The Curious Incident of the Investment in the Market [†]

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Abstract

"Is there any point to which you would wish to draw my attention?" "To the curious incident of the investment in the market." "The agent did nothing in the market." "That was the curious incident." (with apologies to Sir Arthur Conan-Doyle.)

In this paper we study an optimal timing problem for the sale of a non-traded real asset. We solve this problem for a utility maximizing, risk averse manager under two scenarios: firstly when the manager has access to no other investment opportunities, and secondly when they may also invest in a continuously traded financial asset. We construct the model such that the financial asset is uncorrelated with the real asset, so that it cannot be used for hedging the real asset, and such that the financial asset has zero risk premium. In the absence of the real asset, the manager would not include the financial asset in her optimal portfolio.

Although the problem is designed such that naive intuition would imply that the optimal strategy and value functions are the same irrespective of whether the manager is allowed to invest in the financial asset, we find that curiously, for certain parameter values this is not the case. Our work has implications for modeling of portfolio choice problems since seemingly extraneous assets can impact on optimal behavior.

Keywords and Phrases: Real assets, Perpetual options, Optimal stopping, Incomplete markets, Portfolio choice, Real options, Portfolio constraints

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1 Introduction

In this paper we consider the problem facing a risk-averse manager who owns the rights to a real asset, and who wishes to choose the optimal time to sell this asset in order to maximize her expected utility of wealth. Our aim is to take the simplest possible modeling formulation for this problem (CRRA utility, perfect information, constant parameter lognormal dynamics, zero interest rates and an infinite horizon) for which we can obtain explicit solutions for the manager's value function and the optimal selling rule. The reason for this choice is that our interest is in considering what happens when we embed this simple problem within a model in which there is a financial market in which the manager may make additional investments.

We design this market in such a way that it seems unnatural for the manager to be able to take advantage of any of these additional investment opportunities, either for hedging the real asset, or as investments in their own right. Unexpectedly, and most remarkably, we find that for some parameter values the solution of the optimal selling problem is not robust to embedding the problem in the more general setting. The fact that this phenomena exists in such a simple setting means that great care must be taken in interpreting the optimal solution to many classes of optimal control problems.

The general setting of our problem is in the field of real assets, see Vollert [15] for a recent summary and also the classic text of Dixit and Pindyck [3]. The most natural interpretation is to consider a manager who has ownership of the real asset, for example a parcel of land, a patent right, or a family firm, and who wishes to choose the optimal time to sell this asset or disinvest. A key feature is that the manager is not able to trade the real asset in a market, although it has a well defined price which is the amount the manager would receive for selling it immediately. A second key feature is that the real asset is indivisible so that the manager faces a single irreversible decision to sell.

In essence our problem is one of pricing real assets in an incomplete market. Here, following Henderson and Hobson [6] and Musiela and Zariphopoulou [13] it is assumed that the real asset is not traded but there may be a second asset which is traded. (Kahl et al [9] also) Both these assets are driven by exponential Brownian motions which may be correlated. Both [6] and [13] consider the European option case, but the problem here also contains an optimal timing problem and hence shares features with American option problems and especially perpetual American options.

Dixit and Pindyck [3] were amongst the first to consider perpetual real options. However they assumed the real asset was spanned by the market so that the model is complete and the risk preferences are irrelevant. McDonald and Siegel [10] generalized this setting to allow for an incomplete market. However they assumed that the agent needed compensating only for systematic or market risk, and that she was risk-neutral to the idiosyncratic risk. In order to provide a full solution of this problem it is necessary to include the risk preferences of the manager explicitly, and this is the direction taken by Smith and Nau [14], Davis and Zariphopoulou [1] and Henderson [5] each of whom consider option pricing in an incomplete market for agents with exponential utility. Henderson [5] considers a perpetual, incomplete market, American option pricing problem very much like the one we consider here. In a more general setting than ours, Henderson is able to derive the price of call options on the real asset. However in her calculations she makes use of special scaling properties of the exponential utility and, as we shall see, the key phenomenon which interests us here does not occur for exponential utility.

Recall that in this article we study the optimal time to sell a real asset. The key parameters of the problem are the co-efficient of relative risk aversion and the risk premium and volatility of the price process for the real asset. If the risk premium is negative so that the value of the real asset will decrease on average, then it is optimal for the agent to disinvest immediately. Conversely, if the risk premium is strong and positive, then it is never optimal to sell the real asset. The interesting case is when the risk premium is small and positive. In that case the expected growth in future value gives the manager an incentive to continue to hold the real asset but this is counterbalanced by an incentive to sell in order to reduce uncertainty and risk.

We treat the above disinvestment problem in two situations. In the first situation the manager aims to maximize expected utility of wealth where her wealth consists of a fixed initial wealth plus the wealth implicit in holding the real asset. This problem is entirely classical in structure and can be solved in many ways. We postulate the optimal strategy and find the associated value function, at which point it is possible to verify that indeed we have the optimal solution.

Then we extend the problem by embedding it within a slightly larger model. In this model, in addition to the real asset there is a second financial asset, and we assume that the manger is free to invest in this second asset. The market for the financial asset is assumed to be complete and frictionless. There are two immediate reasons why the manager may choose to invest in the financial asset. The first reason is that, if this asset is correlated to the real asset, then the manager could use it to hedge her exposure to the risk implicit in continuing to invest in the real asset. That is, the market risk could be hedged. The second reason is that if the financial asset has non-zero risk premium then it is optimal to hold some units of the asset in order to maximize the rate of growth of expected utility.

In order to remove both of these motivations we assume that the market asset is uncorrelated with the real asset and that it has zero Sharpe ratio. Then it is natural to suppose that the presence of the market is immaterial and the solution for the value problem with the market is the same as that without the market. Unfortunately, as we highlight in Theorem 3.4, this intuition is misguided, and for some parameter values the manager can take advantage of the market to generate greater utility. The problem turns out to be more subtle than expected, and we give a deeper explanation for our results in a discussion section following the main results.

The optimal stopping problems we consider in this paper are sufficiently simple that it is possible to derive explicit solutions, including characterizations of the optimal strategies and expressions for the value functions. However, despite this apparent simplicity there are some counter-intuitive results. In particular, these results urge caution in modeling any optimal stopping problem in isolation. The simple act of omitting uncorrelated assets from a model may have a radical impact upon the form of the solution to a problem.

Modeling of exercise decisions has received much attention recently in the executive stock option literature. Many models in this area (cite ?) value the option to the executive in isolation. Recent papers including those of Ingersoll [7], and also Detemple and Sundaresan [2] have included a market asset in the modeling the exercise decision of an executive. They find ?

In addition to this, the results of our paper show that the omission of market can have an impact on the

2 The Problem with No Market

In this section we consider the optimal behavior of a risk-averse, utility-maximizing manager who has a single unit of an indivisible asset to sell over an infinite horizon, but who does not have access to any other investment opportunities. In order to keep the formulation as simple as possible we assume that interest rates are zero, or equivalently that utility is measured with respect to discounted wealth.

The problem facing the manager is to choose the sale time τ to maximize the expected utility of total wealth, or in other words to find

$$V^N(x,y) = \sup_{\tau} \mathbb{E}[U(x+Y_\tau)|Y_0 = y], \tag{1}$$

where U is logarithmic utility $U(x) = \ln x$. The superscript N on the value function refers to the fact that we are working in the No-market setting. Logarithmic utility is the special case with unit risk aversion of a constant relative risk aversion utility, and we treat the general case in Appendix B. In this section and the next we use logarithmic utility since the resulting formula are slightly more compact.

