Valuing Real Options without a Perfect Spanning Asset

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The real options approach to corporate investment decision making recognizes a firm can delay an investment decision and wait for more information concerning project cashflows. The well known models of McDonald and Siegel (1986) and Dixit and Pindyck (1994) value the investment decision as a perpetual American call option on the project value. The former specifies the equilibrium return via CAPM, whilst the latter, along with much of the literature, uses a replicating portfolio argument. This involves identifying a perfect spanning asset for the project value, often called a "twin security".

In this paper, we instead assume only a partial spanning asset can be found which is imperfectly correlated with project value. This is more realistic, as most real projects can only be partially hedged by traded securities and private risks are common. We find the value of the option to invest and the trigger level are both lowered (compared to the complete model) when the spanning asset is less than perfect, although the option to invest still has value even when there is no spanning asset at all. This implies the firm should wait to invest, but invest earlier than the complete real options model suggests. Investment should also take place earlier under our partial spanning model than under the model of McDonald and Siegel (1986).

Both the McDonald and Siegel (1986) and complete models are special cases of our model, obtained when risk aversion tends to zero and when correlation approaches one, respectively. Despite this, we conclude that approximating via these classic models when correlation is high (or risk aversion is low), may lead to an incorrect investment decision. Thus, by taking private risks into account, the partial spanning model gives a much richer model of corporate investment decisions.

Key words: Real options, partial spanning, perpetual American options, non-traded assets, investment decisions, investment trigger, incomplete model, net present value

1 Introduction

The real options theory of corporate investment, dating back to Myers (1977), recognizes a firm has the option to delay an investment decision and undertake it at some point in the future. When the investment is undertaken, the option to wait is extinguished. However, prior to the decision, the firm has an opportunity cost equal to the value of the option to wait. This way of thinking about corporate investments has received much attention in the last decade with applications to internet and telecommunications, pharmaceutical research and development, energy, investment banking, venture capital and accountancy to name a few. The real options approach was developed by McDonald and Siegel (1986), Dixit (1989), Trigeorgis and Mason (1987), Pindyck (1991) amongst others, and was applied by Brennan and Schwartz (1985) to natural resource investments. These models and many extensions are in the classic text of Dixit and Pindyck (1994). More recently, game theory techniques have been used in real options modeling to account for strategic interactions, see Smit and Trigeorgis (2004).

Some criticism has been directed at the real options theory of investment in recent years due to its unrealistic modeling assumptions. Leppard and Morowitz (2001) claim the real options approach is not yet widely accepted in industry particularly because of the difficulty creating useful and accurate pricing models. Pinches (1998) highlights some of the problems, one of which is that real assets are generally not traded in most real option situations. He suggests that "one avenue for significant future research is that of valuation of options in incomplete markets". This observation is also made by many authors including Amram and Kulatilaka (1999), Borison (2003), Brealey and Myers (2000), Lander and Pinches (1998), Mayor (2001), Myers and Majd (1990) and Dunbar (2000), although these papers do little to address the difficulty. In fact, Myers and Majd (1990) observe that their abandonment option valuation relies on the fact that "capital markets are sufficiently complete...". This paper attempts to address this limitation of the classic real options valuation models and extend real option theory to include incompleteness. We do this by developing a partial spanning model.

The classic continuous time real options models of irreversible investment fall into two main categories. One class of models assumes the existence of a perfect spanning asset, perfectly correlated with the real asset, and therefore assumes a complete market. This is often termed the contingent claims approach, as standard options theory is invoked to obtain the valuation. Models of this type include those of Amram and Kulatilaka (1999), Brennan and Schwartz (1985), Dixit and Pindyck (1994), Mason and Merton (1985), Pindyck (1991) and Trigeorgis and Mason (1987). The second class of models (beginning with McDonald and Siegel (1986)) value the option to invest via an equilibrium style argument, and therefore do not require a perfect hedge, but instead assume equilibrium rates of return are given via the CAPM. Both styles of model can be analyzed with either a finite or infinite investment horizon, although closed form solutions are usually available only for the infinite case. We will discuss only the infinite horizon model here for that reason, see the conclusion of the paper for further discussion concerning the finite case.

It is only under the complete markets assumption of the first class of models, that Black Scholes option theory can be invoked and real options valued as perpetual options on a traded asset. However, there is little evidence in practice that such a perfect spanning asset is available, and little theoretical justification given in the literature for such an assumption.

In this paper, we recognize that perfect spanning assets are a theoretical ideal which often cannot be found in practice, and instead assume only that there is a partial spanning asset available. That is, we assume that there is a traded asset whose returns are partially correlated with those of the real asset.

The partial spanning asset approach implies that the investment comprises both market and private risks, and recognizes that private risk cannot be hedged away via the traded security. Inherent in taking private risks into account in our model is the introduction of risk preferences. A limiting case of our model will recover the complete market style valuation. As the correlation between the traded asset and project tends to one, the valuation under partial spanning approaches the well known perfect spanning valuation.

Returning to the second class of models, the model of McDonald and Siegel (1986) gives a valuation using the correlation between the possibly non-traded project value and the market asset in combination with the CAPM to quantify the difference between the required return in equilibrium and the actual return on the project.[1] Our partial spanning model is also a generalization of their approach, in that we recover the valuation of McDonald and Siegel (1986) when we let risk aversion tend to zero. We do not however require consideration of equilibrium or the CAPM to hold.[2]

As mentioned earlier, risk preferences must be described in our partial spanning model. We assume exponential utility, which allows for a closed form solution for the value of the option to invest and the investment trigger level in our model. Issues of how to value cashflows at intermediate times via utility functions arise and we consider time consistency properties of utilities to arrive at a sensible formulation.[3] A Bellman equation is derived and via a transformation, non-linearities can be removed. The resulting second-order differential equation subject to boundary, value matching and smooth pasting conditions, has a power type solution, in a similar fashion to the complete models of Dixit and Pindyck (1994). Again, similarly to the classic models, we obtain the optimal investment time as the first time the discounted project value reaches a constant level.

The value of the option to invest is found by a certainty equivalence argument. This represents the compensation the firm would require to give up the right to the option. In common with the complete and McDonald and Siegel (1986) models, the value depends on the investment costs and trigger level, which in turn depends on the Sharpe ratios of the project and partial spanning asset. However, unlike the classic models, the partial spanning model valuation depends on risk aversion.[4] In contrast to the complete model, the correlation between the project value and partial spanning asset also impacts on the value of the option to invest. As is the case in the classic models, we can express the option value in our partial spanning model as the solution of an optimal stopping problem. The distinction is that our new representation involves a non-linear function of the option payoff and the expectation

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[1]It is also shown in Proposition 4.4 that their valuation is exactly that arising from a particular pricing measure, namely the minimal martingale measure.

