce the toll rate because it can distribute its cost and profit over the two customers, i.e, the leader and the follower. Keeping the above two factors in mind, the two producers will follow a symmetric, subgame perfect equilibrium entry strategies in which each producer's exercise strategy is value maximizing, while conditional upon the other's exercise strategy. [5] We can have two different exercise models: simultaneous and sequential model.

3.1 Equilibrium under simultaneous exercise: $\pi_t \geq \pi_F$

Define $\pi_F :\rightarrow \pi_F = f(P_F, Q_F)$ as the optimal trigger point of the cash flow that maximizes the follower's option value. As discussed by Grenadier (1996), in the range where $\pi_t \geq \pi_F$, if either producer begins construction at a level of π_t greater than π_F, the other will enter immediately thereafter. In this case, the equilibrium will be reached when one player enters an instant after the other, this strategy is called a simultaneous entry strategy. This simultaneous exercise strategy of two player model has already been discussed in full detail in Steven Grenadier 1996's paper "The Strategic Exercise of Options: Development cascades and Overbuilding in Real Estate Markets". We won't put more effort in it.

3.2 Equilibrium under sequential exercise: $\pi_t \leq \pi_F$

By following the logic of Proposition 2 in Grenadier 1996's paper, we can spell out the optimal strategy of the two producers in the case of $\pi_t \geq \pi_F$ Define $\pi_L :\rightarrow \pi_L = f(P_L, Q_L)$ as the optimal trigger point of the cash flow for the leader to maximize his option value. If $\pi_t < \pi_L$, one producer will wait until the trigger π_L is reached. Any entry before π_t hits π_L will lead to a value strictly less than the optimal value. However, if $\pi_t \in [\pi_L, \pi_F)$, each producer will try to build first. The first mover will win the game and start production, and the slower producer is preempted and has to wait until the trigger π_F is reached.

4 The leader and the follower's cash flow and expected payoff

This paper will focus on the sequential exercise game when $\pi_t \in [\pi_L, \pi_F)$, which means the leader has already exists. The follower is preempted from this stage and will wait until π_t rises to π_F. However, since the leader wants to take the advantage of the network effect N (coming from the reduced toll rate charged by the pipeline company), he wants to induce the follower to start production earlier. Thus, the leader builds a bigger gas plant with excess capacity and try to lease this excess capacity to the follower while charging a leasing rate l. We denote the variable production cost as C_L for the leader and C_F for the follower.

Scenario 1: $\pi_t \in [\pi_F, \infty)$: The leader and the follower build up a gas plant to process their own gas separately. The leader starts at π_L, the follower starts at π_F, which means the leader's cash flow range is $[\pi_L, \infty)$, the follower's cash flow range is $[\pi_F, \infty)$, and $\pi_L < \pi_F$. Their cash flow will be:

$$\pi_L = P_L q_L - (C_L - N) q_L \tag{4.1}$$

$$\pi_F = P_F q_F - (C_F - N) q_F \tag{4.2}$$

The expected payoff to the leader and the follower are:

$$W_L^1(\pi_L)_0 = \widehat{E}_0 \int_0^\infty e^{-rt} \pi_{Lt} dt - K_L(q_L^c) \tag{4.3}$$

$$\text{and } W_F^1(\pi_F)_0 = \widehat{E}_0 \int_0^\infty e^{-rt} \pi_{Ft} dt - K_F(q_F^c) \tag{4.4}$$

where the $\widehat{E}_0$ is the risk-neutral expectations conditional on information available at time 0. Or we can use a certainty equivalent approach to model the firm value as:

$$W_L^1(\pi_L)_0 = \sum_{t=1}^{\infty} \frac{E_0(\pi_{Lt})}{(1+r)^t} - \lambda \sum_{t=1}^{\infty} \sum_{n=0}^{t-1} \frac{b_{nL}}{(1+r)^t} - K_L(q_L^c) \tag{4.5}$$

$$W_F^1(\pi_F)_0 = \sum_{t=1}^{\infty} \frac{E_0(\pi_{Ft})}{(1+r)^t} - \lambda \sum_{t=1}^{\infty} \sum_{n=0}^{t-1} \frac{b_{nF}}{(1+r)^t} - K_F(q_F^c) \tag{4.6}$$

Scenario 2: $\pi_t \in [\pi_L, \pi_F)$
The leader builds up a bigger gas plant and lease the excess processing capacity to the follower while charging a leasing rate l. Thus the construction costs for the leader has to change accordingly in order to build a bigger gas plant which can process the amount $q_L^\Omega \geq q_L + q_F$ per unit of time.
The cash flows for the leader and the follower become:

$$\pi(l)_L = P_L q_L - (C_L - N) q_L + q_F l \tag{4.7}$$

$$\pi(l)_F = P_F q_F - (C_F - N) q_F - q_F l \tag{4.8}$$

The expected payoff to the leader and the follower are:

$$W_L^2(\pi(l)_L)_0 = \widehat{E}_0 \int_0^\infty e^{-rt} \pi(l)_{Lt} dt - K_L(q_L^\Omega) \tag{4.9}$$

$$\text{and } W_F^2(\pi(l)_F)_0 = \widehat{E}_0 \int_0^\infty e^{-rt} \pi(l)_{Ft} dt \tag{4.10}$$

We can also model the payoff using a certainty equivalent approach:

$$W_L^2(\pi_L(l))_0 = \sum_{t=1}^{\infty} \frac{E_0(\pi(l)_{Lt})}{(1+r)^t} - \lambda \sum_{t=1}^{\infty}\sum_{n=0}^{t-1} \frac{b_{nL}}{(1+r)^t} - K_L(q_L^{\Omega}) \quad (4.11)$$

$$W_F^2(\pi_F(l))_0 = \sum_{t=1}^{\infty} \frac{E_0(\pi(l)_{Ft})}{(1+r)^t} - \lambda \sum_{t=1}^{\infty}\sum_{n=0}^{t-1} \frac{b_{nF}}{(1+r)^t} \quad (4.12)$$

