The Optimal Investment Scale and Timing:
A Real Option Approach to Oilfield Development

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Abstract

The oil company holds the investment opportunity to develop a delineated oilfield. The investment plan must be presented until a specific date or the oilfield rights return to the government. The firm considers a set of mutually exclusive alternatives of scale to exploit the oilfield. Larger scale means faster exploitation – increasing the present value of revenues, but also higher investment cost. Oil price uncertainty affects all alternatives. In addition to the scale option, the firm has a timing option and hence this investment opportunity is analog to a finite-lived American call option on the best of multiple assets with the same underlying oil price stochastic process but with different benefits and different exercise prices. We examine both geometric Brownian motion and a mean-reversion process to model oil prices. We obtain the undeveloped oilfield (real option) value and the optimal investment rules, i.e., the optimal timing and the optimal scale thresholds.

Keywords: Real Options, Oilfield Development Investment, Optimal Scale of Projects, American Option on Multiple Assets.

1. Introduction

The real options approach is an effective method of economic analysis of investments in projects or non-financial assets ("real assets") under market and technical uncertainties, because it considers the value of managerial flexibility to react to these changing scenarios. Although relatively recent, the real options approach has been adopted by an increasing number of modern corporations[1].

The presence of managerial flexibility in the decision making process under uncertainty provides important gains in the valuation of the investment opportunity, especially for low net present value (NPV) projects, which are similar to "at-the-money options". These low NPV opportunities are more difficult for decision-makers' analysis about whether to undertake or reject the project[2]. Regarding strategic sequential investments, the NPV rule turns out to be a difficult application and can even fail in trying to quantify the hidden options value provided by the investment[3]. Such investments are precisely those where the real options approach aggregates more economic value.

Finance literature presents several cases of applications of real options to value natural resource investments, such as Tourinho (1979) and Brennan and Schwartz (1985)[4]. Paddock, Siegel and

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[1] A large survey reported in Graham & Harvey (2001) on corporate finance practice, with answers from 392 CFOs of different firms in the USA and Canada, shows that 26.59% of the firms "always or almost always" consider the value of real options in projects. This number is about two times the number of firms that use value-at-risk (VaR) or Monte Carlo simulation. See also at www.realoptions.org articles with recent examples of real options applications for investment decisions in several corporations, such as HP-Compaq, General Motors, Chevron, Texaco, Pfizer Inc., BP-Amoco, Dell Computer Corp., J.P. Morgan,, Boeing and Schering Plough, among others.

[2] Dixit and Pindyck (1994) present a comprehensive explanation of the differences between NPV and real options approach to value investment opportunities under uncertainties.

[3] Trigeorgis (1996) presents some examples such as: optimal timing of an investment, option to expand, abandon or suspend the project, strategic options, option to modularity, learning options, etc.

[4] Dias (2001) gives an overview of real options applications in petroleum. Schwartz and Trigeorgis (2001) present other applications of real options to natural resources investment opportunities.

Smith (1988) present the classical model of real options for exploration and production (E&P) of an oilfield, exploiting the analogy between the concession value and the (financial) American call option.

Usually, after the oilfield delineation at the end of the exploratory phase[5], the E&P firm holds an investment opportunity to develop the oilfield by incurring the development costs. In this phase, uncertainty about the economic value of the reserve, related mainly to future oil prices, is the most relevant since the exploration phases significantly reduce the technical uncertainties. At any time up to the option expiration - established by the government agency - the E&P firm can commit to an investment plan for immediate field development, and therefore holds an equivalent American call option. At the expiration, the firm can return the oilfield to the government if the investment opportunity to develop the field is not attractive. We consider both the value of the underlying asset V (or developed reserve value) and the development cost (D) in present values, so NPV = V − D. Hence, we don't consider the *options* available during the *time to build* the project[6].

This paper follows Dixit (1993a) and Dias (1998, 2001) by calculating the optimal development investment timing of an oilfield through a finite number of mutually exclusive development

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[5] The exploratory phase is also an investment option, where the holder of the track rights can exercise the option to drill the *wildcat* well. In case of success, the holder can delineate the petroleum field by paying the cost of *appraisal* wells. The relevant uncertainties in this phase are mainly oil prices and the *technical uncertainties* related to the existence, size and quality of the petroleum reserve. In bids for tracks rights or in wildcat drilling, the *strategic uncertainty* of the other firms’ behaviour is also relevant.

[6] For applications of the *time to build* see Majd and Pindyck (1987), Pindyck (1993), and the discussion in Dixit and Pindyck (1994, chapter 10). Rocha (1996) provides additional discussions and extensions to the time to build models.

alternatives of scale. We do not consider here the *option to expand* capacity by investing first in a lower scale and then expanding to a higher scale by adding capacity[7].

We examine three mutually exclusive alternatives, but the method can be generalized for n different alternatives of scale. These alternatives have different scale in terms of different numbers of wells, different processing plant capacities, different pipeline diameters, different types of production units, etc. There is a trade-off because higher installed capacity results in higher present value for revenues (higher value of V) but requires higher investment (higher D).

Décamps, Mariotti and Villeneuve (2003) give an important and mathematically rigorous revision of Dixit (1993a). They worked with perpetual options and allowed the *option to switch* from a low-scale to high-scale project.

We work with finite-lived options and do not allow this option to switch. However, their results provide important insight for our model. Particularly, like us they also found intermediate waiting regions that will be discussed in the next topic. In addition, the presence of the option to switch does not change the threshold to invest in a low-scale project, so that the investment rule in high-scale project is *myopic* regarding the possibility to switch to higher scale if the market conditions improve even more.

Another related reference is Capozza & Li (1994). They model a perpetual *option to redevelop* an urban land parcel by choosing both the optimal timing (by looking at market demand) and the *intensity* (or scale) of investment. In their case there is a continuum of investment intensities available to the landowner. Although they use perpetual options, an analytical solution is not available for this case of continuous investment intensities.

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[7] However, PUC and Petrobras (through Pravap-14) developed one research project on alternatives with *option to expand* production through optional wells. See Dias (2001) for the basic starting ideas.

Brodie & Detemple's (1997) analysis of American option on multiple assets seems to be related with our work because the n underlying assets V(k), k = 1, 2…n, are mutually exclusive. In their paper, each asset follows a different but correlated geometric Brownian motion. However, they used the same exercise price (analogous to investment in real options) for all assets, which is not so pertinent for optimal scale choice. Here we use different exercise prices (different investments) for each alternative, and the stochastic process (for oil price P) is the same for all mutually exclusive underlying asset V(P, k) alternatives. Here the difference between alternatives is placed in a single deterministic term in the equation of V (see the next section), in addition to the investment D(k). So, here we model optimal scale with the trade-off between investment D(k) and present value of revenues net of operational costs and taxes V(P, k).

The article of Geltner, Riddiough and Stojanovic (1996) is similar to Broadie & Depemple (1997) because they also assume the same exercise price for the alternatives. However, they studied the case of two alternatives of land use choice in a *perpetual* option framework.

Pindyck (1988) is a classical real options paper on capacity choice, but he considers that the capacity can be incrementally expanded. In our paper we consider that if we install a certain capacity or scale of production, we continue with the same scale until the oil reserve's exhaustion. This is true in a large range of real oilfield development projects, chiefly because the oil production reaches a peak in the beginning of its productive life and declines over time due to *depletion* of reserves. For other industries, Pindyck's case can be more common.

Childs, Ott and Triantis (1998) examine mutually exclusive projects in a real options framework. They consider two interrelated projects, each one with two phases (pilot and implementation). However, they do not address the problem of optimal scale like here and do not consider the option to delay the investment. They consider technical uncertainty and full information revelation with the pilot phase, which affects the choice of alternatives. They concentrate on comparing parallel versus sequential pilot projects and on determining their optimal sequence.