[2]In a related literature, Rubinstein (1976) and Brennan (1979) show risk-neutral valuation relationships hold in a discrete model without assumptions on hedging if strong assumptions are made on the aggregate wealth process and investor preferences. We are not required to model aggregate wealth and thus do not make such strong assumptions.

[3]Other related methods for infinite horizon models are those which maximize the growth rate of a portfolio using the Kelly criterion, see Hakansson (1970).

[4]Note that the model of McDonald and Siegel (1986) assumes CAPM to introduce an aversion to risk. This is different to our useage of risk aversion in the sense of concave utility functions.

involved is no longer risk neutral, both features due to the presence of risk aversion.

We examine the effect of risk aversion and correlation on the value of the option and the investment trigger level in our partial spanning model. Via comparative statics we find that both the investment trigger level and the value of the option to invest are decreasing in risk aversion. A higher risk aversion level encourages earlier investment as the risky option has less value. Comparative statics also shows that as (the absolute value of) correlation is reduced from one, the investment trigger and option value fall. Since private risks exist if correlation is not perfect, risk aversion causes the firm to invest sooner. The option to invest has less value than it would have if correlation were perfect. Even in the limiting case of zero correlation (corresponding to no spanning asset at all), the option to invest still has value and the firm should still wait to invest, in contrast to the recommendation of the NPV rule. The limiting cases of perfect correlation, zero correlation and zero risk aversion obtain the models of Dixit and Pindyck (1986), Kadam et al (2003b) and McDonald and Siegel (1986) respectively. All of these features are illustrated in numerical examples of the option value and investment trigger level in Section 5.

Under the partial spanning model, similarly to the classic real options models, investment will only occur if the project's Sharpe ratio is lower than some critical value. In this case, the value of the option to invest and the investment trigger level are finite. If the Sharpe ratio is large enough, it is optimal to postpone investment and equivalently never exercise the option. Here the value and trigger would be infinite. A striking feature of the partial spanning model is that this critical value differs from those found in the complete model and the McDonald and Siegel (1986) model. The partial spanning asset model is richer because a wider range of parameter values results in a finite investment trigger level and option value, see Theorem 4.4, and how much wider this range is depends on the volatility of the project value. In particular, there are situations where the complete model recommends the firm postpones investment forever, whereas if a highly (but not perfectly) correlated asset were used, the firm should invest at a certain trigger level. Likewise, risk aversion can lead to a finite investment trigger under the partial spanning model, in a situation where the McDonald and Siegel (1986) model would recommend postponing investment indefinitely. In this new region, the value of the option to invest no longer need be everywhere convex in the project value, see Figures 2 and 3. This is caused by the risk aversion dominating the usual option convexity effect.

The main conclusion we can draw from this analysis is first that the use of complete models, or models with a "twin security" for corporate investment decision making is overstating the worth of the option and leading to underinvestment. Likewise, incorporating risk aversion in the partial spanning model leads to a lower valuation and earlier investment time than the model of McDonald and Siegel (1986). Moreover, we conclude that the seemingly common practice of approximating option value and investment decisions via the complete model can lead to the wrong conclusion on investment timing. Our results show that if the complete model concludes the firm should postpone investment, then it not possible to conclude that it is optimal to postpone investment in the partial spanning model. Rather, it may in fact be optimal to invest at some finite trigger level. Similarly, the CAPM style model of McDonald and Siegel (1986) is not a good approximation if risk aversion is not small and again, leads to incorrect investment timing decisions.

A powerful feature of our model is the fact that we can solve explicitly for the value of the option to invest and the optimal investment trigger level. This is useful in a number of ways.

Solving in closed form enables us to characterize situations where it is optimal for the firm to invest at a certain time, and those where investment is postponed indefinitely. This has particular relevance in our partial spanning model since we characterize additional situations in terms of the project's Sharpe ratio leading to a finite investment trigger level and option value. In these situations, if correlation were perfect or risk aversion were negligible, there would be no solution and the firm would wait forever. As mentioned previously, our closed form solutions also allow us to recover the classic valuations as limiting cases.

In addition to the now classic approaches to real option valuation outlined, there are a number of related approaches in the literature. First, dynamic programming and decision trees are popular approaches in industry, however, suffer from the disadvantage that the appropriate discount rate is not specified. Another strand of the literature attempts to justify the use of the complete markets approach, albeit with subjective inputs for the market value of the traded security. Copeland et al (2000) call this the "marketed asset disclaimer", similar arguments are used by Mason and Merton (1985) and Trigeogis (1996). This argument entails using the NPV of the project itself, without flexibility, as an estimate of the market value of the project. The option value using this NPV is claimed to represent the value the project would have if it were traded. Ultimately, this argument is finessing the problem of non-tradability away since completeness is implicitly assumed in using the NPV approach in the first place. Our model provides a framework for dealing with private risks and in doing so, chooses an appropriate discount rate.

The work of Smith and McCardle (1998) building on ideas of Smith and Nau (1995) (and related to the earlier ideas of Constantinides (1978)) is closer in spirit to our partial spanning model. They propose an integrated approach using the hedging arguments of option pricing for risks that are spanned perfectly by traded assets, and decision analysis to value risks that are not spanned. Exponential utility and valuation via certainty equivalence are used in common with our model. That is, both models acknowledge that recognizing private risks introduces risk preferences. However, Smith and McCardle (1998) set up their model in discrete time and study an abandonment problem with a different objective function to ours. A further important distinction is that Smith and McCardle (1998) solve their model numerically in the interesting case where there are private risks. In contrast, we analyze an infinite horizon problem and thus can solve for a closed form expression for the option value and investment trigger, as is the case in the models of McDonald and Siegel (1986) and Dixit and Pindyck (1994). Our partial spanning method, together with the dynamic programming approach and that of Smith and Nau (1995), all recognize the value obtained no longer represents a market valuation, but is the investment's certainty equivalent value based on management's risk preferences.