Scenario 3: The leader builds up a gas plant to produce its gas, but because of the leasing rate l and the construction cost $K(q_F^c)$ is too high, the follower chooses to wait until $\pi_t > \pi_F$. Therefore, the leader can not benefit from the network effect. The cash flow to the leader is:

$$\pi_L^3 = P_L q_L - C_L q_L \quad (4.13)$$

The expected payoff to the leader

$$W_L^3(\pi_L)_0 = \sum_{t=1}^{\infty} \frac{E_0(\pi_L)_t}{(1+r)^t} - \lambda \sum_{t=1}^{\infty}\sum_{n=0}^{t-1} \frac{b_{nL}}{(1+r)^t} - K_L(q_L^c) \quad (4.14)$$

5 The game on three decision variables: the gas plant size (i.e. processing capacity), the production volume and the optimal leasing rate

The next step for this paper is to consider a complete way of modeling the decision process before both producers initiating their investment. In this model, we assume the gas price is purely competitive, and the reserve quantity is random. Thus, the leader and the follower can only take the gas price as given by the market and have no control on the reserve quantity (If there exists asymmetric information about the reserve quantity, then the leader and the follower can play game on this factor). But they can play games on the gas plant size (i.e., the processing capacity), the production volume.

We first consider a simple situation in which the follower has signed a binding contract with the leader. In this contract, the leader and the follower agree that as long as the follower wants to produce, he has to rent the leader's extra capacity, but the leasing rate is negotiable. The decision variables are the plant size: q_L^Ω, q_F^c, which determine the construction costs $K_L(q_L^\Omega)$, $K_F(q_F^c)$ and the production volume: q_L, q_F. (Remember that $q_L = \alpha_L Q_L$, $q_F = \alpha_F Q_F$ and the subscript L and F stands for the leader and the follower respectively.) The parameters are the gas price: P and reserve quantity: Q_L, Q_F.

The leader's production volume is defined as $q_L(q_L^c, \overline{\alpha_L}, Q_L, P)$. q_L^c is the leader's capacity which is determined by the leader's plant size. $\overline{\alpha_L}$ is the leader's production rate which is decided by the technician based on the leader's reserve quality. Since we have assumed that the leader and the follower have signed a binding leasing contract in which the leader promises to offer the follower the processing capacity $q_{F,L}$. Thus, the follower's production volume can not exceed this offered amount, i.e. $q_F \leq q_{F,L}$. Based on above discussion, we have the following relationships:

$$
\begin{aligned}
q_{F,L} &= q_L^c - q_L && (5.1)\\
q_L &= \alpha_L Q_L && (5.2)\\
0 \leq q_L &\leq \overline{q_L} = \overline{\alpha_L} Q_L, \text{ if } \quad \overline{q_L} < q_L^c && (5.3)\\
o \leq q_L &\leq q_L^c, \qquad\qquad \text{if } \quad q_L^c < \overline{q_L} && (5.4)
\end{aligned}
$$

However, we have assumed that the leader is going to build a larger enough gas plant to process both his and the follower's gas. The leader will estimate both producers' need and build a gas plant with capacity $q_L^\Omega \geq q_L^c$. Therefore, the above relationship can be simplified as:

$$
\begin{aligned}
q_{F,L} &= q_L^\Omega - q_L && (5.5)\\
q_L &= \alpha_L Q_L && (5.6)\\
0 \leq q_L &\leq \overline{q_L}, \text{ because } \quad \overline{q_L} < q_L^\Omega && (5.7)
\end{aligned}
$$

In addition, both the leader and the follower will have to consider the effect of leasing rate, l charged by the leader and the reduced toll rate (network effect, N) charged by the pipeline company.

Assume that the follower will use up all the capacity from the lease, i.e. $q_F = q_{F,L}$, we can solve for the optimal trigger price for the follower in terms of q_F, denoted as $P_F^*(q_F, l)$. The leader's objective will be to maximize the leasing rate. Whilst, if the leader and the follower's combined production volume reaches a certain level, the pipeline company will charge them a lower toll rate, which will reduce both producers' variable production cost due to the network effect N. On the other hand, the follower also has the choice between renting the processing capacity from the leader and start producing later. This means that the leader also does not want to drive away the follower because if the follower is not producing, the pipeline company will charge a relatively higher toll rate. Therefore, the leader wants to charge the follower the highest leasing rate up to the point where the follower will start to delay its production. In other word, the leader's task is to find an optimal solution for the leasing rate l, such that it can induce the follower to start production by renting the leader's gas plant at the same time when $P_F^*(q_F, l) = P_L^*$ in order to take the advantage of the network effect N. This will give an upper bound for the leasing rate, denoted as $\widehat{l}$.

Suppose that the leader knows the follower's reserve quantity and production rate Q_F and α_F. The leader, therefore, builds a bigger gas plant which can process the amount $q_L^\Omega \geq q_L + q_F$ per unit of time. It turns out to be convenient to model present value of the leasing fee as the follower's exercise cost.

$$K = PV(q_F l) \tag{5.8}$$

For the lease to happen at the same time when the leader start production, the gas price P_t must exceed the follower's exercise hurdle $P_F^*(q_F, l)$, i.e. $P_t \geq P_F^*(q_F, l)$. For any certain level of l, we can calculate the a critical price or exercise hurdle $P_F^*(q_F, l)$. If $P_F^*(q_F, l) > P_L^*$, indicates l is too high; if $P_F^*(q_F, l) < P_L^*$, indicates l is too low; Only when $P_F^*(q_F, l) = P_L^*$, l is optimal value $\widehat{l}$ for the leader.