In (1) the quantity x is the initial wealth, and Y_t is the price process of the indivisible asset. It is assumed that Y_t is known at time t, so that the problem facing the manager is to choose the optimal disinvestment (or sale) time τ. The price process Y_t is assumed to satisfy constant parameter log-normal dynamics

$$dY_t = Y_t(\sigma dW_t + \mu dt) \tag{2}$$

with volatility σ and drift μ. Since we assume zero interest rates, μ is also the risk premium on Y. This equation has solution $Y_t = y\exp\{\sigma W_t + (\mu - \sigma^2/2)t\}$, where $Y_0 = y$. Note that if $\mu > \sigma^2/2$ then Y_t grows to ∞ whereas, if $\mu < \sigma^2/2$, then Y_t tends to zero, almost surely.

We first discuss the analogous problem if Y were a traded asset. In that case, the manager is facing a complete market and he simply maximizes $\mathbb{E}Y_\tau$ over possible selling times. In such a risk-neutral set-up, the problem is degenerate. The manager either waits indefinitely when Y has positive risk premium or sells immediately if the risk premium is negative. Since the manager is risk-neutral, he simply considers whether (discounted) Y is growing or not. This can also be restated in terms of the parameter $\gamma = 2\mu/\sigma^2$. In the risk-neutral situation, the asset is sold immediately if $\gamma < 0$ and never sold if $\gamma > 0$.

Now we return to the incomplete situation where Y is not traded. It will turn out that since the manager is risk-averse, he will require a greater value of γ to hold onto Y indefinitely, this will be $\gamma > 1$. Recall above that $\gamma = 1$ differentiates between cases where Y tends to infinity or zero. For positive γ the manager's risk aversion means it may be optimal to reduce uncertainty by selling even though the asset has positive risk premium. This will depend on how large Y is compared with wealth. The larger this proportion, the greater the incentive to disinvest.

The problem in this section is very close to that studied in Kadam et al [8]. There are however some differences which make it difficult to compare directly. Their paper assumes $\gamma > 1$ (which in our setup leads to holding onto Y indefinitely) and counterbalances this with an extra subjective discounting factor to artificially encourage sale of the asset.[1]

The problem without a market is amenable to entirely classical optimal stopping techniques. However, we provide an outline of the solution to the problem, to facilitate comparison with the problem we study in the next section. Given the structure and scalings of the problem it is plausible that the optimal stopping rule is of the form to sell the real asset the first time Y_t exceeds a critical value y^*. Further, since initial wealth x is constant we are free to rescale the stopping rule so that it becomes the first time that the ratio of the indivisible asset price to wealth exceeds some level z. With this in mind we consider stopping times of the form

$$\tau_z = \inf_u \{Y_u \geq xz\}.$$

Under this strategy, and assuming $y < xz$, the expected utility of the manager for any level z, is given by

$$F(z) = \mathbb{E}[\ln(x + Y_{\tau_z})] = \mathbb{P}(\tau_z = \infty)\ln x + \mathbb{P}(\tau_z < \infty)\ln(x + xz) = \ln x + \ln(1+z)\mathbb{P}(\tau_z < \infty)$$

Now we maximize over choice of level z. For $\gamma \geq 1$ the real asset grows without bound and $\mathbb{P}(\tau_z < \infty) = 1$, and it is clear in this case that F is maximized over z by taking $z = \infty$. That is, the manager should wait forever and not sell the asset. For $\gamma < 1$ we have (see for example Durrett [4, p359])

$$\mathbb{P}(\tau_z < \infty) = \left(\frac{y}{zx}\right)^{1-\gamma}. \tag{3}$$

In that case the derivative of F becomes

$$F'(z) = \left(\frac{y}{x}\right)^{1-\gamma} z^{\gamma-2}\left[\frac{z}{1+z} - (1-\gamma)\ln(1+z)\right],$$

and since, for $z > 0$, $\ln(1+z) > \frac{z}{(1+z)}$ it follows that for $\gamma \leq 0$ we have $F'(z) \leq 0$, and F is maximized by the choice $z = 0$. In this case the manager sells immediately. The interesting case is when $0 < \gamma < 1$. In that case F is maximized by the unique solution z^* in $(0, \infty)$ to $F'(z) = 0$. Putting these observations together gives

[1] Kadam et al [8] also assume an initial wealth of zero, and hence they restrict to relative risk aversion less than one to avoid utility of negative infinity. We do not need this restriction in the Appendix where we treat general power utilities.

Proposition 2.1 *For $\gamma \leq 0$, $V^N(x,y) = \ln(x+y)$, and for $\gamma \geq 1$, $V^N(x,y) = \infty$.*

In the non-degenerate case $0 < \gamma < 1$, in the exercise region $y \geq xz^*$ we have $V^N(x,y) = \ln(x+y)$, and in the continuation region $y < xz^*$

$$V^N(x,y) = \ln x + \left(\frac{y}{xz^*}\right)^{1-\gamma} \ln(1+z^*). \tag{4}$$

The optimal exercise ratio $z^* = z^*(\gamma)$ is the unique solution in $(0,\infty)$ to

$$(1-\gamma)\ln(1+z) - \frac{z}{1+z} = 0 \tag{5}$$

Proof: A formal proof of the proposition follows on checking that V^N is a supermartingale under any stopping rule, and a true martingale under the stopping rule τ_{z^*}. Given the formulæabove this is a simple exercise in Itô calculus. This proof also verifies that the optimal stopping rule is of the form presupposed above, namely to sell the indivisible asset the first time its price process exceeds xz^*. $\square$

It follows immediately from (5) that the optimal exercise ratio z^* is increasing in γ. Thus, as the risk premium of the real asset increases, the real asset becomes worth more to the manager and he waits longer before selling. Indeed, given the explicit form for the value function, the certainty equivalent value p^N for the real asset Y is given as the solution to $\ln(x + p^N) = V^N(x,y)$. The value p^N is the certain amount the manager would accept in place of the right to sell the real asset. If $\gamma \leq 0$ then $p^N = y$ and if $\gamma \geq 1$ then $p^N = \infty$. Otherwise, for $\gamma \in (0,1)$, the critical exercise ratio exists in $(0,\infty)$ and for $y \geq xz^*$ we have $p^N = y$, and in the continuation region

$$p^N = x\left((1+z^*)^{(y/xz^*)^{1-\gamma}} - 1\right), \qquad y < xz^*. \tag{6}$$

It is clear that this formula is increasing in y, and that (for $y < xz^*$) $p^N < xz^*$. A few lines of algebra also shows that p^N given by (6) satisfies $p^N > y$. That is, the value of the right to sell is greater than it's current value because the asset has a positive risk premium.

Note that in our problem the manager has an option to sell the real asset for strike zero. In the real options complete market setting of Dixit and Pindyck [3], and in the real options risk-neutral (to idiosyncratic risk) setting of McDonald and Siegel [10] the problem of maximizing expected wealth is degenerate since the solution is either to sell instantly (when the real asset has negative risk premium) or never to sell (when the real asset has positive risk premium). It is the combination of a positive risk premium with risk aversion which gives us a non-degenerate problem. In our problem, the manager is risk averse and requires more than a positive risk premium to hold the right to sell indefinitely.

3 The Problem Embedded Within a Market

In this section we revisit the optimal disinvestment problem in a more general setting in which the manager has alternative investment opportunities. For example there may be a financial asset which the manager may trade freely. One very interesting case is when this market asset is strongly correlated to the non-traded asset. However, in this paper we are interested in the opposite situation in which the financial asset is uncorrelated with the non-traded asset.

The problem which the manager now faces is to maximize the expected utility of total wealth

$$V^M(x,y) = \sup_{\tau} \sup_{X_\tau} \mathbb{E}[U(X_\tau + Y_\tau)|X_0 = x, Y_0 = y], \tag{7}$$

for logarithmic utility, where X_t is the manager's trading wealth (from investment in the financial asset) and, as before, Y_t is the price process of the indivisible, non-traded asset and τ is a stopping time chosen by the manager. The trading wealth X is assumed to be non-negative and self-financing.

