" MARKET AND PROCESS UNCERTAINTY IN OPERATIONS"

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ABSTRACT

By adopting a real options framework, we develop and analyze a production control model that jointly incorporates process and market risks. In our model, process risk is typified by random yield variability while market risk is defined through demand uncertainty. The stochastic processes used to depict uncertainty in these state variables reflect a wide variety of distributional forms and are not confined to the traditional processes typically used in the real options literature. In our approach, the production inputs represent renewable, partially renewable or non-renewable resources. Furthermore, the production outputs are treated as non-traded assets, so that the model has a much broader range of applicability beyond that of standard commodities for which futures contracts trade.

Given this setting, techniques of contingent claims analysis and stochastic control theory are employed to obtain value maximizing production policies in a constrained capacity environment. In light of the stochastic nature of the state variables, the rate of production is modeled as an adapted positive real-valued process and analogously evaluated as a sequence of complex real options. As the optimal adjustments to the rate of production also depend on the outputs’ yield, we establish and explore “flexibility triggers” justifying variations to the rate of production over time. This is achieved by providing closed form analytic results in the presence of generalized diffusion processes including mean reverting processes for the state variables to follow. We also use a numerical example to highlight the model’s sensitivity and contingent features.

(I) INTRODUCTION

The integration of uncertainty and management of risk has been, unequivocally, one the most important aspects of recent research in the operations arena. Research on this front essentially addresses two basic, though significant questions. First, how operating uncertainty and its consequent risk affect operating policies? Second, what are the ensuing economic ramifications and insights when uncertainty is explicitly accounted for and related risk(s) are truncated. In this paper, besides addressing these questions within a specific context, we also provide a link between them. In particular, given explicit sources of uncertainty, we explore economic triggers that justify operating flexibility and establish how flexibility in operating policies has economic value.

In most operational environments, commonly encountered sources of uncertainty can be generically classified by time, quantity, and price (or exchange rate). Time uncertainty is primarily due to variations in lead-times, arrival times, or processing times. Traditionally, decision makers have eliminated such risks by early placement of orders, by stockpiling needed material or by installing additional resources, all at the expense of higher inventory, resource acquisition and placement costs. Quantity uncertainty can be either external or internal. The former is typified by forecast errors of future sales or by random demand fluctuations. In the latter case, quality, design, or technology factors and complexities in processing often result in random variations in the actual yields observed. Frequently, managers protect against the downside exposure effects to either type of quantity risk by producing larger lots and through maintaining inventories from which their contractual obligations could be met. However, this too comes at the expense of an increased overhead, as manifest by higher wages, maintenance, production, and inventory costs. Price uncertainty for any input or output in competitive markets results from fluctuations in supply and demand. In situations where the inputs or outputs reflect standard commodities for which (e.g., futures) contracts are traded, their downside risk be blocked by taking a hedged position in the financial markets. Production efforts in mining, memory chip manufacturing (using copper, silver or gold), special metal alloys for use in the aviation and aerospace industries are prime examples of this situation. In almost all other cases, the risk of price uncertainty is borne by the producer. In such cases, the typical protective strategy is an “operational hedge”. Carrying additional input or output inventories is an example; of course, this too comes at the usual expense of increased holding costs. The only exception reflects instances where prices are contractually established and fixed, and in effect, a forward contract is at hand.

In this paper, we exclusively focus on quantity uncertainty, its risk ramifications and impact on production policies. In particular, we consider production settings characterized by both demand and production yield uncertainty. Given this premise, we establish and explore production “flexibility triggers”. That is, the conditions under which altering production policies become economically justifiable. In this context, the value additive nature of such policy revisions is also explored. By assuming that both demand and yield variables follow well behaved and tractable stochastic processes (e.g. Brownian motions), techniques of Contingent Claims Analysis-CCA (real options) and stochastic control theory are employed to properly take into account the opportunity's risk structure and to optimally establish production policies in a manner consistent with a value maximization objective.

Several factors, some interrelated, motivate the objectives of the paper and the approach taken in reaching them. Although, the simultaneous inclusion of demand and yield uncertainty in the operations literature is not a new concept, their concurrent treatment as generalized stochastic processes and within the context of a real options framework is a new contribution. By way of tradition, the operations literature on this front has characterized demand and/or the yield as random variables. This approach is essentially static. Thus, the impact of time on the assumed distributions' moments and the resulting production policies is effectively ignored, or at least dampened. In contrast, the use of stochastic processes in depicting the random behavior of the demand and yield variables obviates these shortcomings. As a consequence, this choice not only accommodates the higher moments' influence on the analysis, it also captures the impact of time dynamics on the resulting optimal policies and their values. Furthermore, if adjustments to production policies are to reflect the management's reaction to process and/or market uncertainty, then the flexibility to adjust policies in response to the uncertainties encountered must be value additive. Clearly, the framework adopted for such a model dictates the soundness of the results. That is, the model capturing the effects of risk truncation and the value of flexibility must be consistent with an equilibrium value structure. Otherwise, any anomalies encountered in the results obtained are unlikely to be due to the specific policies or actions considered. Rather, they will reflect the inconsistencies in the model's equilibrium structure. In this context, the operations literature has typically treated many of the factors that implicitly affect equilibrium as given. These concerns signify our choice of a real options (contingent claims) and control theory approach to the analysis.

Review of the real options literature indicates that the majority of real options (CCA) applications involve production efforts in mining or extraction based projects where outputs reflect traded commodities and with output prices as the typical source of uncertainty. Other measures of market risk including exchange rate or demand uncertainty have, albeit to a lesser extent, also found their way into the current (real options) literature. However, a notable void in this literature concerns process risk typified by reliability issues, lead-time uncertainty, system breakdowns or output yield variability. Another omission is the application of CCA methodology to a broader set of production-based problems where the output(s) are not traded commodities. Specifically, the analysis of manufacturing or other production related projects in the presence of both market and process uncertainty is of tremendous interest. Accordingly, a further objective of this paper is to also broaden the scope of the real options applicability in the operations arena, by specifically including outputs that are not traded commodities.

To that end, we consider the problem of analyzing and valuing production opportunities (primarily manufacturing) characterized by significant "market" and "process" uncertainty. In our analysis, market uncertainty is defined by demand variability[1]. The notion of "process" or "operating" risk is captured

[1] Since the output is not a traded commodity and in that sense its market risk cannot be hedged through other traded assets.

by yield uncertainty: which we define as a random multiplier to the output quantity, reflecting the usable portion of the output levels which are then sold in competitive markets. Here, variability in the output yield is introduced to allow for the inevitable variations that arise in the pattern of output quantities typically due to quality or processing reasons. By incorporating output yield as an uncertain factor in our analysis, we can also explicitly allow for the inherent operating options that may be available to managers in the more severe cases of yield variability. For example, in a manufacturing setting such an option may be manifest as a trigger for system or technology choice alternatives: that is, upgrading, or new facility acquisitions, etc.