The value of the follower's option is the net proceeds from exercising:

$$W(P_F^*)_t = P_{F,t}^* - K \tag{5.9}$$

which is the famous value matching and smooth-pasting conditions.
The solution of the critical value for the follower is given as:

$$P_F^* = \frac{\gamma_+}{\gamma_+ - 1} K \tag{5.10}$$

where,

$$\gamma_{+,-} = \frac{1}{2} + \frac{\delta - r}{\sigma^2} \pm \sqrt{\left(\frac{1}{2} + \frac{\delta - r}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \tag{5.11}$$

To make the discussion clearer, we list the functional relationship between variables as follows (the superscript 1 and 2 stands for the scenario 1 and 2 in section 3 respectively) :

$$\pi_L^1 = \pi_L^1(P, q_L) \tag{5.12}$$
$$W_L^1 = W_L^1(\pi_L^1, q_L^c) = W_L^1(P, q_L, q_L^c) \tag{5.13}$$
$$\pi_L^2 = \pi_L^2(P, q_L, q_F, l) \tag{5.14}$$
$$W_L^2 = W_L^2(\pi_L^2, q_L^\Omega) = W_L^2(P, q_L, q_F, l, q_L^\Omega) * \tag{5.15}$$
$$\pi_F^1 = \pi_F^1(P, q_F) \tag{5.16}$$
$$W_F^1 = W_F^1(\pi_F^1, q_F^c) = W_F^1(P, q_F, q_F^c) \tag{5.17}$$
$$\pi_F^2 = \pi_L^2(P, q_F, l) \tag{5.18}$$
$$W_F^2 = W_F^2(\pi_F^2) = W_F^1(P, q_F, l) \tag{5.19}$$

The key step is the leader's decision on q_L^Ω, and l. Once the two variables are decided, we can solve for the rest. The actual results of the bargaining game will depend on the amount information either the leader or the follower occupies. We will discuss more in this aspect in next version of this paper.

6 One player model for the leader

The leader chooses the optimal time point to invest and start producing, suspend or resume the operation depending on the market price P and remaining reserve quantity Q. Define that

$$\begin{aligned} V(P, Q, m) &= \text{the leader's firm value} \\ \text{where}, m &= 0, \text{if not investing} \\ m &= 1, \text{if producing} \\ m &= 2, \text{if suspending} \end{aligned}$$

Then the free boundary conditions that L's value must satisfy when he starting the investment, suspending and resuming the production are:

$$\begin{aligned} V(P,Q,0) &= V(P,Q,1) - K(q) \\ V(P,Q,1) &= V(P,Q,2) - K_s \\ V(P,Q,2) &= V(P,Q,1) - K_r \\ \text{where, } K_s &= \text{suspend cost} \\ K_r &= \text{resume cost} \end{aligned}$$

Assume the dividend yield on the underlying asset is $\delta(P,t)$ therefore,

$$\mu(P,t) - \lambda\beta(P) = rP - \delta(P,t)$$

where

$$\beta(P) = \frac{cov(dP, df)}{\sqrt{var(dP)var(df)}}$$

and r is the risk free interest rate. Similarly, the risk-neutral drift of Q is:

$$\mu(Q) - q(m) - \lambda_Q\beta(Q) = \mu(Q) - q(m)$$

where

$$\begin{aligned} \beta(Q) &= \frac{cov(dQ, df)}{\sqrt{var(dQ)var(df)}} = 0 \\ if\ m = 0\ or\ 2, q(m) &= 0 = q(0) = q(2) \\ if\ m = 1\ , q(1) &= q \text{ when producing at less than full capacity} \\ or &= q^{max} \text{ when producing at full capacity} \end{aligned}$$

If we assuming the P and Q are not correlated, the firm value must also satisfy the following two dimensional PDEs:
when $m = 0$

$$\frac{1}{2}\left[\sigma^2(Q)V_{QQ}(P,Q,0) + \sigma^2(P)V_{PP}(P,Q,0)\right]$$

$$+V_Q(P,Q,0)\left[\mu(Q) - q(0) - \lambda_Q\beta(Q)\right]$$

$$+V_P(P,Q,0)\left[\mu_P(P) - \lambda_P\beta_P(P)\right] + V_t = rV(P,Q,0)$$

$$\Downarrow$$

$$\frac{1}{2}\left[\sigma^2(Q)V_{QQ}(P,Q,0)+\sigma^2(P)V_{PP}(P,Q,0)\right]+V_Q(P,Q,0)\mu(Q)$$
$$+V_P(P,Q,0)\left[\mu_P(P)-\lambda_P\beta_P(P)\right]+V_t=rV(P,Q,0) \quad (6.1)$$

when $m=1$

$$\frac{1}{2}\left[\sigma^2(Q)V_{QQ}(P,Q,1)+\sigma^2(P)V_{PP}(P,Q,1)\right]+V_Q(P,Q,1)\left[\mu(Q)-q(1)\right]$$
$$+V_P(P,Q,1)\left[\mu_P(P)-\lambda_P\beta_P(P)\right]+V_t+\pi=rV^1(P,Q,1) \quad (6.2)$$

$$\frac{1}{2}\left[\sigma^2(Q)V_{QQ}(P,Q,1)+\sigma^2(P)V_{PP}(P,Q,1)\right]+V_Q(P,Q,1)\left[\mu(Q)-q(1)\right]$$
$$+V_P(P,Q,1)\left[\mu_P(P)-\lambda_P\beta_P(P)\right]+V_t+\pi(l)=rV^2(P,Q,1) \quad (6.3)$$