This paper is organized as follows. Section 2 presents at the conceptual level the investment model under oil price uncertainty with finite time to expiration, and discusses the intermediate waiting regions feature not addressed in the Dixit paper. Section 3 presents the equations for both geometric Brownian motion and mean-reversion models, as well as the stochastic differential equations and their boundary conditions. Section 4 presents numerical examples for both stochastic processes, highlighting the effect of volatility on the optimal timing to develop the oilfield and the optimal scale of output, and discuss the possible issue of discontinuity of the threshold curve for mean-reversion. The last section presents the concluding remarks.

2. The Model on Selection of Mutually Exclusive Alternatives to Develop an Oilfield

Feasibility studies by oil companies typically analyze a set of a few discrete mutually exclusive alternatives to develop an oilfield[8]. Each alternative has a different number of development wells, processing plant capacities, pipeline diameters, etc. In our framework we consider that the only source of uncertainty is the oil price P(t), which changes stochastically over time.

Let there be n = 3 alternatives to develop an oilfield. Denote alternative k by A_k, k = 0, 1, 2, 3, with alternative A_0 being not to develop. Each alternative k has investment D(k) and benefit V(k). The net present value (NPV) of the investment opportunity to develop the oilfield at time t by choosing the optimal alternative k is defined in Eq.(1), where V is the economic value of the *developed* oilfield.

[8] This practical issue was the first reason for Petrobras to develop a research project with PUC-Rio on this theme. The second reason - more theoretical - was some intriguing results related to the existence of *intermediate waiting regions* not addressed in Dixit (1993a) that Dias found by using approximated models with Excel spreadsheets. These waiting regions were confirmed in this research project.

$$NPV(P(t), k) = V(P(t), k) - D(k) \quad \textbf{(1)}$$

We consider the *fiscal regime of concessions* used in about the half the world's countries. For these countries, the *linear* relation between oil prices and the NPV associated with the exercise of the option to develop the oilfield is at least a very good approximation. One linear model[9] is the *proportional model* (V is proportional to P), also called the "business model", expressed as:

$$V(P(t), k) \;=\; P(t) \,.\, q(k) \,.\, B \quad \textbf{(2)}$$

where:

P(t) = oil price per barrel at time t ($/bbl);

q(k) = economic quality of the developed reserve by adopting development alternative k;

B = the reserve volume (or number of barrels in the ground) in millions of barrels (million bbl).

Equation (2) allows that the value of the developed reserve (V) can be conveniently given as proportional to the oil price (P). By using Itô's Lemma, it is easy to show that V follows the same stochastic process as P[10]. Note that since V is the present value, all discounting effects are embedded in the quality factor q.

The proportionality factor q is named the economic quality of the developed reserve because as q increases, so does the specific value of this developed reserve[11] (for the same oil price and volume). This quality factor depends on many variables, such as reservoir rock quality (permo-porosity properties), hydrocarbon quality, country taxes, operational costs, discount rate, infrastructure

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[9] Another linear model is V(k) = [P(t) . q'(k) . B] – C(B), where q' is the economic quality of the reserve for this model and C(B) is a kind of fixed operational cost in present value. In this case, V and P follow different stochastic processes. Proof and details at http://www.puc-rio.br/marco.ind/payoff_model.html

[10] For the proof see: http://www.puc-rio.br/marco.ind/payoff_model.html

[11] Dias (1998) introduced the concept of economic quality of the reserve, suggesting the application of this concept to model the selection of mutually exclusive alternatives under oil prices uncertainty.

proximity, and development capital in place (scale). Only the latter issue is exploited in this paper because the choice of development scale is one of our control variables in this optimization under uncertainty problem. In a more general setting (e.g., non-linear payoff models), the economic quality of the developed reserve for alternative k is defined by:

$$q(P,k) = \frac{1}{B} \frac{\partial V(P,k)}{\partial P} \qquad \textbf{(3)}$$

We will assume that the value of developed reserve V(P) is linear with P, so that the quality parameter will be constant (independent of P), differing only across the alternatives of scale.

The motivation for the name "business model" is drawn from the reserves transactions market. For example, in the United States the average price paid for one barrel of developed reserve is 33% of the wellhead oil price, which Gruy et al. (1982) named the “one-third” rule of thumb[12].

In order to understand the model better, let us examine the case without the option to wait, that is, the case at expiration (the legal constraint set by the government agency).

Figure 1 shows the NPV(P) linear chart for three different development alternatives.

[12] This “one-third” rule of thumb was used also in Paddock, Siegel and Smith (1988) to perform a numerical example. However, they model the option value as a function of F(V), not F(P) as here. In addition, they did not devote any discussion on the properties of the proportionality factor q as here. In our paper, this quality factor has a key role because it captures the differences of alternatives of scale on the benefit side in the simplest way possible.

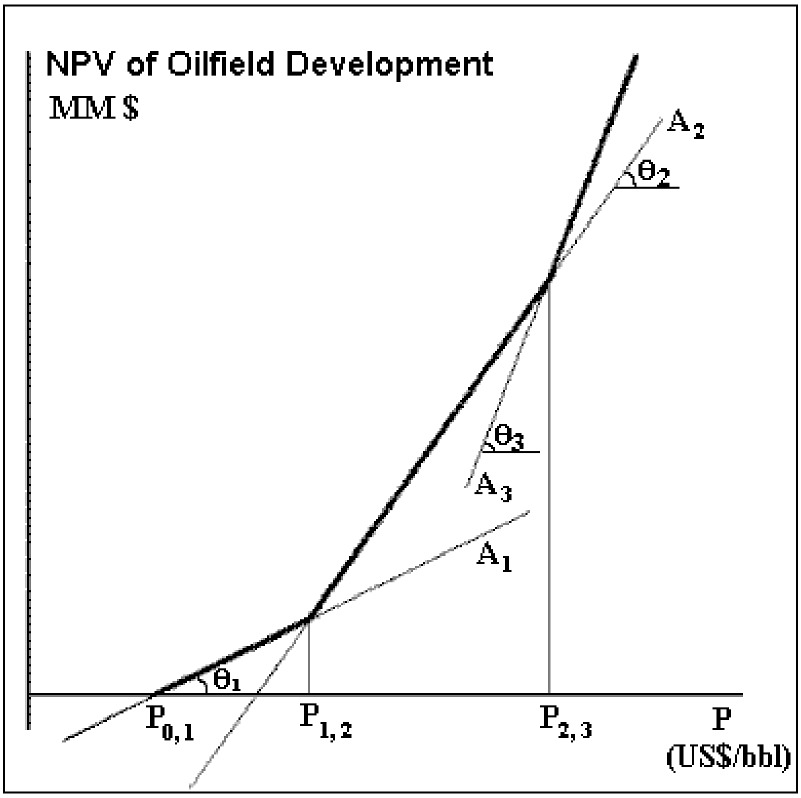

Figure 1 - NPV Functions and the Optimal Exercise at Expiration

Note that the slope of $NPV_k(P)$ is one key difference between the alternatives and it is given by $tg(\theta_k) = q_k\ B$, so that the economic quality of the developed reserve is related to the slope of $NPV_k(P)$. The other difference between the alternatives is the investment, given by the Y-intercept of the NPV lines, that is, $NPV_k(P = 0) = -\ D_k$.

At expiration (t = T), the traditional NPV rule holds, that is, to exercise the alternative with higher NPV. This is presented in Figure 1 above by the envelope NPV function (thicker lines), and will be one boundary condition of our model.

Figure 1 also shows the oil prices at which we would be indifferent between two alternatives. $P_{0,\,1}$ is the indifference point between exercising alternative 1 (A_1) and not investing (alternative 0). $P_{1,\,2}$ is the indifference point between exercising alternatives A_1 and A_2 (equal NPVs for these alternatives). $P_{2,\,3}$ is the indifference point between alternatives A_2 and A_3.

Before expiration (t < T), due to oil price uncertainty, the "wait and see" policy can be optimal even if all alternatives have positive NPVs because the option to delay the exercise can be more valuable than the immediate exercise of any alternative. Before the presentation of the differential equation for the option value, it is useful at this point to discuss conceptually some results in order to understand better the boundary conditions for this model in the next section.