The literature on yield problems in production enjoys a noteworthy and rich heritage. Initially, and motivated by yield variations in agricultural crops, Karlin (1958) considered the inventory implications of uncertain outputs. By considering a multi-period ordering environment and assuming that production yields are binomially distributed, Mazzola, McCoy, and Wagner (1987) derive an EOQ model that allows for backlogging. Given demand uncertainty and defective shipment quantities, Moinzadeh and Lee (1987) provide an analysis of a continuous review model. The inventory implications of a finite horizon, single product production model in the presence of demand and yield uncertainty is due to Gerchak, Vickson and Parlar (1988). By assuming a single period, single product, multistage production setting with random production yields at each stage, Lee and Yano (1988) develop a model indicating that stage dependent costs and a convex function of each stage's production. Bitran and Dasu (1992) develop a model for establishing ordering policies in the presence of stochastic yields and demand substitutability. An extensive and rich review of the random yield literature can be found in Yano and Lee (1995).

As the recent literature in real options (CCA) reveals, most applications typically involve projects with well-defined risk characteristics. Essentially this typifies the class of projects whose costs or revenues directly depend on, or can be linked to, the prices of traded assets or commodities so that data for quantifying their risk is, at least partially, available. For these and other similar type projects, CCA methods can be applied to obtain an arbitrage free valuation model where financial risks may be fully eliminated through proper hedging in the futures market. This arbitrage valuation framework is attractive since in the absence of priced risk elements the model's complexity in terms of parameter estimates and discount rate derivations is substantially reduced. The advantages of a CCA approach to the analysis of real options have been well cited in the literature. Brennan and Schwartz (1985) consider production flexibility issues in mining projects with multiple options to open, close and to subsequently abandon the project. In their paper, the notion of market risk is captured through output prices, which are assumed random in nature. Furthermore, the output is also taken to be homogenous in its composition and therefore, not subject to yield variability. Kamrad and Ernst (2001) extend their elegant analysis to additionally reflect yield variability in the outputs. The general solution to the classical "duration" problem of a long-term renewable resource is provided by Morck, Schwartz and Stangeland (1990). In their production control model, price and the level of inventories reflect the sources of uncertainty. Audreou (1990) provides a model for valuing flexible plant capacity when demand conditions are uncertain whereas He and Pindyck (1992) consider an investment model of flexible production capacity. More recently, Kamrad and Lele (1998) consider price uncertainty and system failure risk and develop an optimal production and maintenance expenditure policy in light of a warranty on shared failure repair costs. Investment mode options and exchange rate uncertainty, is addressed by Kouvelis and Sinha (1994). Exchange rate related work includes Dasu and Li (1997) who develop optimal operating policies; Huchezermeier and Cohen (1996) addressing operational flexibility concerns for the purposes of strategic global manufacturing; and Kogut and Kulatilaka (1994) who consider production shifts among plants in a network of manufacturing centers. Kamrad and Siddique (2003) also consider supply contract valuation problems in the presence of multiple exchange rates and supplier reaction options.

Through adopting a CCA framework, we develop a production control model for the analysis of manufacturing and other production-based projects typified by market and process risk. In what follows, market risk is characterized by output demand uncertainty. The depiction of market risk by demand uncertainty is in part motivated by the fact that for a large class of production based outputs, typically manufactured items, price uncertainty is not a serious risk issue, where as demand uncertainty is critical. In addition, this choice also addresses the general "non-tradability" concern pertaining to manufactured outputs. In this context, our paper extends the conceptual contributions thus far provided by the literature. Process uncertainty is characterized by output yield reflecting the refined or the usable output portions. By incorporating the yield factor into the analysis, we can explicitly account for the inherent heterogeneity problem existing in many production processes. For instance, in the case of mining projects, some reserves may be less accessible and more costly to extract (i.e. the resource to be exploited is non-homogeneous), therefore, inducing an abandonment option consideration. In a manufacturing environment, on the other hand, yield variability may induce a system replacement or repair, or an overhaul option if system calibration fails to regulate the yield problem. Though vastly different from the more traditional models of yield variability encountered in the operations and manufacturing literature, our approach maintains the similarity that the yield variable is modeled as an independent multiplier to the output quantity. In this light, we formulate a production control model maximizing the value of the operations in an environment typified by operating options. For this purpose, the techniques of stochastic control theory are employed to optimally adjust the rate of production in a manner consistent with a value-maximizing criterion. Given appropriate yet straightforward modifications, the yield variability problem may also be modeled as uncertainty in the quality (or usability) of locally supplied inputs in the broader context of a supply chain problem.

The paper is organized as follows. In the next section we define the notation, state the necessary assumptions and develop an options based production control model resulting in a Bellman equation subject to appropriate boundary conditions. Within this framework, we assume there are no finished goods inventories and that the producer, in response to market and process uncertainty, produces at the rate that maximizes the value of production. However, we assume that the producer maintains an inventory of the needed raw materials through which the finished goods are produced. We model this inventory system in such a way to allow for the full characterization of the resource to be exploited. That is, whether or not the resource in question is renewable. This is important since within our setup we can easily distinguish the type of industry within which the production effort lies. For instance, if the raw material inventory is a renewable resource, the production effort represents a manufacturing operation. On the other hand, if the reserves reflect a nonrenewable resource, the case at hand typifies mining or extraction based operations. Our inventory setup also accommodates situations reflecting partially renewable resources: as in timber production with required replanting.

We assume that the risk associated with the output's yield is not priced, and therefore does not induce an additional premium. Here, our model is derived in a general equilibrium context that parallels the findings of Constantinides (1978) and McDonald and Siegel (1985). Since the resulting model (Bellman valuation equation) does not yield an analytic solution, it must be solved numerically to obtain results. Nonetheless, closed form solutions for the optimal production policies are obtained. In the next section, by invoking the Feynman-Kac results, we show how to derive the solution numerically along with the resulting optimal production policy to be followed. To that end, a multinomial lattice approach provides the basis for approximating the stochastic evolution of the state variables. By superimposing a dynamic programming algorithm on the lattice numerical solutions are obtained recursively. This procedure is then illustrated through a stylized example. Section V provides concluding remarks.

The contributions of this paper are as follows. First, it introduces a framework for analyzing production based projects characterized by *both* market (demand) and process (yield) uncertainty. The framework introduced can be easily modified to accommodate other sources of market and process uncertainty. Second, the paper further extends the current literature's findings to a much broader class of production problems. In particular, we examine manufacturing or other non-extraction based production control problems where the inputs may reflect renewable, partially renewable or non-renewable resources. Third, the class of production outputs considered herein reflect non-traded assets, and thus generalize the scope of applicability of this approach beyond that of commodities for which futures and other financial contracts trade. This is important since it fills a void in the existing literature, in light of the set up considered. In that capacity, this paper also provides for future research opportunities in the operations arena using a CCA approach. Fourth, the base model developed draws from generalized stochastic processes (Brownian motion) to depict the sources of uncertainty considered and to obtain closed form results. Finally, and in light of quite robust numerical results, the model presented in this paper is sufficiently adaptable to allow for the inclusion of other sources of market or process uncertainty and the stochastic processes characterizing such uncertainty.