$$\frac{1}{2}\left[\sigma^2(Q)V_{QQ}(P,Q,1)+\sigma^2(P)V_{PP}(P,Q,1)\right]+V_Q(P,Q,1)\left[\mu(Q)-q(1)\right]$$
$$+V_P(P,Q,1)\left[\mu_P(P)-\lambda_P\beta_P(P)\right]+V_t+\pi^3=rV^3(P,Q,1) \quad (6.4)$$

when $m=2$

$$\frac{1}{2}\left[\sigma^2(Q)V_{QQ}(P,Q,2)+\sigma^2(P)V_{PP}(P,Q,2)\right]$$
$$+V_Q(P,Q,2)\left[\mu(Q)-q(2)-\lambda_Q\beta(Q)\right]$$
$$+V_P(P,Q,2)\left[\mu_P(P)-\lambda_P\beta_P(P)\right]+V_t=rV(P,Q,2)$$
$$\Downarrow$$
$$\frac{1}{2}\left[\sigma^2(Q)V_{QQ}(P,Q,2)+\sigma^2(P)V_{PP}(P,Q,2)\right]+V_Q(P,Q,2)\left[\mu(Q)-q(2)\right]$$
$$+V_P(P,Q,2)\left[\mu_P(P)-\lambda_P\beta_P(P)\right]+V_t=rV(P,Q,2) \quad (6.5)$$

7 Solve the one player model

We use explicit finite difference method, Euler methods and symmetric difference. We assume simple log-normal process for both P and Q, which is:

$$dP = \mu_P dt + \sigma_P P dz_P \tag{7.1}$$
$$dQ = \mu_Q dt + \sigma_Q Q dz_Q \tag{7.2}$$

Also, assume a constant dividend yield rate δ_0 which makes $\delta(P,t) = \delta_0 P$. Thus,

$$\mu(P) - \lambda\beta(P) = \mu_P P - \delta_0 P = rP - \delta_0 P \tag{7.3}$$
$$\mu(Q) = \mu_Q Q \tag{7.4}$$
$$\sigma^2(P) = \sigma_P^2 P^2 \tag{7.5}$$
$$\sigma^2(Q) = \sigma_Q^2 Q^2 \tag{7.6}$$

Discretize

$$V_t = \frac{V(P_i, Q_j, t_k) - V(P_i, Q_j, t_{k-1})}{\Delta t} \tag{7.7}$$
$$V_P = \frac{V(P_{i+1}, Q_j, t_k) - V(P_{i-1}, Q_j, t_k)}{2\Delta P} \tag{7.8}$$
$$V_Q = \frac{V(P_i, Q_{j+1}, t_k) - V(P_i, Q_{j-1}, t_k)}{2\Delta Q} \tag{7.9}$$
$$V_{PP} = \frac{V(P_{i+1}, Q_j, t_k) + V(P_{i-1}, Q_j, t_k) - 2V(P_i, Q_j, t_k)}{(\Delta P)^2} \tag{7.10}$$
$$V_{QQ} = \frac{V(P_i, Q_{j+1}, t_k) + V(P_i, Q_{j-1}, t_k) - 2V(P_i, Q_j, t_k)}{(\Delta Q)^2} \tag{7.11}$$

Then, equation 2.1 becomes

$$\frac{1}{2}\sigma_Q^2 j^2 (\Delta Q)^2 \frac{V(P_i, Q_{j+1}, t_k) + V(P_i, Q_{j-1}, t_k) - 2V(P_i, Q_j, t_k)}{(\Delta Q)^2}$$
$$+\frac{1}{2}\sigma_P^2 i^2 (\Delta P)^2 \frac{V(P_{i+1}, Q_j, t_k) + V(P_{i-1}, Q_j, t_k) - 2V(P_i, Q_j, t_k)}{(\Delta P)^2}$$
$$+\mu_Q j \Delta Q \frac{V(P_i, Q_{j+1}, t_k) - V(P_i, Q_{j-1}, t_k)}{2\Delta Q}$$

$$+(r-\delta_0)i\Delta P\frac{V(P_{i+1},Q_j,t_k)+V(P_{i-1},Q_j,t_k)}{2\Delta P}$$

$$+\frac{V(P_i,Q_j,t_k)-V(P_i,Q_j,t_{k-1})}{\Delta t} = rV(P_i,Q_j,t_k) \quad (7.12)$$

$$\Downarrow$$

$$\frac{1}{2}\sigma_Q^2 j^2\left[V(P_i,Q_{j+1},t_k)+V(P_i,Q_{j-1},t_k)-2V(P_i,Q_j,t_k)\right]$$

$$+\frac{1}{2}\sigma_P^2 i^2\left[V(P_{i+1},Q_j,t_k)+V(P_{i-1},Q_j,t_k)-2V(P_i,Q_j,t_k)\right]$$

$$+\frac{1}{2}\mu_Q j\left[V(P_i,Q_{j+1},t_k)-V(P_i,Q_{j-1},t_k)\right]$$

$$+\frac{1}{2}(ri-\delta_0 i)\left[V(P_{i+1},Q_j,t_k)+V(P_{i-1},Q_j,t_k)\right]$$

$$+\frac{V(P_i,Q_j,t_k)-V(P_i,Q_j,t_{k-1})}{\Delta t} = rV(P_i,Q_j,t_k) \quad (7.13)$$

$$\Downarrow$$

Finally, we get the recursive formula:

$$(1-\sigma_Q^2 j^2\Delta t-\sigma_P^2 i^2\Delta t-r\Delta t)V(P_i,Q_j,t_k)+(\frac{1}{2}\sigma_Q^2 j^2\Delta t+\frac{1}{2}\mu_Q j\Delta t)V(P_i,Q_{j+1},t_k)$$

$$(\frac{1}{2}\sigma_Q^2 j^2\Delta t-\frac{1}{2}\mu_Q j\Delta t)V(P_i,Q_{j-1},t_k)+\left(\frac{1}{2}\sigma_P^2 i^2\Delta t+\frac{1}{2}(ri-\delta_0 i)\Delta t\right)V(P_{i+1},Q_j,t_k)$$

$$\left(\frac{1}{2}\sigma_P^2 i^2\Delta t-\frac{1}{2}(ri-\delta_0 i)\Delta t\right)V(P_{i-1},Q_j,t_k) = rV(P_i,Q_j,t_{k-1}) \quad (7.14)$$