The issue of valuing options on non-tradable assets has been addressed in the mathematical finance literature recently, although these papers do not typically focus on real options. European style options on non-traded assets are treated by Henderson (2002), Henderson and Hobson (2002a, 2002b) amongst others. Since these models consider options with a fixed expiry, they cannot give any information about the optimal investment trigger level for the firm. Such information requires American style options. Musiela and Zariphopoulou (2003) examine finite horizon American style options on non-traded assets, however there are no closed form solutions for the option value or exercise boundary and numerics must be carried out.[5] Recently Kadam et al (2003a) valued a perpetual American option under the assumption that no trading could be done in the real asset. Their model assumes there is no correlated traded asset. Values are calculated under power utility with constant relative risk aversion R. The special features of their model forces them to consider only $R < 1$ which is unrealistic in most settings (see Cochrane (2001)). Using this model in practice with $R > 1$ will imply the option to invest has no value. In the context of executive stock options, Kadam et al (2003b) treat the perpetual American option, this time under the assumption of exponential utility and again with no spanning asset. We obtain the valuation of Kadam et al (2003b) as a special case in our model.[6]

This paper combines elements of each of these approaches to value the option to invest. Unlike Henderson (2002), Henderson and Hobson (2002a, 2002b) and Musiela and Zariphopoulou (2003), we value a perpetual American option enabling us to draw conclusions on the optimal timing of investment and to obtain the option value in closed form. In contrast to Kadam et al (2003a), we assume the existence of a correlated traded asset with which partial hedging can be performed. As such, a number of limiting cases can be considered. The case of perfect correlation gives back the complete model of Dixit and Pindyck (1994). Risk aversion tending to zero recovers the McDonald and Siegel (1986) valuation and finally, taking correlation to be zero gives the results of Kadam et al (2003b). In extending these existing frameworks, we obtain a much richer model of corporate investment decisions. Our partial spanning model highlights completely new situations where the firm should invest rather than wait indefinitely. More seriously, the simpler classic models are not good approximations in the realistic cases of only highly correlated assets or risk aversion, and can lead to incorrect investment timing decisions.

The paper is structured as follows. The investment problem considered and our modeling assumptions are outlined in Section 2. In Section 3, two classic approaches to valuing the option to invest are discussed, the perfect spanning approach and the setup of McDonald and Siegel (1986). Our partial spanning asset model is developed in Section 4. The main result is that there are situations where the firm should invest under the partial spanning model, whereas the classic models would recommend postponing investment. We consider three special cases of the partial spanning model, recovering the valuations in the complete, McDonald and Siegel (1986) and Kadam et al (2003b) models. Section 5 analyzes the partial spanning asset model and its implications for investment via comparative statics and numerical examples.

2 The Investment Problem and Modeling Assumptions

Consider a firm facing an irreversible investment decision[7] of when to invest in a single project. The traditional net present value approach to investment concludes a firm should invest immediately if the present value of cash flows from investment exceed investment costs. However, the firm also has the option to wait. Real options theory acknowledges the

[5]Rogers and Scheinkman (2003) also study the finite horizon American option but without a traded correlated asset. Their focus is on the dependence of the optimal exercise policy on the number of options.

[6]In Section 4.4, we show the Kadam et al (2003b) valuation obtains when the correlation between the project value and partial spanning asset tends to zero.

[7]This assumption applies to most firm and industry-specific investments but also arises because of government regulations or institutional arrangements, see Dixit and Pindyck (1994).

possibility of waiting, whereby the decision is delayed until some time in the future. The act of waiting is thought of as an option, and if the investment is eventually undertaken, the option is exercised. The basic message is that there is value to waiting as uncertainty concerning the project's value is being resolved over time.

Following a real options approach, if the investment is undertaken at a future time $\tau \geq t$, the company pays an amount $Ke^{r(\tau-t)}$ to invest in project with cashflows V_τ and so receives $V_\tau - Ke^{r(\tau-t)}$. Since it only makes sense to invest if this is a positive amount, $(V_\tau - Ke^{r(\tau-t)})^+$ is the option payoff if investment takes place at time τ. The problem is to find the value of this option to invest and the optimal investment time.

We assume, in common with the continuous time, irreversible investment models we extend, that the firm has an infinite horizon. This will afford us the luxury of closed form solutions and a clear comparison with the classic models of McDonald and Siegel (1986) and Dixit and Pindyck (1994). As mentioned above, we assume the investment cost grows over time at the riskfree rate, from a constant K. Growth of investment costs at the riskfree rate is also assumed by, amongst others, Trigeorgis and Mason (1987), Copeland et al (2000) and Smith and Nau (1995). This also represents a special case of the setup of McDonald and Siegel (1986) who allow for the possibility that investment costs follow a stochastic process. A simplification of our setup is that used by Dixit and Pindyck (1994)[Chapter 5] where investment costs are constant over time.

We assume the value of the project V follows[8]

$$\frac{dV}{V} = \eta(\xi dt + dW) + rdt \tag{1}$$

where $\xi = \frac{\nu - r}{\eta}$ and ν is the total expected return on the project, η is the volatility of project value and W is a standard Brownian motion. We assume that V is non-traded, however its value is observable.[9] In Section 3, V should be interpreted as the present value of future cashflows from the project. In Section 4, we cannot interpret V in this way since the model is incomplete. We have to make there the assumption that when the firm invests, they receive the value of the project, V at that date, rather than the right to a future stream of cashflows.

Our second modeling assumption is that there is a traded asset P which is correlated with the project value. We call this the partial spanning asset, partial because the correlation may be less than perfect and hence the project cash flows are not spanned. In applications where the investment produces a single commodity and there is a futures market for the commodity, the correlation could be very high between the futures price and the cash flows from the investment, given the quantity of the commodity is known. In such a case, the perfect spanning asset assumption of the standard real options theory may be a good approximation. For an example applying the classic model to commodity futures see Brennan and Schwartz (1985). In general however, a partial spanning asset may be an individual stock, a basket of stocks or relevant industry index, for example. In such cases, there will not be perfect, or even necessarily high, correlation between the asset and the project value. The extreme

[8]The form of our results will hold also for more general models for V of the form

$$\frac{dV}{V} = \eta(t, V_t)(\xi(t, V_t)dt + dW) + rdt$$

although we concentrate on the lognormal model for simplicity and comparison purposes here.

[9]Knowledge of the value of V is essentially assumed in the models with a perfect spanning asset for sizing the project compared with the spanning asset.

case where there is no relevant spanning asset for the project at all is covered in our partial spanning model by taking correlation equal to zero.

The partial spanning asset follows a lognormal process

$$\frac{dP}{P} = \sigma(\lambda dt + dB) + rdt \tag{2}$$

where $\lambda = \frac{\mu - r}{\sigma}$ is its Sharpe ratio. The two driving Brownian motions B and W are correlated with correlation $-1 \leq \rho \leq 1$. If $|\rho| = 1$, the asset P is a perfect spanning asset for the project value. This will be the case in the complete market model of Section 3. If $|\rho| < 1$, P is a partial spanning asset and there is a component of private risk remaining after P is used to hedge the project. In the McDonald and Siegel (1986) equilibrium model of Section 3, P will be interpreted as the market asset, with market Sharpe ratio λ and correlation $|\rho| \leq 1$ between the market and project. In Section 4, we will build a model to value the investment option under the partial spanning assumption of $|\rho| < 1$.