Dixit (1993a), working with perpetual options, stated that if the (upper) option curve smooth pastes the NPV of alternative A_2 at P^*_2, the optimal rule shall be "wait if $P < P^*_2$, and exercise the higher NPV alternative in the opposite case". However, Décamps, Mariotti and Villeneuve (2003) disagree with this conclusion for perpetual options. We also found – at least for the finite-lived options case – that the existence of *intermediate waiting regions* is possible and even common. Figure 2 illustrates this issue, showing an example where intermediate waiting regions appear between the exercise regions of the alternatives A_1, A_2 and A_3.

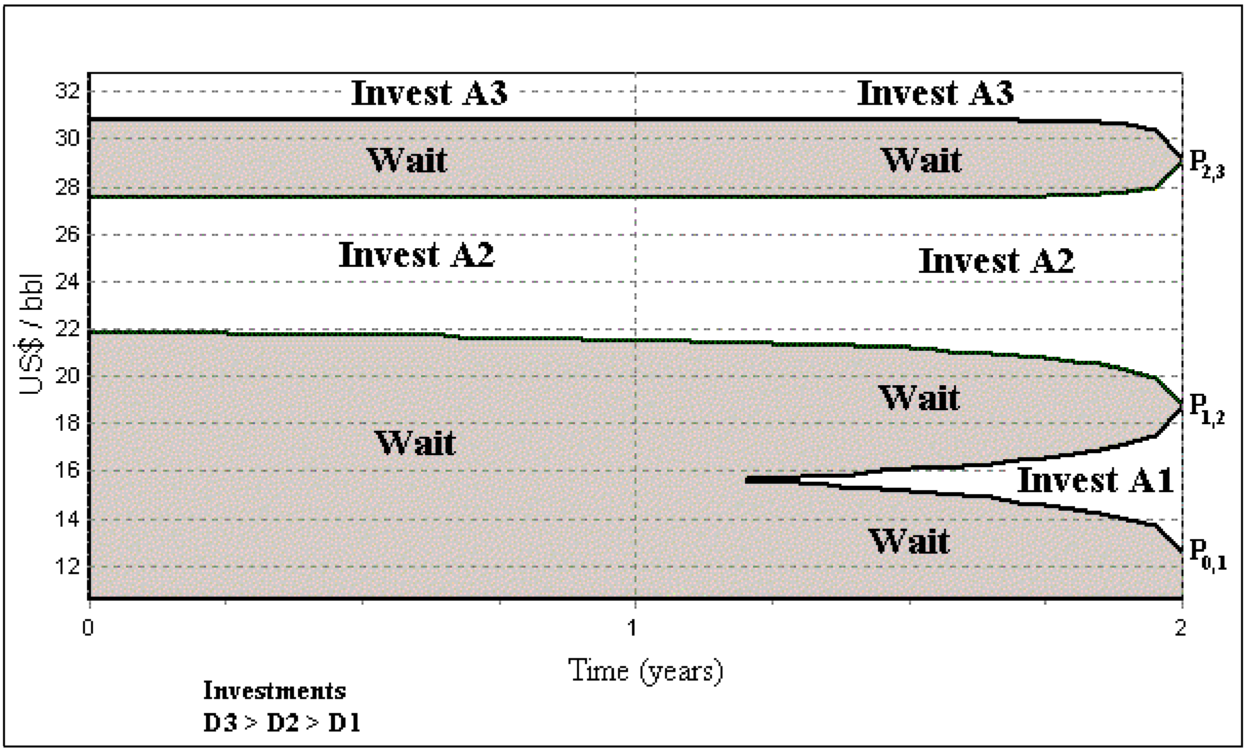

Figure 2 - Optimal Decision Map: Investing and Waiting Regions

The waiting regions are shaded in Figure 2. In this example, the time to expiration is two years. At expiration (t = T = 2 years), the NPV rule holds and the reader can compare the indifference points $P_{0,1}$, $P_{1,2}$, $P_{2,3}$, from Figure 2 with the ones shown in Figure 1. In Figure 2 at t = 0, the decision rule is[13]: wait if $P \in (0, 22)$; invest in alternative A_2 if $P \in [22, 27.6]$; wait again if $P \in (27.6, 31)$; and invest in alternative A_3 if $P \in [31,\infty)$. So, the decision rule is not to invest in the higher NPV alternative if $P \geq P^*_2$ (the threshold to invest in A_2), as in Dixit (1993a). There is an *intermediate waiting region* that in this example at t = 0 is in the range $P \in (27.6, 31)$. The existence of intermediate waiting regions for a given t < T will depend on the parameters, in particular the volatility - higher volatility can make it optimal for us to wait for the higher scale alternative A_3, the intermediate waiting *and* investing regions disappearing.

By observing the asymptotic behavior of this intermediate waiting region in Figure 2, we can conjecture that the existence of this region is possible in the *perpetual* option case as well. Décamps, Mariotti and Villeneuve (2003) worked independently but with perpetual options and proved mathematically that the waiting region around the indifference point between NPV_2 and NPV_3 always occurs (non-empty region) if it is sometimes optimal to invest in alternative 2. They showed that if "*investment in the smaller scale project is sometimes optimal, ... the optimal investment region is dichotomous.*". In their proposition 3.2, they show that the indifference point where $NPV_2 = NPV_3$ (i.e., $P_{2,3}$) *does not belong* to the exercise region. They proved that proposition with the help of the concept of *local time* for the continuous *semimartingale* P applied at this indifference point. This is true even without the option to switch analyzed in that paper. Probably this conclusion from Décamps et al (2003) is even more general, being also true for finite-lived options *except* at the expiration, as our numerical simulations indicate for geometric Brownian

[13] In case of indifference between wait or invest, we assume invest.

motion. However, for the mean-reversion case this can be not true due the discontinuity of the thresholds curve for the high-scale cases (see our section 4.2).

The existence of this intermediate waiting region led to an apparently surprising result. In Figure 2, at t = 0, if the oil price is US$ 30/bbl, the wait and see policy is optimal, but as the price *drops* to US$ 27/bbl, it is optimal to invest in alternative A_2. The intuition in this case is as follows: if the price P is US$ 30/bbl, the hopefulness to invest optimally in the larger scale alternative A_3 is sufficiently high (because P is close to the threshold P^*_3) to offset the value of exercising the "deep-in-the-money" alternative A_2. However, if the price drops to US$ 27/bbl, the probability for the price hitting the threshold P^*_3 before expiration is not high enough to justify the delay in the option exercise of alternative A_2.

Note also in Figure 2 that as time passes, e.g. at t = 1.5 years, a region appears of immediate exercise of alternative A_1. Together with this new exercise region appears a *new* intermediate waiting region. We found that when a new optimal investment region appears, it always divides the previous waiting region into two waiting sub-regions, so the exercise and waiting regions appear together as time goes by. If the oil price follows a geometric Brownian motion, the emergence of new exercise and new waiting regions together is a general result as time approaches expiration for non-dominated scale alternatives[14]. This occurs because the thresholds curves are continuous (no jumps) and at expiration there are three exercise regions (see Figure 1), which comprise the limit (boundary) condition at expiration (t = T). We will see that for the mean-reversion model for oil prices, due to the possibility of discontinuity in the threshold curves at expiration, some intermediate waiting regions can never occur.

[14] In our model, a higher development cost alternative j is non-dominated if it has higher economic quality q_j than all the lower development cost alternatives.

Let us define P^*_k and P^{**}_k, $k < k_{max}$, as the *lower* and *upper* oil price levels (so that $P^*_k < P^{**}_k$) that define the optimal exercise region for alternative A_k during the period this region exists. For the higher scale alternative k_{max}, there is always P^*_{kmax}, but never a finite value of P^{**}_{kmax}. For the other alternatives, P^*_k and P^{**}_k can exist only for time t very *near* expiration. At expiration t = T, P^*_k collapses to $P_{k-1,\,k}$ and P^{**}_k collapses to $P_{k,\,k+1}$.

The existence of waiting regions reflects the real option value of the additional opportunity to invest in a higher scale alternative. Figure 3 below illustrates this point by showing the option value F for t = 0 and the payoffs (NPVs) for immediate exercise in each alternative.