There are three conditions for the input variables to ensure the stability of the solution to the PDEs, for i and j within the recursive formula

$$\begin{aligned}
(1-\sigma_Q^2 j^2\Delta t-\sigma_P^2 i^2\Delta t-r\Delta t) &\geq 0\\
(\frac{1}{2}\sigma_Q^2 j^2\Delta t-\frac{1}{2}\mu_Q j\Delta t) &\geq 0\\
\frac{1}{2}\sigma_P^2 i^2\Delta t+\frac{1}{2}(ri-\delta_0 i)\Delta t &\geq 0\\
\frac{1}{2}\sigma_P^2 i^2\Delta t-\frac{1}{2}(ri-\delta_0 i)\Delta t &\geq 0
\end{aligned}$$

$$\Downarrow$$

$$\frac{1}{\Delta t} - r \geq \sigma_P^2 i_{max}^2 + \sigma_Q^2 j_{max}^2 \tag{7.15}$$

$$j_{min} \geq \frac{\mu_Q}{\sigma_Q^2} \tag{7.16}$$

$$r \geq \delta_0 \tag{7.17}$$

$$i_{min} \geq \frac{r - \delta_0}{\sigma_P^2} \tag{7.18}$$

8 Numerically search the solution

Since it is hard to explicitly solve the PDEs which contain two stochastic variables. We use the numerical method to solve the systems of equations in Section 6 and Section 7.
The solutions are displayed in the following graphs. [1] [6] [5] [10] [8] [9] [2] [3] [7] [4]

[Figure 1 about here.]

[Figure 2 about here.]

[Figure 3 about here.]