(II) Assumptions and Model Development

Let $Z_D(t) \in \mathbb{R}$ and $Z_Y(t) \in \mathbb{R}$ *define* standard Brownian motions that are martingales with respect to the probability space,$(\Omega, \mathcal{F}, \mathfrak{I}, \mathcal{P})$. The filtered probability space, $(\Omega, \mathcal{F}, \mathfrak{I}, \mathcal{P})$ is defined over the pre-established time interval $[0, \tau]$ where the augmented filtration, $\mathfrak{I} = \{\mathcal{F}_t : t \in [0, \tau]\}$, is right-continuous and increasing. In general, let the process depicting uncertainty be defined by $\{X(t): t \geq 0\}$, where its sample path is posited by an Ito differential equation of the form:

$$dX(t) = M(X,t)dt + S(X,t)dZ_X(t) \tag{1}$$

The drift function, $M(X,t)$ denotes the instantaneous change in *X(t).* The volatility function, $S(X,t)$ denotes the standard deviation of the growth rate, and $dZ_X(t)$ is an instantaneous increment to the Brownian motion; $Z_X(t) \in \mathbb{R}$, defined above. We now specify the model assumptions. The output's demand process $\{D(t): t \geq 0\}$ is depicted by the following forms of equation (1):

$$dD(t) = D(t)\{\alpha_D dt + \sigma_D dZ_D(t)\} \qquad D(0) = D_0 > 0 \tag{2a}$$

$$dD(t) = \kappa[\mu_D - D(t)]dt + \sigma_D[D(t)]^{\eta} dZ_D(t) \qquad D(0) = D_0 > 0 \tag{2b}$$

In expression (2a), the constant drift parameter α_D represents the instantaneous expected growth rate in demand; the constant per unit variance of the growth rate is σ_D^2 and the Brownian increment is $dZ_D(t)$ which was defined earlier. The demand process captured by (2a) implies that the conditional distribution of *D(t)* given *D(s)*, $t > s \in [0,\tau]$, is lognormal and that *D(t)* > 0 for all $t \in [0,\tau]$, if $D_0 > 0$. Note too, that a simple log-transform would imply a corresponding conditional normal distribution for the demand process. Expression (2b) reflects a generalized "mean reverting" process. The mean reversion constant, $\kappa \geq 0$ can be thought of an elastic force that pulls *D(t)* to its long run mean μ_D which is assumed to be positive. As $\kappa \geq 0$ becomes smaller, the excursions around the mean become longer. In general, η is arbitrary. For $\eta = 1$, the process is known as an Ornstein – Uhlenbeck process or an inhomogeneous geometric mean reverting process. When $\eta = 1/2$, *D(t)* follows a non-central χ^2 (Chi-squared) distribution with finite mean and variance parameters. The above specifications accommodate a wide range of probability distributions for the demand process depending on the parametric choices.

We assume demand substitutability is not an alternative and that backlogging is not allowed. Furthermore, the producer does not stockpile finished products and hence there are no finished goods inventory concerns. This implies that given the available production capacity, the producer aims to meet as much of the demand as possible. To reduce the potential for the overage costs, we implicitly impose a penalty constraint to that effect. We also assume that the producer's actions do not affect the market demand for the output and that the producer is a value maximizer. In the current context, producer's actions are depicted by the rate of production, $q(D,I,t) \equiv q(t),\ t \in [0,\tau]$ with $q(t) \in (0,Q)$ and where Q defines the current available production capacity. In our set up, $q = \{q(t): t \in [0,\tau]\}$ is an adapted positive real-valued process. The flexibility afforded by having the option to revise operating policies in reaction to both market (i.e.,demand) and process (i.e.,yield) uncertainty is value additive and as such is viewed as a sequence of (real) nested options.

Given that there are no finished good inventories in meeting the demand for the output, the producer simply produces at rate $q(t)$ in a manner that maximizes the operating profits. The producer, however, maintains an inventory of needed raw materials from which finished goods are produced. Let

$I(t),\, t \in [0, \tau]$ define time t level of input inventory. Supposing $I(0) = I_0 > 0$ defines the initial known level of the resource, we have,

$$\frac{dI(t)}{dt} = \phi(q(t)) - q(t) \tag{3}$$

The function $\phi(.)$ is used to determine whether or not the resource in question is renewable. Although, many forms for this function could be examined, in this paper we consider only the simple functional form, $\phi(q(t)) = \xi q(t)$ with $0 \le \xi \le 1.0$. When $\xi = 0$, the resource considered is non-renewable and the RHS of equation (3) simply reduces to $-q(t)$. When $\xi = 1.0$, the resource is instantaneously renewable: in other words, the supply is effectively infinite. All other cases (i.e. $0 < \xi < 1$) present a partially renewable resource with the RHS of equation (3) becoming $(\xi - 1)q(t)$. In this case, the rate of depletion or extraction is faster than the rate of replenishment. Though not used in the context of this paper, it is also possible for $\xi > 1$, implying that rate of inventory replenishment is faster than the depletion rate; this case would be relevant if the capacity Q increased over time. The production cost function, $K(q(t))$ is assumed to be non-linear. In particular, we assume that $K''(q(t)) \ge 0$, depicting increasing marginal cost of producing an additional unit of the output.

The net usable output resulting from production at rate $q(t)$ is defined as $q(t)Y(t)$. The yield variable $\{Y(t),\, t \ge 0\}$ is conceptualized as an independent multiplier to the output rate and is assumed to follow a stochastic process that is also characterized by an Ito differential equation:

$$dY(t) = \mu(Y,t)dt + \sigma_Y(Y,t)dZ_Y(t) \qquad Y(0) = Y_0 > 0 \tag{4}$$

Thus, $Y(t)$ describes the state of the production process at time t and its realization is known at the instant that the incremental production decision is made. Expression (4) fully characterizes the process depending on the choice of the drift function, $\mu(.)$ and the volatility function, $\sigma_Y(.)$. Furthermore, in light of specific functional forms for $\mu(.)$ and $\sigma_Y(.)$, and conditional on time $s \in [0, \tau]$ information, it may be possible to define the probability distribution for $Y(t)$ with $Y(s)$ given, $s < t \in [0, \tau]$. We defer specifying functional forms for $\mu(.)$ and $\sigma_Y(.)$ and address this concern in our results' section and in light of a contextually meaningful distribution for $Y(t)$ to follow. We assume the Brownian increments defined earlier are orthogonal. That is,

$$\rho_{DY} dt = E\left(dZ_D(t) \cdot dZ_Y(t)\right) = 0 \tag{5}$$

Let $\pi(t),\, t \in [0, \tau]$ define the deterministic output price so that the yield-affected revenue resulting from producing at rate $q(t)$ at time t, is $q(t)Y(t)\pi(t)$. To develop the valuation model, let $V(D, Y, I, t; q)$ represent the production value at time t given that the demand is $D(t)$, the yield factor is $Y(t)$, the level of input inventory (or remaining untapped resource level) is $I(t)$, and where the production rate is set at $q(t)$. The function $V(D, Y, I, t; q)$ is taken to be Ito differentiable.