If the financial asset were correlated to the non-traded asset then there would be a hedging motive for trading in the financial asset. This trading would allow the manager to offset some of the risk associated with maintaining investment in the asset Y. Such hedging motives arise in many portfolio choice problems including those with stochastic income, etc, see ?

However, since we assume that the financial asset is uncorrelated with Y, the manager has no incentive for hedging in the market. Furthermore, we suppose that the financial asset has a Sharpe-ratio of zero. Temporarily ignoring the the presence of Y, if the manager were to solve the classical Merton problem [11, 12] over any fixed horizon T, then her optimal portfolio would be to invest nothing in the (risky) financial asset. Thus, in the case of zero correlation and zero Sharpe ratio it seems natural to conjecture that embedding the problem of Section 2 in a larger market would have no effect, and the value function and optimal exercise strategy would be unchanged. However this is *not* the case as we now seek to demonstrate.

Fix $0 < \alpha < \beta \le \infty$ and define a combined investment/exercise strategy as follows. If $Y_t \ge \beta X_t$ then exercise immediately. If $Y_t \le \alpha X_t$ invest nothing in the financial asset, and wait until the first time that $Y_t = \alpha X_t$ if ever. Finally, if $\alpha X_t < Y_t < \beta X_t$, then take a large position in X_t. Since the financial asset has zero Sharpe-ratio it is a martingale under $\mathbb{P}$, and this property is inherited by X. In the limit of larger and larger positions the trading wealth X_t leaves the region $[Y_t/\beta, Y_t/\alpha]$ immediately, with the probabilities of leaving either end determined by the martingale property. If we leave at the left-hand endpoint then $Y_t = \beta X_t$ and we sell the asset Y. Conversely, if we leave at the right-hand endpoint then $Y_t = \alpha X_t$ and we cease investment in X and wait for changes in net wealth Y/X through fluctuations in the value of Y_t.

In the limiting case $\alpha = \beta = z$ we recover the strategies of the previous section.

Proposition 3.1 *Under the above strategy, specified by the thresholds (α, β), the value function is given by*

$$V^M(x,y) = \begin{cases} \ln(x+y) & y \ge \beta x \\ G(x,y) & \alpha x \le y < \beta x \\ H(x,y) & y < \alpha x \end{cases}$$

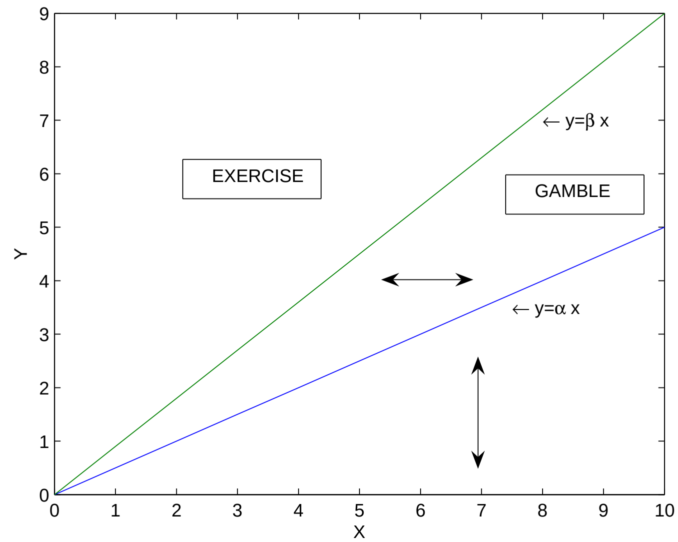


Figure 1:

where

$$G(x,y) = \frac{\alpha}{\beta-\alpha}\left(\frac{\beta x}{y}-1\right)(\ln y - \ln\alpha + \Theta) + \frac{\beta}{\beta-\alpha}\left(1-\frac{\alpha x}{y}\right)(\ln y + \ln(1+1/\beta))$$

and

$$H(x,y) = \ln x + \left(\frac{y}{\alpha x}\right)^{1-\gamma}\Theta$$

with

$$\Theta \equiv \Theta(\alpha,\beta) = \left[\frac{\beta-\alpha+\beta\left(\ln\alpha+\ln(1+1/\beta)\right)}{\beta+(\beta-\alpha)(1-\gamma)}\right]. \tag{8}$$

Proof: The proof of all the results in this section are contained in Appendix A.

It is clear from the expression for H that in order to maximize the value function it is necessary to choose α and β to maximize $\alpha^{\gamma-1}\Theta$. It turns out that for $\gamma \in (0,1)$ there are three distinct ranges of values of γ at which the character of the maxima are different.

Lemma 3.2 *Let γ_- be the unique solution in $(0,1)$ of $\Gamma_-(\gamma)=0$ where*

$$\Gamma_-(\gamma) = (1-\gamma)(2-\gamma)\ln\left(\frac{2-\gamma}{1-\gamma}\right) - 1,$$

and let γ^+ be the unique solution in $(0,1)$ of $\Gamma^+(\gamma)=0$ where

$$\Gamma^+(\gamma) = \frac{(1-\gamma)}{\gamma}\ln\left(\frac{2-\gamma}{1-\gamma}\right) - 1.$$

Then $\gamma_- \sim 0.3492$ and $\gamma^+ \sim 0.5341$.

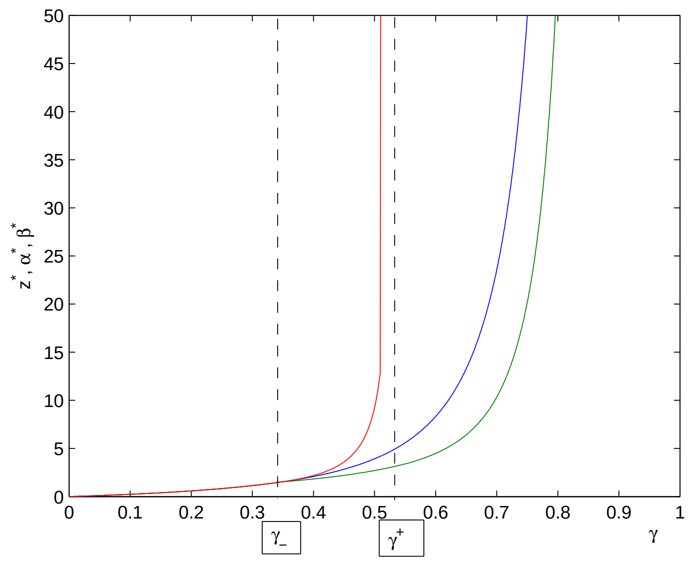


Figure 2:

Consider now the problem of finding the maximum of $\alpha^{\gamma-1}\Theta$ over $\alpha \geq 0$ and $\beta \geq \alpha$.
For $0 < \gamma \leq \gamma_-$ the maximum is attained at $\alpha = z^*$, $\beta = z^*$ where $z^* = z^*(\gamma)$ is the solution to (5).
For $\gamma_- \leq \gamma \leq \gamma^+$ the maximum is attained at $\alpha = \alpha^*$ where

$$\alpha^* = \alpha^*(\gamma) = \frac{2-\gamma}{1-\gamma}\left[\frac{1}{1-\gamma} - \ln\left(\frac{2-\gamma}{1-\gamma}\right)\right] \tag{9}$$

and $\beta = \beta^*$ where

$$\beta^* = \beta^*(\gamma) = \frac{1-(1-\gamma)\ln((2-\gamma)/(1-\gamma))}{(1-\gamma)\ln((2-\gamma)/(1-\gamma)) - \gamma}. \tag{10}$$

Finally, for $\gamma^+ \leq \gamma < 1$ the maximum is attained at $\beta = \infty$ and $\alpha = \alpha^*$ where

$$\alpha^* = \alpha^*(\gamma) = e^{\gamma/(1-\gamma)}.$$

Now we can state the solution to the optimal stopping problem (7). For $\gamma \leq 0$ the optimal stopping rule is to exercise immediately, and for $\gamma \geq 1$, there is no optimal stopping rule (in the sense any candidate stopping rule which is finite can be improved upon by waiting longer). These results are exactly as in the no-market case, and so attention switches to the case $0 < \gamma < 1$. The content of the next proposition is that for γ in this range the optimal stopping rule is of the form described before Proposition 3.1 where the values of α and β are chosen to maximize Θ.