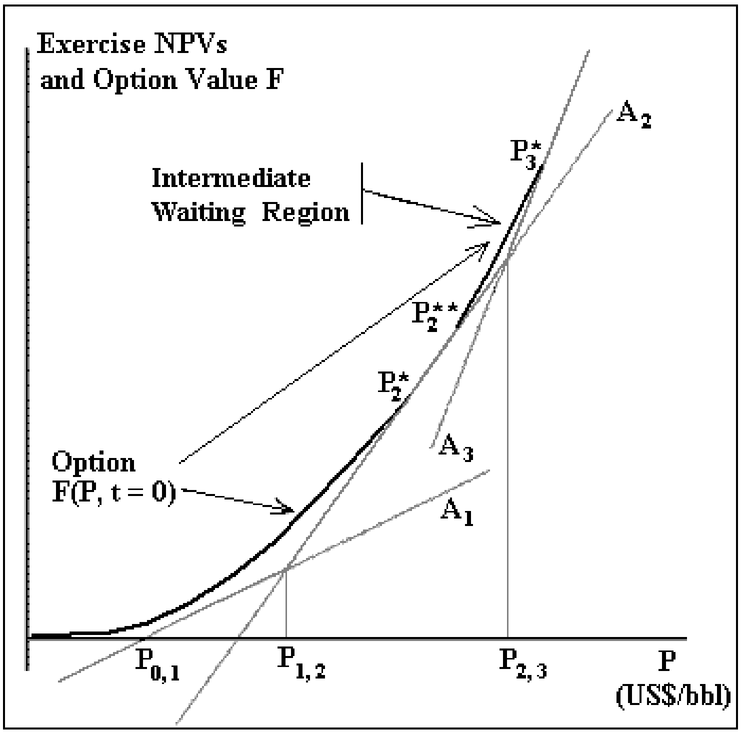

Figure 3 - Option Value at t = 0 and Exercise Payoffs

Note that like in the Figure 2, Figure 3 shows that for t = 0 the threshold P^*_1 does not exist (the option value F is higher than NPV_1 $\forall$ P), there is an exercise region for alternative A_2 (between P_2^* and P_2^{**}), and there is an intermediate waiting region between P_2^{**} and P_3^*. Décamps, Mariotti and Villeneuve (2003) showed for perpetual options that this region always exists (if sometimes it is optimal to exercise alternative 2, as occurs in the figure). Numerically, the error can be small if exercising the option (higher NPV as suggested by Dixit in 1993) rather than waiting in this intermediate optimal waiting region. So, even not being strictly correct, the error from using Dixit's insight can be small in practice. However, there can be a large range to the optimal waiting interval (between P_2** and P_3*).

Another insight from Figure 3 is that both the *value-matching* and *smooth-pasting* properties exist at the optimal investment threshold not only for P_k* but for the upper threshold P_k** as well, if they exist. For example, at P_2** in Figure 3 the option curve F has the same slope as the straight payoff line NPV_2. This is not surprising because the case of waiting with $P > P_2^{**}$ and exercising when P drops to $P = P_2^{**}$, is like an *option to abandon* the opportunity to invest in the higher scale alternative A_3, by taking the irreversible decision to invest in A_2. The smooth-pasting and value-matching properties at optimal thresholds occur in practically every real options problem – see Dixit & Pindyck (1994) and Dixit (1993b).

In addition to be optimal "wait and see" policy in this intermediate waiting region, the option curve value in this region can also indicate something about an optimal project scale. Imagine that the feasibility study team is willing to study another scale alternative but is looking for an immediate optimal investment. At $t = 0$, if P belongs to this intermediate waiting region, what is the optimal scale project for an *immediate investment exercise*? It must have a quality q* equal to the derivative of this option value, that is, q* must be equal to the derivative of the option value F(P, t) with respect to P. This conclusion is derived from the smooth-pasting property at the optimal option exercise. This project does not exist in the figure above, but the optimal scale in this case is an intermediate scale between alternatives 2 and 3 with a payoff curve slope (given by the economic quality q*) so that $q_2 < q^* < q_3$. So, there is an interesting practical link between the concept of quality q and the smooth-pasting property for the optimal scale option exercise.

The solution shown in the Figures 2 and 3 must be found numerically, for example by using a binomial approach and working backwards. In this setting we compare the option value of waiting with the payoff from exercising the higher NPV alternative for every time t and every oil-price value in the binomial tree. We will work with the *partial differential equation approach* for the option value as function of the stochastic oil price and with the appropriate boundary conditions. The discussion above facilitates understanding of the boundary conditions.

3. Geometric Brownian Motion, Mean-Reversion and the Differential Equations

3.1) Geometric Brownian Motion

By knowing the current oil price and the stochastic differential equation that governs its future evolution we can determine both the optimal decision rule and the option value of the undeveloped oilfield (the option to develop). The real option value is conditional on the optimal decision rule. The optimal decision rule considers *two control variables*, the optimal *timing* to develop the oilfield and the optimal production *scale*.

In addition to oil prices (P), the other *state variable* is the time (t), because we have here a finite-lived real option, with t ∈ [0, T]. After the expiration time T, in case of non-development, the oilfield returns to the government agency and is worth nothing.

Let F(P,t) be the real option value to develop and let the oil price (P) be given by the following stochastic differential equation, known as geometric Brownian motion:

$$\frac{dP}{P} = \alpha\, dt + \sigma\, dz \qquad \textbf{(4)}$$

where, α is the drift of the process, σ is the volatility parameter and dz is the Wiener increment defined as: $dz = \varepsilon \sqrt{dt} \quad \varepsilon \approx N(0,1)$, with N(0,1) being the standard normal distribution.

Lemma: Let F(P, t) be the real option to develop the oilfield by choosing timing and scale from a set of n investment alternatives, k = 0, 1, … n, and $NPV_k(P)$ the net present value by exercising the alternative of scale k (being NPV(k = 0) = 0). The following inequality holds:

$$F(P, t) \geq NPV_k(P) \quad \forall \; k \in \{0, 1, \ldots n\}, \forall \; t \in [0, T] \qquad \textbf{(5)}$$

This lemma is derived by construction, given that the oil firm has the freedom to choose the development timing (constrained by the legal limit T) and the scale from a set of alternatives found in the feasibility study. Because the firm *can* exercise the higher NPV alternative, the option value cannot be lower than the NPV of any alternative. This lemma will help in the boundary conditions to define the existence of investment thresholds (see below).

Assuming that the oil market is sufficiently complete and that there are no arbitrage opportunities in equilibrium, by using Ito's Lemma and contingent claims approach it can be shown that[15] the option value follows the following partial differential equation (PDE), where r is the risk-free interest rate and δ is the convenience yield of the commodity:

$$\frac{1}{2}\sigma^2 P^2 F_{PP} + (r - \delta) P F_P + F_t = r F \qquad \textbf{(6)}$$

where the subscripts (P, PP, and t) denote partial derivatives. Let the scale alternatives be ordered, i.e., $D_1 < D_2 < \ldots < D_n$. Eq. (6) is subject to the following boundary conditions Eq.(7 – 12):

$$F(0, t) = 0 \qquad \textbf{(7)}$$

$$F(P, T) = Max[NPV_k(P)], \text{ for all } k = 0, 1, \ldots n \qquad \textbf{(8)}$$

$$F(P_k^*(t), t) = NPV_k(P_k^*, t), \text{ for } t < T \text{ and for all } k \text{ which exist } P_k^*(t) \qquad \textbf{(9)}$$

[15] See Black and Scholes (1973) or Merton (1973) for a financial option pricing methodology and Dixit and Pindyck (1994) for a real options approach using *contingent claims*. The partial differential equation is the same as that of Black-Scholes-Merton, the differences are in the boundary conditions.