3 Classic Real Options Models: The McDonald and Siegel (1986) model and the perfect spanning model

In this section we consider the value of the option to invest under the setup and assumptions of Section 2. We consider two well known approaches to this problem, firstly, the complete markets approach, and secondly, the approach of McDonald and Siegel (1986). The complete markets approach assumes the spanning asset P is perfectly correlated with V (hence $\rho = 1$) or equivalently, V itself is traded. In this case, the option to invest can be valued via risk neutral pricing, see amongst others, Brennan and Schwartz (1985), Dixit and Pindyck (1994), Trigeorgis and Mason (1987), Mason and Merton (1985), Pindyck (1991) and Trigeorgis (1996). In a second approach, McDonald and Siegel (1986) present a valuation of the investment option, drawing on the CAPM to relate the equilibrium Sharpe ratio on the project to the Sharpe ratio of the market. This approach is valid when V is not traded, and therefore is a way of obtaining an option value in this more realistic situation. Of course, one is relying on equilibrium CAPM relationships called upon in their model.

First consider the perfect spanning method whereby the existence of a "twin security" is assumed. This twin security, P, is perfectly correlated with the project value, V. The perfect spanning assumption ensures the market is complete and the investment opportunity is analogous to a perpetual American call option on a dividend paying asset.[10] The value of the option $p^{(1)}(v)$ can be expressed as the expectation of the discounted value of the payoff under unique risk neutral probabilities (denoted by expectation $\mathbb{E}^{\mathbb{Q}}$), maximized over investment times τ:

$$p^{(1)}(v) = \sup_{t \leq \tau} \mathbb{E}_t^{\mathbb{Q}}[e^{-r(\tau - t)}(V_\tau - Ke^{r(\tau - t)})^+ | V_t = v]. \tag{3}$$

[10] This problem was first solved by McKean (1965) in an appendix to Samuelson (1965). The investment decision is simply the exercise decision in the financial option. Implicit in valuing real options via a contingent claims approach is the fact that the company should be hedging in the perfect spanning asset. Only then does the value obtained via a contingent claim approach truly represent the value of the investment opportunity, in the same way that a financial option is only worth the Black and Scholes (1973) value if delta hedging is taking place.

The corresponding Bellman equation is given by

$$\frac{1}{2}\eta^2 v^2 p_{vv}^{(1)}(v) + \eta(\xi - \lambda) v p_v^{(1)}(v) = 0 \tag{4}$$

with boundary, value matching and smooth pasting conditions

$$\begin{aligned} p^{(1)}(0) &= 0 && (5)\\ p^{(1)}(\tilde{V}^{(1)}) &= \tilde{V}^{(1)} - K && (6)\\ p_v^{(1)}(\tilde{V}^{(1)}) &= 1. && (7) \end{aligned}$$

The optimal investment time τ^* is given by

$$\tau^* = \inf\left\{u \geq t : V_u = \tilde{V}^{(1)} e^{r(u-t)}\right\}, \tag{8}$$

the first time the project value reaches a level given by an exponential boundary, starting at the constant level $\tilde{V}^{(1)}$ today. Equivalently, it is the first time the discounted value $S_u = V_u e^{-r(u-t)}$ reaches the constant level $\tilde{V}^{(1)}$. We characterize $\tilde{V}^{(1)}$ shortly.

Proposing a solution of the general form $L^{(1)} v^{\beta^{(1)}}$, (where $L^{(1)}$ is a constant to be determined), results in the quadratic

$$\frac{1}{2}\eta^2 \beta^{(1)}(\beta^{(1)} - 1) + \eta(\xi - \lambda)\beta^{(1)} = 0 \tag{9}$$

with solutions[11]

$$\beta_1^{(1)} = 1 - \frac{2(\xi - \lambda)}{\eta}, \qquad \beta_2^{(1)} = 0.$$

Given (5), the choice $\beta_2^{(1)} = 0$ can be rejected, and the solution is of the form $p^{(1)}(v) = L^{(1)} v^{\beta_1^{(1)}}$. There are two possibilities. If $\beta_1^{(1)} \leq 1$ (corresponding to $\xi \geq \lambda$), smooth pasting fails and there is no solution. The firm always postpones investment in this case. Otherwise, if $\beta_1^{(1)} > 1$ (corresponding to $\xi < \lambda$),

$$p^{(1)}(v) = (\tilde{V}^{(1)} - K)\left(\frac{v}{\tilde{V}^{(1)}}\right)^{\beta_1^{(1)}} \tag{10}$$

and

$$\tilde{V}^{(1)} = \frac{\beta_1^{(1)}}{\beta_1^{(1)} - 1} K. \tag{11}$$

The model leads to a decision rule whereby: if $\xi \geq \lambda$ always postpone investment whereas if $\xi < \lambda$, investment occurs when V reaches a threshold level $\tilde{V}^{(1)} e^{r(u-t)}$ which starts at the constant $\tilde{V}^{(1)}$ today and grows over time by the riskless rate. It can be seen from (11) that the threshold $\tilde{V}^{(1)}$ is in excess of K. Thus, in contrast to the NPV rule, uncertainty drives a wedge between $\tilde{V}^{(1)}$ and K.

The condition $\beta_1^{(1)} > 1$ or $\xi < \lambda$ says the Sharpe ratio on the spanning asset is greater than that on the project and thus there is an opportunity cost to keeping the option alive

[11] In the case where (9) has repeated roots at zero, the solutions to (4) are $p^{(1)}(v) = B + L^{(1)} \ln v$. In this case it is not possible to achieve a smooth fit.

represented by $\lambda - \xi > 0$. If $\lambda = \xi$ (or $\xi > \lambda$) then there is no such opportunity cost and the option should be kept alive whilst the firm keeps waiting. This interpretation is found in Dixit and Pindyck (1994)[Chapter 5].[12]

Now consider the second approach. Under the modeling assumptions in Section 2 and with the interpretation that P is the market asset, McDonald and Siegel (1986) value the option using a CAPM approach. Our exposition in this section agrees with the special case of McDonald and Siegel (1986) where the investment costs are growing at the riskfree rate over time. In this case, the value of the option to invest, $p^{(\rho)}(v)$ is expressed as

$$p^{(\rho)}(v) = \sup_{t\leq\tau} \mathbb{E}_t[e^{-\mu^e(\tau-t)}(V_\tau - Ke^{r(\tau-t)})^+|V_t = v] \tag{12}$$

where μ^e is the equilibrium expected rate of return on the investment. The equilibrium approach requires the option earns a rate of return commensurate with the risk involved in holding the option as an asset. The corresponding Bellman equation is given by

$$\frac{1}{2}\eta^2 v^2 p^{(\rho)}_{vv}(v) + (\nu - r)vp^{(\rho)}_v(v) + (r - \mu^e)p^{(\rho)}(v) = 0 \tag{13}$$

with the same boundary, value matching and smooth pasting conditions as in the complete model earlier. The optimal investment time τ^* is given by the first time the discounted value of V reaches a constant level $\tilde{V}^{(\rho)}$:

$$\tau^* = \inf\left\{u \geq t : V_u = \tilde{V}^{(\rho)}e^{r(u-t)}\right\}, \tag{14}$$

where again, we characterize this unknown constant level shortly.

