Optimal Scrapping and Technology Adoption under Uncertainty

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Abstract

A firm has to decide sequentially to replace its technology and to implement a new one chosen among an increasing range over time under technological and market uncertainty. The optimal decision rule is a (s, S) *style* policy where the trigger and target technology levels are positively correlated with boom persistence and negatively related with recession persistence and technological uncertainty. The average time between two adoptions is governed by several factors. Bounded technological progress imposes a limitation on the best grade available, which can accelerate updating when the firm wishes to continue to operate an advanced technology. Technological uncertainty reinforces depreciation and thus hastens replacement. Moreover, both types of uncertainty have an impact on the scrapping and upgrading levels. Overall, adoption is more frequent for economies spending a large fraction of time in booms. The likelihood of switching during a recession is negatively affected by the arrival rate of booms. The end of a recession can trigger updating since the firm will want to operate an efficient technology in order to seize the high cash flows associated with the forthcoming boom. This result implies that investment spikes are procyclical.

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1. INTRODUCTION

In the May 1999 issue of "Communications of the ACM", the computer science magazine tried to explore the following dilemma: how often should a firm buy a new computer and what type of machine should it buy? The article reached the conclusion that a firm should replace its PC at regular intervals using two dominating strategies: either buy high-end machines every 36 months for organizations seeking substantial computer performance or buy intermediate-level computers every 36 months, a cheaper alternative. Changing configurations and declining prices lead to an important characteristic of the PC market: computers must be replaced at regular intervals.

This paper investigates the effects of both technological and market uncertainty on the size and frequency of adoption of technology at the plant level.

There are two fundamental aspects in the adoption of a new technology:

1. lumpiness: firms occasionally adjust their capital in discrete bursts. Doms and Dunne (1998) study capital adjustments over a sample of 13,702 US manufacturing plants focusing on the lumpy nature of investment. They find that half of the plant's total investment over the 1973-1988 period occurred in just three years. Cooper, Haltiwanger and Power (1999) estimate that the largest investment episode for each plant contributes to 17% of cumulative aggregate investment, and the top five investments to more than 50% of the latter. Cooper and Haltiwanger (1993) stress the importance of non-convexity in machine replacement by automobile producers inducing investment bursts. Goolsbee and Gross (1997) estimate the adjustment cost function for the US airline industry and present clear evidence of important non-convexities: 60% of the total acquisitions of an aircraft type by a given airline occur in the largest two-year investment episode and desired output must differ from actual output by between 10% and 40% to trigger investment or disinvestment;

2. Uncertainty is a key element in the process of selecting and adopting a technology. The literature has examined several aspects of the role of uncertainty:

   - demand uncertainty. Market uncertainty may both delay adoption and affect the size of innovations. Why improve one production tool if markets are expected to be depressed?

   - innovation uncertainty. There are many kinds of uncertainty surrounding a new technology. For instance, the speed of arrival and the size of future innovations significantly matter. Rosenberg (1976) and Farzin, Huisman and Kort (1998) point out that since the sunk cost of investing prematurely in a given technology is usually unrecoverable, a manager expecting a major technological breakthrough may choose to delay adoption as she tries to avoid to lock herself in. Alternatively, uncertainty may lie in the quality of the new technology or, more generally, in its profitability. The moment when a technological curiosity becomes a commercial one is hard to define. Mansfield (1968) mentions that in the case of a new piece of equipment, both the supplier and the user often take a considerable risk.

These observations raise an interesting set of questions. How do different sources of uncertainty in the economy affect scrapping of obsolete technologies? What are their impacts on the type of technologies adopted? Does uncertainty increase or decrease the frequency of adoptions? Is the adoption of new technologies more likely to occur in recessions or in periods of boom?

This paper constructs a model in which a firm (plant), confronted with market and technological uncertainty, must choose sequentially when to scrap its old technology and the size of its technology leap forward.

1.1. Related Literature

A common feature of all technology adoption models is the trade-off between waiting and upgrading. An update in technology is costly and usually irreversible, so a natural concern for the manager is: how will the market evolve and how fast will technological progress occur? When adoption is decided, the manager may hesitate over the type of new technology to implement: Does the new piece of equipment require specific knowledge to be operated properly? How large will the gains in efficiency be? Adoption of a new technology is by no means a simple issue to study so the literature has tried to disentangle independently the role of several factors.

A large class of models focus on the complementarity between technology and skills. There is a trade-off between improving expertise and experience by continuing to operate a given technology (learning by doing) and switching to a more profitable production process that is not fully mastered by the firm right after adoption (Jovanovic and Nyarko (1996), Chari and Hopenhayn (1991)). Parente (1994) proposes a model where learning exhibiting decreasing returns takes time and switching technology induces a loss in know how. These authors emphasize the link between the low pace of diffusion of a technology and the time required to acquire the skills to use it.

Another class of models tries to capture the uncertainty surrounding the arrival of a new technology. Does new necessarily mean more efficient, and if yes for how long? To overcome the first difficulty, Jensen (1982) proposes a model in which the plant manager observes signals from which she can infer the quality of the technology and, therefore, updates her beliefs over time. Similarly, Jensen (1983) presents a firm undertaking trials to evaluate the quality of two competing innovations. Both Balcer and Lippman (1984) and Farzin, Huisman and Kort (1997) examine the optimal timing of technology adoption in a context of uncertainty regarding the arrival speed of innovations and the efficiency of improvements. They show that significant technological improvements and a high rate of innovations delay adoption.

The link between technology adoption and the business cycle has also been examined in the literature. Klenow (1998) develops a model where technology updates are more likely in a boom than in a recession since learning (by doing) about the new technology rises faster in periods of high production. He demonstrates that the sensitivity of technology upgrades to demand shocks depends on the persistence of the shocks. In particular, when shocks are persistent, firms may prefer to upgrade in recessions. Cooper and Haltiwanger (1993) analyze machine replacement in the automobile industry within a framework of *deterministic* cycles and show that retooling is likely to occur at the end of the downturn whereas new machines begin operation at the beginning of the boom following (for sure) the trough of the cycle. Goolsbee (1998), investigating the airline industry, concludes that capital retirement affects in priority old vintages; older planes are much likely to be scrapped in recessions when the cost of capital is low. Goolsbee and Gross (1997) show that for the airline industry capital retirement is associated with significant adjustment costs and rises during recessions whereas companies concentrate purchases of new planes during expansions.

One important restriction of these models is that the firm has no choice but adopting the cutting edge technology. Few attempts have been made to relax this assumption. Jovanovic and Rob (1998) construct a deterministic general equilibrium model in which a manager can choose to upgrade among an increasing range of vintages as technological progress continues. Yet since the production function considered exhibits constant returns to scale, the state of the art technology is always purchased. Bar-Ilan and Mainon (1993) introduce a stochastic environment in which the firm must adjust its technological level with respect to the frontier technology. Indeed, in reality, managers pay attention to what type of technology to implement. Why adopt the frontier technology in a recession time?

We propose a tractable model in which a firm can select the size of its technological leap among an increasing range over time without being certain of the innovation's profitability. This analysis is justified for two reasons. First, the economic environment may change: demand for the product can be low (high) as in a recession (boom); as adoption is irreversible, this induces a risk of adopting an under(over)performing technology depending on the evolution of the market. We aim at studying the impact of the duration of each regime on the firm's adoption decisions when the manager of the firm can identify these regimes in *real time*. Second, the efficiency gains from the new technology are uncertain: the technology itself may not be reliable over time or the firm may not have the necessary human capital to operate it properly. Our approach focuses on the option value of waiting to adopt a suitable technology as there is uncertainty and the decision taken is irreversible. We lie in the vein of models developed by Abel and Eberly (1996) and (1998), Abel et al. (1996), Bertola and Caballero (1994), Dixit and Pindyck (1994) or in a context of indivisible durable goods by Grossman and Laroque (1990). Some of the central issues of this paper are related to the work by Cooper, Haltiwanger and Power (1999), who investigate the links between machine replacement and the business cycle. While they consider both investment at the plant and aggregate levels, we restrict our attention to the plant level but allow implementation of any technology available at the adoption time, in particular, inferior technologies.

The main contribution of the paper is to provide an analytical solution to the technology adoption problem in the presence of bounded technological progress under both demand and idiosyncratic uncertainty. We are thus able to analyze and disentangle the specific effects of the two types of uncertainty on the decision to scrap technology, on the size of the upgrading and on the timing and frequency of adoption of new technologies. The predictions of our model seem consistent with some empirical evidence.

1.2. Results

Adoption of new technologies is the result of both physical and economic depreciations as new technologies are exogenously introduced in the economy. We show that the manager of the firm chooses (optimally) a (s, S) *style* policy. More precisely, in an economy experiencing recessions (state 1) and booms (state 2), she uses the following policy: Whatever the market conditions, she remains passive as long as the technology is above a scrapping level a_2^*. If the economy is in a boom when the technology level reaches a_2^*, she updates to a target level b_2^* that may (but need not) be the cutting edge technology. If a recession prevails, she decides to continue operating the same technology for a while. As soon as the recession ends and a boom occurs, she switches technology on the spot: *booms can trigger technology adoption.* If the economy remains in recession, ultimately she scraps at level $a_1^* < a_2^*$ and upgrades to a target level b_1^*. As a recession reduces current cash flows, the firm has less of a need for highly performing technology. Consequently, the manager tends to delay scrapping and adopts a less sophisticated technology during recessions, $b_1^* \leq b_2^*$. The scrapping and target thresholds depend in particular on three parameters: the arrival rate of booms ϕ_1, the arrival rate of recessions ϕ_2 and the technological uncertainty σ. Non-lasting booms provide few incentives to the manager for implementing a sophisticated technology since a recession is looming. Conversely, adopting in a non-lasting recession, the manager would like to seize the opportunity of forthcoming high cash flows by implementing a fairly good technology. The purchase of new technologies depends not only on market conditions, but also on the persistence of the cycles.

In an attempt to answer Doms and Dunne's question - how lumpy is investment? - we analyze the impact of regime persistence on the frequency of switches. We find that for economies where recessions are persistent, machine replacement is slower than for economies enjoying lasting expansions. In addition, when we impose a common rate ϕ of arrival of regimes, economies experiencing more stability (low ϕ) display more frequent adoptions. Technological uncertainty plays two distinct roles. One *direct effect* is to alter the features of technology itself. In the presence of decreasing returns, the instantaneous expected operating profit and, therefore, the value of operating the same technology forever are reduced. More technological uncertainty hastens adoption; a higher technological uncertainty increases the chances that the technology level of the firm reaches its scrapping thresholds sooner by amplifying the technological process fluctuations. In short, technological uncertainty reinforces physical depreciation. One *indirect effect* is to shift the band $[a_i^*, b_i^*]$ as it lowers both thresholds. The firm has a wary attitude: if technological uncertainty is significant, it buys a moderate technology and chooses to keep on operating a more depreciated technology. The magnitude of the indirect effect is captured by the ratio $\frac{b_i^*}{a_i^*}$ of the upgrading level b_i^* over the scrapping level a_i^*. Numerical simulations show that this ratio rises with technological uncertainty, indicating that the scrapping level is relatively more sensitive. As a consequence, the indirect effect tends to slow down adoption of new technologies. Surprisingly (or not), the overall effect of technological uncertainty is to accelerate adoption. One implication is that firms with a low ability in operating new technologies should buy intermediate technologies and update often.

In an attempt to shed some light on the ongoing debate, “Are technology adoptions more likely to take place in recessions versus booms?”, the model turns out to be tractable enough so that the probability to update in a recession versus boom can be computed explicitly. Since scrapping occurs at an earlier stage during an expansion, the arrival rate of booms is a key factor in determining whether adoption of technology is more likely in a boom or in a recession. Unless the arrival rate of recessions is quite significant with respect to the arrival rate of expansions, investment spikes are likely to be procyclical. In particular, this result is true when the economy spends on average an even fraction of time in a recession and in a boom.

The paper is organized as follows. Section 2 describes the economic setting and provides some insights on the structure of the optimal decision rule. Section 3 contains the derivation of the optimal policy and outlines the relationships between uncertainty, the frequency of adoptions and the likelihood of upgrading during recessions. Section 4 presents numerical simulations showing the effects of market and technological uncertainty on the decisions and the timing of adoption as well as the chances of switching during a recession versus an expansion. Section 5 concludes.

2. THE GENERAL ECONOMIC SETTING

We consider an infinite horizon economy in which a firm has to decide sequentially the quality of the technology it should operate.

2.1. Uncertainty and Information Structure

Uncertainty in this paper arises from two independent sources:

- The firm environment (demand for the product) represented by the variable X_i may change: recessions follow booms and vice versa. Index i refers to the state of the economy, with $i = 1$ for a recession and $i = 2$ for an expansion;

- On average, the firm can only expect to adopt a better performing technology. Buying a superior technology is not the same thing as having the know-how to operate it efficiently. For instance, workers may not be able to master fully the newly implemented technology potentially due to some organizational issues or, simply, a lack of human capital. Campbell (1998) outlines that different plants implement the same technology with varying degrees of success as they are subject to idiosyncratic productivity shocks. Another aspect is the reliability of the new technology. Manufacturing defects or unexpected complications may arise. For example, running the machine for a certain amount of time without interruption may cause overheating and then require it to be stopped for a while. This obviously slows down one production line, inducing delays and reducing the operating profit. Alternatively, a new version of a computer software may contain some bugs or may not be fully compatible with the existing code used by the firm. More generally, we denote by “technological uncertainty” the fact that the contribution of technology to operating cash flows is not perfectly known at the time of adoption and can fluctuate over time. The relative technological position of the firm with respect to the frontier technology u follows a geometric *one*-dimensional Brownian Motion that is independent of the state of the economy.

We assume that the manager takes decisions at time t based only on the information available to her at time t. All the stochastic processes considered in this paper are Markov processes. Therefore, the only relevant information for the manager is *the current relative level of the firm technology u and* the *current state i of the economy,* i.e., an element of $\mathbb{R}_+ \times \{1,2\}$.[1]

2.2. Technological Progress

Technology is embodied in new capital goods. A single variable $a \geq 0$ captures all the relevant attributes of a production process to the operating cash flow. Roughly speaking, a represents the grade of the technology. $A(t)$ denotes the state of the art technology available in the economy at time t. This frontier technology evolves exogenously at a constant rate μ according to the following deterministic law of motion

$$dA(t) = \mu A(t)dt, \tag{2.1}$$

with $A(0) > 0$.

The firm can only operate *one* technology at one time. $a_k(t)$ denotes the grade *in efficiency units* of the kth technology adopted at time t_k operated at time t, with $t_k < t \leq t_{k+1}$. As the contribution of technology to cash flows is uncertain, $a_k(t)$ is a random variable that follows a geometric Brownian Motion

$$da_k(t) = a_k(t)\left[-\kappa dt + \sigma dw(t)\right], \tag{2.2}$$

where $dw(t)$ is the increment of a standard Wiener process, κ represents the rate of physical depreciation and the parameter σ captures the magnitude of the technological uncertainty.[2]

When switching technology at time τ, the feasible set of available technologies is $\mathcal{A}(\tau) = \{a(\tau) \in \mathbb{R}_+,\, a(\tau) \leq A(\tau)\}$. The relative position of the firm with respect to the frontier technology at time t is $u_k(t) = \frac{a_k(t)}{A(t)}$. Considering equations (2.1) and (2.2), u_k follows the law of motion

$$du_k(t) = u_k(t)\left[-(\mu + \kappa)dt + \sigma dw(t)\right]. \tag{2.3}$$

Define the overall rate of depreciation $\lambda = \mu + \kappa$. In this model, a technology becomes obsolete because it physically depreciates at rate κ *and* it economically depreciates at rate μ, as the introduction of more advanced technologies renders existing ones obsolete technologies.

[1] More formally, we define a probability space $(\mathbb{R}_+ \times \{1,2\}, \mathcal{F}, \mathbb{F}, P)$ where $\mathcal{F}$ denotes the tribe of subsets of the state space $\mathbb{R}_+ \times \{1,2\}$ that are events over which the probability measure P is assigned. The filtration $\mathbb{F} = \{\mathcal{F}_t,\, t \geq 0\}$ represents how the information is revealed over time. At time t, the information available is $\mathcal{F}_t = \sigma\left((u(s), X(s)); 0 \leq s \leq t\right) \subset \mathcal{F}$, the smallest sigma field generated by observations of $u(s)$ and $X(s)$ up to time t. If $t \leq s$, the information set $\mathcal{F}_t$ is included in $\mathcal{F}_s$: information is not forgotten over time. To be more precise, we should consider the completion of $(\mathbb{R}_+ \times \{1,2\}, \mathcal{F}, \mathbb{F}, P)$.