$$\frac{\partial\left[F(P_k^*(t),t)\right]}{\partial P} = \frac{\partial\left[NPV_k(P_k^*,t)\right]}{\partial P} = q_k \,.\, B\,, \quad \textit{for } t < T \textit{ and for all k which exist } P_k{*}(t) \tag{10}$$

$$F(P_k{**}(t),\ t) = NPV_k(P_k{**},\ t), \quad \textit{for } t < T \textit{ and for all } k < n \textit{ which exist } P_k{**}(t) \tag{11}$$

$$\frac{\partial\left[F(P_k^{**}(t),t)\right]}{\partial P} = \frac{\partial\left[NPV_k(P_k^{**},t)\right]}{\partial P} = q_k \,.\, B, \quad \textit{for } t < T \textit{ and for all k which exist } P_k{**}(t) \tag{12}$$

Eq.(7) is usual in option pricing and says that P = 0 is an absorbing barrier so that the option value is worthless for P = 0. Eq.(8) is the option expiration condition, when the option value is either to commit to the investment for the higher positive NPV given by the alternative of development $k \geq 1$ or return the oilfield (k = 0) earning a zero option value otherwise. Eq.(9) and Eq.(10) are the value-matching and smooth-pasting conditions for the lower threshold P_k* if this threshold exists for a given t < T. These conditions set the continuity of the option value and its derivative at the optimal price P_k*. Recall our discussion in the previous section showing that depending on the problem parameters, the threshold to exercise high-scale alternatives may not exist for a given t, e.g., in Figures 2 and 3, at t = 0, P_1* does not exist. The value matching condition is equivalent to the lemma for the equality case, so that the equality case in the lemma sets the existence or not of this lower threshold. Eq.(11) and Eq.(12) are also value-matching and smooth-pasting conditions, but for the upper threshold P_k** if this threshold exists for a given t < T. Note that for k = n, this threshold does not exist, but P_k** can exist if it the associated P_k* exists for k < n. Again, see the previous section for the intuition behind this point. The four boundary conditions (Eqs. 8-12) assume by definition that $P*_k < P**_k$ when they exist (see the previous section).

This option-pricing problem described by Eq (6) and its boundary conditions is a free-boundary problem of optimal stopping time, usual in option pricing theory. The option was solved numerically by applying the finite difference methods in the explicit form[16] with an optimization procedure as shown in Appendix A.

Note that for a number n of alternatives for oilfield development, there are at most (2n − 1) threshold curves for the oil price to calculate. Each region between these threshold curves $P_k^*(t)$ and $P_k^{**}(t)$ for the time interval where these regions exist, corresponds to one optimal alternative k for development during the option's lifetime.

Figure 4 shows another example of the investment and waiting regions, considering that the oil price follows a geometric Brownian motion for development rights that expire in 2 years and for n = 3 alternatives (A_1, A_2, or A_3) of development scale in increasing order.

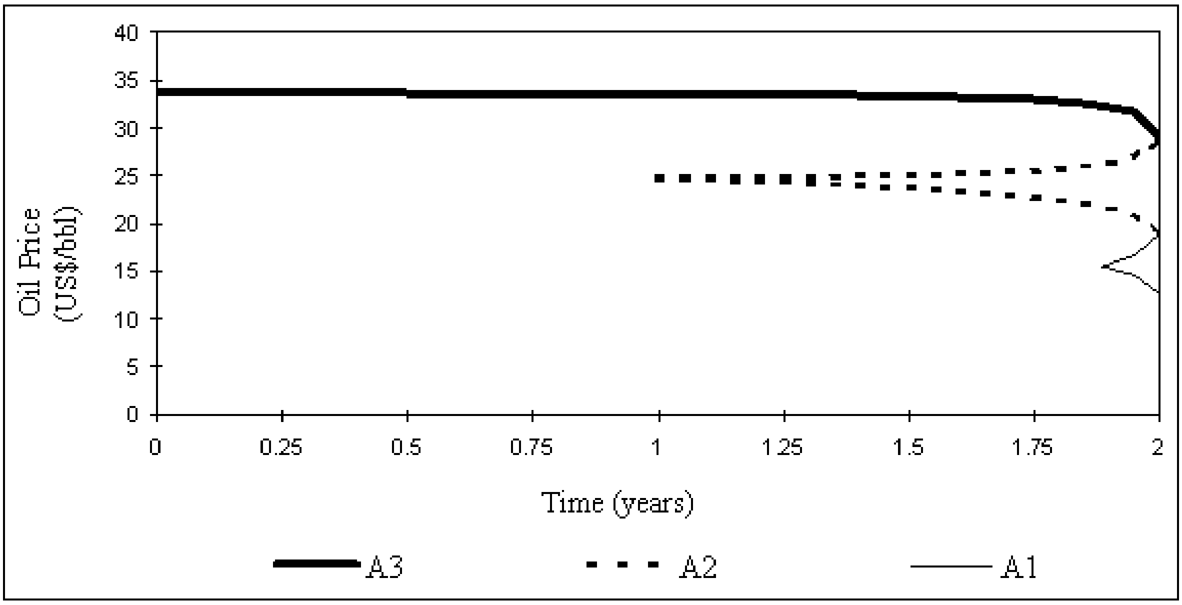

Figure 4 - Investment and Waiting Regions: Geometric Brownian Motion

When compared with Figure 2, the example in Figure 4 shows for t = 0 a case where we either exercise the higher scale alternative A_3 (if $P \geq 33.5$ \$/bbl) or "wait and see" otherwise. There are no intermediate exercise or waiting regions due to the higher volatility used in this example (25% p.a.).

[16] For an application of the finite difference methods to option pricing, see Brennan and Schwartz (1978). More details about the methodology can be found in Ames (1977) or Smith (1971).

Since we are working numerically with three alternatives, there are at most five optimal threshold curves to determine, performing three areas of option exercise. Again, this maximum number of threshold curves occurs only near of the expiration.

Figure 5 shows the option value at the current (t = 0) and at the expiration time (t = 2 years), for the numerical parameters used in Figure 4 and presented in Section 4 below. Note again that at expiration, we have three different areas for exercise, each one representing a certain development alternative for the oilfield. The option at current time is only exercised for alternative A_3 with oil prices equal to or above US$33.50/bbl, since the intermediate waiting and investment regions do not exist for t = 0.

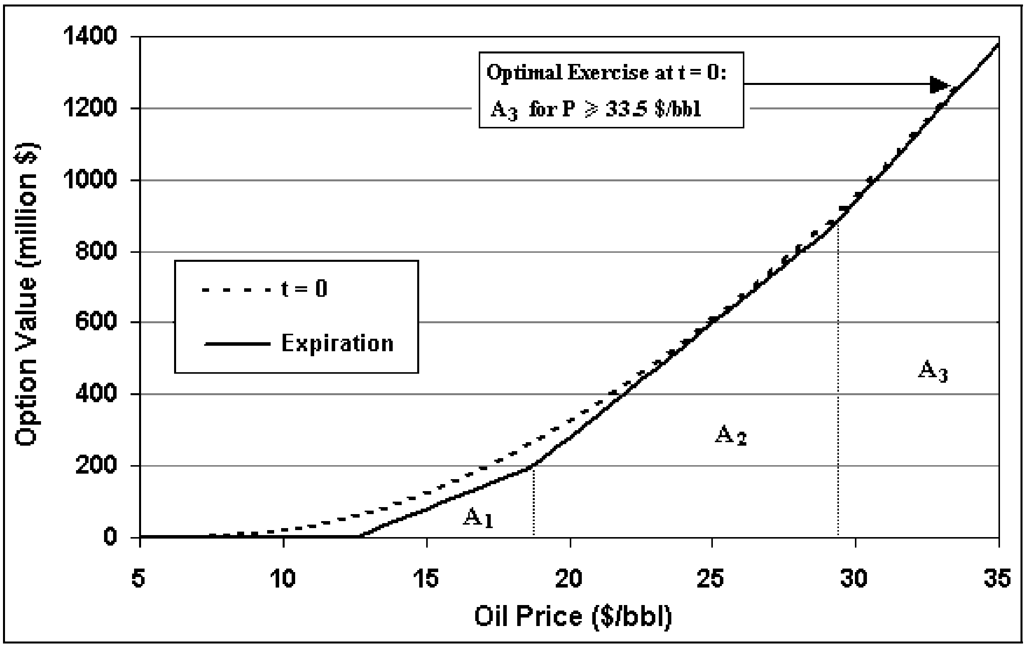

Figure 5 - Option Value of the Undeveloped Oilfield: Geometric Brownian Motion

3.2) Mean Reversion Model

Consider the mean reversion hypothesis, frequently used for commodities, where oil prices (P) evolve as the following stochastic process, known as inhomogeneous geometric Brownian motion, or as Battacharya's (1978) mean-reverting process[17]:

$$dP = \eta(\overline{P} - P)dt + \sigma P dz \tag{13}$$

where, η is the reversion speed of the process, σ is the volatility parameter, $\overline{P}$ is the long-run equilibrium mean and dz is the Wiener increment defined as before: $dz = \varepsilon\sqrt{dt}$, $\varepsilon \approx N(0,1)$.