Again, proposing a solution of the form $L^{(\rho)}v^{\beta^{(\rho)}}$ gives an option value of the form

$$p^{(\rho)}(v) = (\tilde{V}^{(\rho)} - K)\left(\frac{v}{\tilde{V}^{(\rho)}}\right)^{\beta^{(\rho)}} \tag{15}$$

and a quadratic for $\beta^{(\rho)}$ involving the unknown equilibrium rate of return μ^e:

$$\frac{1}{2}\eta^2\beta^{(\rho)}(\beta^{(\rho)} - 1) + \beta^{(\rho)}(\nu - r) + (r - \mu^e) = 0. \tag{16}$$

McDonald and Siegel (1986) obtain a characterization for μ^e via the CAPM. Applying Ito's formula to $X = p^{(\rho)}(v)$ gives

$$\frac{dX}{X} = \left(\beta^{(\rho)}\nu + \frac{1}{2}\beta^{(\rho)}(\beta^{(\rho)} - 1)\eta^2 + (1 - \beta^{(\rho)})r\right)dt + \beta^{(\rho)}\eta dW. \tag{17}$$

Under the CAPM, investors require compensating only for systematic or market risk of projects. Hence, the equilibrium required rate of return on the project is

$$\hat{\nu} = r + \lambda\rho\eta$$

[12] Although the solution presented here differs slightly from Dixit and Pindyck (1994) as they assume investment costs are constant over time. This causes optimal investment to take place in their model when V itself reaches a constant level. These differences are minor and the form of the value of the option to invest in (10) and trigger level in (11) are the same.

where λ is the Sharpe ratio on the market asset. Under this assumption, the equilibrium expected rate of return on the investment opportunity μ^e can be derived by equating the Sharpe ratios of the investment opportunity with the project

$$\frac{\mu^e - r}{\beta^{(\rho)}\eta} = \frac{\hat{\nu} - r}{\eta}$$

so that

$$\mu^e = r + \beta^{(\rho)}(\hat{\nu} - r) = r + \beta^{(\rho)}\lambda\rho\eta. \tag{18}$$

Equating the expression for μ^e above with the expected rate of return on the investment opportunity in (17), or equivalently substituting into (16) gives

$$\frac{1}{2}\beta^{(\rho)}(\beta^{(\rho)} - 1)\eta^2 + \beta^{(\rho)}(\nu - r - \lambda\rho\eta) = 0.$$

Let $\delta = \hat{\nu} - \nu = r + \lambda\rho\eta - \nu$, the difference in the equilibrium expected rate of return and the expected return on the project. We will commment on this later. Using (18) and the form of the option value in (15), we can rewrite the Bellman equation in (13) as

$$\frac{1}{2}\eta^2 v^2 p_{vv}^{(\rho)}(v) + \eta(\xi - \lambda\rho)v p_v^{(\rho)}(v) = 0. \tag{19}$$

We can also reexpress the above quadratic for $\beta^{(\rho)}$ in terms of Sharpe ratios,

$$\frac{1}{2}\beta^{(\rho)}(\beta^{(\rho)} - 1)\eta^2 + \beta^{(\rho)}\eta(\xi - \lambda\rho) = 0 \tag{20}$$

with solutions[13]

$$\beta_1^{(\rho)} = 1 - \frac{2(\xi - \lambda\rho)}{\eta}, \quad \beta_2^{(\rho)} = 0. \tag{21}$$

Rejecting $\beta_2^{(\rho)} = 0$ as before, and considering $\beta_1^{(\rho)} > 1$ gives the value of the option to invest as

$$p^{(\rho)}(v) = (\tilde{V}^{(\rho)} - K)\left(\frac{v}{\tilde{V}^{(\rho)}}\right)^{\beta_1^{(\rho)}} \tag{22}$$

with

$$\tilde{V}^{(\rho)} = \frac{\beta_1^{(\rho)}}{\beta_1^{(\rho)} - 1}K. \tag{23}$$

In this model, we see a similar decision rule to the perfect spanning model earlier: invest when the discounted value of V first reaches a constant level $\tilde{V}^{(\rho)}$. The only difference in the two valuations is the form of the root $\beta_1^{(\rho)}$ versus $\beta_1^{(1)}$ of the perfect spanning model. The root in the McDonald and Siegel (1986) model depends upon the correlation factor ρ, whilst this is one in the complete model.

The value of $\beta_1^{(\rho)}$ governs again whether it is optimal for the firm to ever invest. The condition $\beta_1^{(\rho)} > 1$ corresponds to $\xi < \lambda\rho$, or $\nu < \hat{\nu}$. The interpretation is that provided the expected capital gain on the project is lower than the equilibrium rate (which includes a risk

[13]See Footnote 10.

premium), then there is an opportunity cost to keeping the option alive. This means it is optimal for the firm to invest at some point in the future.

However, if $\beta_1^{(\rho)} \leq 1$ (equivalently $\xi \geq \lambda\rho$, or $\nu \geq \hat{\nu}$), the firm postpones investment, since holding the option is the best strategy. Since $\delta \leq 0$, there is no opportunity cost to keeping the option alive.

Since investment is only ever optimal when $\nu < \hat{\nu}$, or equivalently $\delta > 0$, McDonald and Siegel (1986) (and earlier McDonald and Siegel (1984) and Constantinides (1978)) refer to δ as a "below equilibrium return shortfall". We later recover the two models presented in this section as special cases of the partial spanning model, described in Section 4.

4 The Partial Spanning Asset Investment model

Both of the models presented in the previous section made strong assumptions in deriving the value of the option to invest. The complete market model makes the assumption that a perfect spanning asset or twin security exists, one which is often not justifiable in practice. Alternatively, the model of McDonald and Siegel (1986) assumes CAPM to quantify how risk is compensated.

4.1 The Model Framework

In this section we develop the partial spanning asset model. In doing so, we assume that the returns from asset P do not necessarily perfectly span the project value, and so allow for $-1 < \rho < 1$. We will show that in the special case of $|\rho| = 1$, our model will recover the classic complete markets valuation presented in Section 3.