[2] In a discrete time framework, an equivalent formulation is $a_{k,\, t+1} = (1-\kappa)a_{k,\, t} + a_{k,\, t}\sigma\varepsilon_{t+1}$, where ε_{t+1} is i.i.d. and normally distributed.

Remark 1. *Note that $u_k(t_k)$ cannot be more than 1 but for $t \geq t_k$, according to (2.3), $u_k(t)$ may be greater than 1. We now provide a justification for the apparent striking feature of the model. At the time of adoption t_k, $a_k(t_k)$ is the grade, but when the technology is used by the firm for $t \geq t_k$, $a_k(t)$ measures the effective contribution of the kth technology to the production process. For instance, a firm familiar with operating an old vintage may have a higher productivity than if using the less familiar state of the art technology. However, note that on average u declines to zero.*

2.3. Market Structure

2.3.1. Demand Side

As in Abel and Eberly (1996) and Caballero and Dixit (1992), the demand for the product is taken to be isoelastic

$$Q^d(t) = z(t)p(t)^{-\varepsilon}, \tag{2.4}$$

where $z(t)$ is an exogenous stochastic process that captures demand shocks and $\varepsilon > 1$ is the price elasticity. For convenience, we define $X \equiv \frac{1}{\varepsilon}(1 - \frac{1}{\varepsilon})^{\varepsilon-1}z$ and since X is simply proportional to z, we will refer to X as the level of demand in what follows.

2.3.2. Supply Side

A monopolist produces the same good over time. Technology is cost reducing. When using technology of grade a, the cost of producing q units of output is $a^{-\chi}q$ with $\chi > 0$. Using equation (2.4) and dropping the time dependence, the instantaneous profit π is simply

$$\pi = \max_{q \geq 0} z^{1/\varepsilon} q^{1-\frac{1}{\varepsilon}} - a^{-\chi} q.$$

After maximization, we obtain that

$$\pi = X a^{\alpha},$$

where $\alpha = \chi(\varepsilon - 1)$. We assume $\alpha < 1$.

2.3.3. A Closer Look at the Market Uncertainty

In order to keep things as simple as possible, we assume that X can only take two values. When the economy is experiencing a recession, $X = X_1$, and when the economy is experiencing a boom, $X = X_2$ with $X_1 < X_2$. Such a framework allows the manager to identify recessions and booms in *real time*. Regime changes are driven by independent exponential probability laws. Being in a recession, ϕ_1 denotes the instantaneous arrival rate of a boom and being in a boom, ϕ_2 denotes the instantaneous arrival rate of a recession. The parameters ϕ_1 and ϕ_2 measure the persistence of the states: The lower the parameter ϕ_1 (ϕ_2), the more persistent are recessions (booms) and the expected duration of regime i is $\frac{1}{\phi_i}$. More precisely, between time t and $t+h$, the change of states can be described by the following transition matrix

time $t+h$

time t		X_1	X_2
	X_1	$1-\phi_1 h+o(h)$	$\phi_1 h+o(h)$
	X_2	$\phi_2 h+o(h)$	$1-\phi_2 h+o(h)$

The corresponding conditional probability $P_{ij}(t) = P(X(t) = X_j | X(0) = X_i)$ is given by

$$P_{11}(t) = 1 - \frac{\phi_1}{\phi_2+\phi_1}\left[1 - e^{-(\phi_2+\phi_1)t}\right] \text{ and } P_{21}(t) = \frac{\phi_2}{\phi_2+\phi_1}\left[1 - e^{-(\phi_2+\phi_1)t}\right].$$

Using these analytical expressions, it can be shown that the average fraction of time spent in regime i is $\frac{\phi_j}{\phi_i+\phi_j}$. Our framework shares some common features with the one developed by Hasset and Metcalf (1994) in a context of investment with uncertain tax policy where changes occur according to some Poisson processes. However, in their model, the firm can only select *one* project whereas we allow multiple technology adoptions here. The variables u, A and X enable us to rewrite the profit

$$\pi(u, X, A) = A^{\alpha} X u^{\alpha}.$$

2.4. Timing of Adoption

We follow Jovanovic and Rob (1998). Denoting one particular adoption time by τ, switching technology requires two steps:

- At time τ^-, the firm has to scrap its old technology $a(\tau^-)$. The underlying idea is that technologies are fully incompatible. We assume thin markets for used machines: the firm activity may be so specific that capital resales only occur at heavy discounts. In our case, the resale price is simply *zero and scrapping is costless.* In some industry, capital retirement can be associated with significant financial burden: Goolsbee and Gross (1997) point out that firing costs of pilot per plane can represent more than 15% of the cost of the average plane of their sample.

- At time τ^+, the firm decides which technology to adopt $a(\tau^+)$ in $[0, A(\tau)]$. Obviously, the manager will always pick up $a(\tau^+) > a(\tau^-)$. Cooley, Greenwood and Yorukoglu (1997) as well as Greenwood, Hercowitz and Krusell (1997) outline that the relative price of an efficiency unit of equipment dropped off steadily and rapidly in the postwar US economy. In our set-up, the price of one efficiency unit of technology declines at a constant rate ψ, $p(t) = p_0 e^{-\psi t}$ where p_0 is a constant.

2.5. The Firm's Problem

An alternative and equivalent approach to the problem is to let the manager decide of the relative technological position of the firm u with an upgrading level $u(\tau^+)$ in $[0, 1]$. This formulation turns out to be easier to analyze.

Switching technology implies giving up the cumulative discounted profit at the interest rate ρ that could have been realized with the technology already in use. The manager is therefore facing an opportunity cost: upgrading cannot be continuous across time. Here, we capture the lumpiness of technology adoption. The firm optimally chooses an increasing sequence of stopping times[3] $\{\tau_k\}_{k=0}^{\infty}$ and a sequence of positive random variables $\{v_k\}_{k=0}^{\infty} \in [0,1]$, where v_k represents the relative level of the kth technology adopted at τ_k. By convention, $\tau_0 = 0$ and $v_0 = u_0$. This is a typical *impulse control problem* (see Harisson, Sellke and Taylor (1983) and Brekke and Oksendal (1994)).

Denote by E_0^i the expectation operator conditional on the initial state to be (u_0, i) and $J_i(u_0, A_0, p_0)$ the value of the firm at the initial date $t = 0$ when the initial state of the world is (u_0, A_0, p_0, i). The firm's problem can be formalized as follows

$$\begin{aligned} J_i(u_0, A_0, p_0) \quad = \quad & \sup_{(\{\tau_k \geq 0,\ v_k \in [0,1]\}_{k=1}^{\infty})} E_0^i \left[\int_0^{\tau_1} X(s) A(s)^{\alpha} u_0(s)^{\alpha} e^{-\rho s} ds \right. \\ & \left. + \sum_{k=1}^{\infty} \left[\int_{\tau_k}^{\tau_{k+1}} A(s)^{\alpha} X(s) u_k(s)^{\alpha} e^{-\rho s} ds - p(\tau_k) A(\tau_k) v_k e^{-\rho \tau_k} \right] \right]. \end{aligned}$$

ASSUMPTION: $\psi = (1-\alpha)\mu$

It seems reasonable to think that the faster technological progress is, the larger the economic depreciation of existing technologies. This implies that the growth rate of technological progress μ and the rate of decline in the price of the technology ψ must be positively related. In order to avoid a race between these two rates which, as time passes, would lead to a degenerate decision rule such as the cutting edge technology is always or never implemented when adoption takes place, we assume that $\psi = (1-\alpha)\mu$. This assumption allows us to express scrapping and adoption decisions in terms of relative position with respect to the technological frontier.

For all time t, the quantity $p(t)A(t)^{1-\alpha}$ is now a constant denoted c. Recall that the profit function is $A(t)^{\alpha} X(t) u(t)^{\alpha}$ and the cost of adoption of relative technology $v(t)$ can be rewritten $cA(t)^{\alpha} v(t)$. Exploiting the linearity of the law of motion of the process A as well as the homogeneity of degree α of the profit and cost functions in A, we can define a new value function V_i such that $J_i(u, A) = A^{\alpha} V_i(u)$. The firm's problem can be reformulated using the new value function V_i characterized by

[3] A stopping time τ is a measurable function from the state space $(\mathbb{R}_+ \times \{1,2\},\ \mathbb{F})$ to $\mathbb{R}_+$ such that $\{(a,i) \in \mathbb{R}_+ \times \{1,2\},\ \tau(a,i) \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. It means that the stopping rule is a non anticipated strategy or in other terms the decision of switching technology only depends on the information available up to the time of the adoption.

$$\begin{aligned} V_i(u_0) &= \sup_{(\{\tau_k \geq 0,\ v_k \in [0,1]\}_{k=1}^{\infty})} E_0^i \left[\int_0^{\tau_1} X(s) u_0(s)^\alpha e^{-rs} ds \right. \\ & \qquad \left. + \sum_{k=1}^{\infty} \left[\int_{\tau_k}^{\tau_{k+1}} X(s) u_k(s)^\alpha e^{-rs} ds - c v_k e^{-r\tau_k} \right] \right] \\ \text{s.t. } du_k(t) &= u_k(t) \left[-\lambda dt + \sigma dw(t) \right] \text{ and } u_k(\tau_k^+) = v_k \\ X(t) &\in \{X_1, X_2\}, \end{aligned}$$

with $r = \rho - \alpha\mu$. We assume $r > 0$ to guarantee that the value function takes finite values.

In the following, somewhat abusing language, we will refer to u as the grade of the technology instead of the relative grade with respect to the frontier and c as the adoption cost of technology per efficiency unit.

3. RESOLUTION OF THE MODEL

3.1. Inaction Region and Conjecture of the Optimal Policy

Due to the Markovian structure of the problem and the infinite time horizon, the optimal policies are stationary functions of the current state of the world depicted by a couple (u, i).

The key point to notice is that whatever technology is operated, the profit of the firm remains positive. When scrapping its technology the firm actually incurs a cost equal to the cumulative expected discounted profit that could have been generated by operating the old technology. If the state of the world is (u_0, i), we can define an opportunity cost of scrapping at time τ $SC_i(u_0) = E_0^i \left[\int_\tau^\infty X(s) u_0(s)^\alpha e^{-rs} ds \right]$. As in Grossman and Laroque (1990) who assume a linear transaction cost for home purchases, this acts like a *fixed* cost in an optimal stopping problem. This opportunity cost implies that it cannot be optimal for the firm to switch technology *continuously.* Adoption of technology displays some lumpy features. Using Bellman's principle, we have the following recursive formulation

$$V_i(u_0) = \sup_{(\tau \geq 0,\ v(\tau) \in [0,1],\ j \in \{1,2\})} E_0^i \left[\int_0^\tau X(s) u_0(s)^\alpha e^{-rs} ds + e^{-r\tau} \left[V_j(v(\tau)) - c v(\tau) \right] \right], \quad (3.1)$$

where τ is the first stopping time from date 0. Note that j may be different than i if the first switch occurs in a state different state than the initial one. The relationship (3.1) is useful to understand how the manager decides to remain inactive. Given the state of the economy i, there exists an inaction region IR_i defined as the range of technology levels for which the firm has no incentive to switch

$$IR_i = \left\{ u \in \mathbb{R}_+,\ V_i(u) > \sup_{v \in [0,1]} V_i(v) - cv \right\}. \quad (3.2)$$

We conjecture that each inaction region IR_i is connected and its boundary ∂IR_i is characterized by a critical value u_i^*. Given the state i of the economy, the firm remains passive as long as its technology level u exceeds u_i^*. Economic intuition suggests that during a boom, the firm should have more incentives to operate performing technologies and to scrap less depreciated technologies than it does during a recession in order to enjoy higher operating cash flows. Thus we expect $u_1^* < u_2^*$ and therefore $IR_2 \subset IR_1$. We also anticipate that the firm adopts a better technology during a boom than it does during a recession, i.e., $v_1^* \leq v_2^*$. This intuition is confirmed by studying the limiting case of an economy continuously experiencing a recession or a boom, i.e., X is constant over time. Both scrapping and upgrading thresholds are found to be increasing with X.[4] So at least, for small values of ϕ_1 and ϕ_2, relying on a continuity argument, the threshold values should remain in a neighborhood of their corresponding unique regime values.

We now study and interpret some properties of the value function in order to characterize the optimal decision rule.

3.2. Derivation of the Optimal Policy

3.2.1. Some Properties of the Value Function and Option Value

The value of the firm V_i has two components: operating a certain technology generates some cash flows and in addition the manager has the option to update the firm's technology, which is also valuable. The value of operating forever the same technology u_0 adopted at time 0 when the initial state of the economy is i is

$$G_i(u_0) = E_0^i \int_0^\infty X(s) u_0(s)^\alpha e^{-rs} ds.$$

The option value is

$$F_i(u_0) = V_i(u_0) - G_i(u_0).$$

It represents the cumulative discounted gap between cash flows obtained after sequentially implementing new technologies minus the cost of adoption and cash flows that would have been realized by sticking to the initial technology.

Proposition 3.1. *The value of the firm V_i takes finite values and switching occurs within a finite time almost surely. V_i and the value of operating forever the same technology G_i are increasing in the technology level u. The option value function F_i is decreasing in u. Moreover, given u, the value of the firm, the option value and the value of operating forever the same technology when starting in a boom is higher than when starting in recession, i.e., $V_1 \leq V_2$, $F_1 \leq F_2$ and $G_1 \leq G_2$.*

Proof. See appendix 1. ■

The value of operating forever the same technology G_i is increasing in u and the option value F_i is decreasing in u since when u is high there is less room for benefits from improving technology. There is a trade-off between G_i and F_i. Right after the adoption of new

[4] See appendix 3

technology and for a while, G_i is "dominating" F_i. As time passes, technology level u declines and, at some point, F_i will "dominate" G_i. A new adoption will occur.

We now characterize the inaction region IR_i.

Proposition 3.2. *Given the state of the world i, the inaction region IR_i is connected and characterized by*

$$IR_i = \left\{u \in \mathbb{R}_+,\ 0 < V_i'(u)\right\} \ .$$

The manager does not upgrade as long as the marginal value of operating the technology remains positive. Moreover, in the inaction region, the value function V_i satisfies the usual no arbitrage relationship

$$rV_i(u) = \pi\left(u, X_i\right) + \frac{E_t^i\left[dV\right]}{dt} \ .$$

The return from investing in a firm of value V_i at the risk free rate r during a period of time dt is the sum of the instantaneous dividend, or profit, $\pi\left(u, X_i\right) dt$ plus the expected capital gain $E_t^i\left[dV\right]$.

Proof. See appendix 1. ■

Proposition 3.3. *Scrapping occurs at an earlier stage in a boom with respect to a recession, i.e., $u_1^* < u_2^*$.*

Proof. See appendix 1. ■

From the instantaneous profit function Xu^α, it is easy to see that the marginal profit $\alpha X u^{\alpha-1}$ is higher in a boom than in a recession. Thus, booms provide more incentives to operate a technology close to the frontier, leading to scrap at an earlier stage.

We now decompose the analysis of the firm's problem in two parts:

1. We investigate the shape of the value function V_i in three regions: the first one is when $u \in IR_1 \cap IR_2$, i.e., $u > u_2^*$, the second one is when $u \in IR_1 \backslash IR_2$, i.e., $u_1^* < u \leq u_2^*$ and the last one is when $u \notin IR_1 \cup IR_2$, i.e., $u \leq u_1^*$.