Proposition 3.3 *(i) For $\gamma \leq 0$, $V^M(x,y) = \ln(x+y)$.*
(ii) For $0 < \gamma \leq \gamma_-$ the value function is given by $V^M(x,y) = \ln(x+y)$ in the exercise region $y \geq xz^*$,

and in the continuation region $y < xz^*$

$$V^M(x,y) = \ln x + \left(\frac{y}{xz^*}\right)^{1-\gamma} \ln(1+z^*)$$

where z^* solves (5).

(iii) For $\gamma_- < \gamma < \gamma_+$ the value function is given by, in the exercise region $y \geq x\beta^*(\gamma)$, $V^M(x,y) = \ln(x+y)$, for $x\alpha^*(\gamma) < y < x\beta^*(\gamma)$,

$$V^M(x,y) = \frac{\alpha^*}{\beta^* - \alpha^*}\left(\frac{\beta^* x}{y} - 1\right)(\ln y - \ln\alpha^* + \Theta^*) + \frac{\beta^*}{\beta^* - \alpha^*}\left(1 - \frac{\alpha^* x}{y}\right)(\ln y + \ln(1 + 1/\beta^*))$$

and for $y \leq x\alpha^*(\gamma)$

$$V^M(x,y) = \ln x + \left(\frac{y}{\alpha^* x}\right)^{1-\gamma} \Theta^*$$

where α^* and β^* are given by (9) and (10) and $\Theta^* = \Theta(\alpha^*, \beta^*)$.

(iv) For $\gamma_+ \leq \gamma < 1$ the value function is given by, for $xe^{\gamma/(1-\gamma)} < y$

$$V^M(x,y) = \frac{e^{\gamma/(1-\gamma)}x}{y}\left(\ln y + \frac{1-\gamma}{2-\gamma}\right) + \left(1 - \frac{e^{\gamma/(1-\gamma)}x}{y}\right)\ln y$$

and for $y \leq xe^{\gamma/(1-\gamma)}$

$$V^M(x,y) = \ln x + \left(\frac{y}{x}\right)^{1-\gamma} \frac{e^{-\gamma}}{(2-\gamma)(1-\gamma)}$$

(v) For $\gamma \geq 1$ the value function is given by $V^M(x,y) = \infty$.

Given the value function V^M it is possible to define the manager's certainty equivalent value p^M for the real asset via $p^M = e^{V^M(x,y)} - x$ and we return to this later. However, the precise forms of the value function for the problem stated in (7) are less interesting than how they relate to the value function for the corresponding problem (1) with no market.

Theorem 3.4 *For $\gamma \leq \gamma_-$ and $\gamma \geq 1$ we have that $V^N(x,y) = V^M(x,y)$ and the solution with the market is identical to the solution of the optimal stopping problem with no market given in Section 2.*

For $\gamma_- < \gamma < 1$ we have that $V^N(x,y) < V^M(x,y)$ and the solution to the optimal stopping problem (7) is different to the solution of (1).

In particular, for $\gamma_- < \gamma < 1$, even though the market asset is uncorrelated with Y and serves no purpose in hedging the risks associated with the real asset, and the market asset has zero Sharpe-ratio so that the optimal solution of the Merton problem is to invest nothing in the financial asset, the manager is able to take advantage of the market to improve her expected utility. Further, for $\gamma^+ < \gamma$ the manager without the market investment opportunities will always sell the real asset at some finite ratio, whereas it is never optimal for the manager with the additional market opportunities to sell the real asset.

We delay the interpretation and intuitive explanation of this result until after we have discussed a discrete-time example.

4 A One-period Binomial Model

In this section we show how the same phenomena we described above in the continuous model can arise in a stylized, single period, binomial model. The continuous time models have a much richer behavior and it is much easier to give an interpretation to the key parameters, but it remains the case that in a binomial model a manager with access to a financial market can outperform a manager with no access to outside opportunities.

Consider a manager with initial wealth x and with ownership of a real asset with current price y. After one time-period the price of the real asset will change to either yu or yd where $u > d$. We assume that these two events have equal probability, and this yields a significant simplification of the subsequent formulas. We suppose that $u + d > 2$, so that in expectation the price of the real asset is rising, but $ud < 1$ so that the expected logarithm of the price is falling.

The problem facing the manager in the no-market situation is either to sell the asset immediately at time zero, or to wait until time 1. We suppose the manager has logarithmic utility. In the first case the utility of the agent is $\ln(x + y)$ whereas in the second case the utility is $(\ln(x + yu) + \ln(x + yd))/2$. It turns out that it is optimal for the manager to wait until time 1 to sell the real asset provided $y < xz^*$ where z^* is the solution to

$$(1 + z^*)^2 = (1 + z^*u)(1 + z^*d) \tag{11}$$

or equivalently, $z^* = (u + d - 2)/(1 - ud)$. Note that the parameters u and d have been chosen to ensure that (11) has a unique positive solution. This choice of parameters is similar to the choice $0 < \gamma < 1$ in previous sections.

Now suppose that at time 0, but before the manager decides whether to sell the real asset, the manager is offered the chance to make a small bet of size ϵ which results in net gains of $\pm\epsilon$ with probability one-half in each case. She may postpone the decision about selling the real asset until after the outcome of this bet is known. The outcome of the bet is independent of the behavior of the real asset. If the manager did not own the real asset then the concavity of her utility function would mean that she would choose not to accept the bet, further since the outcome is uncorrelated with the behavior of the real asset the bet cannot be used to hedge the fluctuations in the price of the real asset. However we will show that there are circumstances under which it is optimal for the manager to accept the gamble.

Suppose the initial value of the real asset is $y = xz^*$, and that the manager chooses to accept the bet. If the bet is successful then the value of the real asset $y = xz^*$ is below the exercise ratio threshold $x(1 + \epsilon)z^*$, and she will choose to keep the real asset until time one, otherwise, if the bet is unsuccessful, she will sell the real asset at time 0. Her expected utility is

$$\frac{1}{4}\left\{\ln(x + \epsilon + xz^*u) + \ln(x + \epsilon + xz^*d)\right\} + \frac{1}{2}\ln(x - \epsilon + xz^*).$$