Since the project value, V, is not perfectly spanned by the traded asset P, private risks will exist which cannot be hedged with P. The valuation must take into account risk preferences and thus our model allows for incompleteness. Our approach is to assume these preferences are summarized by the exponential utility function $U(x) = -\frac{1}{\gamma}e^{-\gamma x}, \;\; \gamma > 0$ which exhibits constant absolute risk aversion. As such, the valuation no longer represents a financial market valuation but rather the value to the firm of the investment. This is inevitable since the firm must assess the private risk of the investment according to its risk preferences. In fact, any model recognizing private risk must make an equivalent assumption, for example, Smith and Nau (1995).

The second special case of the partial spanning model occurs when $\gamma \to 0$. In this case, our model recovers the valuation of McDonald and Siegel (1986) in Section 3. Their valuation makes strong assumptions and we show it is equivalent to just one possible valuation in our partial spanning asset model. Varying the risk aversion results in different valuations to the McDonald and Siegel (1986) model. Thus the partial spanning model of this paper generalizes both of the existing valuation approaches presented in Section 3.

Our setup is as outlined in Section 2. The option payoff if investment is undertaken at $\tau \geq t$ is given by $(V_\tau - Ke^{r(\tau-t)})^+$. As was the case in the classic models, we will find the firm invests if the project value V climbs high enough compared with the investment costs. Our model will give a characterization of the level V needs to reach and the value of the option to invest in closed form. These will be compared with the valuation and optimal investment trigger found in Section 3 for the complete and McDonald and Siegel (1986) models.

An assumption we are required to make in our incomplete model is that when the company invests at time τ they immediately receive the cash amount $V_\tau - Ke^{r(\tau-t)}$ rather than the right to a future stream of cashflows over time. The cash amount is then invested optimally over the infinite horizon. In exactly the same way as in the complete model (although this is not usually made explicit) the firm hedges in the partial spanning asset P (and invests cash in the bank account) to give wealth X. Changes in wealth are given by

$$dX = \theta \frac{dP}{P} + r(X-\theta)dt \tag{24}$$

where θ is the cash amount invested in the partial spanning asset P. In the complete model of Section 3, this hedging is perfect. When the firm only has a partial spanning asset however, risk preferences are used to assess the private risk involved.

At the time of investment τ, the company has generated wealth X_τ from trading the partial spanning asset P (and investing in the bank account), and receives amount $(V_\tau - Ke^{r(\tau-t)})$, the difference in the project value and investment costs. The firm's investment problem can be expressed in terms of a utility maximization problem: maximize expected utility of wealth over an infinite horizon. That is, the firm should optimally choose the investment time τ, and partial hedge in P to maximize expected utility from receiving cash amount $X_\tau + (V_\tau - Ke^{r(\tau-t)})^+$. Inherent in this approach is that the amount at τ is then optimally invested in P (and the bank account) over the infinite horizon. The utility maximization we propose requires valuing a cashflow at the investment time $\tau < \infty$ which is not usually considered in standard utility maximization problems. We need to consider time consistency of utility functions, which is discussed in Appendix 7.1.

The firm's value function for the partial spanning model is given by the optimal stopping problem:

$$G(x,v) = \sup_{t\le\tau} \sup_{\theta_u, t\le u\le\tau} \mathbb{E}_t \left[U_\tau \left(X_\tau + (V_\tau - Ke^{r(\tau-t)})^+ \right) | X_t = x, V_t = v \right]$$

where the appropriate time-τ consistent exponential utility function $U_\tau(x)$ is given in the following proposition.

Proposition 4.1 *The time-τ consistent exponential utility function is given by*

$$U_\tau(x) = -\frac{A}{\gamma} e^{-\gamma e^{-r(\tau-t)} x} e^{\frac{1}{2}\lambda^2(\tau-t)},$$

where A is a constant and γ is the constant absolute risk aversion of the firm at the fixed time t. The value function for the firm under the partial spanning model can be written as

$$G(x,v) = \sup_{t\le\tau} \sup_{\theta_u, t\le u\le\tau} \mathbb{E}_t \left[-\frac{A}{\gamma} e^{\frac{1}{2}\lambda^2(\tau-t)} e^{-\gamma e^{-r(\tau-t)}(X_\tau + (V_\tau - Ke^{r(\tau-t)})^+)} | X_t = x, V_t = v \right]$$

Proof: Appendix 7.1 contains the details of the time consistent utility, which is given in (46). The reformulation of the value function is then immediate. □

4.2 The Solution

A Bellman equation can now be derived (see Appendix 7.2) for $G(X_t, V_t)$ from which we can solve for the optimal investment trigger level τ^* and holding in the partial spanning asset θ^*. In the stopping region,

$$G(x,v) = -\frac{A}{\gamma}e^{-\gamma(x+(v-K)^+)}.$$

In the continuation region, $G(x,v) > -\frac{A}{\gamma}e^{-\gamma(x+(v-K)^+)}$ and

$$0 = \frac{1}{2}\lambda^2 G + \xi\eta v G_v + \frac{1}{2}\eta^2 v^2 G_{vv} - \frac{1}{2}\frac{(\lambda G_x + \rho\eta v G_{xv})^2}{G_{xx}} \tag{25}$$

with boundary, value matching and smooth pasting conditions

$$\begin{aligned} G(x,0) &= -\frac{A}{\gamma}e^{-\gamma x} \\ G(x,\tilde{V}^{(\rho,\gamma)}) &= -\frac{A}{\gamma}e^{-\gamma(x+(\tilde{V}^{(\rho,\gamma)}-K)^+)} \\ G_v(x,\tilde{V}^{(\rho,\gamma)}) &= AI_{\{\tilde{V}^{(\rho,\gamma)}>K\}}e^{-\gamma(x+(\tilde{V}^{(\rho,\gamma)}-K)^+)}. \end{aligned}$$

The optimal investment time τ^* is given by

$$\tau^* = \inf\left\{u \geq t : V_u = \tilde{V}^{(\rho,\gamma)}e^{r(u-t)}\right\}$$

so investment takes place when the discounted project value reaches some constant level $\tilde{V}^{(\rho,\gamma)}$, where we characterize this constant shortly. This investment criteria is of the same form as that in the classic models of Section 3.