As a preliminary to developing our valuation model, we allow for the existence of a financial security that has the same covariance with market return as does demand[2]. Suppose $W(t)$ depicts the price of this traded security at time $t \in [0,\tau]$ and that the equilibrium growth rate on this financial asset is α_W. We assume that the instantaneous change in the price level of this security is characterized by the following stochastic differential equation,

$$dW(t) = W(t)\alpha_W dt + W(t)\sigma_W dZ_W(t) \tag{6}$$

Expression (6) defines a geometric Brownian motion, with α_W and σ_W reflecting the constant drift (expected rate of return) and volatility (standard deviation of rate of return) parameters. Here, α_W represents the equilibrium rate of return on a financial security having the same covariance with the market return as does the demand. Let α_M and σ_M define the instantaneous expected rate of return (drift) and the standard deviation of the rate of return on the market, respectively. The unexpected rate of return component defined by, $\sigma_M dZ_M(t)$, with $Z_M(t) \in \mathbb{R}$ as a standard Brownian motion that is also a martingale with respect to the probability space, $(\Omega, \mathcal{F}, \mathfrak{I}, \mathcal{P})$. The constant and riskless rate of return is depicted by *r*. Employing Merton's (1973) Intertemporal Capital Asset Pricing Model, the market premium on this financial security is given by $\lambda\rho_{WM}\sigma_W$, which for valuation purposes is equivalent to $\lambda\rho_{DM}\sigma_D$. Here, ρ_{WM} and ρ_{DM} define the instantaneous correlation on returns between the financial security and the market and that of the demand and the market, respectively[3]. By definition,

$$\lambda = \frac{\alpha_M - r}{\sigma_M} \tag{7}$$

Furthermore, let the rate of return shortfall be defined by $\psi = \alpha_W - \alpha_D$, with ψ unrestricted in sign. By employing an intertemporal CAPM approach, the equilibrium rate of return on the financial security must reflect an adjustment for systematic risk. In this context, we have:

$$\alpha_W = r + \lambda\rho_{DM}\sigma_D \tag{8}$$

Recall, by definition, $\lambda\rho_{WM}\sigma_W = \lambda\rho_{DM}\sigma_D$ [4]. Given this setup, we can obtain *V*(.) using a replicating portfolio approach. In particular, consider portfolio *G*(*t*) consisting of a long position in *V*(.) together with a short position of δ units in security, $W(t)$. The instantaneous change in the value of this portfolio, in light of the necessary cash flow adjustment is,

$$dG(t) = dV(t) - \delta dW(t) + \{q(t)Y(t)\pi(t) - K(q(t))\}dt \tag{9}$$

The expression in curly brackets denotes the reward function $R(q(t))$. To ensure the existence of only diversifiable risks, set $\delta = (D/W)\cdot(\partial V(\cdot)/\partial D(t))$. Absent arbitrage opportunities, this implies that the expected return on this portfolio, $E(dG(t))$ should be the riskless rate so that,

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2 We will use this condition to arrive at equations (8 and 10).

3 By definition $\rho_{DM}\sigma_M\sigma_D = \rho_{WM}\sigma_M\sigma_W$, since the financial security has the same return covariance with the market as does the demand. Thus, $\rho_{DM}\sigma_D = \rho_{WM}\sigma_W$. That is, $E(dZ_M(t)\cdot dZ_D(t))\sigma_D = E(dZ_M(t)\cdot dZ_W(t))\sigma_W$. We use this to obtain expression (8) using a CAPM framework.

4 See also Constantinides (1979) and, McDonald and Siegel (1985). Equation (8) is consistent with their findings.

$$E\left(dG(t)\right) = rV(t)dt \tag{10}$$

Through applying Ito's lemma to the right hand side of equation (9), taking the resulting expectations, and equating it to expression (10) we obtain the desired Bellman valuation equation. Using equation (2a), it follows without loss of generality that [5],

$$\underset{q\in[0,Q]}{Max}\left\{\frac{\partial V}{\partial D}D(r-\psi)+\frac{\partial V}{\partial Y}\mu(Y,t)+\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial D^2}D^2\sigma_D^2+\frac{1}{2}\frac{\partial^2 V}{\partial Y^2}\sigma_Y^2(Y,t)+q[Y\pi-\frac{\partial V}{\partial I}(1-\xi)]-K(q)-rV\right\}=0 \tag{11}$$

s.t.

$$\lim_{Y\to 0} V(D,Y,I,t;q)=0 \tag{12a}$$

$$\lim_{D\to 0} V(D,Y,I,t;q)=0 \tag{12b}$$

$$\lim_{D\to\infty}\frac{V(D,Y,I,t;q)}{D}<\infty \tag{12c}$$

$$q(t)\in[0,\ Q] \tag{12d}$$

$$V(D,Y,0,t;q)=0 \tag{12e}$$

$$V(D,Y,I,\tau;q)=C(I,\tau) \tag{12f}$$

$$V(D,Y,I,t;q)+P(I,t)\geq 0 \tag{12g}$$

Equations (12a-g) characterize the constraints to the above Bellman equation (11). First, we assume that the value goes to zero as the yield goes to zero. That is, $Y(t)=0$ is an absorbing state, since the production effort becomes valueless (12a). Likewise, $D(t)=0$ is an implied absorbing state given equation (2): i.e. equation (12b). We further assume that the value function is bounded above by $cD(t)$ where $c<\infty$ is a constant (12c). This regularity condition is stated for completeness; it is trivially satisfied since the production capacity is bounded above by *Q*, as indicated by equation (12d). We also assume that the value drops to zero when the input level inventory falls to zero. The terminal value at the close of the project is defined by function $C(I,\tau)$ via equation (12f). To account for shutdown as a flexibility option, equation (12g) ensures that the operating value of the production effort exceeds the corresponding shutdown cost, as reflected by $P(I,t)$. Other constraints, besides expressions (12a-g), may also be relevant. For instance, penalty constraints on excessive production overage or underage or switching constraints on output level changes reflect a few examples. To illustrate their impact these constraints will be incorporated into our model when we solve it numerically. Several issues regarding equation (11) are noteworthy. In particular, consider the first term in equation (11) reflecting the quantity $(r-\psi)$. This quantity has effectively replaced the original drift term of the demand process as a result of a replicating strategy barring arbitrage opportunities (see equations (9) and (10)). In the current context, $(r-\psi)$ is an equivalent martingale representing the average growth rate for the demand process. Specifically, $(r-\psi)$ characterizes the "market" adjusted instantaneous growth rate of demand with ψ unrestricted in its sign and where $\psi=\alpha_W-\alpha_D$. Recall that, α_W is the expected rate of return on a financial asset having the same (financial) risk as the demand variable. When $\psi<0$, it implies that the expected growth rate of demand is greater than the equilibrium rate of return on a security (here, proxied by *W*) that has the same financial risk in the market as the demand. In contrast, when $\psi > 0$, the expected growth rate of demand is less than the aforementioned equilibrium rate of return[6]. The above discussion and findings are consistent with results obtained by Constantinides (1978) and McDonald and Siegel (1985). The latter paper also includes a more detailed and intuitive discussion[7]. In light of the above setup and results, for valuation purposes the demand process can be depicted by,