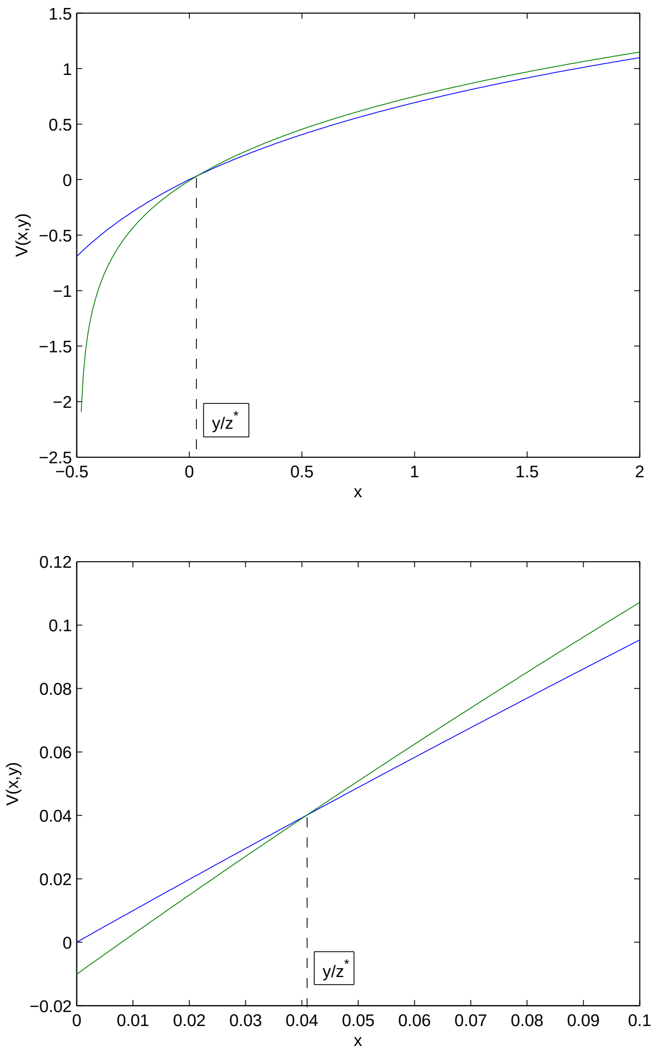


Figure 3: *The two graphs plot the value functions in the no-market case against initial wealth x, with $y = 1$. In the top panel,* $\ln(x + y)$ *and () are plotted, with their intersection at $y/z^* = 0.041$. Parameter values are $u = 2.0, d = 0.49$. The lower panel zooms in on the two functions to focus on the point where the manager is indifferent between selling and waiting.*

Writing $\epsilon = x\tilde{\epsilon}$ and using (11) this becomes

$$\ln x + \ln(1+z^*) + \frac{1}{4}\left\{\ln\left(1+\frac{\tilde{\epsilon}}{1+z^*u}\right) + \ln\left(1+\frac{\tilde{\epsilon}}{1+z^*d}\right) + 2\ln\left(1-\frac{\tilde{\epsilon}}{1+z^*}\right)\right\}$$

This utility exceeds $\ln x + \ln(1+z^*)$ if the final bracket is positive, and expanding this term in $\tilde{\epsilon}$ we see that this happens (for some positive ϵ) provided

$$0 < \frac{1}{1+z^*u} + \frac{1}{1+z^*d} - \frac{2}{1+z^*} = \frac{z^*}{(1+z^*)^2}(u+d-2).$$

However, as noted above, we chose $u+d>2$ to ensure that the asset-sale problem in the no-market setting had a non-degenerate solution. Hence, the value function for a manager with the additional investment opportunity always exceeds that of a manager with no such opportunities for a side-bet.

This one-period example can be extended in many ways. For example, one could discuss the optimal (fair) side bet that the manager would wish to enter at time 0, and one could discuss examples in which the up and down probabilities of the real asset are unequal. It is also possible to extend the model to a multiple period setting. However, the interpretation of all these extensions is more complicated than in the continuous time setting where there is a single key parameter which determines the form of the optimal strategy for the manager, and the optimal strategy is specified by at most two free boundary levels.

5 Discussion

The main contribution of this paper is to give a relatively simple example of a problem where the naive intuition that uncorrelated assets can be omitted from the model breaks down. The optimal stopping problems described in Sections 2 and 3 are designed such that there is no incentive to hold the market asset, in the sense that it cannot be used to hedge the fluctuations in value of the real asset, and since the financial asset is a martingale, a risk-averse manager would not normally include it in her portfolio, and yet the presence of the market asset changes the solution.

Theorem 3.4 shows that for sufficiently large values of γ the equivalence between the no-market and market problems breaks down. The reason why for these parameter values the manager can take advantage of the market is as follows. The real asset has a relatively large growth rate, and hence the manager would like to hold it for as long as possible. However, when this asset forms a large part of her total wealth, maintaining a position in the real asset becomes too risky. If the real asset could be sold in small parts then she might choose to reduce her holdings by a partial sale of the real asset, but in our model the real asset is indivisible.

However, there is one way in which the manager can (slightly) reduce the proportion of her wealth that she has invested in the real asset (under some scenarios), and that is by trading on the market. If she trades successfully then the proportion of her wealth invested in the real asset drops, it is optimal for her to continue to hold the real asset, and she benefits from the expected future growth. If she trades unsuccessfully, then it will be optimal for her to sell the real asset and, given her trading wealth has dropped, she is worse off than before. The overall benefit from trading on the market depends on the balance between the benefit from growth in both trading wealth together with potential future growth in the price of the real asset, and the loss of utility from a loss of trading wealth. When γ is sufficiently large the first effect dominates. This is precisely what we saw in the one-period model.

The key point is that although for fixed wealth x the value function of the manager who cannot trade in the market is concave in the price level y of the the real asset, it is not concave in x. This is illustrated in Figure **??** in the case of a continuous time model, but would also be the case in the one-period no-market problem of Section 4. As a result the manager can increase her utility with an investment in a fair game.

The fact that for certain parameters the value function changes when we introduce (or omit) the market asset, means that the certainty equivalence value of the real asset changes when we move from one setting to the other. In Figure **??** we present the certainty equivalence value for the real asset for the risk averse manager with logarithmic utility under the two scenarios. As described above these prices are given by the solutions to $\ln(x + p^N) = V^N(x, y)$ and $\ln(x + p^M) = V^M(x, y)$.

Note $p^N = p^M$ in first figure. Also $p^N = y$ for $y \geq z^*$, similarly $p^M = y$ for $y \geq \beta^*$. Are price differences big anywhere? Ratio of price differences? Note, prices are not convex.

The analysis of this paper has been completed under the assumption of logarithmic utility. However it is interesting to consider the extent to which the conclusions are robust to a change in utility. Although logarithmic utility is very convenient for calculations and leads to closed form formulæ for many of the quantities of interest, there is every reason to expect that for a large class of utility functions the introduction of an extra trading opportunity (albeit an opportunity in which it appears a manager would choose not to invest) would have an impact upon the value function.

To illustrate this claim, in Appendix B we present results for constant relative risk aversion with risk aversion parameter R. It turns out that the analysis of Sections 2 and 3 carries through to this more general case with only minor modifications. The key quantities of interest are the functions $\gamma_-(R)$ and $\gamma^+(R)$ which denote the critical values at which firstly, the omission of the market asset begins to have an effect, and secondly, the real asset is never sold in the market setting.

Notes: $\gamma_-(R)$ and $\gamma^+(R)$ are increasing and satisfy $0 < \gamma_-(R) < \gamma^+(R) < R \wedge 1$.
Note, in the limit $R \downarrow 0$, critical values are zero. In the risk-neutral case (cf Dixit and Pindyck [3]) get uninteresting answer, issue only arises when have incompleteness and risk aversion.
Note, in the limit $R \uparrow \infty$, critical values are 1. This limit is exponential utility. Hence do not get this effect for exponential utility. (Can see this directly, for exponential utility, wealth factors out, so no-market value function inherits concavity from utility function.)