We now want to solve the non-linear pde (25). Proposing a solution of the form $G(x,v) = -\frac{A}{\gamma}e^{-\gamma x}J(v)$ and setting $J(v) = \Gamma(v)^g$ gives

$$0 = \left[v\Gamma_v\eta\,(\xi - \lambda\rho) + \frac{1}{2}\eta^2 v^2\Gamma_{vv} + \frac{1}{2}\frac{\Gamma_v^2}{\Gamma}\eta^2 v^2(g(1-\rho^2)-1)\right]. \tag{26}$$

Choosing $g = \frac{1}{1-\rho^2}$ eliminates the non-linear term completely, leaving

$$0 = \left[v\Gamma_v\eta\,(\xi - \lambda\rho) + \frac{1}{2}\eta^2 v^2\Gamma_{vv}\right] \tag{27}$$

with corresponding conditions on $\Gamma(v)$

$$\Gamma(0) = 1 \tag{28}$$

$$\Gamma(\tilde{V}^{(\rho,\gamma)}) = e^{-\gamma(\tilde{V}^{(\rho,\gamma)}-K)^+(1-\rho^2)} \tag{29}$$

$$\frac{\Gamma_v(\tilde{V}^{(\rho,\gamma)})}{\Gamma(\tilde{V}^{(\rho,\gamma)})} = -\gamma I_{\{\tilde{V}^{(\rho,\gamma)}>K\}}(1-\rho^2) \tag{30}$$

Similarly to the classic models of Section 3, we propose a solution of the form $\Gamma(v) = L^{(\rho,\gamma)}v^{\psi}$, which gives[14]

$$0 = \psi(\psi - \beta_1^{(\rho,\gamma)})$$

[14]The notational dependence of $\beta_1^{(\rho,\gamma)}$ on γ is simply to distinguish it from the other models.

where $\beta_1^{(\rho,\gamma)} = 1 - \frac{2(\xi-\lambda\rho)}{\eta}$. Solutions are

$$\psi = \beta_1^{(\rho,\gamma)} = 1 - \frac{2(\xi - \lambda\rho)}{\eta}, \qquad \psi = 0. \tag{31}$$

It can be seen that $\Gamma(v) = L^{(\rho,\gamma)} v^{\beta_1^{(\rho,\gamma)}} + B$, and (28) gives $B = 1$. There are two possibilities. If $\beta_1^{(\rho,\gamma)} \leq 0$ (or equivalently $\xi \geq \lambda\rho + \frac{\eta}{2}$) then $L^{(\rho,\gamma)} = 0$, smooth pasting fails and there is no solution.[15] In this case, the firm postpones investment. If $\beta_1^{(\rho,\gamma)} > 0$ (correspondingly $\xi < \lambda\rho + \frac{\eta}{2}$), the firm will invest at time τ^*.

In the case $\beta_1^{(\rho,\gamma)} > 0$, (29) gives an expression for $L^{(\rho,\gamma)}$ and via (30) we solve for the optimal investment trigger, $\tilde{V}^{(\rho,\gamma)}$, as the solution to

$$\tilde{V}^{(\rho,\gamma)} - K = \frac{1}{\gamma(1-\rho^2)} \ln\left[1 + \frac{\gamma \tilde{V}^{(\rho,\gamma)}(1-\rho^2)}{\beta_1^{(\rho,\gamma)}}\right]. \tag{32}$$

We also obtain

$$\Gamma(v) = L^{(\rho,\gamma)} v^{\psi} + 1 = 1 - \left(1 - e^{-\gamma(\tilde{V}^{(\rho,\gamma)}-K)(1-\rho^2)}\right)\left(\frac{v}{\tilde{V}^{(\rho,\gamma)}}\right)^{\beta_1^{(\rho,\gamma)}}. \tag{33}$$

Recalling the form of the value function $G(x, v)$, we find the solution is of the form

$$\tau^* = \inf\left\{u \geq t, V_u \geq \tilde{V}^{(\rho,\gamma)} e^{r(u-t)}\right\},$$

$$G(x,v) = \begin{cases} -\frac{1}{\gamma} e^{-\gamma x}\left[1 - (1 - e^{-\gamma(\tilde{V}^{(\rho,\gamma)}-K)(1-\rho^2)})\left(\frac{v}{\tilde{V}^{(\rho,\gamma)}}\right)^{\beta_1^{(\rho,\gamma)}}\right]^{\frac{1}{1-\rho^2}} & v \in [0, \tilde{V}^{(\rho,\gamma)}) \\ -\frac{1}{\gamma} e^{-\gamma x} e^{-\gamma(v-K)} & v \in [\tilde{V}^{(\rho,\gamma)}, \infty) \end{cases} \tag{34}$$

4.3 The Value of the Option to Invest

We are interested in the value of the option to invest, having solved for the optimal investment time, and the value function thus far. The value of the option to invest can be found by a certainty equivalence argument. This establishes the certain amount at which the firm would be indifferent between owing the investment option and selling the option for a certain amount. Equivalently, it represents the compensation required by the firm for giving up the right to the option payoff. As we will see later, the certainty equivalence value and the complete market value coincide when the correlation is perfect.

We evaluate the certainty equivalent amount by comparing the value achievable by investing in P and the riskfree asset and receiving the amount $p^{(\rho,\gamma)}(v)$ for the option, to the value achievable by having the option. Equating the two values and solving for $p^{(\rho,\gamma)}(v)$ gives

[15] If $\beta_1^{(\rho,\gamma)} = 0$ then (27) becomes

$$0 = \left[v\Gamma_v + v^2\Gamma_{vv}\right]\frac{\eta^2}{2}$$

with solution $\Gamma(v) = L^{(\rho,\gamma)} \ln v + B$. Again $\Gamma(0) = 1$ forces $\Gamma(v) = 1$ and smooth pasting fails.

Proposition 4.2 *The value of the option to invest in the partial spanning asset model is given by*

$$p^{(\rho,\gamma)}(v) = -\frac{1}{\gamma(1-\rho^2)} \ln \left(1 - (1 - e^{-\gamma(\tilde{V}^{(\rho,\gamma)} - K)(1-\rho^2)}) \left(\frac{v}{\tilde{V}^{(\rho,\gamma)}} \right)^{\beta_1^{(\rho,\gamma)}} \right). \tag{35}$$

This is the firm's valuation of the real investment opportunity. We can compare this valuation to that obtained in the complete and McDonald and Siegel (1986) models of Section 3. As in the complete model, the value (35) depends on the level of (in discounted terms) investment costs K and the investment trigger level $\tilde{V}^{(\rho,\gamma)}$ which in turn depends on the Sharpe ratios of the project and spanning asset. In contrast to the complete model valuation in (10), the value in (35) also depends on the correlation between the project value and the value of the partial spanning asset. The valuation in the McDonald and Siegel (1986) model (22) also depends on correlation, interpreted to be the correlation between the project value and the market. In contrast to both of the models in Section 3, the value under the partial spanning model above depends on the risk aversion γ. We will see later in Section 5 how these differences alter the value of the option to invest in numerical examples.