$$dD(t) = D(t)\{(r-\psi)dt + \sigma_D dZ_D(t)\} \qquad D(0) = D_0 > 0 \tag{13}$$

In what follows, closed form results regarding the optimal policies are provided. However, the value of the project, as characterized by equation (11), must be solved for numerically (as we demonstrate in section III.2) since the Bellman valuation equation does not yield closed form 'value' results. We note that an alternative approach would be to formulate the valuation problem as a stochastic dynamic program, subject to the aforementioned or other relevant constraints. Indeed, in section III.2 we adopt such an approach and include practical constraints to obtain results numerically. However, we believe that our CCA formulation provides additional and complementary insights as shown below.

(III) <u>Results</u>

This section provides closed form results for the optimal operating policies. Later, numerical results for the project's value are addressed and reviewed. To this end, we assume that the production cost function $K(q(t))$ is non-decreasing (monotone) in the rate of production, $q(t)$. In particular, we assume that the production cost function is quadratic, having the functional form:

$$K(q(t)) = k_0 + k_1 q(t) + k_2 q^2(t) \tag{14}$$

where the monoticity conditions imply that $k_1, k_2 \geq 0$. We further assume that the drift and volatility functions to the yield process are defined by,

$$\mu(Y,t) = \mu \;\; and \;\; \sigma_Y(Y,t) = \sigma_Y$$

where μ and σ are constant parameters. That is,

$$dY(t) = \mu dt + \sigma_Y dZ_Y(t) \qquad Y(0) = Y_0 > 0 \tag{15}$$

thereby implying that the distribution of the yield variable *Y*(*t*), given *Y*(s) with $t > s$, and $t, s \in [0, \tau]$ follows a doubly truncated normal distribution with zero as an absorbing barrier and with one as a reflecting barrier for *Y*(*t*). Furthermore, in some situations, a mean reverting process is a more appropriate description for the demand variable. Our model can easily accommodate this adjustment. In particular, if the demand process were defined according to expression (2b), then the resulting Bellman valuation equation is:

$$\underset{q\in[0,Q]}{Max}\left\{\frac{\partial V}{\partial D}[D(r-\tilde{\psi})] + \frac{\partial V}{\partial Y}\mu(Y,t) + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial D^2}D^2\sigma_D^2 + \frac{1}{2}\frac{\partial^2 V}{\partial Y^2}\sigma_Y^2(Y,t) + q[Y\pi - \frac{\partial V}{\partial I}(1-\xi)] - K(q) - rV\right\} = 0 \tag{16}$$

---

[5] Note that changes in the economy are assumed independent of the overall economy and therefore not priced.

[6] If instead of demand uncertainty we were to represent market uncertainty through stochastic prices for the output and had it been the case that the output were a commodity for which futures contracts traded, then ψ would represent the commodity's convenience yield instead. See Brennan and Schwartz (1985).

subject to the same constraints as before: that is (12a-g). Note too that the only difference between the Bellman equations (11) and (16) is the first term inside the brackets involving the partial w.r.t. $D(t)$ with;

$$\tilde{\psi} = \tilde{\alpha}_W - \tilde{\alpha}_D \quad \text{with} \quad \tilde{\alpha}_D = (\frac{\kappa}{D})\mu_D \quad \text{and} \quad \tilde{\alpha}_W = \alpha_W + \kappa \tag{17}$$

where as in equation (8), $\alpha_W = r + \lambda\rho_{DM}\sigma_D$

(III.1) <u>Optimal Production Policies</u>

If the input inventory level at time $t \in [0,\tau]$ drops to zero, so will the value function as constraint (12e) indicates. To ensure production remains a viable option, we assume that (input) inventory is strictly positive during the planning horizon. If this were not the case, we would simply devise an optimal policy over time until the inventory is fully exhausted, and that would complete the analysis. Thus, we assume that the input inventory level remains positive over the period $[0,\tau]$. A sufficient condition for this assumption is that $I(0) > \tau Q(1-\xi)$.

<u>Theorem 1:</u>

Let $I(0) > \tau Q(1-\xi)$ with $0 \le \xi < 1$. Suppose that the demand process is defined by either expression (2a) or (2b). Also assume that the production cost function, $K(q(t))$ is an increasing convex quadratic function in the production rate, $q(t)$. The optimal production policy $\{q^*(t),\ q^* \in [0,\ Q]\},\ t \in [0,\tau]$ is given by:

$$q^*(t) = Min[q_C(t), \frac{D(t)}{Y(t)}] \tag{18}$$

where

$$q_C(t) = \begin{cases} Q & \textit{if} \quad Y(t) \ge \bar{Y}(t) \\ \tilde{q}(t) & \textit{if} \quad Y^*(t) < Y(t) < \bar{Y}(t) \\ 0 & \textit{if} \quad Y(t) \le Y^*(t) \end{cases}$$

with

$$\tilde{q}(t) = \frac{\pi(t)Y(t) - (k_1 + \frac{\partial V(\cdot)}{\partial I(t)}(1-\xi))}{2k_2} \tag{19}$$

$$\bar{Y}(t) = \frac{2k_2 Q + (k_1 + \frac{\partial V(\cdot)}{\partial I(t)}(1-\xi))}{\pi(t)} \tag{20}$$

$$Y^*(t) = \frac{(k_1 + \frac{\partial V(\cdot)}{\partial I(t)}(1-\xi)) + 2(k_0 k_2)^{1/2}}{\pi(t)} \tag{21}$$

<u>Proof:</u> See Appendix A

Note that the demand process provides an effective upper bound for production. The inventory level affects the production decision through the term $\partial V(\cdot)/\partial I(t)$. For example, when the relative price of raw materials rises sharply, the firm may exercise its shutdown option and sell the input. When $\xi = 1$, the supply of the input material is effectively infinite and the only relevant costs are the processing cost, which is absorbed into $K(q(t))$.