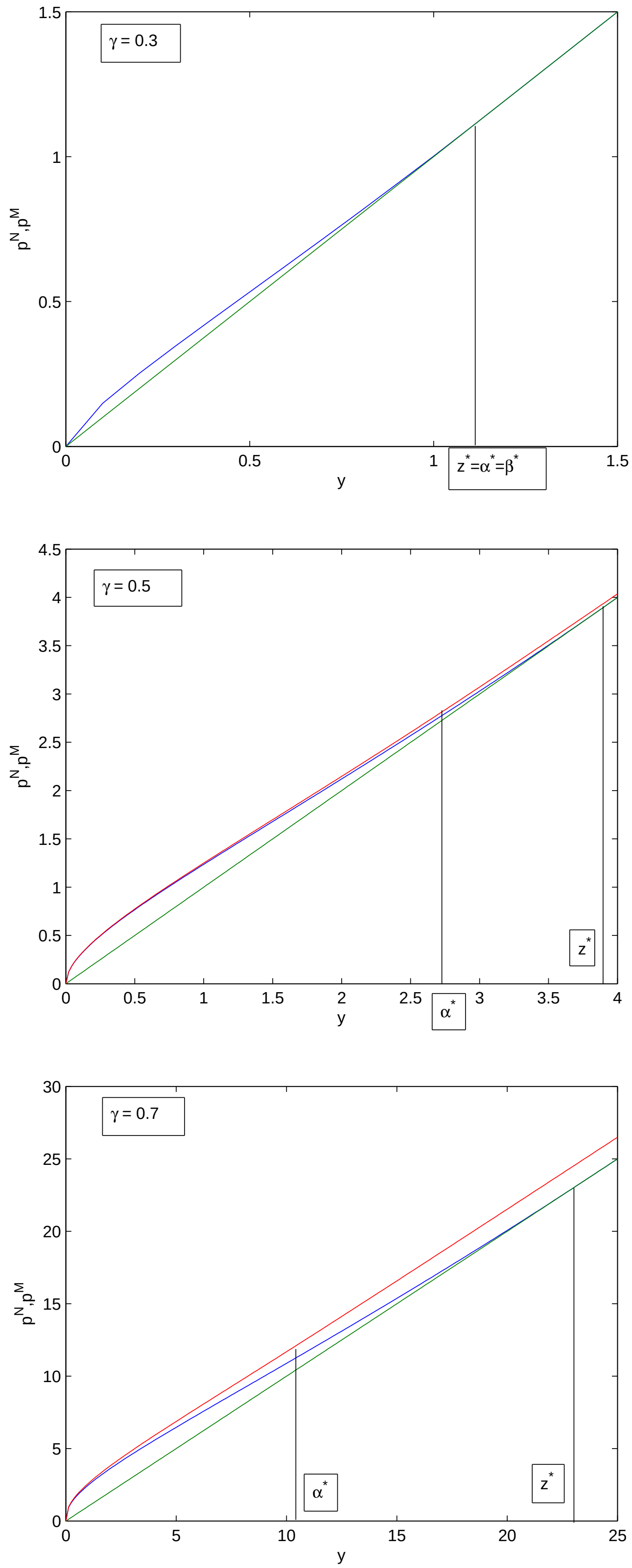
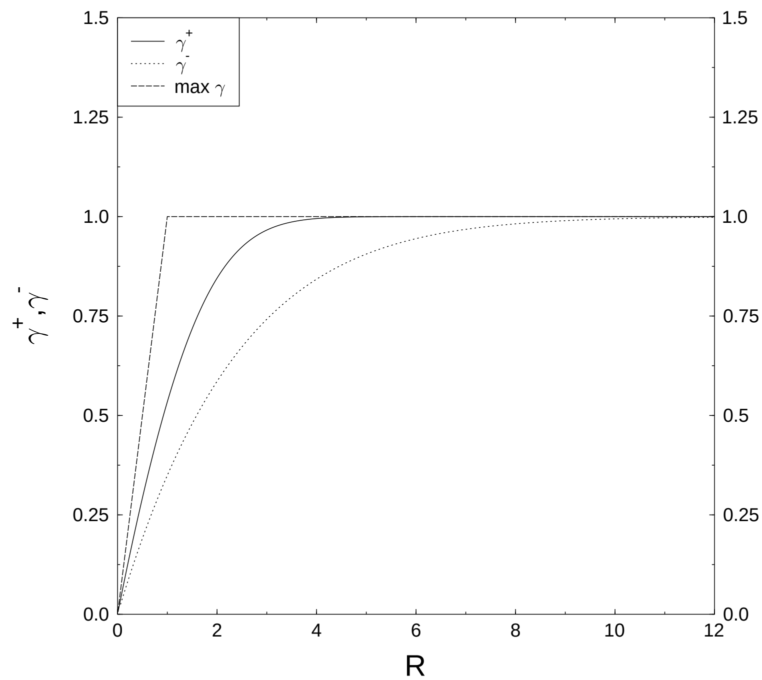


Figure 4: *The manager's certainty equivalent value in the no-market versus market case. Each panel uses a different value of* γ*. We take* $x = 1$ *for all graphs. The first graph takes* $\gamma = 0.3$*, so* $z^* = \alpha^* = \beta^* = 1.14$ *and* $p^N = p^M$*. The second graph uses* $\gamma = 0.5$ *giving* $z^* = 3.92$ *and* $\alpha^* = 2.7, \beta^* = 9.14$*. The final graph is for* $\gamma = 0.7$ *and* $z^* = 23.46$*,* $\alpha^* = 10.31$*.*



γ^+
γ^-
max γ
γ^+, γ^-
R
1.5
1.25
1.0
0.75
0.5
0.25
0.0
0
2
4
6
8
10
12

Figure 5:

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A Proofs

Proof of Proposition 3.1:

For $y \geq \beta x$ we exercise immediately and we have $V^M(x,y) = U(x+y) = \ln(x+y)$. For $y/\beta < x < y/\alpha$, the strategy is to invest in the market until wealth reaches the extremes of this range. In the limit the investment positions are chosen so that this happens before the value of Y changes. Thus, using the martingale property for the investments in x,

$$G(x,y) = \frac{x - y/\beta}{y/\alpha - y/\beta} G(y/\alpha, y) + \frac{y/\alpha - x}{y/\alpha - y/\beta} G(y/\beta, y),$$

and, by value matching on the boundaries $y = \alpha x$, $y = \beta x$,

$$G(x,y) = \frac{\alpha}{\beta - \alpha}\left(\frac{\beta x}{y} - 1\right) H(y/\alpha, y) + \frac{\beta}{\beta - \alpha}\left(1 - \frac{\alpha x}{y}\right) \ln(y + y/\beta).$$

To complete the proof it is sufficient to show that

$$H(x,y) = \ln x + \left(\frac{y}{\alpha x}\right)^{1-\gamma} \Theta$$

where Θ is as given in (8).

We have that for $y < \alpha x$

$$H(x,y) = \ln x \, \mathbb{P}(Y_t \text{ never reaches } \alpha x | Y_0 = y)) + H(x, \alpha x) \mathbb{P}(Y_t \text{ hits } \alpha x | Y_0 = y)$$

and, at least in the case $\gamma \leq 1$, by (3) this becomes

$$H(x,y) = \ln x + [H(x,\alpha x) - \ln x] \left(\frac{y}{\alpha x}\right)^{1-\gamma}.$$

It remains to prove that $H(x, \alpha x) - \ln x = \Theta$.