Following the same *contingent claims* approach described in the previous section, it can be shown that[18] the option value F(P,t) follows the partial differential equation shown below, where r is the risk-free interest rate and ρ is the risk-adjusted discount rate for the underlying (oil price) variable:

$$\frac{1}{2}\sigma^2 P^2 F_{PP} + \left[r - \left(\rho - \frac{\eta(\overline{P} - P)}{P} \right) \right] P F_P + F_t = rF \tag{14}$$

Again, the subscripts (P, PP, and t) denote partial derivatives. Eq. (14) is subject to the same six boundary conditions (Eqs.7-12) described in the previous sub-section.

Note that for mean-reverting processes, the convenience yield of the commodity is not constant, it is a function of the oil price:

$$\delta(P) = \left(\rho - \frac{\eta(\overline{P} - P)}{P} \right) \tag{15}$$

This non-constant convenience yield is a usual characteristic of mean reversion processes. The parameter δ is endogenous in our model, and from a market point of view, is used in the sense

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[17] This equation corresponds to Eq.(4.9) in Kloeden & Platen (1992, p.119). See also an interesting discussion of this process and comparison with other mean-reverting processes in Robel (2001).

[18] See Dixit and Pindyck (1994) chapters 5 and 12 for similar examples of geometric mean reverting process.

described by Schwartz (1997, p.2): "*In practice, the convenience yield is the adjustment needed in the drift of the spot price process to properly price existing futures prices*". Note also that for the mean-reversion model, the risk-adjusted discount rate ρ appears, even assuming complete markets. This feature does not occur with geometric Brownian motion because the convenience yield is constant in that model. Here the convenience yield is a function of the risk-adjusted discount rate for oil price risk.

See also Dias & Rocha (1999) for a discussion of the value of the convenience yield in mean-reverting models highlighting the possibility of this parameter's becoming negative for low oil prices. Eq.(15) shows that $\delta(P) < 0$ is possible for low values of P. In this case, there is the possibility of discontinuity in the threshold curves at expiration because from the literature of American options we know that the *earlier* option exercise is never optimal when the convenience yield is negative or zero. We will address this question in Topic 4.2.

Figure 6 shows the option value at the current ($t = 0$) and at the expiration time, using volatility of 25% p.a. and for the numerical parameters for the mean reversion process presented in Section 4 below. Note again that at expiration we have three different exercise areas, each representing a certain development alternative for the oilfield. The option at current time is exercised for the alternative A_2 if the oil price is between US$ 22.90 and 28.30/bbl, there is a thin intermediate waiting region between US$ 28.30 and 29.90/bbl, and for $P \geq$ US$ 29.90/bbl, it is optimal to exercise scale alternative A_3.

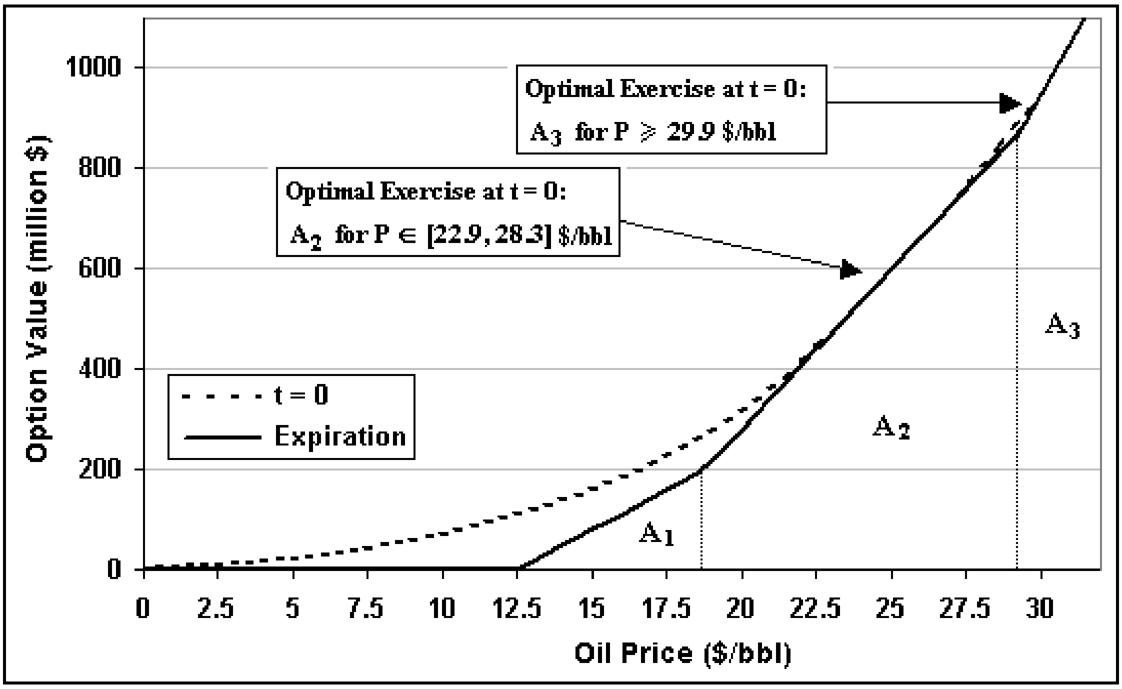

Figure 6 - Option Value of the Undeveloped Oilfield: Mean Reversion Model

4. Numerical Simulations and the Effect of Volatility

4.1) Geometric Brownian Motion

Consider three oilfield development alternatives, A_1, A_2 and A_3. The investment opportunity to develop the oilfield expires in T = 2 years and the manager has to decide which is the optimal timing for development as well the optimal production scale that maximize the investment option value.

Let the following be the parameters for the base case[19] (MM = million): $q_1 = 0.08$, $q_2 = 0.16$, $q_3 = 0.22$, D_1 = US$ 400 MM, D_2 = US$ 1000 MM, D_3 = US$ 1700 MM, B = 400 MM bbl, r = 8% p.a., δ = 8% p.a., σ = 25% p.a.[20] and P_0 = US$ 20/bbl. With these parameters, the NPVs for immediate exercise are NPV_1 = US$ 240 MM; NPV_2 = US$ 280 MM; and NPV_3 = US$ 60 MM.

[19] Some values were estimated using available data about oil prices or using available related literature.

[20] Based on the volatility estimation of Dias and Rocha (1999).

Table 1 shows the managerial flexibility value for having a certain number of development alternatives. Note that the option value increases with the number of available alternatives, reflecting the option value of additional managerial freedom in choosing the investment scale. However, the rate of additional value is decreasing with the number of alternatives, so that in practice after 3 alternatives the additional value may not compensate the time spent by the feasibility study team in detailing additional scale alternatives.

Table 1: Managerial Flexibility Value

Development Alternatives : Production Scale	Option Value (US$ MM)
k = 2 : Medium Scale	310.98
k = 1 or 2 : Small or Medium Scale	322.65
k = 1 or 2 or 3: Small, Medium or Large Scale	323.33

Note also that these values are higher than the highest NPV alternative (NPV_2 = US$ 280 MM).

Figure 7 compares the oil price threshold curves with a change in the volatility parameters (15% p.a. and 25% p.a.). A lower volatility means lower probability of changes in oil price evolution. In this situation the option is exercised sooner (note how the exercise area for the development alternatives becomes higher) because the option of waiting become less valuable. At t = 0, the intermediate waiting and exercise regions appear only in the low volatility case. These regions appear even for very high volatility values, but only at times near or very near expiration.