2. We solve the free boundary problem, i.e., determine the boundary ∂IR_i of the inaction region IR_i for $i = 1, 2$.

3.2.2. Derivation of the Value Function

Region $u_2^* < u$. Starting in the state (u, i) with u in $IR_1 \cap IR_2$, the value function V_i over a small internal of time dt needs to satisfy the following Bellman equation

$$V_i(u)e^{rdt} = \pi(u, X_i)dt + (1 - \phi_i dt)E_u\left[V_i(u + du)\right] + (\phi_i dt)E_u\left[V_j(u + du)\right] \ . \tag{3.3}$$

Expanding the right hand side of (3.3) using Ito's lemma and retaining only terms in dt, after simplification, V_1 and V_2 are characterized by the following system of differential equations:

$$\begin{aligned} rV_1(u) &= X_1 u^\alpha - \lambda u V_1'(u) + \frac{\sigma^2}{2} u^2 V_1''(u) + \phi_1\left[V_2(u) - V_1(u)\right] \\ rV_2(u) &= X_2 u^\alpha - \lambda u V_2'(u) + \frac{\sigma^2}{2} u^2 V_2''(u) + \phi_2\left[V_1(u) - V_2(u)\right] \ . \end{aligned}$$

Details of the derivation of the analytical expressions for V_1 and V_2 are presented in appendix 1. We obtain:

$$\begin{aligned} V_1(u) &= B_1 u^{\alpha} + H u^{-\gamma} - \phi_1 K u^{-\delta} && (3.4) \\ V_2(u) &= B_2 u^{\alpha} + H u^{-\gamma} + \phi_2 K u^{-\delta} \ . && (3.5) \end{aligned}$$

where

$$\begin{aligned} B_1 &= \frac{1}{\phi_2 + \phi_1} \left[(\phi_2 B + \phi_1 A) X_1 + \phi_1 (B - A) X_2 \right] \\ B_2 &= \frac{1}{\phi_2 + \phi_1} \left[\phi_2 (B - A) X_1 + (\phi_1 B + \phi_2 A) X_2 \right] \ , \end{aligned}$$

$$A = \frac{1}{r + \phi_1 + \phi_2 + \alpha\lambda + \frac{\alpha(1-\alpha)\sigma^2}{2}} \text{ and } B = \frac{1}{r + \alpha\lambda + \frac{\alpha(1-\alpha)\sigma^2}{2}} \ ,$$

γ and δ are the positive roots of the following quadratic equations

$$\begin{aligned} \frac{\sigma^2}{2}\gamma^2 + (\frac{\sigma^2}{2} + \lambda)\gamma &= r \\ \frac{\sigma^2}{2}\delta^2 + (\frac{\sigma^2}{2} + \lambda)\delta &= r + \phi_1 + \phi_2 \ , \end{aligned}$$

and finally, (K, H) are two constants to be determined.

As mentioned, the value function is the sum of the value of operating forever the same technology G_i and the option value F_i. By identification, we obtain

$$\begin{aligned} G_1(u) &= B_1 u^{\alpha} && (3.6) \\ G_2(u) &= B_2 u^{\alpha} \ . && (3.7) \end{aligned}$$

From (3.6) and (3.7), we verify that $G_2(u) > G_1(u)$ for all u as $B_2 > B_1$.

The option value is

$$\begin{aligned} F_1(u) &= H u^{-\gamma} - \phi_1 K u^{-\delta} && (3.8) \\ F_2(u) &= H u^{-\gamma} + \phi_2 K u^{-\delta} \ . && (3.9) \end{aligned}$$

Since $\gamma < \delta$, when u becomes large, $F_i(u)$ behaves as $Hu^{-\gamma}$. Since the option value of the firm has to be positive, H must also be positive. In addition, since $F_1 \leq F_2$, from (3.8) and (3.9), it follows that $K \geq 0$.

Region $u_1^* < u \leq u_2^*$. The evolution of V_2 is now different. In fact, for all $u \leq u_2^*$, $V_2(u)$ is a constant equal to $V_2(u_2^*)$. To see this, note that if $u \leq u_2^*$ and the economy is experiencing a boom, the adoption of a new technology occurs on the spot. As the value of the firm before adoption $V_2(u)$ has to be equal to the value of the firm after adoption minus the cost of adoption, we must have $V_2(v_2^*) - cv_2^* = V_2(u_2^*)$. This implies that for all $u \leq u_2^*$,

$V_2(u) = V_2(u_2^*)$. In this sense, the optimal policy differs from regular (s, S) policies. There is a whole range of technologies $[u_1^*, u_2^*]$ within which switching is optimal as soon as the economy turns into a boom. Hence, V_1 evolves as follows

$$rV_1(u) = X_1 u^\alpha - \lambda u V_1'(u) + \frac{\sigma^2}{2} u^2 V_1''(u) + \phi_1 \left[V_2(u_2^*) - V_1(u)\right] \ . \tag{3.10}$$

The general solution to the differential equation (3.10) is

$$V_1(u) = \frac{\phi_1 V_2(u_2^*)}{r + \phi_1} + C_1 u^\alpha + D u^\theta + E u^\eta \ , \tag{3.11}$$

where θ and η are respectively the positive and the negative roots of the following quadratic

$$\frac{\sigma^2}{2} x^2 - (\frac{\sigma^2}{2} + \lambda) x = r + \phi_1 \ ,$$

$$C_1 = \frac{X_1}{r + \phi_1 + \alpha\lambda + \frac{\alpha(1-\alpha)\sigma^2}{2}} \ ,$$

and D and E are two constants to be determined.

Region $u \leq u_1^*$. In this region, whatever the state of the economy is, scrapping occurs on the spot and by continuity V_1 and V_2 are given by

$$\begin{aligned} V_1(u) &= V_1(u_1^*) \\ V_2(u) &= V_2(u_2^*) \ . \end{aligned}$$

It remains to determine the four constants D, E, H, K and the four thresholds u_1^*, v_1^*, u_2^* and v_2^*.

3.2.3. Derivation of the Boundaries

There are eight unknowns to be determined: the four thresholds and the four constants. We start to derive the optimal conditions for the scrapping and target levels of technology.

Proposition 3.4. *For all the possible states $i \in \{1, 2\}$ of the economy, the decision of adopting a new technology is characterized by a scrapping threshold u_i^* and a target upgrading level v_i^* satisfying the following optimal conditions*

$$\begin{aligned} u_i^* &\in \arg\min_{u \in [0,1]} V_i(u) \\ v_i^* &\in \arg\max_{v \in [0,1]} V_i(v) - cv \\ V_i(u_i^*) &= V_i(v_i^*) - c v_i^* \ . \end{aligned}$$

The first and second conditions are called smooth pasting conditions. The first condition actually means that the manager decides to upgrade exactly when the operated technology

u_i^* *achieved a minimum for* V_i, *thus* $G_i'(u_i^*)+F_i'(u_i^*)=0$. *The interpretation goes as follows: the marginal benefit of continuing to operate the same technology forever,* $G_i'(u_i^*)$, *is equal to the marginal opportunity cost of scrapping,* $-F_i'(u_i^*)$.[5] *The second condition is a "natural" maximization condition and for further details, see Harisson, Sellke and Taylor (1983)). The third condition is the* matching condition *and simply says that when switching technology occurs, the value of the firm must satisfy the relationship* $\min_{u\in[0,1]} V_i(u) = \max_{v\in[0,1]} V_i(v) - cv$, *which is a straightforward condition given equation (3.2) defining the inaction region* IR_i.

Proof. See appendix 2. ■

So far, we have six equations (two times three optimal conditions for each regime). We need two more relationships. At the point u_2^*, we require V_1 to be continuously differentiable (matching and smooth pasting conditions) using the analytical expressions found in (3.4) and (3.11). Consequently, we obtain a non linear system of *eight* equations with *eight* unknowns.

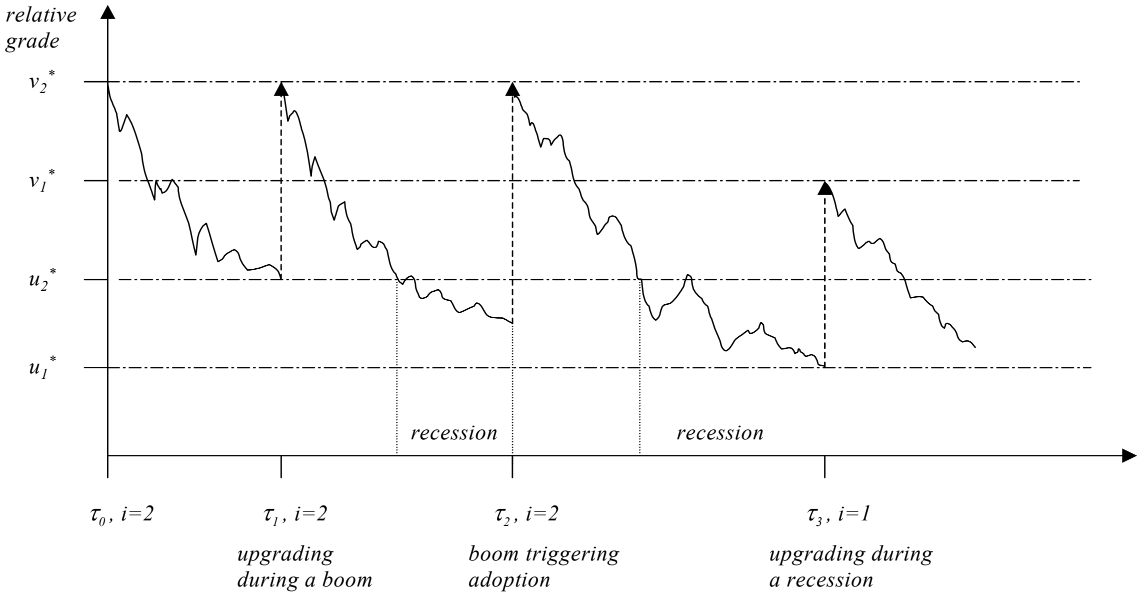
The next paragraph summarizes the results obtained and provides a complete description of the optimal policy.

3.2.4. The Optimal Decision Rule

We first characterize the optimal policy of the firm. Then, we stress the existence of two cases depending on whether the upgrading level in recession v_1^* is greater or lower than the scrapping level in boom u_2^*. We interpret these two cases in terms of relative magnitude of fluctuations of the cycles.

Proposition 3.5. *Denoting by* $u=\frac{a}{A}$ *the relative technological position of the firm, the manager chooses the following optimal policy illustrated by the graph below:*

[5] Recall that the opportunity cost of scrapping at date 0 is $SC_i(u_0) = E_0^i\left[\int_0^\infty \pi\left(u(u_0,s),X(s)\right)e^{-rs}ds\right]$, where for convenient reasons we denote here $u_0(s)$ by $u(u_0,s)$ and the current profit $X(s)u_0^\alpha(s)$ by $\pi\left(u(u_0,s),X(s)\right)$. Thus, the marginal opportunity cost at $u_0=u_i^*$ is: $MSC_i(u_i^*) = E_0^i\left[\int_0^\infty \pi_1\left(u(u_i^*,s),X(s)\right)\frac{\partial u(u_i^*,s)}{\partial u_0}e^{-rs}ds\right]$ and as shown in appendix 1, when $\tau^*=0$, $F_i'(u_i^*) = -E_0^i\left[\int_0^\infty \pi_1\left(u(u_i^*,s),X(s)\right)\frac{\partial u(u_i^*,s)}{\partial u_0}e^{-rs}ds\right]$.



Whatever market conditions are, the manager remains inactive as long as the technology of the firm is above a threshold u_2^*. When the level of technology operated reaches u_2^*, she upgrades the firm's relative technology to v_2^* providing the economy is in a boom (upgrading during a boom on the graph), otherwise she waits. If the economy is in a recession and the level of technology operated is in the interval $[u_1^*, u_2^*]$, switching occurs at any level as soon as the economy moves into a boom (boom triggering adoption on the graph). Finally, if the recession continues, the manager upgrades to level v_1^* when the technology level reaches u_1^* (upgrading during a recession on the graph). Moreover, the four thresholds $(u_1^*, v_1^*, u_2^*, v_2^*)$ are fully characterized by the following system

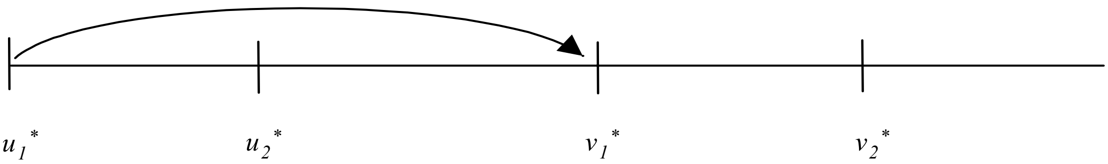
$V_1'(u_1^*) = 0$	(1)	$V_2'(u_2^*) = 0$	(4)	$V_1(u_2^{*-}) = V_1(u_2^{*+})$	(7)
$V_1'(v_1^*) = c$ or $v_1^* = 1$	(2)	$V_2'(v_2^*) = c$ or $v_2^* = 1$	(5)	$V_1'(u_2^{*-}) = V_1'(u_2^{*+})$.	(8)
$V_1(u_1^*) = V_1(v_1^*) - cv_1^*$	(3)	$V_2(u_2^*) = V_2(v_2^*) - cv_2^*$	(6)		

Proof.

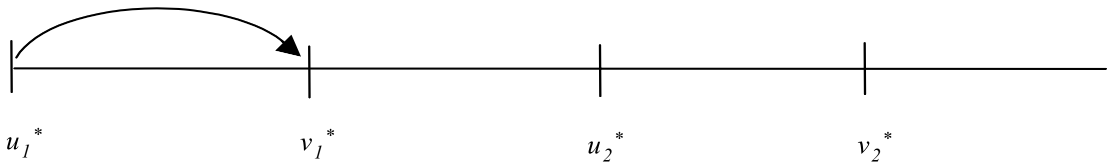
Using relationships (3.4), (3.5) and (3.11), it suffices to write the six optimal conditions and the two smooth pasting conditions for V_1 at u_2^*. The corresponding analytical equations are reported in appendix 2. ■

Equation (1) and (2) characterize the scrapping level u_1^* and upgrading level v_1^* relevant for recessions, with potentially adoption of the cutting edge technology. Equation (3) simply means that the value of the firm before switching in a recession plus the cost of switching is equal to the value of the firm after upgrading. Equations (4), (5) and (6) are mirror conditions for the case when upgrading occurs in a boom. Equations (7) and (8) formalize the smooth pasting conditions for the value of the firm V_1 at the point u_2^*.

We now examine the two possible cases mentioned above. Since V_1 has different analytical expressions when $u \leq u_2^*$ (equation (3.11)) and $u \geq u_2^*$ (equation (3.4)), we need to be specific in the way of writing the optimal condition (2) regarding the quality of the technology adopted in a recession. The case $v_1^* < u_2^*$ implies that the manager of the firm is not willing to operate a technology even recently adopted in a recession if a boom occurs. This means that the demand level must be much higher in a boom than in a recession so it is worth incurring again the opportunity cost of scrapping technology. It is convenient to define a new parameter $m = \frac{X_2}{X_1} > 1$ that captures the relative magnitude of booms with respect to recessions. The case $v_1^* < u_2^*$ occur when m is very large. We have chosen to focus on the second case where $v_1^* > u_2^*$ or equivalently when m is not too large as it seems more plausible from an empirical point of view.[6] The following diagram illustrates the two cases.



Cycles of low magnitude: small m



Cycles of high magnitude: large m

We now examine some sufficient conditions to ensure $v_1^* > u_2^*$. Denoting by $\overline{u}_1$ and $\overline{v}_1$ ($\overline{u}_2$ and $\overline{v}_2$) the scrapping and upgrading thresholds for **an economy always in a recession (boom)** respectively, one way to ensure the desired condition is to choose the parameters of the model in such a way that $\overline{u}_1 < \overline{u}_2 < \overline{v}_1 < \overline{v}_2$. To see this, recall that we have $\overline{u}_1 < u_1^*$, $u_2^* < \overline{u}_2$, $\overline{v}_1 < v_1^*$ and $v_2^* < \overline{v}_2$. Indeed, it implies $v_1^* > u_2^*$. A *necessary* condition for $u_2^* < v_1^*$ is given by the following lemma.

Lemma 3.6. *Assume that $v_2^* < 1$. Define implicitly the number $x > 1$ as the unique root in the interval $[1, \infty)$ of the equation*

$$\gamma(1-\alpha)x^{\alpha+\gamma} - (\alpha+\gamma)x^{\gamma} + \alpha(1+\gamma) = 0.$$

When the ratio of the demand levels $m = \frac{X_2}{X_1}$ satisfies

$$m < x^{1-\alpha},$$

[6] As reported in appendix 2, the optimal condition (2) is therefore obtained using relationship (3.4) or is $v_1^* = 1$ in the corner solution case.

then $\overline{u}_2 < \overline{v}_1$ *and therefore* $u_2^* < v_1^*$. *When* $v_2^* = 1$, *a necessary condition is*

$$\Phi(\left[\frac{(\alpha+\gamma)B\left[1-x^{-\alpha}\right]}{(1+\gamma)c}X_1\right]^{\frac{1}{1-\alpha}}) < 0,$$

with

$$\Phi(y) = \alpha y^{\alpha+\gamma} - (\alpha+\gamma)y^{\alpha} + \gamma(1 - \frac{c}{BX_2}).$$

Proof. See appendix 3. ■

In section 4, numerical simulations are run for the case when m is small. We briefly indicate how the comparative static results greatly differ in the case when m is very large.