Corollary 1:

Assume further that $\xi = 1$. *In this case, the optimal production policy* $\{q^*(t),\ q^* \in [0,Q]\},\ t \in [0,\tau]$ *is given by:*

$$q^*(t) = Min[q_C(t), \frac{D(t)}{Y(t)}] \tag{22}$$

where
$$q_C(t) = \begin{cases} Q & if \quad Y(t) \geq \bar{Y}(t) \\ \tilde{q}(t) & if \quad Y^*(t) < Y(t) < \bar{Y}(t) \\ 0 & if \quad Y(t) \leq Y^*(t) \end{cases}$$

with
$$\tilde{q}(t) = \frac{\pi Y(t) - k_1}{2k_2} \tag{23}$$

$$\bar{Y}(t) = \frac{2k_2 Q + k_1}{\pi(t)} \tag{24}$$

$$Y^*(t) = \frac{k_1 + 2(k_0 k_2)^{1/2}}{\pi(t)} \tag{25}$$

Proof: See Appendix A

The above Corollary applies more appropriately to manufacturing opportunities wherein the infinite supply of resource levels cannot be fully depleted during the venture's life due to typically limited production capacities. As such, the problem may be analogously viewed and managed as an infinite resource case.

Theorem 2:

Let $k_2 = 0$, and *assume* $I(0) > \tau Q(1-\xi)$ *with* $0 \leq \xi < 1$. *Then, the optimal production policy* $\{q^*(t),\ q^* \in [0,\ Q]\},\ t \in [0,\tau]$ *is given by:*

$$q^*(t) = Min[q_C(t), \frac{D(t)}{Y(t)}] \tag{26}$$

where
$$q_C(t) = \begin{cases} Q & if \quad Y(t) \geq Y^*(t) \\ 0 & if \quad Y(t) < Y^*(t) \end{cases}$$

with the critical yield factor,

$$Y^* = \frac{\frac{\partial V(\cdot)}{\partial I(t)}(1-\xi) + k_1}{\pi(t)} \tag{27}$$

Proof: See Appendix A

Equations (26) and (27) characterize a "bang-bang" production policy in that we produce at the maximum feasible rate of production, Q only if the yield factor exceeds the "profit adjusted" variable cost of raw material inventory and production. To offer additional insight, recall that low realizations of the yield factor reduce profits relative to higher yield realizations. In effect, only when yield adjusted revenues exceed the variable cost of inventory and production, as shown by the numerator of equation (26), it becomes profitable enough to produce at the maximum rate; otherwise, a no-production mode is optimal. In the current context, the level of the raw material inventory is considered finite and therefore, the variable inventory cost or more appropriately "shadow price", $\partial V(\cdot)/\partial I(t)$ has a direct bearing on the optimal production policy. As $I(t)$ defines the level of a renewable resource, its shadow price in this framework can be interpreted as the holding (or carrying) cost rate of the raw materials' inventory, excluding the opportunity cost of capital. The opportunity cost of capital has been indirectly incorporated into the analysis when considering the return shortfall rate, $\psi = \alpha_W - \alpha_D$, or $\tilde{\psi} = \tilde{\alpha}_W - \tilde{\alpha}_D$ depending on which stochastic process is used to represent the demand uncertainty.

Corollary 2:

Assume further that $\xi = 1$. *Then, the optimal production policy* $\{q^*(t),\ q^* \in [0,\ Q]\},\ t \in [0, \tau]$ *is :*

$$q^*(t) = Min[q_c(t), \frac{D(t)}{Y(t)}] \tag{28}$$

where
$$q_C(t) = \begin{cases} Q & if \quad Y(t) \ge Y^*(t) \\ 0 & if \quad Y(t) < Y^*(t) \end{cases}$$

with the critical yield factor,
$$Y^*(t) = \frac{k_1}{\pi(t)} \tag{29}$$

Proof: Follows the proof of Corollary 1 and Theorem 2.

Theorem 3:

Let, the demand process be defined by either stochastic process (2a) or (2b). Also, let $V^* = \underset{q}{Max}[V(D,Y,I,t:q)]$ *with* $q(t) \in (0,Q)$. V^* *is unique.*

Proof: See Appendix B

The approach taken in this paper may be viewed as complementary to that taken in Boone, Ganeshan, Guo and Ord (2000). Those authors develop guidelines to determine suitable run times in the presence of imperfect production processes. Thus, the Boone et al. results are applicable when set-up costs are considerable. In the present research, we also recognize that production processes are imperfect, but operate in continuous time, making the decision at each point in time whether to continue production. In the manufacturing context, when the decision is to cease production, the opportunity exists to "retool". That is, the production facility may be repaired or restored to a better level of operation, and production restarted under the guidelines provided by our solutions.

(III.2) Numerical Results

Effectively, different numerical techniques can be used to obtain results. Given our set up, the most appropriate techniques employed are either the finite difference methods or the lattice techniques. The former methodology obtains numerical solutions through approximating the partial differential equation, which in our paper is the Bellman valuation equation. The latter, the lattice techniques, provide results by approximating the stochastic evolution of the underlying sources of uncertainty. To obtain solutions numerically, and in light of two stochastic sources of variability, we adopt a multinomial lattice approach. This provides the basis to approximate the stochastic evolution of the demand and the yield processes. To that end, note that the stochastic process used here for defining the demand variable is given by expression (2a) which implies that conditional distribution of demand is lognormal. The yield process, as remarked earlier, implies a doubly truncated normal distribution for the yield factor. To employ the intended multinomial lattice approach appropriately, a simple log-transformation is needed and thereafter the multinomial lattice can be constructed easily. The relationship between these two processes is well established in Karlin and Taylor (1981). With two state variables, two types of multinomial lattices may be used. Here, we use the 4-jump model of Boyle, Evnine and Gibbs (1989). An alternative approach is the 5-jump lattice model of Kamrad and Ritchken (1991). In general, a 5-jump lattice provides more accurate results. However, for our purposes a 4-jump lattice is sufficiently accurate for the analysis considered. As stated earlier, a backward recursion is used to dynamically superimpose our production control problem on a 4-jumps lattice.[8] We control for the upper and lower bound values on the yield process by establishing appropriate barriers at zero and one.

Since there is no finished goods inventory and that conditional on demand information the producer's objective is to produce at a rate that maximizes profits, we impose a penalty cost to regulate overages or underages. For the purposes of this example, we use τ = 1.0 year and n = 5 production periods. Table (2) below depicts the case parameters and functional forms. We should like to make the following observation with regard to this example. In particular, some of the constraints (i.e. 12a-g) differ slightly between the theoretical model and the numerical example. This action was taken deliberately to demonstrate the more pragmatic features of the model when solving for practical solutions numerically. As such, and as stated earlier, two sets of constraints are introduced. First, since the changes in the production level are costly, we introduce a switching cost to smooth out the production plan. Of course, this cost could have been introduced earlier, but would have complicated the theoretical results without gaining much by way of additional insight. Second, we introduce a penalty function to limit excess or deficient production. This is instead of using demand as an upper limit. For simplicity we take this function to be quadratic. It also helps smooth out production.

[8] Due to space limitation, lattice details are omitted. However, they are available upon request.