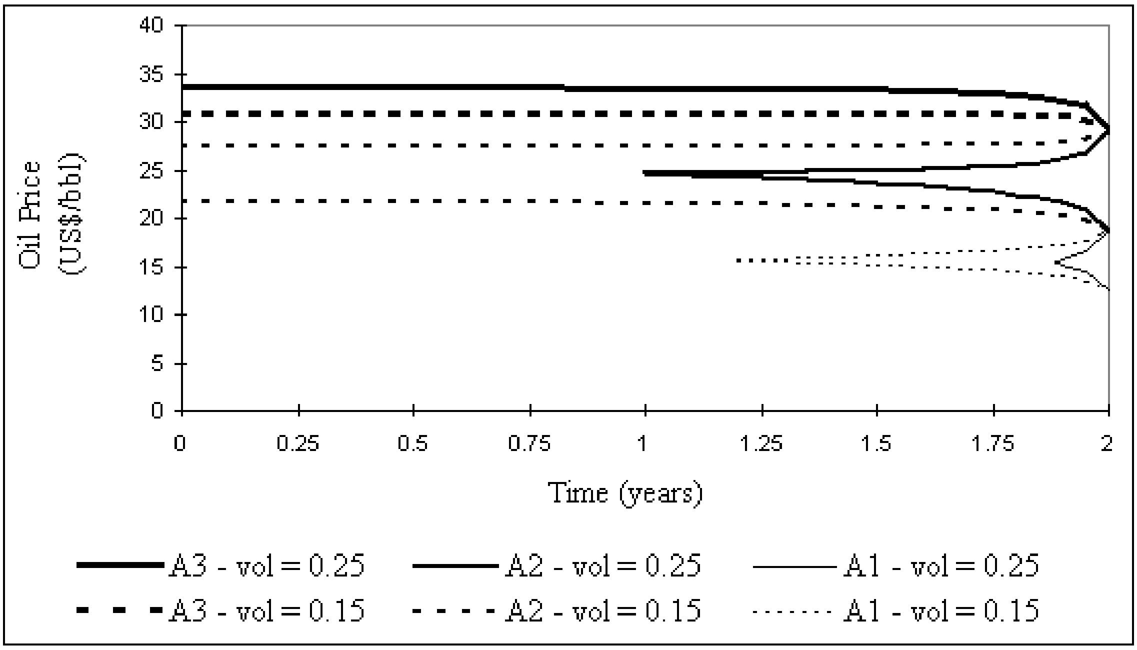

Figure 7 - Geometric Brownian Motion: The Effect of Volatility on the Thresholds

Table 2 shows the real option values at t = 0 and the optimal investment rule for different volatilities (including an intermediate volatility not shown in Figure 7) and different current oil prices.

Table 2: Option to Develop Values (US$ MM) and Optimal Action for Geometric Brownian Motion

Volatility (% p.a.)	Current Oil Price (US$/bbl)		
	15	25	30
15	85.89 “wait”	600 “exercise A_2”	942.21 “wait”
20	102.55 “wait”	600 “exercise A_2”	948.65 “wait”
25	122.29 “wait”	605.21 “wait”	958.72 “wait”

4.2) Mean-Reversion Process

We use the base case parameters from the previous section (except δ) plus the specific parameters of the mean reverting process: ρ = 12% p.a., η = 0.3466[21], $\overline{P}$ = US$20/bbl[22].

[21] This reversion speed implies a half-life of about 2 years. Bradley (1998, p.59) finds a half-life of 1.39 years.

Before showing the threshold chart, it is important to calculate the break-even prices (for which $NPV_k = 0$), and the oil price that makes the convenience yield equal to zero, due to the possibility of discontinuity in the threshold curves at expiration, as explained below.

The break-even prices for the three alternatives are: P_{be1} = US$ 12.50/bbl; P_{be2} = US$ 15.635/bbl; and P_{be3} = US$ 19.32/bbl, calculated with the equation $P_{be(k)} = D_k/(q_k\ B)$.

By using the Eq.(15) and the base-case parameters, it is easy to find that the oil price that makes the convenience yield equal to zero is P = US$ 14.86/bbl. So, for oil prices equal to or lower than US$ 14.86/bbl, it is never optimal to exercise the option *before* expiration. However, exactly at expiration, if the oil price is in the range $12.50 \leq P \leq 14.86$, even with $\delta \leq 0$, it is optimal to exercise alternative A_1 because we have a non-negative NPV_1 and so exercising the option increases the shareholder value if $P > 12.5$ in this numerical example. This means that there is a discontinuity in the threshold curve at least for alternative A_1: the threshold to invest in this alternative at expiration is US$ 12.5/bbl, but at an infinitesimal time before the threshold for this alternative jumps to a much higher value in order to avoid exercise with negative, zero or too low values for the oil convenience yield $\delta(P)$.

In the presence of other scale alternatives, this individual behavior of alternative A_1 can make both the intermediate exercise and waiting regions vanish even very shortly before expiration. This will occur in our example for alternative A_1.

Figure 8 compares the threshold curves for oil price and volatility effect (25% p.a. e 45% p.a.). Note that the region for investment in alternative A_1 no longer exists except at expiration, even for a

[22] Baker et al (1998, p.129) estimate the long run oil price as $18.86/bbl (in 1995 dollars) and used (pp.138-140) US$20/bbl as the initial long run level in their model. This value is also adopted in Bradley (1998, pp.59-61) and shown in Cortazar & Schwartz (1996, Figure 4). Econometric tests including more recent data point to a long-run level higher than US$ 20/bbl.

time very *near* expiration. This is caused by the discontinuity of the threshold curve for this alternative at expiration, as explained in the previous paragraphs. For alternative A_1, the oil prices region for optimal exercise *at expiration* implies either a negative convenience yield or values very near zero. So, for alternative A_1 *alone*, a very small time before expiration requires a threshold much higher than the upper oil price value that makes this alternative optimal at expiration. However, at expiration, it is optimal to exercise alternative A_1 for a range of prices of $12.50 \leq P < 18.75$, shown as a vertical line at t = T.

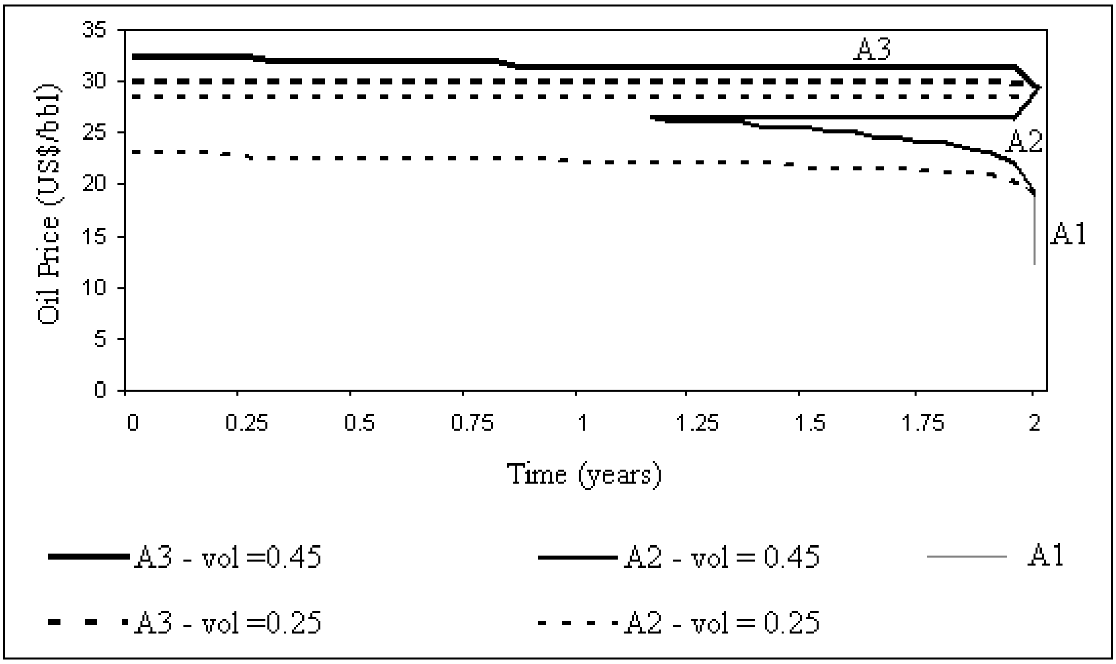

Figure 8 - Investment and Waiting Regions for the Mean-Reversion Model

This issue of threshold curve discontinuity never occurs with geometric Brownian motion because the convenience yield is constant (and positive) in that model. So, intermediate exercise and waiting regions always occur at least for times very near expiration in that model, but the same is not true in the mean-reversion model. Note also in Figure 8 that for the base-case volatility of 25% p.a., at t = 0, there is a thin intermediate waiting region between US$ 28.30 and 29.90/bbl and a region for exercising alternative A_2 between US$ 22.90 and 28.30/bbl. Compare these regions from Figure 8 with the option chart in Figure 6.