3.2.5. Benchmark Case: Unique Regime

This case is interesting *per se* as it provides a good approximation for economies where a regime is very persistent. As we impose $X_1 = X_2 = X$, it is easy to see that $V_1(u) = V_2(u) \triangleq V(u)$ and $u_1^* = u_2^* \triangleq u^*$, $v_1^* = v_2^* \triangleq v^*$. It follows that $V(u) = BXu^{\alpha} + Hu^{-\gamma}$. Economies in recessions differ from economies in expansion as the frontier technology may not be adopted when the market demand is too weak. We distinguish two cases.

Recessions When the demand is low, the firm may not have enough incentives to implement the cutting edge technology.[7] Formally, we have an interior solution and the thresholds (u^*, v^*) are characterized by the following system

$$(\alpha+\gamma)BX\left[(v^*)^{\alpha} - (u^*)^{\alpha}\right] = (1+\gamma)cv^* \tag{3.12}$$

$$\alpha BX\left[(v^*)^{\alpha+\gamma} - (u^*)^{\alpha+\gamma}\right] = c(v^*)^{1+\gamma} \ . \tag{3.13}$$

Expansions Conversely when the demand is high enough ($X \geq \overline{X}$), the firm buys the leading technology. We have a corner solution, $v^* = 1$ and the scrapping threshold u^* is the unique root in $(0,1)$ of the equation

$$\alpha BXu^{\alpha+\gamma} - (\alpha+\gamma)BXu^{\alpha} + \gamma BX - \gamma c = 0 \ . \tag{3.14}$$

During periods of expansion, the firm aims at operating a technology close to the top of the line in order to enjoy high profits. The implication is that the frequency of adoptions is enhanced for economies experiencing booms. In appendix 3, existence and uniqueness of the thresholds for both recessions and expansions are proved and we show that both scrapping and upgrading levels are non decreasing in the demand level X and non increasing in the adoption cost c. Jovanovic and Rob (1998) also obtain that the price of machines lowers the scrapping trigger.

[7] More precisely, the recession case corresponds to a market in which the demand X is less than an upper bound $\overline{X}$ obtained by setting $v^* = 1$ and eliminating u^* in equations (3.12) and (3.13).

Remark 2. *In the extreme case where* $\phi_1 = \phi_2 = \infty$, *since the operators* E_0^1 *and* E_0^2 *are the same, then* V_1 *and* V_2 *are equal and the scrapping thresholds and adoption levels are also the same. This implies* $K = 0$ *and* $G(u) = \frac{1}{2}(X_1 + X_2)Bu^{\alpha}$. *It follows that* $V(u) = \frac{1}{2}B(X_1 + X_2)u^{\alpha} + Hu^{-\gamma}$. *The resolution of the problem is similar to the unique regime case.*

We now aim at examining the effects of uncertainty on the frequency of adoptions. Our next step is to determine the expected time between two consecutive adoptions.

3.3. Expected Time between Two Adoptions

Starting from an upgrading level v_i^*, the expected time between two adoptions is the average time it takes to the technology process to reach a scrapping level in $[u_1^*, u_2^*]$. The frequency of adoption is governed by three factors:

- Technological uncertainty affects the law of motion of the technology process u altering the velocity at which technology declines from the upgrading level down to the scrapping level. We will refer to this change in velocity as the *direct effect*;

- Both market and technological uncertainty affect the range of operation of a given technology by having an impact on the scrapping and upgrading levels. As we consider a geometric Brownian Motion for modeling technology, only the geometric distance, i.e., the ratio between the upgrading and scrapping levels actually matters. Since scrapping does not occur at a fixed level, this effect is hard to access directly. Nevertheless, determining the effects of uncertainty on the ratio $\frac{v_i^*}{u_i^*}$ is a good proxy for estimating the strength of this factor. We will refer to the shift in the operation range as the *indirect effect*;

- Finally, since technological progress is limited in our framework, we may encounter a corner effect when the manager is constrained to implement the cutting edge technology even though she wishes that a more advanced technology had been invented. This effect is likely to arise when adoption takes place during a persistent boom. As a consequence, economies experiencing lasting periods of high activity should display frequent adoptions as the firm aims at operating a technology close to the cutting edge technology to enjoy high profits levels. The firm can at most adopt the frontier technology. If the frontier is not moving fast enough, the firm may have incentives to upgrade frequently. Cooper, Haltiwanger and Power (1999) establish some empirical evidence of a higher frequency of investment bursts during period of expansions.

Unlike most of the models dealing with adoption of technology developed in the literature such as Cooley, Greenwood and Yorukoglu (1997), Caballero and Hammour (1994), Farzin, Huisman and Kort (1998), Goolsbee (1998) among others, here the manager is *free to choose what type of new technology to implement*, provided it has been invented at the time of adoption. Abel and Eberly (1998) modeling demand uncertainty by a geometric Brownian Motion show that the direct and the indirect effects of market uncertainty are to increase the expected time between two investments but do not impose any restriction on the size of capital purchases. We now express the average time between two adoptions.

Proposition 3.7. *Adopting a new technology v_1^* in a recession, respectively v_2^* in a boom, the expected time until the next adoption $E_1(\tau)$, respectively $E_2(\tau)$, has the following expression*

$$E_1(\tau) = \frac{\ln\frac{v_1^*}{u_2^*}}{\lambda+\frac{\sigma^2}{2}} + \phi_1\Lambda\left(\frac{\phi_2}{\phi_1} + (\frac{v_1^*}{u_2^*})^{-\beta}\right) \tag{3.15}$$

$$E_2(\tau) = \frac{\ln\frac{v_2^*}{u_2^*}}{\lambda+\frac{\sigma^2}{2}} + \phi_2\Lambda\left(1 - (\frac{v_2^*}{u_2^*})^{-\beta}\right) , \tag{3.16}$$

with β being the positive root of the quadratic

$$\frac{\sigma^2}{2}x^2 + (\lambda + \frac{\sigma^2}{2})x - (\phi_1+\phi_2) = 0 ,$$

ξ and ζ the roots of the quadratic

$$\frac{\sigma^2}{2}x^2 - (\lambda + \frac{\sigma^2}{2})x - \phi_1 = 0 ,$$

and

$$\Lambda = \frac{1}{\phi_1}\left[\frac{1}{\phi_2+\phi_1} + \frac{(\xi-\zeta)+\phi_1\left(\frac{1}{\lambda+\frac{\sigma^2}{2}} - \frac{\beta}{\phi_2+\phi_1}\right)\left((\frac{u_1^*}{u_2^*})^{\xi} - (\frac{u_1^*}{u_2^*})^{\zeta}\right)}{((\phi_2+\phi_1)\xi+\beta\phi_1)\,(\frac{u_1^*}{u_2^*})^{\zeta} - ((\phi_2+\phi_1)\zeta+\beta\phi_1)\,(\frac{u_1^*}{u_2^*})^{\xi}}\right] \geq 0 .$$

Proof. The central idea of the proof relies on the construction of an appropriate martingale in order to be able to use the *Optional Stopping Theorem.* A complete proof is provided in appendix 4. ■

From relationships (3.15) and (3.16), we observe that the expected time $E_i(\tau)$ has two components. The first term $\frac{\ln\frac{v_i^*}{u_2^*}}{\lambda+\frac{\sigma^2}{2}}$ represents the average time between adoption at level v_i^* and scrapping at levels u_2^* relevant for booms. The second term is some additional time since scrapping may occur at a lower level than u_2^* if the economy is in a recession when u reaches u_2^*. Recall that we show that the inaction region IR_i is an interval with lower bound u_i^* for $i = 1, 2$. Conditional on holding technology u, the average time until the next adoption must be increasing in u which implies that the older the technology operated, the sooner the firm will upgrade. Goolsbee (1998) finds that older planes are more likely to be retired that younger ones.

Usually, empirical data on the frequency of technology adoptions conditional on the initial state of the economy are not available. Recall that the fraction of time spent in regime i is $\frac{\phi_j}{\phi_i+\phi_j}$. The overall average time between two consecutive adoptions is[8]

[8] This relationship can also be obtained in a more formal way using the ergodicity of the Markov chain X and applying the Mean Ergodic Theorem.

$$E(\tau) = \frac{\phi_2}{\phi_1 + \phi_2} E_1(\tau) + \frac{\phi_1}{\phi_1 + \phi_2} E_2(\tau) \ .$$

In the unique regime case, $\Lambda = 0$ and we have

$$E(\tau) = \frac{\ln \frac{v^*}{u^*}}{\lambda + \frac{\sigma^2}{2}} \ .$$

In this case, $E(\tau)$ increases with the level of technology adopted v^* and decreases with the scrapping threshold u^*. The direct effect is encapsulated through the term in σ^2 at the denominator and the ratio $\frac{v^*}{u^*}$ captures the indirect effect.

The next paragraph deals with the timing of adoption and the business cycle. To address the issue, we compute the probability that the next adoption takes place in a recession versus a boom.

3.4. Likelihood that the Next Adoption Occurs in a Recession

Answering the question –are replacement and adoption of technologies more likely to take place in a recession versus an expansion?– is, indeed, a hard task. Using data from the Automobile industry over the period 1978-1985, Cooper and Haltiwanger (1994) develop and test a model in which retooling mainly occurs at the end of a downturn when the opportunity cost of labor is low, i.e., when low seasonal demand (or high value of leisure) and just prior to upturns where the benefits of replacement are high. They assume cycles to be deterministic and the study mostly focuses on seasonal cycles rather than business cycles which are more persistent. In a companion paper joined with Power (1999), using a stochastic dynamic framework, they establish that the higher fixed adjustment costs and shocks persistence, the more likely replacement investment is to be procyclical. Caballero and Hammour (1994) introduce a vintage model of "creative destruction" showing that job destruction is more cyclically responsive than job creation. They emphasize the *cleansing effect of recessions*, period of times when outdated or relatively unprofitable techniques exit the productive system. In Klenow (1998), learning by doing is faster when output is high inducing firms to upgrade preferably during booms. Obviously, the more persistent a state is, the higher will be the chances to adopt during such a state. It may also be the case that companies choose recession times to get rid off their old pieces of equipment, take advantage of low economic activity for restructuring, and concentrate the purchase of new vintage during good times. Campbell (1998) finds positive correlation between the exit rate and recessions and between the entry rate and expansions. Given the persistence of the states, since in our model scrapping occurs earlier during a boom, it seems more likely that adoption occurs during a boom. Nevertheless, if physical depreciation (due to technological uncertainty) or economical depreciation (obsolescence) is significant, it may be the case that having upgraded during a recession, the firm upgrades again in the *same recession*. Two additional important factors on which our model remains silent need to be outlined. The first one is *time to build* and the second one is *time to learn*. If there is a lag between adoption and production at an efficient level, this reduces discounted benefits from updating which might induces plants to scrap before periods of high economic activity to be ready for good times. Bar Ilan and Strange (1996) discuss the effects of investment lags. They find that with a short lag, a rise in uncertainty postpones investment. Conversely, with a long lag (plant construction, specific training ...), it is possible that more uncertainty hastens investment.

We know express the probability that the next adoption occurs in a recession.

Proposition 3.8. *Being in a recession (respectively a boom) and operating technology level u, the probability that the next switch occurs in a recession $p_{11}(u)$ (respectively $p_{12}(u)$) is given by:*

For $u_1^* \leq u \leq u_2^*$,

$$p_{11}(u) = \frac{((\phi_2+\phi_1)\xi+\beta\phi_1)\,(\frac{u}{u_2^*})^{\zeta} - ((\phi_2+\phi_1)\zeta+\beta\phi_1)\,(\frac{u}{u_2^*})^{\xi}}{((\phi_2+\phi_1)\xi+\beta\phi_1)\,(\frac{u_1^*}{u_2^*})^{\zeta} - ((\phi_2+\phi_1)\zeta+\beta\phi_1)\,(\frac{u_1^*}{u_2^*})^{\xi}} \tag{3.17}$$

$$p_{12}(u) = 0 \tag{3.18}$$

and for $u_2^* \leq u$,

$$p_{11}(u) = \frac{(\xi-\zeta)\left(\phi_2+\phi_1(\frac{u}{u_2^*})^{-\beta}\right)}{((\phi_2+\phi_1)\xi+\beta\phi_1)\,(\frac{u_1^*}{u_2^*})^{\zeta} - ((\phi_2+\phi_1)\zeta+\beta\phi_1)\,(\frac{u_1^*}{u_2^*})^{\xi}} \tag{3.19}$$

$$p_{12}(u) = \frac{\phi_2(\xi-\zeta)\left(1-(\frac{u}{u_2^*})^{-\beta}\right)}{((\phi_2+\phi_1)\xi+\beta\phi_1)\,(\frac{u_1^*}{u_2^*})^{\zeta} - ((\phi_2+\phi_1)\zeta+\beta\phi_1)\,(\frac{u_1^*}{u_2^*})^{\xi}} \tag{3.20}$$

Proof. See appendix 5. ■

We can also compute an unconditional probability that the next adoption occurs in a recession, operating today technology level u,

$$p_1(u) = \frac{\phi_2}{\phi_1+\phi_2}p_{11}(u) + \frac{\phi_1}{\phi_1+\phi_2}p_{12}(u) \ .$$

From equations (3.17) – (3.20), it is easy to show that for $u_2^* \leq u$, p_1 is independent of u (u lies in the intersection of the inaction regions). For $u_1^* \leq u \leq u_2^*$, p_1 is decreasing in u: a boom can trigger replacement and the farther u is from u_1^*, the more likely such an event. Moreover, for $u \leq u_1^*$, $p_1(u) = \frac{\phi_2}{\phi_2+\phi_1}$.

In the next section, we present some numerical simulations displaying the effects of the two types of uncertainty on the optimal policies and the frequency of adoptions.

4. COMPARATIVE STATICS AND NUMERICAL SIMULATIONS

In this section, we aim at investigating the effects of changes in regime persistence and technological uncertainty on the decisions and frequency of adoption of new technologies. We start by examining the effects of changes of uncertainty on the expected current profit in order to get some intuition about the mechanisms of adoption decisions.

4.1. Effects on the Regime Persistence and Technology Uncertainty on Instantaneous Expected Profit

Given an initial state of the economy, we analyze how current expected profit responds to changes in market and technological uncertainty. We first examine the effects of independent change in the arrival rate of booms and recessions ϕ_1 and ϕ_2. Then, we impose an equal arrival rate of regimes ϕ and investigate changes in current expected cash flows. Finally, we let technological uncertainty σ vary.

Proposition 4.1. *Ceteris paribus, when the arrival rate of a boom ϕ_1 (recession ϕ_2) rises, the current expected profit rises (falls); when the arrival rate of recessions and booms are equal and rises, being in a recession, the current expected profit rises and being in a boom, the current expected profit decreases. Finally, ceteris paribus, due to the decreasing return in the technology grade of the profit function, when the technological uncertainty σ rises, the current expected profit decreases.*

Proof. See appendix 6. ■

These effects also apply to the value of operating the same technology G_i forever. Intuitively, one can expect that any change in the uncertainty parameter that increases the current expected profit provides more incentives for the manager to invest in technology, i.e., the firm should be willing to operate a sophisticated technology to enjoy high cash flows. In particular, scrapping should occur early and the quality of the new technology adopted should be increased.

We now display and comment some numerical simulations obtained for the following set of parameters:

Rate of technological progress:	$\mu = 0.025$
Interest rate:	$\rho = 0.06125$
Adoption cost per efficiency unit:	$c = 2$
Rate of physical depreciation:	$\kappa = 0.075$
Parameter:	$\alpha = 0.5$
Level of demand in recessions:	$X_1 = 1$
Relative magnitude of cycles:	$m = \sqrt{2}$

This implies $\lambda = 0.1$ and $r = 0.05$. The rate of technological progress is the same as in Klenow (1998). Other parameters have been selected such that during booms, the firm adopts the frontier technology and during recessions, the firm upgrades to an intermediate technology in general. All the results obtained making vary the arrival rate of recessions ϕ_2 are reversed results obtained when the arrival rate of booms ϕ_1 varies, the economic interpretation remaining valid. For the sake of completeness, corresponding graphs are displayed at the end of the paper. The methodology used for the numerical computations is described at the end of appendix 6.