Table (2) : Base Case Parameters and Function Coefficients

Production Cost	$K(q)$	$k_0 = 100.00$; $k_1 = 15.00$; $k_2 = 5.00$
Price	π	π = \$300.00 per unit
Initial Inventory	I	$I_0 = 20$ units
Initial Demand	D_0	$D_0 = 10$ units
Initial Yield	Y_0	$Y_0 = 0.70$ per annum
Demand Volatility	σ_D	$\sigma_D = 0.30$ per annum
Yield Volatility	σ_Y	$\sigma_Y = 0.2$ per annum
Interest Rate	R	$r = 0.08$ per annum
Average Output Yield	μ	$\mu = 0.10$ per annum
Adjustments to drift shortfall	ψ	$\psi = 0.03$ per annum
Production Capacity	Q	$Q = 5$ units per period
Renewable Resource constant	ξ	$\xi = 0$
Switching Cost Function	$1(q_i - q_{i-1})^2$	
Penalty Cost Function	$5.0\,(D_i - Y_i q_i)^2$	
Shutdown Cost Function: $P(I, t_i) = 0$, for $i = 0, 1, 2, ..., n$		
Salvage Cost Function: $C(I, \tau_n) = 0.50 I_n Y_n \pi_n$ given the state at time t_n		

The effect of increasing production capacity on the value function is shown in Figure-1 which demonstrates precisely the anticipated results. Namely, that the capacity constraint is relaxed (non-binding) beyond a certain point. The impact of increasing in the average yield on project value is depicted by Figure-2, showing the expected steady growth.

[Figure-1] and [Figure-2]

Of particular interest is the case where increased volatility in the yield process produces a corresponding increase in the value function. As σ_Y increases, so will the upside potential while the downside risks are truncated through a "no-production" or shutdown option. Thus, on an average basis the project's value improves as the yield volatility increases, as shown in Figure-3. However, when the volatility of the demand is increased, the project's value diminishes, as seen in Figure-4. This too is logical in light of the situation at hand. As σ_D increases, it becomes harder to satisfy demand due to limited production capacity and the non-existence of finished goods inventory. Beyond this capacity, any discrepancy between demand realization and production levels are also penalized as constrained in Table (2).

[Figure-3] and [Figure-4]

In Figure-5, the effect of ψ on the value function is shown. As ψ increases, the implied average growth rate of the demand process drops. All things being equal, and in light of our capacity constraint, it becomes that much easier to meet demand. This reduction in the total operating costs is manifested by a corresponding increase in the project's value.

[Figure-5]

V. CONCLUSIONS

By adopting a real options framework and by focusing exclusively on quantity risk, we have developed and analyzed a production-based valuation model in a constrained capacity environment. Given this setting, we have established and explored "flexibility triggers" justifying variations to the rate of production over time. In our approach, production rates are modeled as an adapted positive real-valued process and analogously evaluated as a sequence of real nested options.

This paper contributes to the literature in the following ways. First, it introduces a framework for analyzing production based projects characterized by *both* market (demand) and process (yield) uncertainty. Second, the paper further extends the current literature's findings to a much broader class of production problems in two specific ways. *(i)* To the analysis of manufacturing and other non-extraction based production control problems where the inputs are renewable, partially renewable, or nonrenewable resources. *(ii)* To applications wherein the production outputs reflect non-traded commodities or assets. *(iii)* In light of quite robust numerical results, the models presented in this paper are sufficiently flexible to allow for capturing other sources of market or process uncertainty. *(iv)* Our formulation covers a broad range of stochastic models, including both the geometric and arithmetic Brownian motion and a class of mean reverting processes.

Appendix A

Proof of Theorem 1:

We have from equation (11),

$$\frac{\partial V}{\partial D}D(r-\psi)+\frac{\partial V}{\partial Y}\mu+\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial D^2}D^2\sigma_D^2+\frac{1}{2}\frac{\partial^2 V}{\partial Y^2}\sigma_Y^2+q(\pi Y-\frac{\partial V}{\partial I}(1-\xi))-K(q)=rV \tag{A1}$$

From the above equation (A1) the necessary and sufficient conditions imply that for maximization purposes,

$$\pi Y(t)-k_1-2k_2q-\frac{\partial V(.)}{\partial I(t)}(1-\xi)=0 \tag{A2}$$

Solving for q we obtain,

$$\tilde{q}(t)=\frac{\pi Y(t)-(k_1+\frac{\partial V(\cdot)}{\partial I(t)}(1-\xi))}{2k_2} \tag{A3}$$

At $q(t)=Q$, (A2) becomes,

$$2k_2Q=\pi Y(t)-k_1-\frac{\partial V(.)}{\partial I(t)}(1-\xi) \tag{A4}$$

implying that the minimum yield level to induce production at capacity is,

$$\bar{Y}(t)=\frac{2k_2Q+(k_1+\frac{\partial V(\cdot)}{\partial I}(1-\xi)}{\pi} \tag{A5}$$

Solving (A1) with $\tilde{q}(t)$ results in

$$Y^*(t)=\frac{(k_1+\frac{\partial V(\cdot)}{\partial I}(1-\xi))+2(k_0k_2)^{1/2}}{\pi} \tag{A6}$$

This completes the proof.

Proof of Corollary 1:

The proof follows from the above in a straightforward manner. Specifically,

$$\int_0^\tau q(t)dt\le\tau Q<I_0=\infty\ \Rightarrow\ \frac{\partial V(\cdot)}{\partial I(t)}=0 \tag{A7}$$

Substituting equation (A7) into equations (17)-(19) obtains the desired results.

Proof of Theorem 2:

From the Bellman equation (11), it follows that $V(D,Y,I,t:q)$ is maximized if $q(t)$ is either zero or at maximum Q since by assumption the production cost function, $K(q(t))$ is linear. Specifically, $q^*(t)=Q$ so long as:

$$\pi Y(t)\ge k_1+\frac{\partial V(.)}{\partial I(t)}(1-\xi) \tag{A8}$$

which results in equation (25). However, if the marginal operating revenues are less than the corresponding operating costs then, $q^*(t)=0$.

Appendix B

To prove ν^* is unique, we will prove the concavity of the value function $V(\cdot)$ in q. Differentiating Bellman equation (11) successively with respect to $q(t)$ we obtain,

$$r\frac{\partial V(\cdot)}{\partial q} = \pi Y - (\frac{\partial K(q)}{\partial q} + \frac{\partial V(\cdot)}{\partial I}(1-\xi)) \tag{B1}$$

$$r\frac{\partial^2 V(\cdot)}{\partial q^2} = -\frac{\partial^2 K(q)}{\partial q^2} \tag{B2}$$

The second derivative is negative by definition of $K(q(t))$. Consider (B1) where in perfect competition,

$$\pi Y = \frac{\partial K(q)}{\partial q} + \frac{\partial V(\cdot)}{\partial I}(1-\xi)$$

and in the case of monopoly,

$$\pi Y > \frac{\partial K(q)}{\partial q} + \frac{\partial V(\cdot)}{\partial I}(1-\xi)$$

Therefore, the RHS of (B1) is at least zero. Therefore, it can have at most one real maximum, that is ν^*.