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List of Figures

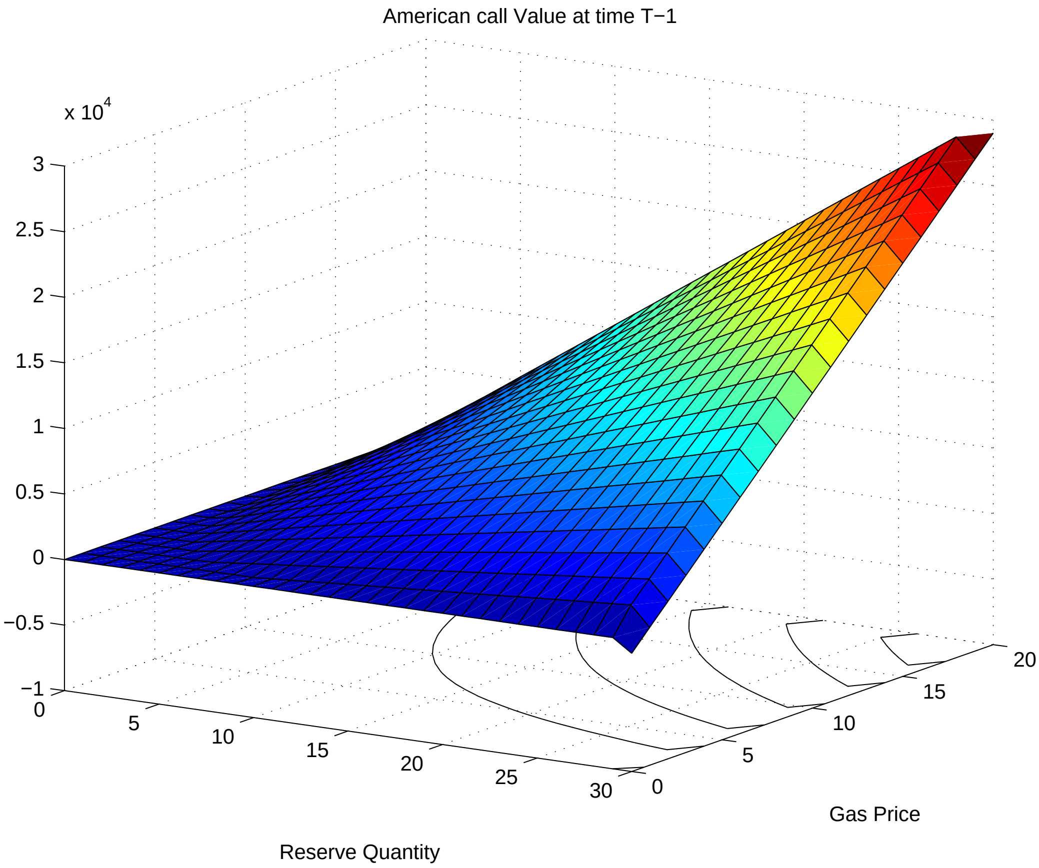


Figure 1: American Call01

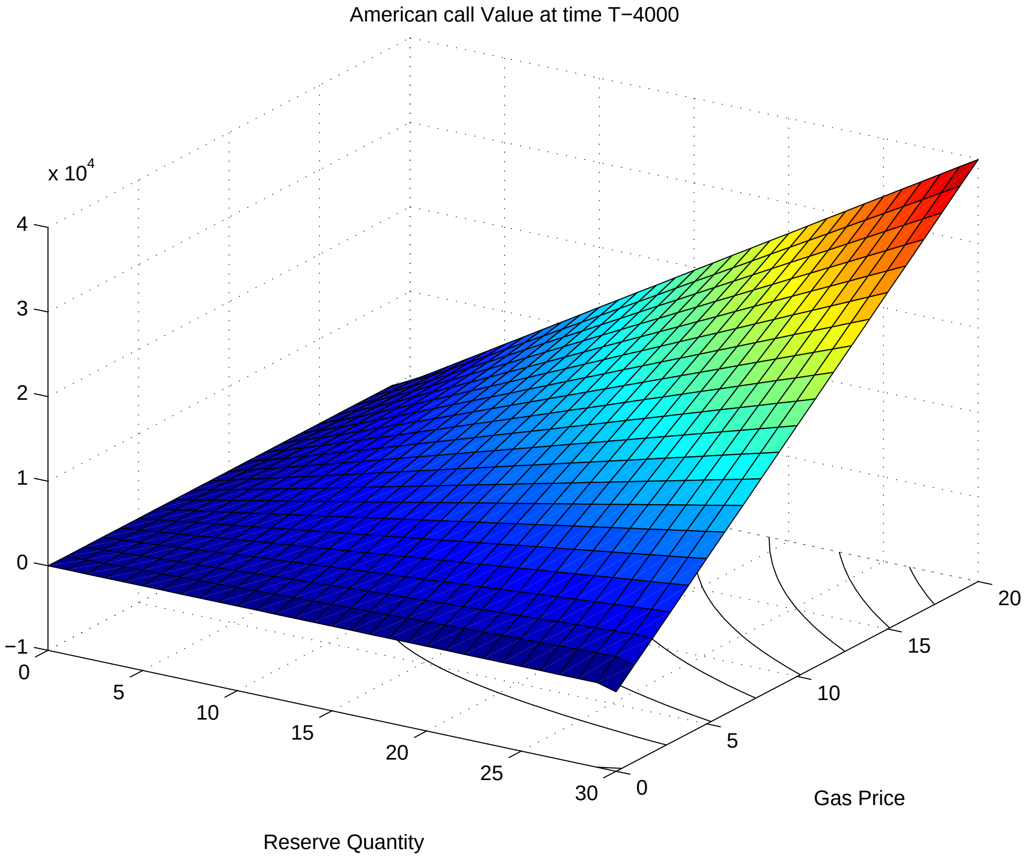


Figure 2: American Call02

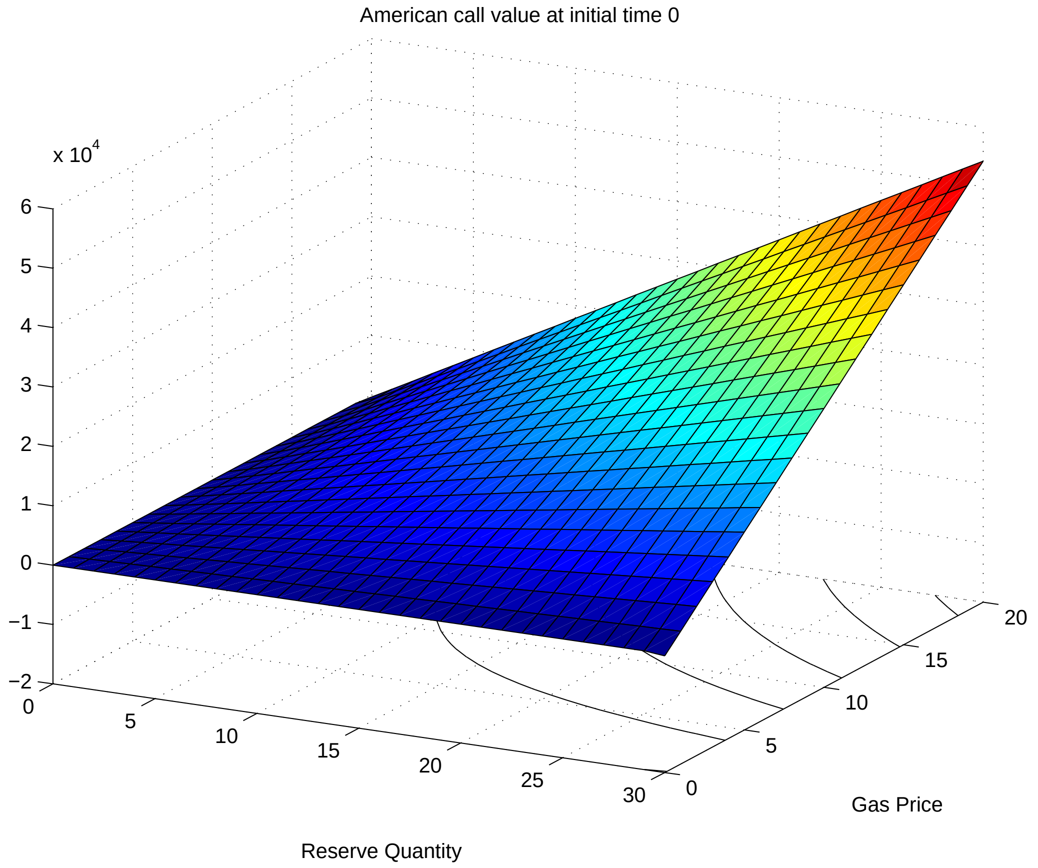