By the homogeneity of the problem we must have $V^M(\lambda x, \lambda y) = \ln \lambda + V^M(x,y)$. Applying this to H we deduce

$$x\frac{\partial H}{\partial x} + y \frac{\partial H}{\partial y} = 1. \tag{12}$$

The derivative with respect to y is easy to calculate:

$$y\frac{\partial H}{\partial y} = (1-\gamma)[H(x,\alpha x) - \ln x] \left(\frac{y}{\alpha x}\right)^{1-\gamma}. \tag{13}$$

Further, by smooth pasting at $y = \alpha x$

$$x \frac{\partial H}{\partial x}\bigg|_{(x,\alpha x)} = x \frac{\partial G}{\partial x}\bigg|_{(x,\alpha x)} = \frac{\beta}{\beta - \alpha}[H(x,\alpha x) - \ln x - \ln \alpha - \ln(1 + 1/\beta)]$$

Collecting these expressions together, and evaluating (12) and (13) at $(x, \alpha x)$ we obtain

$$\frac{\beta}{\beta - \alpha}[H(x,\alpha x) - \ln x - \ln \alpha - \ln(1+1/\beta)] + (1-\gamma)[H(x,\alpha x) - \ln x] = 1,$$

and finally

$$H(x, \alpha x) - \ln x = \frac{\beta - \alpha + \beta[\ln\alpha + \ln(1 + 1/\beta)]}{\beta + (1-\gamma)(\beta - \alpha)}.$$

□

Proof of Lemma 3.2:

Both Γ_- and Γ^+ are monotonic functions on $(0, 1)$ with a positive value near zero, and a negative value at 1. Hence they have unique roots in $(0, 1)$.

For fixed $\gamma \in (0, 1)$ we want to maximise $\alpha^{\gamma-1}\Theta(\alpha, \beta)$. It is convenient to reparameterize the independent variables as α and $\delta = (\beta - \alpha)/\alpha$. Then, if $\Phi(\alpha, \delta) = \alpha^{\gamma-1}\Theta(\alpha, \beta)$, we have

$$\Phi(\alpha, \delta) = \alpha^{\gamma-1}\left[\frac{(1+\delta)\ln(\alpha + 1/(1+\delta)) + \delta}{1 + \delta(2-\gamma)}\right]$$

with $0 \leq \alpha \leq \infty$ and $0 \leq \delta \leq \infty$. For fixed δ it is clear that the maximum of Φ over α is attained at an interior point. However, this need not be the case for fixed α, and the maximum value of Φ may occur at $\delta = 0$ or $\delta = \infty$. Thus we have to investigate the possibility of maxima of Φ which occur on the boundaries. We have

$$\frac{\partial\Phi}{\partial\alpha} = \frac{\alpha^{\gamma-1}}{1+\delta(2-\gamma)}\left[\frac{(1+\delta)^2}{1+\alpha(1+\delta)} - \frac{(1-\gamma)}{\alpha}[(1+\delta)\ln(\alpha + 1/(1+\delta)) + \delta]\right] \tag{14}$$

and

$$\frac{\partial\Phi}{\partial\delta} = \frac{\alpha^{\gamma-1}}{(1+\delta(2-\gamma))^2}\left[\frac{\alpha + \delta(\alpha+\gamma-2)}{1+\alpha(1+\delta)} - (1-\gamma)\ln(\alpha + 1/(1+\delta))\right] \tag{15}$$

Setting both expressions equal to zero and eliminating the logarithmic term we find that at a turning point

$$\delta\{\alpha(1+\delta)(1-\gamma) - (1+\delta(2-\gamma))\} = 0$$

Hence, for fixed α, there are at most two turning points given by

$$\delta_1 = 0 \qquad \text{and} \qquad \delta_2 = \frac{\alpha(1-\gamma) - 1}{(2-\gamma) - \alpha(1-\gamma)}.$$

Note that $0 < \delta_2 < \infty$ only if $1 < (1-\gamma)\alpha < (2-\gamma)$.

Consider the turning point corresponding to $\delta_1 = 0$. For $\delta = 0$,

$$\frac{\partial\Phi}{\partial\alpha} = \alpha^{\gamma-1}\left[\frac{1}{1+\alpha} - \frac{(1-\gamma)}{\alpha}\ln(1+\alpha)\right].$$

Hence, recall (5), $(\alpha = z^*(\gamma), \delta = 0)$ is a turning point of Φ. In order to determine whether this point is a (local) maximum it is necessary to consider the Hessian matrix of second derivatives. This is given by

$$\Phi''|_{(z^*,0)} = \frac{(z^*)^{\gamma-1}}{(1+z^*)^2}\begin{bmatrix} (\gamma(1+z^*)/z^*) - 1 & 1 - (1-\gamma)(1+z^*) \\ 1 - (1-\gamma)(1+z^*) & -1 \end{bmatrix}.$$

This matrix is negative definite, and hence the turning point is a local maximum, provided $1 - \gamma(1 + z^*)/z^* > (1 - (1-\gamma)(1+z^*))^2$, or equivalently, on substituting for γ using (5), $(z^*)^2 < (1+z^*)\ln(1+z^*)$.

Using the definition of Γ_- this condition translates to $\gamma < \gamma_-$. Otherwise, for $\gamma > \gamma_-$, $(z^*(\gamma), 0)$ is a saddle point.

Now consider the large δ boundary. In the limit $\delta \uparrow \infty$, $\Phi(\alpha, \delta)$ has limit

$$\Phi(\alpha) = \frac{\alpha^{\gamma-1}}{(2-\gamma)}(1 + \ln \alpha).$$

This is maximized by $\alpha = e^{\gamma/(1-\gamma)}$. Further

$$\lim_{\delta \uparrow \infty} \delta^2 \frac{\partial \Psi}{\partial \delta} = \frac{\alpha^{\gamma-2}}{(2-\gamma)^2}[(\alpha + \gamma - 2) - \alpha(1-\gamma)\ln \alpha]$$

and this is positive at $\alpha = e^{\gamma/(1-\gamma)}$ provided $e^{\gamma/(1-\gamma)} > (2-\gamma)/(1-\gamma)$, or equivalently $\Gamma^+(\gamma) < 0$. Hence, for $\gamma > \gamma^+$, $(\alpha = e^{\gamma/(1-\gamma)}, \delta = \infty)$ is a local maximum.

Finally consider the value $\delta = (\alpha(1-\gamma) - 1)/((2-\gamma) - \alpha(1-\gamma))$. Substituting this expression into $\partial\Phi/\partial\alpha = 0$ we find

$$\alpha = \frac{2-\gamma}{1-\gamma}\left[\frac{1}{1-\gamma} - \ln\left(\frac{2-\gamma}{1-\gamma}\right)\right]$$

and then

$$\delta = \frac{1 - (1-\gamma)(2-\gamma)\ln\left(\frac{2-\gamma}{1-\gamma}\right)}{(2-\gamma)\left[(1-\gamma)\ln\left(\frac{2-\gamma}{1-\gamma}\right) - \gamma\right]} \tag{16}$$

For $\delta \in (0, \infty)$ we need $(1-\gamma)(2-\gamma)\ln((2-\gamma)/(1-\gamma)) < 1$, or equivalently $\gamma > \gamma_-$ and $(1-\gamma)\ln((2-\gamma)/(1-\gamma)) > \gamma$ so that $\gamma < \gamma^+$. Hence for δ_2 to correspond to a feasible turning point we must have $\gamma_- < \gamma < \gamma^+$. In particular, for each value of γ, at most one of $\delta_1 \equiv 0$, δ_2 and $\delta = \infty$ yields a local maximum. Hence each of the local maxima we have found is in fact a global maximum.

The value of δ given in (16) translates to (10). □

Proof of Proposition 3.3:

Proposition 3.3 gives the value function for a general form of candidate strategy. In order to prove that the specific form of this strategy given by the optimizing values of α and β from Lemma 3.2is optimal we need to show that the associated value function is maximal. It is sufficient to show that $V^M(x, y) \geq \ln(x + y)$ with equality in the stopping region, that $V^M(X_t, Y_t)$ is a supermartingale, and that $V^M(X_t, Y_t)$ is a martingale in the continuation region. These last properties are a straight-forward exercise in stochastic calculus, using the fact that by definition the trading wealth X_t must be a martingale. Crucial in the proof is the fact that for the optimal α^*, β^* we have smooth fit at $y = \beta^* x$.