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Figure - 1
Effect of Average Yield on Project Value

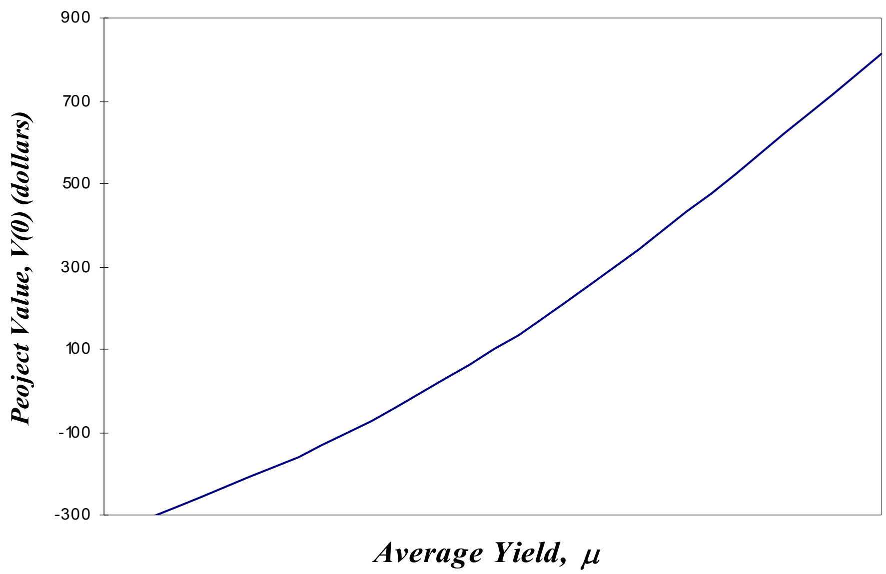

Figure - 2
Effect of Production Capacity on Project Value

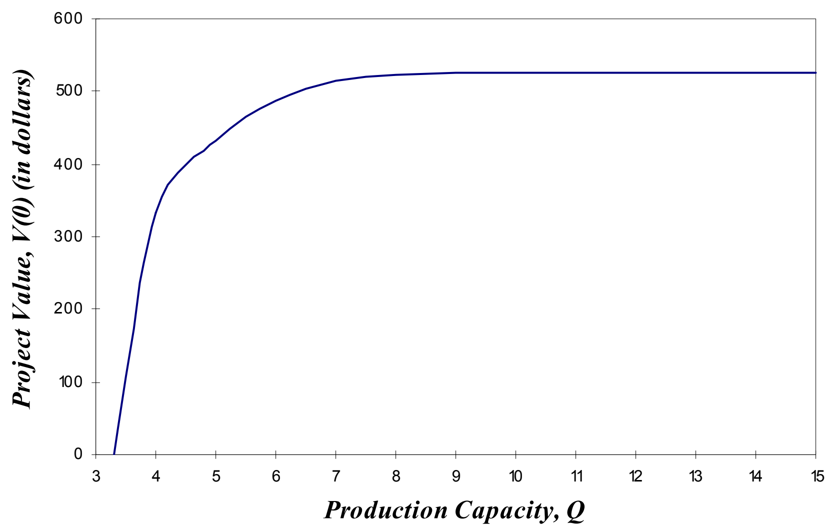

Figure - 3
Effect of Yield Volatility on Project Value

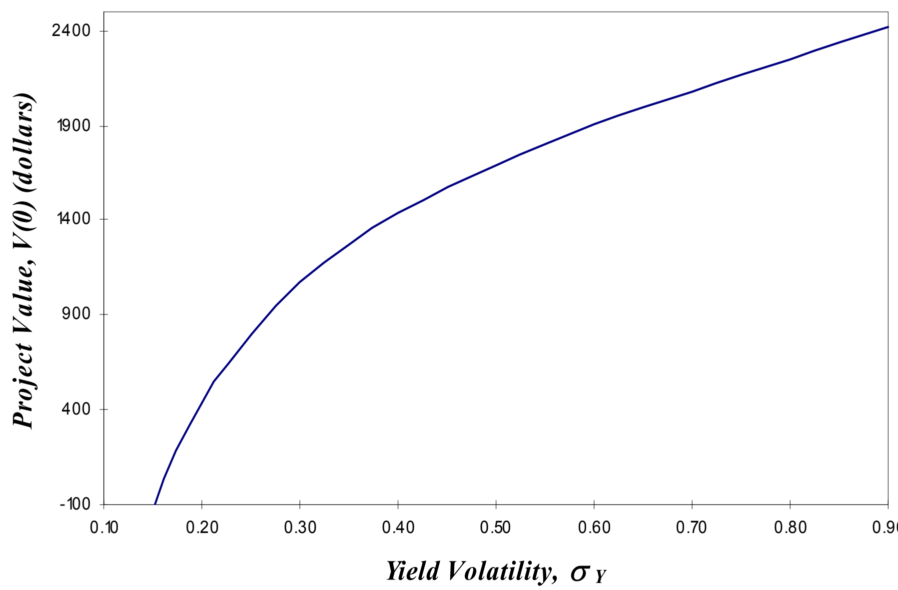

Figure - 4
Effect of Demand Volatility on Project Value

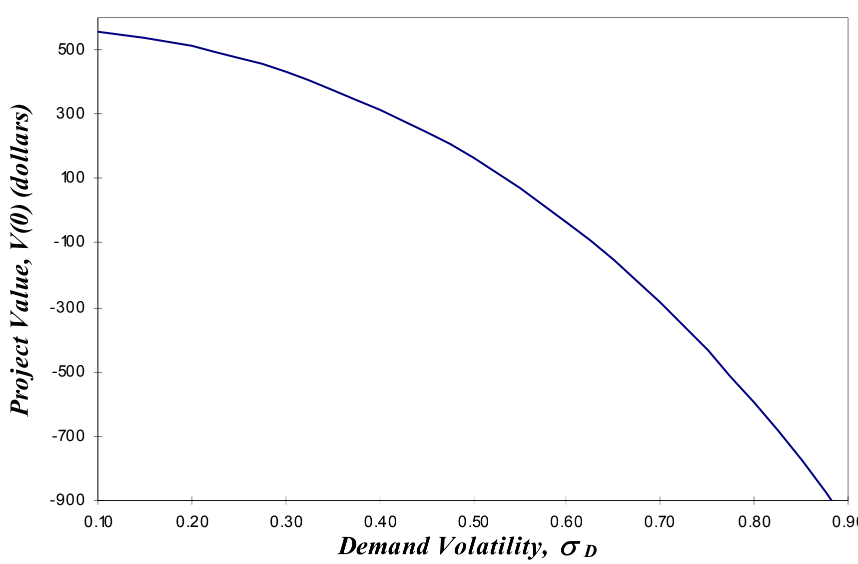

Figure - 5
Effect of ψ on Project Value