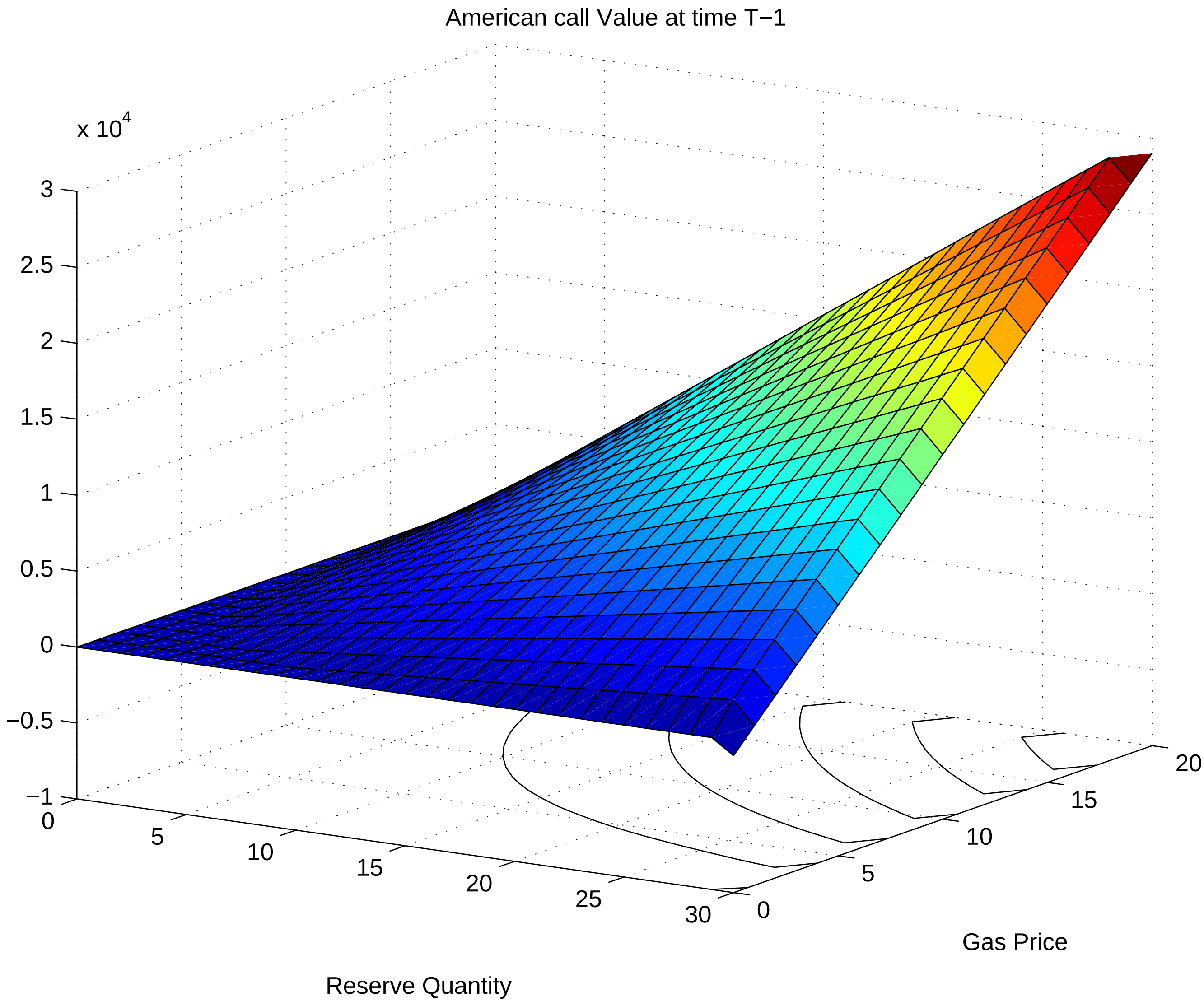
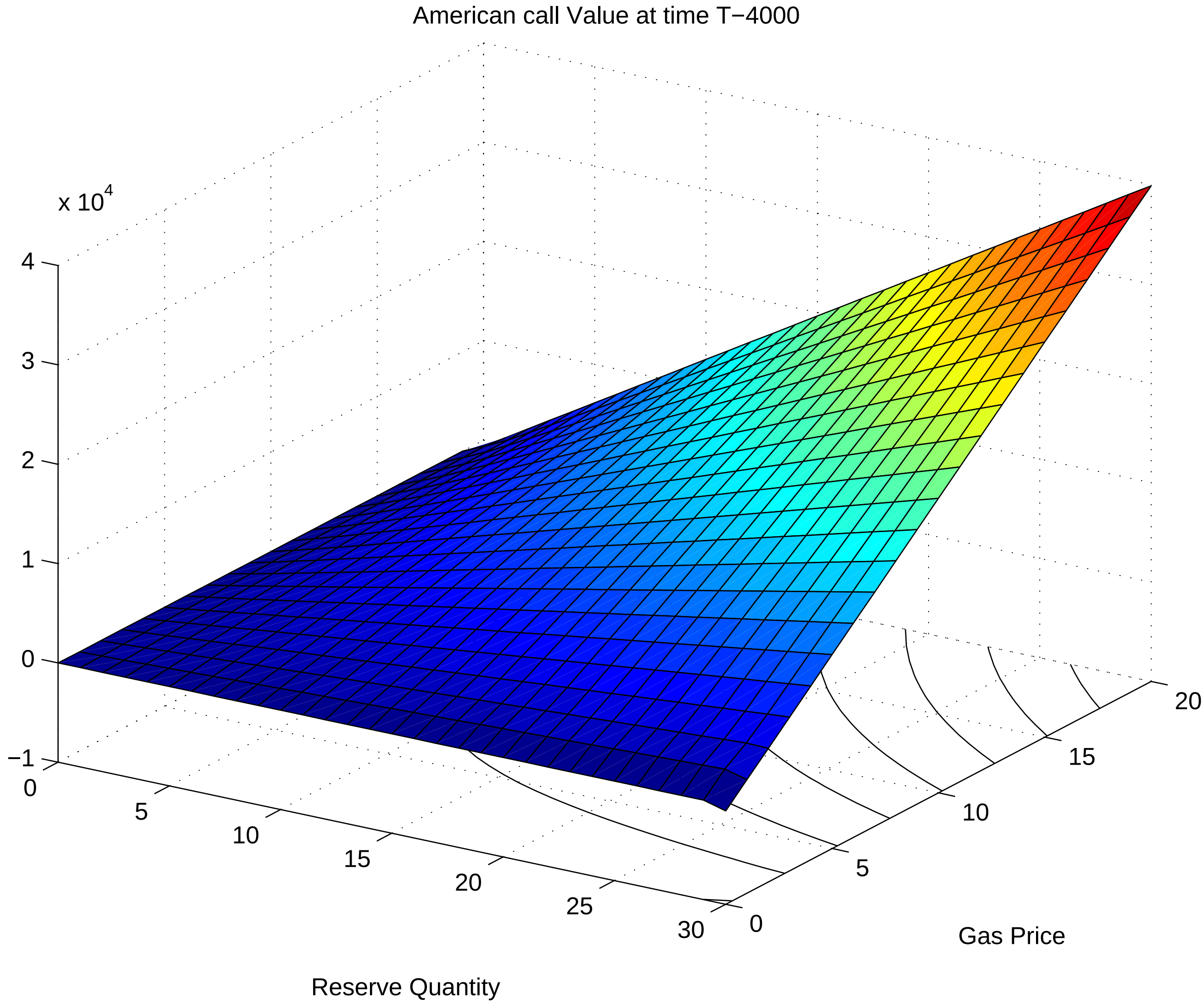
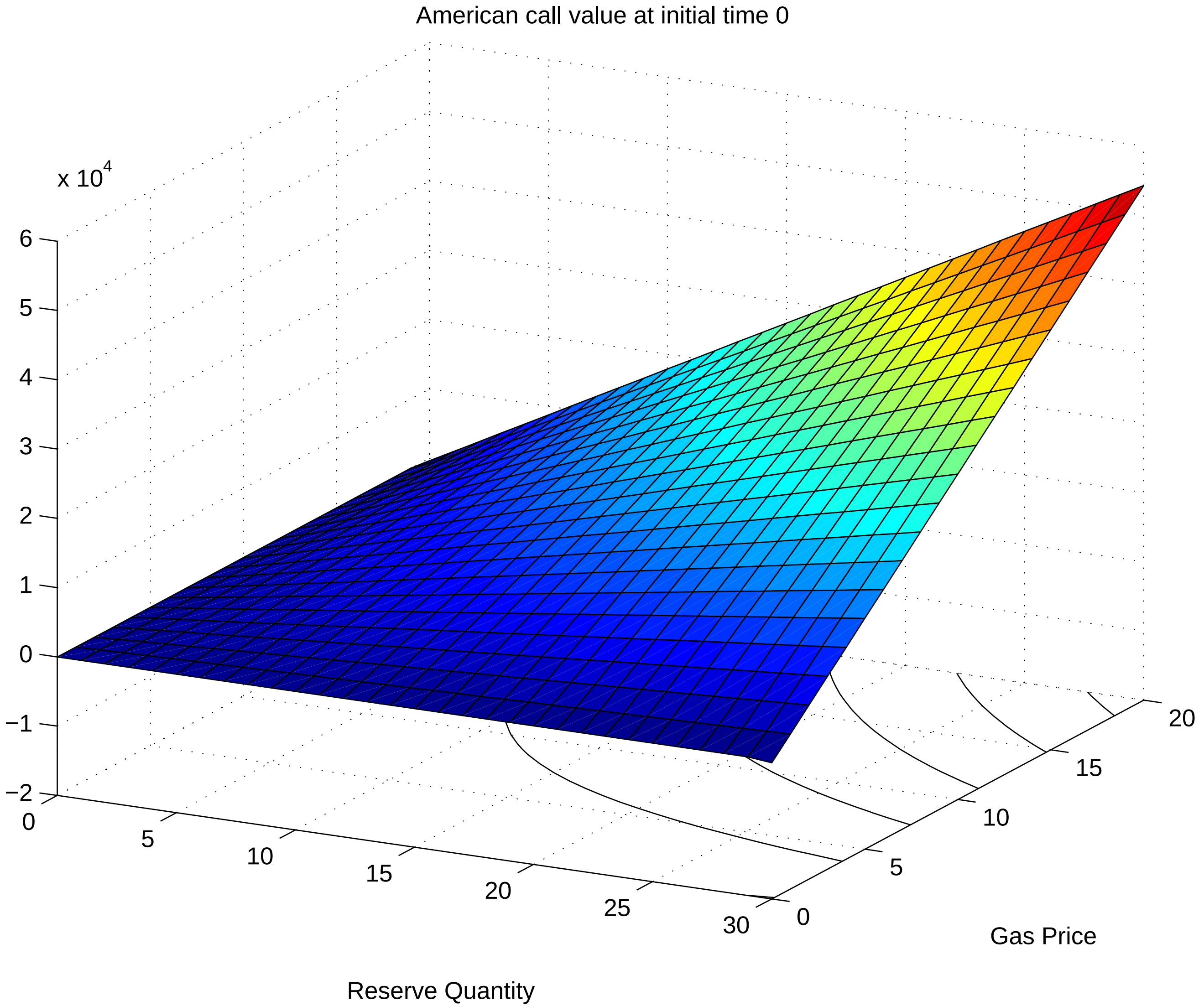


Figure 3: American Call03



American call Value at time T−1
x 10^4
3
2.5
2
1.5
1
0.5
0
−0.5
−1
0
5
10
15
20
25
30
0
5
10
15
20
Gas Price
Reserve Quantity


American call Value at time T−4000
x 10^4
4
3
2
1
0
−1
0
5
10
15
20
25
30
Reserve Quantity
0
5
10
15
20
Gas Price


American call value at initial time 0
x 10^4
6
5
4
3
2
1
0
−1
−2
0
5
10
15
20
25
30
Reserve Quantity
0
5
10
15
20
Gas Price