4.2. Effects of Changes in Uncertainty on Adoption Decisions

4.2.1. Effects of Regime Persistence

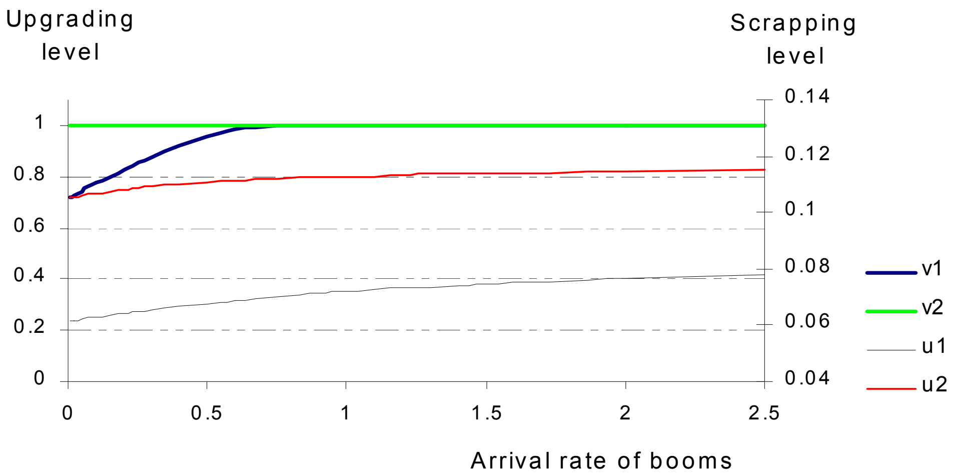


Figure 1: Effects of the arrival rate of booms ϕ_1; $\phi_2 = 0.25$ and $\sigma = 1$.

Ceteris paribus, when the arrival rate of booms ϕ_1 increases, both scrapping and upgrading levels rise. Since the expected profit is higher on average, the firm is concerned with operating a highly performing technology which can be the cutting edge technology if booms are frequent enough.

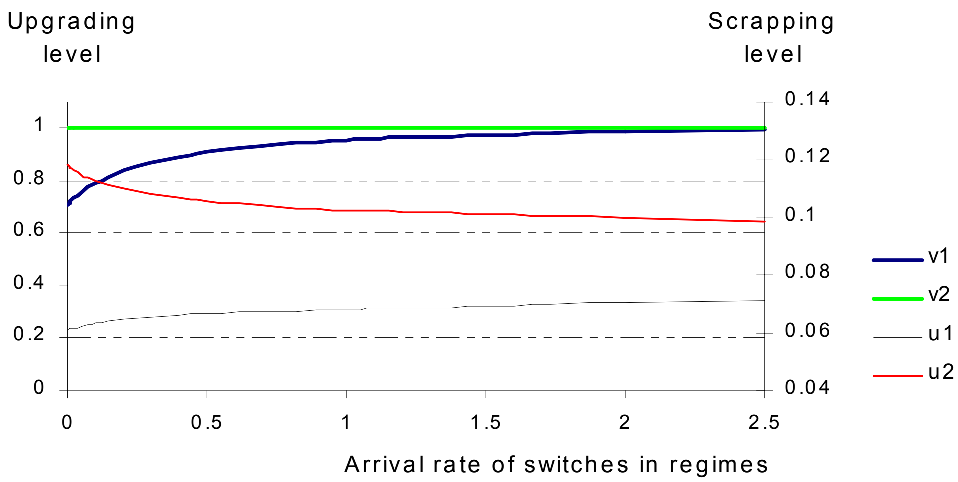


Figure 2: Effects of equal arrival rates of regimes ϕ; $\sigma = 1$.

Two antagonistic effects arise: less persistent booms give fewer incentives for adopting a good technology whereas less persistent recessions provide more incentives. When ϕ increases, scrapping occurs in a recession at an earlier stage and the quality of the new technology adopted is higher as the manager expects a better economic activity. Conversely, scrapping occurs in a boom at a later stage and the quality of the new technology adopted may be reduced as the manager expects a drop of the cash flows caused by less persistent booms. Indeed, discounting plays an important role: the manager is mainly concerned with the degree of persistence of the state of the economy in which adoption takes place. For instance, even though the manager knows that booms never last for very long, she demonstrates a more optimistic behavior when switching in a non lasting recession.

4.2.2. Effects of Technological Uncertainty

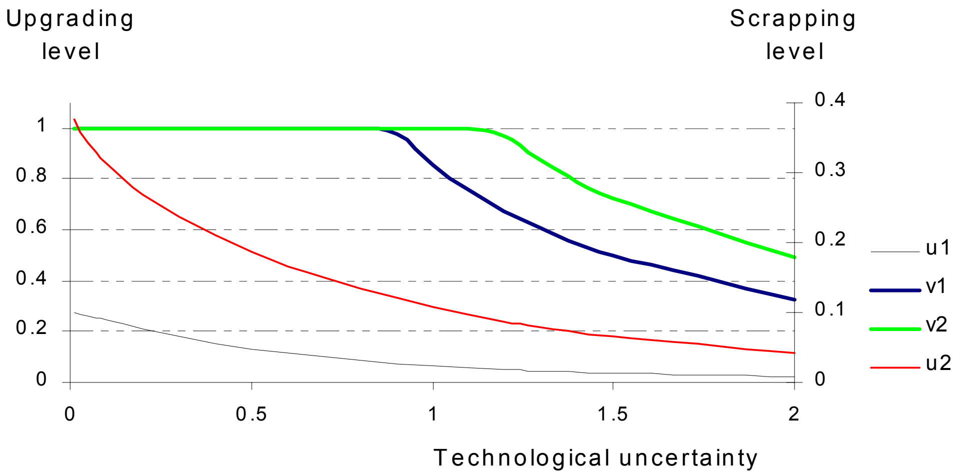


Figure 3: Effects of technological uncertainty σ; $\phi_1 = \phi_2 = 0.25$.

An increase in technological uncertainty σ reduced the current expected profit which provides fewer incentives to the manager to invest in a good technology. Consequently, she decides to delay scrapping and in general adopt a less sophisticated technology. Recall that we interpret σ as a measure of the firm's human capital: a low value for σ corresponds to a firm employing high skilled workers. Many studies have outlined the technology-skill complementarity. Doms, Dunne and Troske (1997) show that US. manufacturing plants utilizing more advanced technologies employ more educated workers.

Numerical simulations are indeed consistent with the comparative static results depicted in the previous paragraph: changes in parameters that increase current expected profit lead to adoption of a more advanced technology and earlier scrapping.

Remark 3. *We have also run some simulations when m is equal to* 4, *i.e., when cycles display a high relative magnitude. In this case, it is possible to have $u_1^* < v_1^* < u_2^* < v_2^*$. The main difference with what precedes is that if recessions are not persistent enough,* ***the firm becomes very reluctant to upgrade technology within a recession****. This does not imply that it never does. Simply, the firm has incentives in delaying scrapping and* *adopting a low grade technology when choosing to upgrade within a recession. The optimal policy consists of slightly improving the grade of the technology operated in order to wait and switching again as soon as a boom occurs. As technologies are fully incompatible and it is not optimal to keep on operating a technology adopted in a recession when the economy turns into a boom, the firm tries to minimize the opportunity cost induced by scrapping. When the arrival rate of booms ϕ_1 increases, both scrapping level u_1^* and adoption level v_1^* go down. On the contrary, the levels u_2^* and v_2^* go up as the manager of the firm becomes more optimistic when adopting during a boom.*

4.3. Effects of Changes in Uncertainty on the Frequency of Adoptions

Numerical simulations (not displayed here) indicate that in general the conditional average times $E_1(\tau)$ and $E_2(\tau)$ do not respond in a monotonic way to changes in uncertainty parameters and, both cases $E_1(\tau) < E_2(\tau)$ and $E_1(\tau) > E_2(\tau)$ can be encountered. As already mentioned, we use as a proxy the ratio of the upgrading level divided by the scrapping level corresponding to the state in which scrapping is most likely to occur. As a general trend, when the cutting edge technology is not adopted, the scrapping threshold u_j^* is more sensitive to uncertainty than the upgrading level v_i^*, i.e., $\left|\frac{\Delta u_j^*}{u_j^*}\right| > \left|\frac{\Delta v_i^*}{v_i^*}\right|$. Consequently, the indirect effect moves in the same direction as the scrapping level.[9]

We now concentrate our analysis on the overall average time $E(\tau)$ since empirical data are only available for this variable.

4.3.1. Effects of Regime Persistence (Figures 4., 5., 6.)

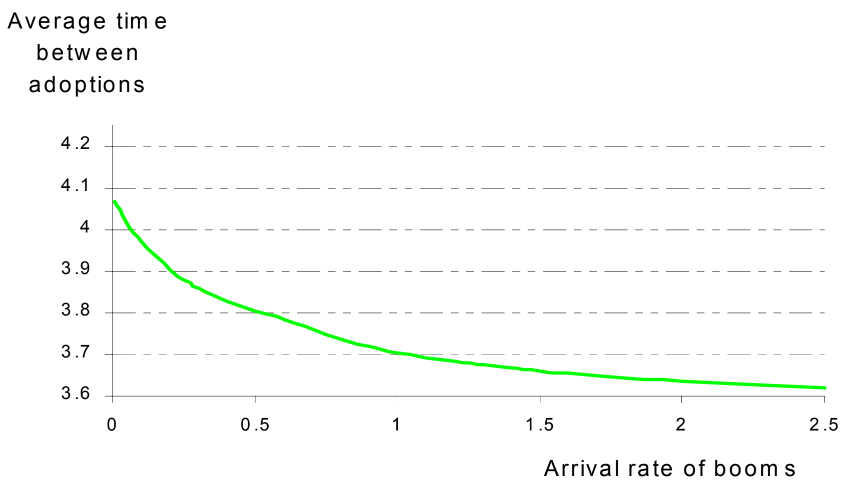


Figure 4: Effects of the arrival rate of booms ϕ_1; $\phi_2 = 0.25$ and $\sigma = 1$.

[9] This result can be proved formally in the case of the unique regime case: the indirect effect of more technological uncertainty is to rise the ratio $\frac{v^*}{u^*}$, delaying adoption.

Figure 4. shows that when the economy experiences more booms, the frequency of adoptions increases since $E(\tau)$ shrinks. We know that both scrapping and upgrading thresholds go up but as mentioned above, the scrapping threshold rises (in relative terms) more than the upgrading one which drives down the average time between two adoptions.

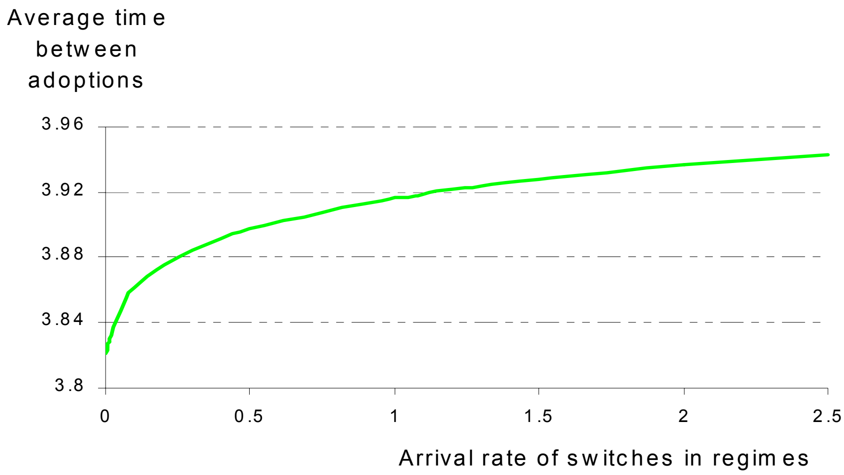


Figure 5: Effects of equal arrival rates of regimes ϕ; $\sigma = 1$.

Recall that in this case, on average the economy is half of the time in a recession and half of the time in a boom. Figure 5. shows that when the frequency of regime switches increases, the average time between two adoptions rises. The fall in boom persistence which increases the average time (see Figure 11. displayed at the end of the paper) has a stronger impact than the antagonistic effect due to the drop in recession persistence (Figure 4.). When ϕ is large, recessions are not persistent and since scrapping occurs at an earlier stage in a boom, it is likely that adoption takes place in a boom as confirmed by Figure 8.. Alternatively, when ϕ is large, booms are not persistent. Therefore the frequency of adoptions is reduced.

4.3.2. Effects of Technological Uncertainty

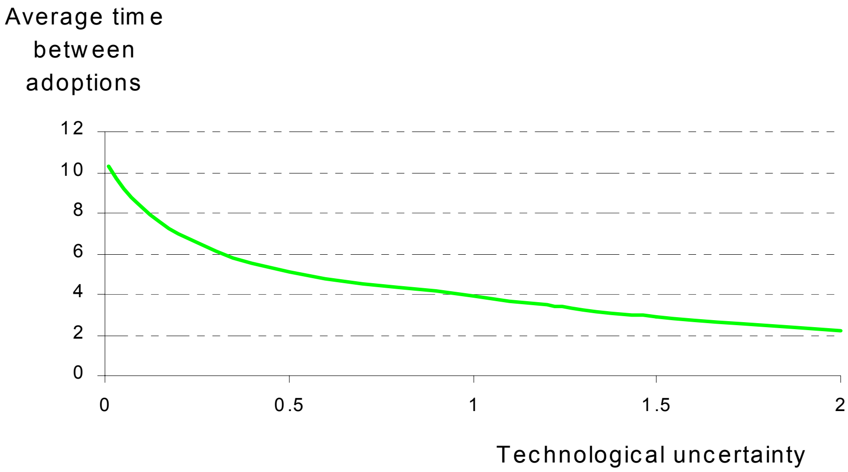


Figure 6: Effects of technological uncertainty σ; $\phi_1 = \phi_2 = 0.25$.

When recessions and booms are equally likely, simulations show that more technological uncertainty hastens the adoption of new technologies. Fixing threshold levels at their unique regime values, the direct effect of more technological uncertainty is a reduction in the average time. The interpretation is the following. **Higher technological uncertainty σ amplifies the fluctuations of the technology process**. Since the optimal switching rule of the firm is -scrap as soon as the technology level goes below a given level – chances are that this level may be hit earlier when fluctuations are high and consequently, adoption is hastened. It can be shown that the ratio $\frac{v_i^*}{u_i^*}$, $i = 1, 2$ rises indicating that the indirect effect of technological uncertainty is to delay adoption. Overall, the direct effect is stronger that the indirect one: more technological uncertainty accelerates adoption. One implication is that firms whose ability to master new performing technologies is low should buy intermediate technologies but update more frequently.

4.4. Effects of Changes in Uncertainty on the Likelihood of Updating in a Recession

We examine the combined effects of the two types of uncertainty on the relationship between replacement and the business cycle. The velocity of the decline of the relative position u and the arrival rate of booms ϕ_1 play a key role. We have chosen to display simulations only about the unconditional probability p_1 that the next updating occurs in a recession since we aim at obtaining a global estimation of the likelihood of upgrading in a recession versus a boom. We focus on the case $u_2^* \leq u$. We have also run simulations for the case $u_1^* \leq u \leq u_2^*$ and the results obtained are similar to the one commented below.

4.4.1. Effects of Regime Persistence (Figures 7., 8., 9.)

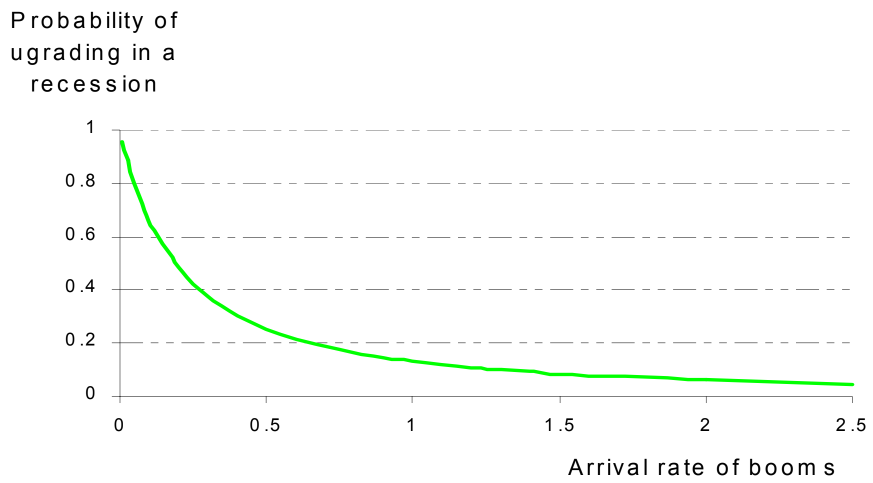


Figure 7: Effects of the arrival rate of booms ϕ_1; $\phi_2 = 0.25$ and $\sigma = 1$.

Decrease in recession persistence reduces the probability of switching in a recession next time, not surprisingly.