The existence of these intermediate waiting regions between the optimal exercise of alternatives 2 and 3 (for σ = 25% at t = 0 and for σ = 45% at t > 1.2 in Figure 8, and in the other simulations we performed) indicates that the conclusion is more general – valid for other stochastic processes – that these waiting regions always occur if it is sometimes optimal to invest in alternative 2 (intermediate scale alternative). This is a strong conjecture demonstrated only for the geometric Brownian motion and perpetual options by Décamps et al. (2003).

The real option to develop, considering all three scales options, is worth US$ 313.86 MM if the current oil price is US$ 20/bbl. Table 3 shows the option values at t = 0 and the optimal investment rule for different volatilities and different oil prices.

Table 3: Option Value (US$ MM) and Optimal Action for Mean-Reverting Process

Volatility (% p.a.)	Current Oil Price (US$/bbl)		
	15	25	30
15	126.21 "wait"	600 "exercise A_2"	940 "exercise A_3"
20	140.92 "wait"	600 "exercise A_2"	940 "exercise A_3"
25	158.45 "wait"	600 "exercise A_2"	940 "exercise A_3"

Comparing Tables 2 and 3, we can see the differences in the option values due to the hypothesis about the oil price evolution. Options values for mean-reversion are higher for low initial prices (lower than $\overline{P}$) and slightly lower for high initial oil prices.

Note in Table 3 that the option under the mean-reverting hypothesis is exercised immediately when the current oil price is above the long-run equilibrium mean (US$20/bbl). This is due to the expectation of the mean-reverting process, leading to a small probability for oil prices to be too far from the mean[23], decreasing the value of waiting for a higher scale alternative that has a threshold

[23] The mean-reverting process is a stationary process with a bounded variance. The reversion speed is the force that pulls back the oil price towards the long-run equilibrium mean. As far is the oil price from the equilibrium as higher is the reversion force, like a spring force.

very far from the long-run equilibrium price to which the prices are attracted. Immediate exercise, in this case, maximizes the option value.

However, for oil prices lower than the long-run equilibrium mean, the option is not exercised even for the lower volatility case. In this case, the expectation that prices will revert to the mean increases the option to wait. These results vary according to the parameters of the model (speed of reversion, current price, long-run mean, volatility, and time to expiration) and should be taken with caution. For example, at t = 0, Figure 8 shows that for the very high volatility case of 45% p.a., waiting is the optimal policy even for prices of US$ 25 or 30/bbl.

5. Concluding Remarks

This paper analyses the investment opportunity of developing an oilfield. It calculates the investment option value and also the optimal developing timing and production scale, considering oil price uncertainties and applying the real options approach.

The investment opportunity in the oilfield is analogous to an American call option on several assets but with the same underlying oil price stochastic process and with different exercise prices and different payoffs (different oil price functions) from the different scale alternatives. The payoff (NPV) of exercising a specific alternative k is the value of the developed reserve using this scale alternative minus the development cost of the alternative. The economic quality of the reserve parameter distinguishes the scale alternatives on the benefit side.

The presence of managerial flexibilities, i.e., the options to choose when to develop the oilfield and to set the optimal production scale, increases the investment option value.

We determine the oil price threshold curves for the exercising options. Instead of one threshold curve as in standard (financial or real) American call options, we have regions for option exercise and waiting regions. The existence of intermediate waiting regions has not been addressed before in the real options literature and can occur with both stochastic processes that we analyzed.

According to oil price volatility and the stochastic process for the oil price, the area for exercising the option can degenerate and the option to wait can dominate immediate development. An increase (decrease) in the volatility parameter decreases (increases) the area for exercising the option for both stochastic processes: geometric Brownian motion and mean reversion. This is due to the fact that the option to wait for better conditions to commit to the investment (waiting for a higher scale) is higher (lower) in such cases.

Finally, the mean-reverting process - frequently used in commodities modeling - presents different results than geometric Brownian motion. Specific production scales can never be optimal before expiration. This is caused by discontinuity of the threshold curves at expiration if the break-even oil prices for these alternatives imply a negative or zero convenience yield. For geometric Brownian motion, this issue never occurs because the convenience yield is constant (independent of the oil price) and commonly assumed to be positive.

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Appendix A: Finite Difference Method in Explicit Form for Numerically Solving the Partial Diferential Equation (PDE)

The finite difference method transforms the partial differential equation, Eq. (6) in the text, and its respective boundary conditions into a difference equation that can be solved numerically.

By using a specific discretization mesh (time step, and oil price step), the explicit form converges to the exact solution of Eq. (6) and it is easier and faster (especially for a low number of state variables) than the implicit forms or the Monte Carlo simulations techniques associated with optimization procedures.

Regarding the free-boundary problems defined in Eq. (6), the explicit form and the discretization mesh can easily handle the optimization algorithm used to solve those optimal stopping time problems, like the "backward induction" style of a stochastic dynamic programming approach. Implicit forms, however, have to deal with a simultaneous system of equations together with the optimization procedure.

Numerical solutions for partial differential equations can be found in Ames (1977) or Smith (1971). Dixit and Pindyck (1994), Chapter 10, applies the same procedure (explicit form together with an optimization algorithm) to solve an option-pricing problem about sequential investment.

Let F(P,t) at point (P,t) be represented by $F_{i,j}$, where P = iΔP for $i \in (0,m)$ and t = jΔt for $j \in (0,n)$

Assume the following partial derivative approximations:

$$F_{PP} \approx [F_{i+1,j+1} - 2F_{i,j+1} + F_{i-1,j+1}] / (\Delta P)^2; \quad F_P \approx [F_{i+1,j+1} - F_{i-1,j+1}] / 2\Delta P; \quad F_t \approx [F_{i,j+1} - F_{i,j}] / \Delta t \qquad \textbf{(A1)}$$

We use the central difference approximation to variable price (P), and forward difference approximation to variable time (t). Applying these expressions in Eq.(6), we have the following difference-equation:

$$F_{i,j} = p_i^+ F_{i+1,j+1} + p_i^- F_{i-1,j+1} + p_i^0 F_{i,j+1} \qquad \textbf{(A2)}$$

$$p_i^{+} = \frac{\frac{1}{2}\sigma_P^2 i^2 + \frac{(r-\delta)i}{2}}{r + \frac{1}{\Delta t}} \qquad p_i^{-} = \frac{\frac{1}{2}\sigma_P^2 i^2 - \frac{(r-\delta)i}{2}}{r + \frac{1}{\Delta t}} \qquad p_i^{0} = \frac{-\sigma_P^2 i^2 + \frac{1}{\Delta t}}{r + \frac{1}{\Delta t}} \qquad \textbf{(A3)}$$

We can apply the same procedure to the boundary conditions of Eq. (6). It can be shown[24] that the solution to equation (A2) converges to the solution of Eq.(6) if all the "p_i" coefficients in Eq.(A3) are non-negative numbers. Therefore, we have to choose a discretization time-step, and a price-step in order to guarantee that condition.

[24] See the theorem in Ames (1977), page 65.