Note that in case (iv) the formulæarise in the limit $\beta \uparrow \infty$ of case (iii), on substitution of the optimal value of α. We know from Section 2 that when $\gamma \geq 1$ there is a strategy for which wealth is constant and for which the value function is infinite. Since this strategy remains feasible in the case with a market, the value function must be infinite in this case also. □

B Constant Relative Risk Aversion Utilities

In this section we extend the results for logarithmic utility to power law utilities of the form

$$U(x) = U_R(x) = \frac{x^{1-R} - 1}{1-R}, \qquad R \in (0,\infty), R \neq 1.$$

The slightly non-standard form of the utility function is chosen so that in the limit $R \to 1$ we recover logarithmic utility: recall that $\lim_{R\to 1}(u^{1-R} - 1)/(1-R) = \ln u$. As a result the logarithmic case of Sections 2 and 3 can be recovered immediately from the results of this section, in the limit $R \to 1$. In all cases the ideas behind the proofs are identical to those in the logarithmic case so we omit them.

In the no-market case the expected utility of the agent using the stopping rule τ_z and power utility is

$$\begin{aligned} F(z) = \mathbb{E}[U(x + Y_{\tau_z})] &= \frac{(x^{1-R} - 1)}{1-R}\mathbb{P}(\tau_z = \infty) + \frac{x^{1-R}(1+z)^{1-R} - 1}{1-R}\mathbb{P}(\tau_z < \infty) \\ &= \frac{x^{1-R}[1 + \{(1+z)^{1-R} - 1\}\mathbb{P}(\tau_z < \infty)] - 1}{1-R}. \end{aligned}$$

For $\gamma \geq 1$, $\mathbb{P}(\tau_z < \infty) = 1$, and it is clear in this case that F is maximized over z by taking $z = \infty$. Otherwise, using the formula (3)

$$F'(z) = \left(\frac{y}{x}\right)^{1-\gamma} x^{1-R} z^{\gamma-2}\left[z(1+z)^{-R} - (1-\gamma)\frac{(1+z)^{1-R} - 1}{1-R}\right]$$

Since, for $z > 0$

$$\frac{(1+z)^{1-R} - 1}{1-R} > z(1+z)^{-R} > (1+z)^{1-R} - 1$$

it follows that for $\gamma \leq 0$ there is no solution to $F'(z) = 0$ and F is maximized by the choice $z = 0$, and for $\gamma \geq R$, there is again no solution to $F'(z) = 0$ and F is maximized by the choice $z = \infty$. The interesting case is when $0 < \gamma < R \wedge 1$. In that case F is maximized by the unique solution z^* in $(0,\infty)$ to $F'(z) = 0$. The following proposition describes the optimal behavior and value function for a manager with power utility.

Proposition B.1 *For all $R \neq 1$ and $\gamma \leq 0$, $V^N(x,y) = ((x+y)^{1-R} - 1)/(1-R)$, for $R < 1$ and $\gamma \geq R$, $V^N(x,y) = \infty$ and for $R > 1$ and $\gamma \geq 1$, $V^N(x,y) = 1/(R-1)$.*

In the non-degenerate cases $0 < \gamma < R \wedge 1$, $V^N(x,y) = ((x+y)^{1-R} - 1)/(1-R)$ in the exercise region $y \geq xz^*$, and in the continuation region $y < xz^*$

$$V^N(x,y) = \frac{x^{1-R} - 1}{1-R} + \left(\frac{y}{xz^*}\right)^{1-\gamma} x^{1-R}\frac{[(1+z^*)^{1-R} - 1]}{1-R}. \tag{17}$$

The optimal exercise ratio z^* solves

$$(1-\gamma)\frac{(1+z)^{1-R} - 1}{1-R} - z(1+z)^{-R} = 0 \tag{18}$$

Note that the formulæ (4) and (5) follow immediately on taking the limit $R = 1$ in (17) and (18).

One feature of the problem with relative risk aversion R is the fact that the optimal stopping problem has a degenerate solution for $\gamma \geq R \wedge 1$. When $\gamma > 1$ the real asset drifts to plus infinity, and so it is clearly never optimal to exercise at any finite threshold, the real asset is simply too good an investment. However in the case $R < \gamma \leq 1$, even though the value of the real asset will converge to zero almost surely, the expected value of $(Y^{1-R} - 1)/(1-R)$ tends to plus infinity, and the real asset is worth an infinite amount.

Now consider the optimization problem embedded in a market in the sense of Section 3.

Proposition B.2 *Under the strategy described before Proposition 3.1, and specified by α, β, the value function is given by*

$$V^M(x,y) = \begin{cases} U(x+y) & y \geq \beta x \\ G(x,y) & \alpha x \leq y < \beta x \\ H(x,y) & y < \alpha x \end{cases}$$

where

$$G(x,y) = \left(\frac{\beta x}{y} - 1\right) \frac{\alpha}{\beta - \alpha} H(y/\alpha, y) + \left(1 - \frac{\alpha x}{y}\right) \frac{\beta}{\beta - \alpha} \left[\frac{y^{1-R}(1+1/\beta)^{1-R} - 1}{1-R}\right]$$

and

$$H(x,y) = \frac{x^{1-R} - 1}{1-R} + y^{1-\gamma} x^{\gamma - R} \alpha^{\gamma - 1} \Theta$$

with

$$\Theta \equiv \Theta(\alpha, \beta) = \left[\frac{\beta - \alpha + \beta\left(\frac{\alpha^{1-R}(1+1/\beta)^{1-R} - 1}{(1-R)}\right)}{\beta + (\beta - \alpha)(R - \gamma)}\right].$$

In order to find the optimal strategy from stopping rules of this class we need to find the maximum of $\alpha^{\gamma-1}\Theta$.

Lemma B.3 *Let $\gamma_-(R)$ be the unique solution in $(0, R \wedge 1)$ of*

$$(R-\gamma)^R (R+1-\gamma) = (2R-\gamma)^R (1-\gamma)$$

and let $\gamma^+(R)$ be the unique solution in $(0, R \wedge 1)$ to

$$(R-\gamma)^R (R+1-\gamma)^{1-R} = R^R (1-\gamma).$$

Then $\gamma_-(R) < \gamma^+(R)$.

For $0 < \gamma \leq \gamma_-(R)$ the maximium of $\alpha^{\gamma-1}\Theta$ is attained at $\alpha^*(\gamma, R) = \beta^*(\gamma, R) = z^*$ where z^* is the solution to (18).

For $\gamma_-(R) \leq \gamma \leq \gamma^+(R)$ the maximum of $\alpha^{\gamma-1}\Theta$ is attained at $(\alpha^*, \beta^*) \in \{(\alpha, \beta) : 0 < \alpha < \beta < \infty\}$ which are given by

$$\alpha^* = \frac{1}{(R-\gamma)(1-R)} \left[R(R+1-\gamma) - R(R-\gamma)\left(\frac{1+R-\gamma}{1-\gamma}\right)^{1/R}\right]$$

and

$$\beta^* = \frac{\alpha^* R(R-\gamma)}{R(R+1-\gamma) - \alpha^*(R-\gamma)}.$$

For $\gamma^+(R) \leq \gamma \leq R \wedge 1$ the maximium of Θ is attained at $\beta = \infty$ and

$$\alpha = \alpha^*(\gamma, R) = \left(\frac{R(1-\gamma)}{R-\gamma} \right)^{1/(1-R)}.$$

This lemma allows us to state the analog of Theorem 3.4.

Theorem B.4 *For $\gamma \leq \gamma_-(R)$ the solution with the market is identical to the solution of the optimal stopping problem with no market given in Section 2.*

For $\gamma_-(R) < \gamma < R \wedge 1$ the solution to the optimal stopping problem (7) with the market asset is different to the solution of (1) without the market asset.