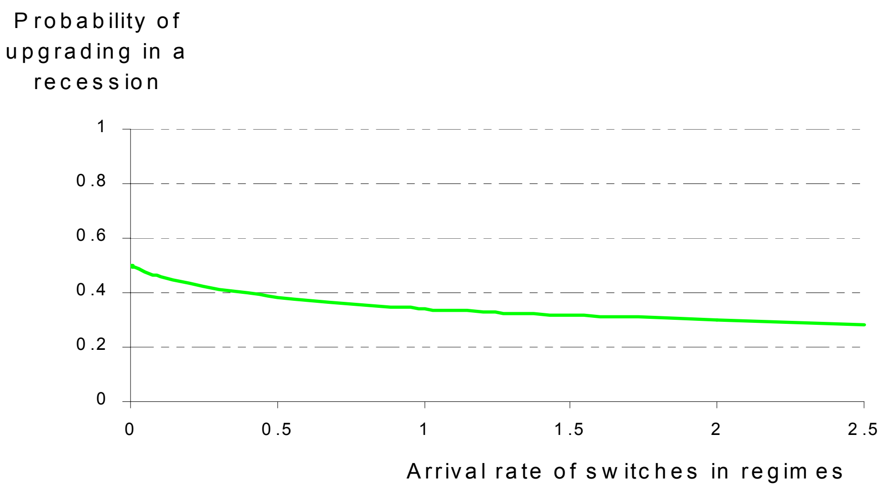


Figure 8: Effects of equal arrival rates of regimes ϕ; $\sigma = 1$.

Figure 8. indicates that adoption is more likely to occur in a boom and this likelihood increases as the frequency of switches rises. This result is interesting as in this case, *the economy spends on average an even fraction of time in a recession and in a boom* and still upgrading is more likely to take in a boom. The key factor in the analysis is to bear in mind that scrapping occurs at an earlier stage during a boom.

4.4.2. Effects of Technological Uncertainty

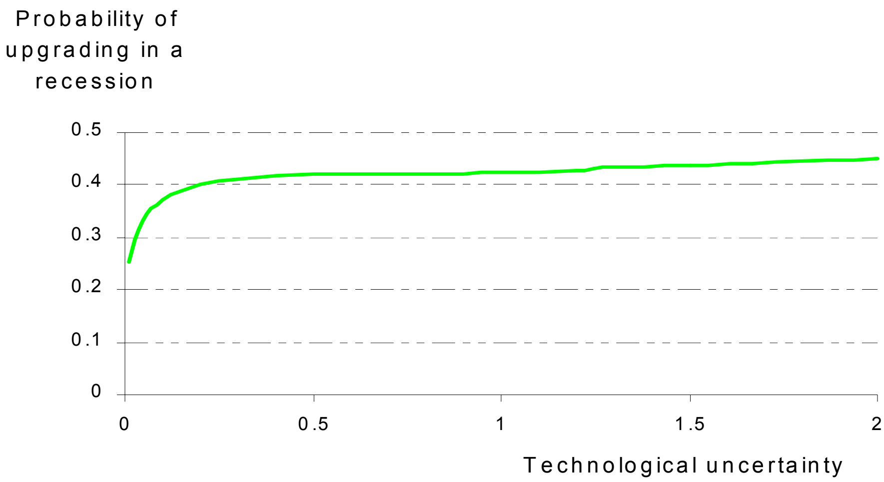


Figure 9: Effects of technological uncertainty σ; $\phi_1 = \phi_2 = 0.25$.

Figure 9 shows that an increase in technological uncertainty raises the probability to upgrade in a recession but the latter remains less than $\frac{1}{2}$. More technological uncertainty accelerates adoption, which increases the chances to upgrade in the *same* state as the current one. Recall that upgrading always occurs at an earlier stage in a boom. Moreover, when σ is small, the expected time between two adoptions is large. This reduces the chance to update in the same state and therefore adoption is more likely to occur in a boom. As σ rises, simulations (not displayed here) show that p_{11} goes up to 1 whereas p_{12} goes down to 0. Consequently, as σ increases, the probability to upgrade in a recession increases to $\frac{1}{2}$.

5. CONCLUSION

This paper analyzes the combined effects of market and technological uncertainty on the decisions of scrapping and upgrading technologies. For the sake of simplicity, much of the literature dealing with technology adoption in a dynamic framework chose to examine the special case where the cutting edge technology is systematically purchased. We relax this assumption and the firm is free to implement *any* technology available within an increasing range across time due to exogenous technological progress. Upgrading is the result of physical and economical depreciations. The two types of uncertainty play different roles. Technological uncertainty affects the behavior of the law of motion of the technology process to be controlled by the manager. Both technological uncertainty and demand uncertainty have an impact on the optimal scrapping and upgrading levels and, therefore, alter the range of operation of a given grade. Parameters increasing the expected profit provide more incentives to scrap early and adopt a sophisticated technology in order to enjoy high cash flows. In particular, more technological uncertainty discourages adoption of advanced technologies and delays scrapping.

We investigate the effects of uncertainty on the frequency of adoptions as well as the likelihood of upgrading in a recession versus an expansion. Bounded technological progress limits the range of available technologies, restricting the firm to adopt the best already invented grade. During periods of high economic activity, the firm is eager to operate a highly performing technology, and to achieve this, is willing to upgrade more often. Direct and indirect effects can be antagonistic. Overall, economies spending a large fraction of time in booms display more frequent adoptions. In contrast to what common intuition might at first suggest, technological uncertainty σ hastens adoption because it reinforces depreciation. Interpreting the parameter σ as the ability of a firm to operate properly a technology, one implication of the model is that plants with less ability should buy intermediate technologies and update frequently.

In an attempt to shed some light on the timing of adoption and the business cycle, we find that technological uncertainty increases the chances to adopt in the same state as it hastens upgrading. Consistent with Cooper and Haltiwanger (1993), scrapping can take place at the end of a recession: booms trigger adoption. Recessions appear to be periods of time where the firm delays replacement as cash flows are low (Goolsbee 1998) whereas booms give incentives for adopting of a better technology as a promise of high benefits. Unless the economy is in a recession most of the time, investment spikes at the plant level are likely to be procyclical.

We have considered an extreme case where the new technology implemented is more productive right after the adoption date. Lag effects such as *time to build* or *time to learn* can have a significant impact of the timing of adoption of a new technology. In addition, updating decisions are based on expectations about future available technologies. We have taken the arrival of new grades as exogenous. A general equilibrium model would allow us to endogenize it. This is left for further research.

6. APPENDIX

Recall that $u_0(s)$ is the level of the initial relative technology at time s. For convenience, we denote $u_0(s)$ by $u(u_0, s)$ and the instantaneous profit made at time s, $X(s)u_0(s)^\alpha$, by $\pi\left(u(u_0, s), X(s)\right)$ whenever needed.

6.1. APPENDIX 1

6.1.1. V_i takes finite values

Proof.

We observe that the maximum relative grade that can be adopted is 1. Therefore, at *any* time t, the profit of a firm operating a technology level u is less than $X(t)(1+u(t)^\alpha) \leq X(t)u(t)^\alpha + X_2$. Moreover the value of the firm is less than if adoption was free. We obtain that for all $i \in \{1, 2\}$,

$$\begin{aligned} V_i(u) &\leq E_0^i \int_0^\infty X(s)(1+u(s)^\alpha)e^{-rs}ds \\ &\leq B_i u^\alpha + \frac{X_2}{r} < \infty \end{aligned}$$

■

6.1.2. Switching occurs within a finite time almost surely

Proof.

The value of a firm operating forever the same technology is $V_i(u) = B_i u^\alpha$. If the manager decides to upgrade *only once*, the value of firm is $\max\limits_{v\in[0,1]} B_i v^\alpha - cv$. It is never optimal to upgrade if and only if for all u, we have $B_i u^\alpha > \max\limits_{v\in[0,1]} B_i v^\alpha - cv$. We can compute explicitly the RHS of the inequality and we obtain:

$$RHS = \begin{cases} \alpha B_i - c + (1-\alpha)B_i \text{ if } c \leq \alpha B_i \\ (1-\alpha)B_i \left(\frac{\alpha B_i}{c}\right)^{\frac{\alpha}{1-\alpha}} \text{ if } c \geq \alpha B_i \end{cases}$$

We observe that the RHS is bounded away from zero. As time passes, since the process u has a negative drift, it will eventually reach any arbitrary positive value $\varepsilon > 0$, almost surely. Therefore, the LHS goes to zero as time passes. This implies that the inequality *cannot always* be satisfied and after some time switching becomes optimal. Since the problem is stationary, the firm will actually upgrade an infinite number of times. ■

6.1.3. V_i is increasing in u and $V_1 \leq V_2$.

Proof.

We start proving that V_i is strictly increasing in its inaction region IR_i. If u_0 is in IR_i, using relationship (3.1), by the Envelope Theorem, we have

$$V_i'(u_0) = E_0^i \left[\int_0^{\tau^*} \pi_1 \left(u(u_0, s), X(s)\right) \frac{\partial u(u_0, s)}{\partial u_0} e^{-rs} ds \right] , \tag{6.1}$$

where τ^* denotes the first optimal stopping time from date 0. It follows immediately that $V_i'(u_0) > 0$ and $V_i'(u) = 0$ exactly when $\tau^* = 0$, i.e., exactly at the moment of the switch in the state i. Another way to see that V_i is increasing is the following. Consider a given initial level of technology u_0 and its optimal strategy associated $(\{\tau_k \geq 0,\ v_k \in [0,1]\}_{k=1}^{\infty})$. For any initial level of technology $u_0' > u_0$, we have $u(u_0', s) > u(u_0, s)$ for all $s > 0$ almost surely. Therefore, $\pi\left(u(u_0', s), X(s)\right) > \pi\left(u(u_0, s), X(s)\right)$ and $E_0^i \left[\int_0^{\tau_1} \pi\left(u(u_0', s), X(s)\right) e^{-rs} ds\right] \geq E_0^i \left[\int_0^{\tau_1} \pi\left(u(u_0, s), X(s)\right) e^{-rs} ds\right]$. Moreover for the initial level of technology u_0', the strategy $(\{\tau_k \geq 0,\ v_k \in [0,1]\}_{k=1}^{\infty})$ is feasible though it may not be optimal. It follows easily that $V_i(u_0') \geq V_i(u_0)$. In order to prove that $V_1 \leq V_2$, we show that $G_1 \leq G_2$ and $F_1 \leq F_2$. ■

6.1.4. G_i is increasing in u and $G_1 \leq G_2$.

Proof.

The properties of G_i follow directly from the properties of π and the process u. Direct computations displayed in the main body of the paper show that $G_1 \leq G_2$. ■

6.1.5. The option value F_i is decreasing in u and $F_1 < F_2$.

Proof.

Recall that $F_i(u_0) = V_i(u_0) - E_0^i \int_0^{\infty} \pi(u(u_0, s), X(s)) e^{-rs} ds$. It follows directly that

$$F_i'(u_0) = -E_0^i \int_{\tau^*}^{\infty} \pi_1(u(u_0, s), X(s)) \frac{\partial u(u_0, s)}{\partial u_0} e^{-rs} ds \leq 0 \ .$$

In order to show that for all u, $F_2(u) \geq F_1(u)$, consider the optimal strategy $\{\tau_k^* \geq 0,\ v_k \in [0,1]\}_{k=1}^{\infty}$ of firm 1 starting in a *recession* ($i = 1$) with an initial level of technology u_0. Since $F_1(u) = V_1(u) - G_1(u)$, breaking into pieces the value of operated the same technology, we can write

$$\begin{aligned}
F_1(u_0) &= E_0^1 \left[\int_0^{\tau_1^*} X(s) u_0(s)^{\alpha} e^{-rs} ds + \sum_{k=1}^{\infty} \left[\int_{\tau_k^*}^{\tau_{k+1}^*} X(s) u_k(s)^{\alpha} e^{-rs} ds - c v_k e^{-r\tau_k^*} \right] \right] \\
&\quad - E_0^1 \left[\int_0^{\infty} X(s) u_0(s)^{\alpha} e^{-rs} ds \right] \\
&= E_0^1 \left[\sum_{k=1}^{\infty} \left[\int_{\tau_k^*}^{\tau_{k+1}^*} \left(u_k(s)^{\alpha} - u_0(s)^{\alpha}\right) X(s) e^{-rs} ds - c v_k e^{-r\tau_k^*} \right] \right] \ .
\end{aligned}$$

Consider firm 2 that starts in a boom ($i = 2$) with the same initial technology u_0, the manager can use the *same* strategy $\{\tau_k^* \geq 0,\, v_k \in [0,1]\}_{k=1}^{\infty}$, which may not be optimal. It follows that

$$F_2(u_0) \geq E_0^2 \left[\sum_{k=1}^{\infty} \left[\int_{\tau_k^*}^{\tau_{k+1}^*} \left(u_k(s)^\alpha - u_0(s)^\alpha\right) X(s) e^{-rs} ds - c v_k e^{-r\tau_k^*} \right] \right] .$$

Define the random variable $Y_i(u_0) = \sum_{k=1}^{\infty} \left[\int_{\tau_k^*}^{\tau_{k+1}^*} \left(u_k(s)^\alpha - u_0(s)^\alpha\right) X(s) e^{-rs} ds - c v_k e^{-r\tau_k^*} \right]$ when the initial state is (u_0, i). Recall that the processes u and X are independent. We define two other stopping times: $\tau_{1,2}$ as the first time starting at $X(0) = X_1$, the process X switches to the value X_2 and $\tau_{2,1}$ the first time starting at $X(0) = X_2$, the process X switches to the value X_1. Then set $\theta = \tau_{1,2} \wedge \tau_{2,1}$ the first time the two firms are in the same state of the world. Before $\tau_1^* \wedge \theta$, $Y_2(u_0) \geq Y_1(u_0)$; after $\tau_1^* \wedge \theta$, by the Markov property of the process X, the two random variables $Y_2(u_0)$ and $Y_1(u_0)$ have the same distribution. It follows that $E_0^2 \left[Y_2(u_0)\right] \geq E_0^1 \left[Y_1(u_0)\right]$, which exactly means $F_2(u_0) \geq F_1(u_0)$. ■

6.1.6. The inaction region IR_i is connected.

Proof.

From relationship (6.1), since $\pi_1 > 0$ and $\frac{\partial u(u_0,s)}{\partial u_0} > 0$ for all s, then $V_i'(u_0) > 0$ iff $\tau^* > 0$. Scrapping at level u_i^* means exactly $\tau^* = 0$: it follows immediately that $V_i'(u_i^*) = 0$. In addition, we have proved in appendix 1 that V_i is (at least!) weakly increasing. So according to (3.2), if u is in IR_i then $V_i(u) > \sup_{v \in [0,1]} V_i(v) - cv$ and for $u' > u$, $V_i(u') \geq V_i(u)$ which implies $V_i(u') > \sup_{v \in [0,1]} V_i(v) - cv$, i.e., u' is also in IR_i. This implies that the inaction region is an interval is so is connected. We conclude that u is in the inaction region IR_i exactly means $V_i'(u) > 0$ and the frontier of the inaction region ∂IR_i is characterized by a lower bound u_i^* such that $V_i'(u_i^*) = 0$. The remaining part of the proposition is simply an arbitrage condition. ■

6.1.7. $u_1^* < u_2^*$.

Proof.

We start showing by contradiction that $u_2^* \neq u_1^*$. Assume that $u_2^* = u_1^* \triangleq u^*$. Starting at an initial state i at time 0 , the assumption means that the optimal stopping time τ^* is independent of the state i. Writing symbolically $E_0^i = E_{u_0 \times} E^i$ and using (6.1), we obtain: $V_i'(u_0) = E_{u_0} \left[\int_0^{\tau^*} E^i \left(X(s)\right) \frac{\partial u(u_0,s)}{\partial u_0} e^{-rs} ds \right]$. Recall that $P_{11}(t) = \frac{\phi_2}{\phi_2+\phi_1} + \frac{\phi_1}{\phi_2+\phi_1} e^{-(\phi_2+\phi_1)t}$, it follows easily that:

$$\begin{aligned}
V_1'(u_0) &= E_{u_0}\left[\int_0^{\tau^*}\left(\frac{\phi_2 X_1+\phi_1 X_2}{\phi_2+\phi_1}+\phi_1\frac{X_1-X_2}{\phi_2+\phi_1}e^{-(\phi_2+\phi_1)s}\right)\frac{\partial u(u_0,s)}{\partial u_0}e^{-rs}ds\right] \\
&= \frac{\phi_2 X_1+\phi_1 X_2}{\phi_2+\phi_1}E_{u_0}\left[\int_0^{\tau^*}\frac{\partial u(u_0,s)}{\partial u_0}e^{-rs}ds\right] \\
&\quad -\phi_1\frac{X_2-X_1}{\phi_2+\phi_1}E_{u_0}\left[\int_0^{\tau^*}\frac{\partial u(u_0,s)}{\partial u_0}e^{-(\phi_2+\phi_1+r)s}ds\right]
\end{aligned}$$

In the same fashion, permuting indexes 1 and 2, we obtain:

$$\begin{aligned}
V_2'(u_0) &= \frac{\phi_2 X_1+\phi_1 X_2}{\phi_2+\phi_1}E_{u_0}\left[\int_0^{\tau^*}\frac{\partial u(u_0,s)}{\partial u_0}e^{-rs}ds\right] \\
&\quad +\phi_2\frac{X_2-X_1}{\phi_2+\phi_1}E_{u_0}\left[\int_0^{\tau^*}\frac{\partial u(u_0,s)}{\partial u_0}e^{-(\phi_2+\phi_1+r)s}ds\right]
\end{aligned}$$

This implies that there exist two positive real valued functions Φ and Ψ such that:

$$\begin{aligned}
V_1'(u) &= \Phi(u)-\phi_1\Psi(u) \\
V_2'(u) &= \Phi(u)+\phi_2\Psi(u).
\end{aligned}$$

Such a decomposition is incompatible with the analytical expressions of V_1 and V_2 given by equations (3.4) and (3.5) when u is in the intersection of the inaction regions. Therefore, $u_2^* \neq u_1^*$ whatever the values of the parameters of the model, in particular ϕ_1 and ϕ_2. Moreover, the thresholds u_1^* and u_2^* are smooth (in particular continuous) functions of the parameters of the model; this can be shown using the Implicit Function Theorem applied to the system of nine equations characterizing u_1^* and u_2^* displayed in appendix 3. Thus, we must *always* have either $u_2^* < u_1^*$ or $u_2^* > u_1^*$. As proved in appendix 3 for the unique regime case, the scrapping threshold is strictly increasing in the demand level. We conclude using a continuity argument that $u_2^* > u_1^*$. ■

6.1.8. Analytical expressions for V_1 and V_2 when $u \leq u_2^*$.

Recall that V_1 and V_2 satisfy the following ODE system:

$$\begin{aligned}
rV_1(u) &= X_1 u^\alpha - \lambda u V_1'(u) + \frac{\sigma^2}{2}u^2 V_1''(u) + \phi_1\left[V_2(u)-V_1(u)\right] \\
rV_2(u) &= X_2 u^\alpha - \lambda u V_2'(u) + \frac{\sigma^2}{2}u^2 V_2''(u) + \phi_2\left[V_1(u)-V_2(u)\right]\ .
\end{aligned}$$

We define two auxiliary functions $\Delta \equiv V_2 - V_1$ and $\Theta \equiv \phi_2 V_1 + \phi_1 V_2$. They satisfy

$$\begin{aligned}
r\Delta(u) &= (X_2 - X_1)u^\alpha - \lambda u \Delta'(u) + \frac{\sigma^2}{2}u^2\Delta''(u) - (\phi_1 + \phi_2)\Delta(u) \\
r\Theta(u) &= (\phi_2 X_1 + \phi_1 X_2)u^\alpha - \lambda u \Theta'(u) + \frac{\sigma^2}{2}u^2\Theta''(u) \ .
\end{aligned}$$

The general solutions for Δ and Θ are

$$\begin{aligned}
\Delta(u) &= A(X_2 - X_1)u^\alpha + Ku^{-\delta} + K'u^{-\delta'} \\
\Theta(u) &= B(\phi_2 X_1 + \phi_1 X_2)u^\alpha + Hu^{-\gamma} + H'u^{-\gamma'} \ ,
\end{aligned}$$

where

$$A = \frac{1}{r + \phi_1 + \phi_2 + \alpha\lambda + \frac{\alpha(1-\alpha)\sigma^2}{2}} \text{ and } B = \frac{1}{r + \alpha\lambda + \frac{\alpha(1-\alpha)\sigma^2}{2}} \ ,$$

(γ, γ') and (δ, δ') are respectively the positive and negative roots of the following quadratic equations

$$\begin{aligned}
\frac{\sigma^2}{2}\gamma^2 + (\frac{\sigma^2}{2} + \lambda)\gamma &= r \\
\frac{\sigma^2}{2}\delta^2 + (\frac{\sigma^2}{2} + \lambda)\delta &= r + \phi_1 + \phi_2 \ ,
\end{aligned}$$

and (K, H, K', H') are constants to be determined. To shorten notation, we set.

$$\begin{aligned}
B_1 &= \frac{1}{\phi_2 + \phi_1}\left[(\phi_2 B + \phi_1 A)X_1 + \phi_1(B - A)X_2\right] \\
B_2 &= \frac{1}{\phi_2 + \phi_1}\left[\phi_2(B - A)X_1 + (\phi_1 B + \phi_2 A)X_2\right] \ .
\end{aligned}$$

Using the same notation for the constants, we obtain the following expressions for $V_1(u)$ and $V_2(u)$

$$\begin{aligned}
V_1(u) &= B_1 u^\alpha + Hu^{-\gamma} + H'u^{-\gamma'} - \phi_1 K u^{-\delta} - \phi_1 K' u^{-\delta'} \\
V_2(u) &= B_2 u^\alpha + Hu^{-\gamma} + H'u^{-\gamma'} + \phi_2 K u^{-\delta} + \phi_2 K' u^{-\delta'} \ .
\end{aligned}$$

Using the decomposition $V_i(u) = G_i(u) + F_i(u)$, by identification, we obtain that the option value F_i is given by:

$$\begin{aligned}
F_1(u) &= Hu^{-\gamma} + H'u^{-\gamma'} - \phi_1 K u^{-\delta} - \phi_1 K' u^{-\delta'} \\
F_2(u) &= Hu^{-\gamma} + H'u^{-\gamma'} + \phi_2 K u^{-\delta} + \phi_2 K' u^{-\delta'} \ .
\end{aligned}$$

Recall that the option value must be decreasing in u. Moreover, economic intuition suggests that when u goes to infinity, the option value must goes to zero. This implies $K' = H' = 0$.

6.2. APPENDIX 2

6.2.1. Condition 1.

Proof.
Recall that from equation (6.1), we obtain $V_i'(u_0) = E_0^i \left[\int_0^{\tau^*} \pi_1\left(u(u_0,s), X(s)\right) \frac{\partial u(u_0,s)}{\partial u_0} e^{-rs} ds\right]$. Switching exactly means $\tau^* = 0$, it follows immediately that $V_i'(u_i^*) = 0$. In addition for $u > u_i^*$, $\tau^* > 0$, so $V_i'(u) > 0$. Henceforth, for all $u \geq u_i^*$, $V_i(u) \geq V_i(u_i^*)$, which exactly means that u_i^* is a minimum. ■

6.2.2. Condition 2.

Proof.
At the adoption date, since the set of relative feasible technologies is independent of time, it is simply optimal to maximize over v the quantity $V_i(v) - cv$. ■

6.2.3. Condition 3.

Proof.
Recall that $V_i(u_0) = \sup\limits_{(\tau,\, v\in[0,1],\, j\in\{1,2\})} E_0^i \left[\int_0^{\tau} \pi\left(u(u_0,s), X(s)\right) e^{-rs} ds + e^{-r\tau}\left[V_j(u(v,\tau)) - cv\right]\right]$. Just before the date of the switch, the state of the world is (u_i^*, i) and right after (v_i^*, i). In addition, at the date of adoption, $\tau^* = 0$. Using the recursive formulation of the value function immediately yields the desired result. ■

6.2.4. Analytical expressions characterizing the optimal policy.

Using the analytical expressions (3.5), (3.4) and (3.11) of the value function, the four thresholds $(u_1^*, v_1^*, u_2^*, v_2^*)$ are characterized by the following system:

$$
\begin{array}{ll}
\alpha C_1(u_1^*)^{\alpha-1} + \theta D(u_1^*)^{\theta-1} + \eta E(u_1^*)^{\eta-1} = 0 & (Eq.\ 1.)\\
\alpha B_1(v_1^*)^{\alpha-1} - \gamma H(v_1^*)^{-\gamma-1} + \delta\phi_1 K(v_1^*)^{-\delta-1} = c \text{ or } v_1^* = 1 & (Eq.\ 2.)\\
\frac{\phi_1 V_2(u_2^*)}{r+\phi_1} + C_1(u_1^*)^{\alpha} + D(u_1^*)^{\theta} + E(u_1^*)^{\eta} + cv_1^* = B_1(v_1^*)^{\alpha} + H(u_1^*)^{-\gamma} - \phi_1 K(u_1^*)^{-\delta} & (Eq.\ 3.)\\
\alpha B_2(u_2^*)^{\alpha-1} - \gamma H(u_2^*)^{-\gamma-1} - \delta\phi_2 K(u_2^*)^{-\delta-1} = 0 & (Eq.\ 4.)\\
\alpha B_2(v_2^*)^{\alpha-1} - \gamma H(v_2^*)^{-\gamma-1} - \delta\phi_2 K(v_2^*)^{-\delta-1} = c \text{ or } v_2^* = 1 & (Eq.\ 5.)\\
B_2(u_2^*)^{\alpha} + H(u_2^*)^{-\gamma} + \phi_2 K(u_2^*)^{-\delta} + cv_2^* = B_2(v_2^*)^{\alpha} + H(v_2^*)^{-\gamma} + \phi_2 K(v_2^*)^{-\delta} & (Eq.\ 6.)\\
\frac{\phi_1 V_2(u_2^*)}{r+\phi_1} + C_1(u_2^*)^{\alpha} + D(u_2^*)^{\theta} + E(u_2^*)^{\eta} = B_1(u_2^*)^{\alpha} + H(u_2^*)^{-\gamma} - \phi_1 K(u_2^*)^{-\delta} & (Eq.\ 7.)\\
\alpha C_1(u_2^*)^{\alpha-1} + \theta D(u_2^*)^{\theta-1} + \eta E(u_2^*)^{\eta-1} = \alpha B_1(u_2^*)^{\alpha-1} - \gamma H(u_2^*)^{-\gamma-1} + \delta\phi_1 K(u_2^*)^{-\delta-1} & (Eq.\ 8.)\\
V_2(u_2^*) = B_2(u_2^*)^{\alpha} + H(u_2^*)^{-\gamma} + \phi_2 K(u_2^*)^{-\delta} & (Eq.\ 9.)\ .
\end{array}
$$

6.3. APPENDIX 3

We start to show existence and uniqueness of the trigger and target thresholds (u^*, v^*) in the recession case.

Recessions. Define $x = \frac{v^*}{u^*} > 1$. Manipulating equations (3.12) and (3.13), it is easy to show that x has to satisfy the relationship $\gamma(1-\alpha)x^{\alpha+\gamma} - (\alpha+\gamma)x^{\gamma} + \alpha(1+\gamma) = 0$. For $y \geq 1$, define $\varphi(y) = \gamma(1-\alpha)y^{\alpha+\gamma} - (\alpha+\gamma)y^{\gamma} + \alpha(1+\gamma)$. Note that $\varphi(1) = 0$ and $\lim_{+\infty}\varphi(y) = \infty$. Moreover, $\varphi'(y) = \gamma(\alpha+\gamma)y^{\gamma-1}\left[(1-\alpha)y^{\alpha} - 1\right]$. Thus φ is decreasing on the interval $[1, y^*]$ and increasing on the interval $[y^*, +\infty)$, with $y^* = \left(\frac{1}{1-\alpha}\right)^{\frac{1}{\alpha}}$. Since $\varphi(1) = 0$, φ is continuous and $\lim_{+\infty}\varphi(y) = \infty$, there must be a unique $x > 1$ such that $\varphi(x) = 0$. Note that x is independent of the cost c and the demand level X. Once we have existence and uniqueness of $x = \frac{v^*}{u^*}$, using (3.12), we get a unique $v^* = \left[\frac{(\alpha+\gamma)B\left[1-x^{-\alpha}\right]}{(1+\gamma)c}X\right]^{\frac{1}{1-\alpha}}$ and $u^* = \frac{v^*}{x}$. Both u^* and v^* are increasing in X and decreasing in c.

Expansions. We want to show existence and uniqueness of u^* as the unique root in $(0,1)$ of the equation (3.14). For $0 \leq y \leq 1$, define $\psi(y) = \alpha y^{\alpha+\gamma} - (\alpha+\gamma)y^{\alpha} + \gamma - \gamma cB^{-1}X$. Note $\psi(0) = \gamma(1 - cB^{-1}X)$ and $\psi(1) = -\gamma cB^{-1}X < 0$. It can be shown that $\overline{X} > \frac{c}{\alpha B}$ which implies $\psi(0) > 0$ as $X_1 \geq \overline{X} > \frac{c}{B}$. Moreover, $\psi'(y) = \alpha(\alpha+\gamma)y^{\alpha-1}\left[y^{\gamma} - 1\right] < 0$. As ψ is continuous, there is a unique value u in $(0,1)$ such that $\psi(u) = 0$. Since ψ' is negative, by the Implicit Function Theorem, we can write $u^*(X)$ and totally differentiating with respect to X relationship (3.14), we obtain $\frac{\partial u^*(X)}{\partial X} = -\frac{\gamma cB^{-1}}{\psi'(u^*(X))} > 0$. In the same fashion, we can show that u^* is decreasing in c.

6.4. APPENDIX 4

We define

$$\tau(v,k) = \inf\{t \geq 0 : (u(t), X(t)) = (u_1^*, X_1) \text{ or } (u, X_2) \text{ with } u \leq u_2^* |\ u(0) = v, X(0) = X_k\},$$

the stopping time that describes the date of the next adoption. We assume that $P(\tau < \infty) = 1$. One way of obtaining the expected time between two adoptions is to define a suitable martingale $\mathcal{M}$ and exploit the following martingale property: $E_0\left[\mathcal{M}(\tau)\right] = \mathcal{M}(0)$. This is the central idea of the **Optional Stopping Theorem**.

Starting right after adoption in state (v_i^*, i), we are looking for a martingale $\mathcal{M}$ of the form $\mathcal{M}(t) = f(u(t), i) + t$. Since $i \in \{1, 2\}$, for convenient reasons, we write $\mathcal{M}(t) = f_i(u(t)) + t$ and our goal is to determine two suitable functions f_1 and f_2. In addition, we impose the following boundary conditions

$$f_i(u_i^*) = 0,\ i = 1, 2. \tag{6.2}$$

Now, if the initial state is (v_i^*, i), *given the boundary conditions* $f_i(u(\tau)) = 0$ *and the martingale property*, we have we would like to be able to write $E_0^i\left[\mathcal{M}(\tau)\right] = \mathcal{M}(0)$ i.e., $E_0^i\left[\tau\right] = f_i(v_i^*)$ since τ is a stopping time. Unfortunately, this is not true for *any* martingale $\mathcal{M}$. We need to impose some limiting growth conditions on the function f_i as u can take unbounded values.[10]

[10] The Optional Stopping Theorem may fail in this case if u can take arbitrary large values. One way to

Case 1: $u_2^* \le u$. One necessary condition for M to be a martingale is[11]

$$\begin{aligned} -\lambda u f_1'(u) + \frac{\sigma^2}{2} u^2 f_1''(u) + \phi_1 \left(f_2(u) - f_1(u)\right) &= -1 \\ -\lambda u f_2'(u) + \frac{\sigma^2}{2} u^2 f_2''(u) + \phi_2 \left(f_1(u) - f_2(u)\right) &= -1 \; . \end{aligned}$$

We consider two auxiliary functions $\Delta \equiv f_2 - f_1$ and $\Gamma \equiv \phi_1 f_2 + \phi_2 f_1$ that must satisfy:

$$\begin{aligned} -\lambda u \Delta'(u) + \frac{\sigma^2}{2} u^2 \Delta''(u) - (\phi_1 + \phi_2)\Delta(u) &= 0 \\ -\lambda u \Gamma'(u) + \frac{\sigma^2}{2} u^2 \Gamma''(u) &= -(\phi_2 + \phi_1) \; . \end{aligned}$$

Solving these equations leads to

$$\begin{aligned} \Delta(u) &= -M u^{-\beta} + \widetilde{M} u^{\widetilde{\beta}} \\ \Gamma(u) &= \frac{\phi_2 + \phi_1}{\lambda + \frac{\sigma^2}{2}} \ln u + \widetilde{N} u^{\frac{2\lambda}{\sigma^2}+1} + N \; , \end{aligned}$$

where β and $\widetilde{\beta}$ are respectively the positive and the negative roots of the quadratic equation $\frac{\sigma^2}{2}\beta^2 + (\lambda + \frac{\sigma^2}{2})\beta - (\phi_1 + \phi_2) = 0$. As limiting growth conditions, we impose $\widetilde{M} = \widetilde{N} = 0$ and therefore

$$\begin{aligned} \Delta(u) &= -M u^{-\beta} \\ \Gamma(u) &= \frac{\phi_2 + \phi_1}{\lambda + \frac{\sigma^2}{2}} \ln u + N \; . \end{aligned}$$

As $f_2(u_2^*) = 0$, we ultimately obtain

$$\begin{aligned} f_1(u) &= \frac{\ln \frac{u}{u_2^*}}{\lambda + \frac{\sigma^2}{2}} + \frac{M}{\phi_2 + \phi_1} \left(\phi_1 u^{-\beta} + \phi_2 (u_2^*)^{-\beta}\right) &\quad (6.3) \\ f_2(u) &= \frac{\ln \frac{u}{u_2^*}}{\lambda + \frac{\sigma^2}{2}} + \frac{\phi_2 M}{\phi_2 + \phi_1} \left((u_2^*)^{-\beta} - u^{-\beta}\right) . &\quad (6.4) \end{aligned}$$

Case 2: $u_1^* \le u \le u_2^*$. In this case, we know that $\tau(u, 2) = 0$, so we choose $f_2(u) = 0$. It follows that

$$-\lambda u f_1'(u) + \frac{\sigma^2}{2} u^2 f_1''(u) - \phi_1 f_1(u) = -1 \; .$$

circumvent this difficulty is to consider the same problem and add an arbitrary upper bound $\overline{U}$ on u. Define a new stopping time T being the first time u hits a scrapping level or u hits $\overline{U}$. In this case, u remains in a bounded domain and we can use safely the Optional Stopping Theorem. Then, let $\overline{U}$ goes to ∞.

[11] To be more formal, this is just imposing the infinitesimal generator of $\mathcal{M}$ to be equal to 0.

The general solution is

$$f_1(u) = \frac{1}{\phi_1} + Pu^{\xi} + Qu^{\zeta} \ , \tag{6.5}$$

where ξ and ζ are the roots of the quadratic $\frac{\sigma^2}{2}x^2 - (\lambda + \frac{\sigma^2}{2})x - \phi_1 = 0$. It remains to determine the three constants (M, P, Q). We use the boundary condition on f_1 (6.2) and the fact that f_1 has to be continuously differentiable at u_2^* (matching and smooth pasting conditions) relying on the relationships (6.3) and (6.5). This leads to the following system

$$\begin{aligned}
(u_1^*)^{\xi}P + (u_1^*)^{\zeta}Q + \frac{1}{\phi_1} &= 0 \\
(u_2^*)^{\xi}P + (u_2^*)^{\zeta}Q + \frac{1}{\phi_1} &= (u_2^*)^{-\beta}M \\
\xi(u_2^*)^{\xi}P + \zeta(u_2^*)^{\zeta}Q &= \frac{1}{\lambda + \frac{\sigma^2}{2}} - \frac{\phi_1\beta(u_2^*)^{-\beta}}{\phi_2 + \phi_1}M \ .
\end{aligned}$$

Solving this linear system and setting $\Lambda = M(u_2^*)^{-\beta}$ provides the desired result.

6.5. APPENDIX 5

Define

$$\begin{aligned}
\tau_1(v,k) &= \inf\{t \geq 0 : (u(t), X(t)) = (u_1^*, X_1) |\ (u(0) = v, X(0) = X_k)\} \qquad \text{and} \\
\tau_2(v,k) &= \inf\{t \geq 0 : (u(t), X(t)) = (u, X_2) \text{ and } u \leq u_2^* \mid (u(0) = v, X(0) = X_k)\} \ ,
\end{aligned}$$

the two stopping times that describe respectively the time of the next switch occurring in a recession (τ_1) or in a boom (τ_2), starting with an initial state of the world (v, k). Notice that $\tau = \min\{\tau_1, \tau_2\}$ is the random variable that describes the date of the next adoption. Here, we are interested in τ_1. We want to compute the probability $p_1(u, i)$ that being in the state (u, i), the *next adoption occurs in a recession.* Again, we write $p_1(u, i) = p_{1i}(u)$ for $i \in \{1, 2\}$ and our goal is to determine the two functions p_{11} and p_{12}. We impose the following boundary conditions $p_{11}(u_1^*) = 1$ and $p_{12}(u_2^*) = 0$.

Case 1: $u_2^* \leq u$. It can be shown using the Markovian structure of the problem that p is a martingale[12] so

$$\begin{aligned}
-\lambda u p_{11}'(u) + \frac{\sigma^2}{2}u^2 p_{11}''(u) + \phi_1\left(p_{12}(u) - p_{11}(u)\right) &= 0 \\
-\lambda u p_{12}'(u) + \frac{\sigma^2}{2}u^2 p_{12}''(u) + \phi_2\left(p_{11}(u) - p_{12}(u)\right) &= 0 \ .
\end{aligned}$$

[12] To be more formal, this is just imposing the infinitesimal generator of p to be equal to 0.

We solve this problem in the same way as for the derivation of average time displayed in appendix 4. Using the boundary condition $p_{12}(u_2^*) = 0$, we obtain

$$\begin{aligned} p_{11}(u) &= \frac{M'}{\phi_2 + \phi_1} \left(\phi_1 u^{-\beta} + \phi_2 (u_2^*)^{-\beta} \right) && (6.6) \\ p_{12}(u) &= \frac{\phi_2 M'}{\phi_2 + \phi_1} \left((u_2^*)^{-\beta} - u^{-\beta} \right) , && (6.7) \end{aligned}$$

where $M' \geq 0$ is a constant to be determined.

Case 2: $u_1^* \leq u \leq u_2^*$. In this case, we have $p_{12}(u) = 0$ as the switch occurs in a boom for sure. It follows that

$$-\lambda u p'_{11}(u) + \frac{\sigma^2}{2} u^2 p''_{11}(u) - \phi_1 p_{11}(u) = 0 .$$

The general solution is

$$p_{11}(u) = P' u^{\xi} + Q' u^{\zeta} , \tag{6.8}$$

where ξ and ζ are the roots of the following quadratic

$$\frac{\sigma^2}{2} x^2 - (\lambda + \frac{\sigma^2}{2}) x - \phi_1 = 0 .$$

It remains to determine the three constants (M', P', Q'). We use the boundary condition on p_{11} and the matching and smooth pasting conditions for p_{11} at u_2^* relying on the relationships (6.6) and (6.8). This leads to the following system

$$\begin{aligned} (u_1^*)^{\xi} P' + (u_1^*)^{\zeta} Q' &= 1 \\ (u_2^*)^{\xi} P' + (u_2^*)^{\zeta} Q' &= (u_2^*)^{-\beta} M' \\ \xi (u_2^*)^{\xi} P' + \zeta (u_2^*)^{\zeta} Q' &= -\frac{\phi_1 \beta (u_2^*)^{-\beta}}{\phi_2 + \phi_1} M' . \end{aligned}$$

Solving this linear system provides the desired result.

6.6. APPENDIX 6

6.6.1. Effect of uncertainty parameters on current expected profit.

We need to discriminate according to the initial state of the economy. As the processes u and X are assumed to be independent, we can write symbolically $E_0^i = E_{u_0 \times} E^i$.

Initial state 1**:** $P_{11}(t) = \frac{\phi_2}{\phi_2+\phi_1} + \frac{\phi_1}{\phi_2+\phi_1} e^{-(\phi_2+\phi_1)t}$.

We compute

$$\begin{aligned} \frac{\partial P_{11}(t)}{\partial \phi_1} &= -\frac{\phi_2}{(\phi_2+\phi_1)^2} \left[1 - e^{-(\phi_2+\phi_1)t} \right] - \frac{t e^{-(\phi_2+\phi_1)t}}{\phi_2 + \phi_1} < 0 \\ \frac{\partial P_{11}(t)}{\partial \phi_2} &= \frac{\phi_1}{(\phi_2+\phi_1)^2} \left[1 - e^{-(\phi_2+\phi_1)t} - t(\phi_2+\phi_1) e^{-(\phi_2+\phi_1)t} \right] . \end{aligned}$$

Set $x = t(\phi_2 + \phi_1) > 0$ and $\Gamma(x) = 1 - e^{-x} - xe^{-x}$. Since $\Gamma(0) = 0$ and $\Gamma'(x) = xe^{-x} > 0$. We can conclude that $\Gamma(x) > 0$, hence $\frac{\partial P_{11}(t)}{\partial \phi_2} > 0$.

Given u, $E^1[\pi(u(t), X(t))] = P_{11}(t)X_1u(t)^\alpha + (1 - P_{11}(t))X_2u(t)^\alpha$. Therefore, $\cdot\frac{\partial}{\partial \phi_1}\left(E^1[\pi(u(t), X(t))]\right) = u(t)^\alpha(X_1 - X_2)\frac{\partial P_{11}(t)}{\partial \phi_1} > 0$, as $X_1 < X_2$. In the same way, $\frac{\partial}{\partial \phi_2}\left(E^1[\pi(u(t), X(t))]\right) = u(t)^\alpha(X_1 - X_2)\frac{\partial P_{11}(t)}{\partial \phi_2} < 0$, as $X_1 < X_2$.

Initial state 2: $P_{21}(t) = \frac{\phi_2}{\phi_2+\phi_1}\left[1 - e^{-(\phi_2+\phi_1)t}\right]$**.**

Permuting index 1 and 2, we obtain $\frac{\partial P_{22}(t)}{\partial \phi_1} > 0$ and $\frac{\partial P_{22}(t)}{\partial \phi_2} < 0$. Given u, $E^2[\pi(u(t), X(t))] = P_{12}(t)X_1u(t)^\alpha + (1 - P_{22}(t))X_2u(t)^\alpha$. Therefore, $\frac{\partial}{\partial \phi_1}\left(E^2[\pi(u(t), X(t))]\right) = u(t)^\alpha(X_2 - X_1)\frac{\partial P_{22}(t)}{\partial \phi_1} > 0$, as $X_1 < X_2$. In the same way, $\frac{\partial}{\partial \phi_2}\left(E^2[\pi(u(t), X(t))]\right) = u(t)^\alpha(X_2 - X_1)\frac{\partial P_{22}(t)}{\partial \phi_2} < 0$, as $X_1 < X_2$.

Equal arrival rates: $\phi_2 = \phi_1 = \phi$

Case 3.1.: initial state 1 so $P_{11}(t) = \frac{1}{2} + \frac{1}{2}e^{-2\phi t}$.

We obtain $\frac{\partial P_{11}(t)}{\partial \phi} < 0$. It is easy to conclude that $\frac{\partial}{\partial \phi}\left(E^1[\pi(u(t), X(t))]\right) = u(t)^\alpha(X_1 - X_2)\frac{\partial P_{11}(t)}{\partial \phi} > 0$ since $X_1 < X_2$.

Case 3.2.: initial state 2 so $P_{21}(t) = \frac{1}{2}\left[1 - e^{-2\phi t}\right]$.

We obtain $\frac{\partial P_{21}(t)}{\partial \phi} > 0$. It is easy to conclude that $\frac{\partial}{\partial \phi}\left(E^2[\pi(u(t), X(t))]\right) = u(t)^\alpha(X_1 - X_2)\frac{\partial P_{21}(t)}{\partial \phi} < 0$ since $X_1 < X_2$.

Case 4: effects of σ**.**

Set $y = u^\alpha$. Using Ito's lemma, $dy(t) = y(t)\left[\left(-\alpha\lambda - \frac{1}{2}\alpha(1-\alpha)\sigma^2\right)dt + \sigma dw(t)\right]$. Thus taking conditional expectation E_{u_0} of both sides yields

$$dE_{u_0}y(t) = -\left(\alpha\lambda + \frac{1}{2}\alpha(1-\alpha)\sigma^2\right)E_{u_0}y(t)dt\ .$$

It follows that $E_{u_0}y(t) = y_0e^{-(\alpha\lambda+\frac{1}{2}\alpha(1-\alpha)\sigma^2)t}$. Therefore, given $X(t)$, the conditional expected instantaneous profit is given by

$$E_{u_0}(\pi(u(t), X(t)) = u_0^\alpha X_1(t)e^{-(\alpha\lambda+\frac{1}{2}\alpha(1-\alpha)\sigma^2)t}\ .$$

Recall that $0 < \alpha < 1$, it is then easy to see that an increase in σ decreases the expected current profit.

6.6.2. Methodology used for numerical simulations

We have used MATHEMATICA® 3.0. to solve numerically the eight by eight system of equations. Since there is no guaranty to obtain a unique solution, we have proceeded in the following way. Given a value for σ, we first determine the scrapping and upgrading levels for economies always in recession and economies always in expansion. We thus solve a two by two non linear system that yields a unique solution for the range of values we are interested. Given the smoothness of the relationships characterizing the scrapping and upgrading levels, by the Implicit Function Theorem, scrapping and upgrading levels can be expressed as smooth functions of (ϕ_1, ϕ_2, σ). Starting in the neighborhood of an economy always in a unique regime, we increment little by little the value of the parameter ϕ_1 (or ϕ_2), each time giving some relevant initial conditions for the values of the unknowns in order to be sure that the algorithm used by MATHEMATICA® 3.0. converges to the desired solution. To study the effect of the technological,uncertainty σ, we use the same approach starting in the neighborhood of the values obtained for $\phi_1 = \phi_2 = 0.25$ and $\sigma = 1$.

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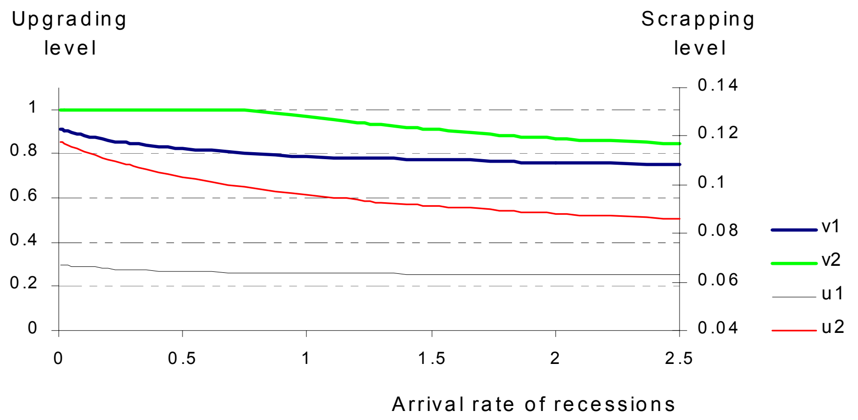

Figure 10: Effects of the arrival rate of recessions ϕ_2; $\phi_1 = 0.25$ and $\sigma = 1$.

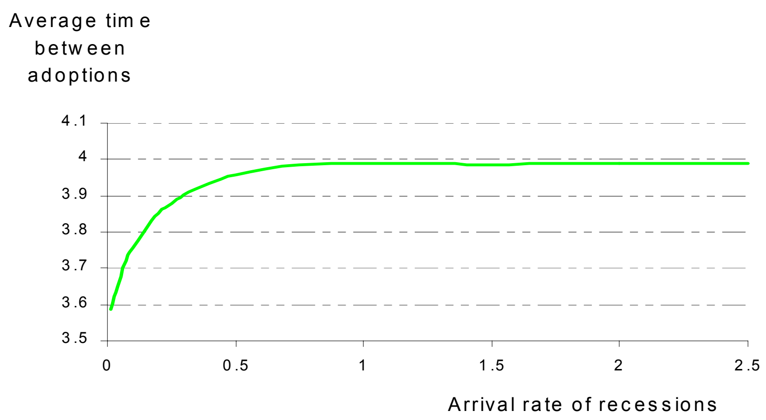

Figure 11: Effects of the arrival rate of recessions ϕ_2; $\phi_1 = 0.25$ and $\sigma = 1$.

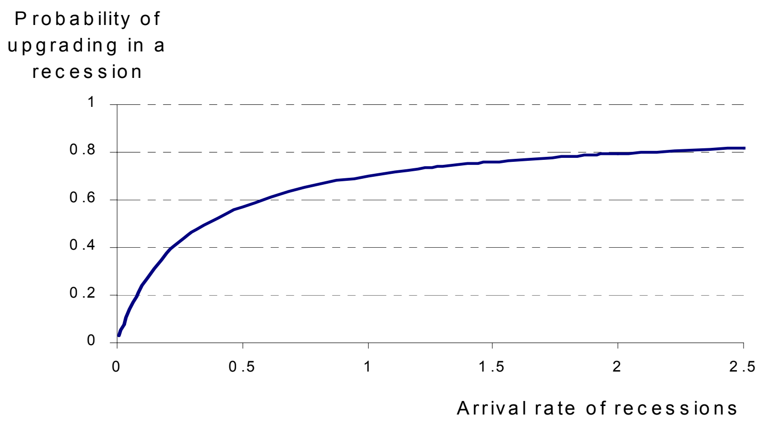

Figure 12: Effects of the arrival rate of recessions ϕ_2; $\phi_1 = 0.25$ and $\sigma = 1$.