Recall in Section 3, the complete model value (see (3)) was expressed as the risk neutral expectation of the discounted value of the payoff, maximized over investment times. A similar representation held for the McDonald and Siegel (1986) value in (12), albeit involving the equilibrium expected rate of return and original $\mathbb{P}$ expectation. It turns out that we can also write the partial spanning value in (35) as an optimal stopping problem, analogous to these earlier representations.

Proposition 4.3 *The value of the option to invest in the partial spanning model can be represented as*

$$p^{(\rho,\gamma)}(v) = \sup_{\tau<\infty} -\frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}^{\mathbb{Q}^0}(e^{-\gamma(1-\rho^2)e^{-r(\tau-t)}(V_\tau - Ke^{r(\tau-t)})^+} | V_t = v).$$

where $\mathbb{Q}^0$ *denotes a new measure, defined in (53).*

Proof: The details of the proof are relegated to Appendix 7.3. □

We can relate the representation for the value of the option to invest given above to the values in the complete model (3), and the McDonald and Siegel model (12), earlier. All are represented as optimal stopping problems, the firm chooses when to optimally invest. The difference is that the value in the partial spanning model becomes a more complicated, non-linear function of the payoff. This is due to the incorporation of risk preferences via the exponential utility.

It is straightforward[16] to derive the following pde for $p^{(\rho,\gamma)}(v)$

$$0 = -vp_v^{(\rho,\gamma)}\eta(\xi - \lambda\rho) - \frac{1}{2}\eta^2 v^2 (p_{vv}^{(\rho,\gamma)} - (p_v^{(\rho,\gamma)})^2 \gamma(1-\rho^2)) \tag{36}$$

[16] Use the representation (51) together with the pde for $\Gamma(v)$ given in (27).

with

$$\begin{aligned} p^{(\rho,\gamma)}(0) &= 0 & (37)\\ p^{(\rho,\gamma)}(\tilde{V}^{(\rho,\gamma)}) &= \tilde{V}^{(\rho,\gamma)} - K & (38)\\ p_v^{(\rho,\gamma)}(\tilde{V}^{(\rho,\gamma)}) &= 1. & (39) \end{aligned}$$

These representations for the value of the option to invest under our partial spanning model will prove useful later in the paper. In the coming sections, we show some special cases of the partial spanning model, and also highlight a striking difference between the partial spanning model and the classic models of Section 3.

4.4 The Special Case of No Spanning Asset

A special case of the partial spanning model is obtained when the correlation between the project's value and the spanning asset P is zero. In this case, there is effectively no spanning asset and, assuming $\beta_1^{(0,\gamma)} > 0$, the value of the investment opportunity (from (35)) is

$$p^{(0,\gamma)}(v) = -\frac{1}{\gamma} \ln \left(1 - (1 - e^{-\gamma(\tilde{V}^{(0,\gamma)} - K)}) \left(\frac{v}{\tilde{V}^{(0,\gamma)}} \right)^{\beta_1^{(0,\gamma)}} \right)$$

where $\beta_1^{(0,\gamma)} = 1 - \frac{2\xi}{\eta}$ and $\tilde{V}^{(0,\gamma)}$ solves

$$\tilde{V}^{(0,\gamma)} - K = \frac{1}{\gamma} \ln \left(1 + \frac{\gamma \tilde{V}^{(0,\gamma)}}{\beta_1^{(0,\gamma)}} \right).$$

Observe that the investment trigger level $\tilde{V}^{(0,\gamma)}$ is always greater than the investment cost K. That is, even when there is no spanning asset, the option to invest still has value and it is optimal to wait to invest until the discounted project value reaches $\tilde{V}^{(0,\gamma)} > K$. Again, if $\beta_1^{(0,\gamma)} \le 0$, the firm postpones investment. Modulo differences due to discounting, this solution corresponds to that obtained in Kadam et al (2003b) in the context of executive stock options.

4.5 Recovering the McDonald and Siegel & Complete Model Valuations

In this section, we show that both the valuation of McDonald and Siegel (1986) and the complete market valuation of Dixit and Pindyck (1994) (and many others), are special cases of the partial spanning model. The complete market valuation obtains when we let $\rho \to 1$, whilst the McDonald and Siegel (1986) valuation is recovered by taking $\gamma \to 0$. This is expressed in the following result. The result says that in the limit as, first, the partial spanning asset becomes perfect or, second, the risk aversion becomes negligible, the value of the investment option, $\beta_1^{(\rho,\gamma)}$, and the investment trigger all collapse to their complete or McDonald and Siegel model counterparts respectively.

Proposition 4.4 *Two special cases of the partial spanning model are:*
(A) As $\rho \to 1$,

$$\text{(i) } \beta_1^{(\rho,\gamma)} \to \beta_1^{(1)}; \qquad \text{(ii) } \tilde{V}^{(\rho,\gamma)} \to \tilde{V}^{(1)}; \qquad \text{(iii) } p^{(\rho,\gamma)}(v) \to p^{(1)}(v).$$

(B) As $\gamma \to 0$,

$$\text{(i) } \beta_1^{(\rho,\gamma)} \to \beta_1^{(\rho)}; \qquad \text{(ii) } \tilde{V}^{(\rho,\gamma)} \to \tilde{V}^{(\rho)}; \qquad \text{(iii) } p^{(\rho,\gamma)}(v) \to p^{(\rho)}(v)$$

Proof: Parts (i) and (ii) in both (A) and (B) are straightforward from the definitions. An intuitive approach to parts (iii) can be shown via Bellman equations as follows. The pde (36) with $\rho = 1$ simplifies to

$$\frac{1}{2}\eta^2 v^2 p_{vv}^{(1,\gamma)} + \eta(\xi - \lambda) v p_v^{(1,\gamma)} = 0$$

with boundary conditions (37)-(39). This is exactly the Bellman equation (4) obtained in the complete model and thus $p^{(1,\gamma)}(v) = p^{(1)}(v)$, where $p^{(1)}(v)$ is the option value in the complete model, given in (10). Similarly, letting $\gamma \to 0$ in (36) gives

$$\frac{1}{2}\eta^2 v^2 p_{vv}^{(\rho,0)} + \eta(\xi - \lambda\rho) v p_v^{(\rho,0)} = 0$$

with the same boundary conditions (37)-(39). Again, since this is the Bellman equation we obtained earlier in (19) for the McDonald and Siegel model, $p^{(\rho,0)}(v) = p^{(\rho)}(v)$ where $p^{(\rho)}(v)$ is the valuation given in (22).

The above approach only shows the values converge in the situation where $\beta_1^{(\rho,\gamma)} > 1$ and all the models give a finite value and investment trigger. It will also be instructive to consider showing the result via the optimal stopping representation in Proposition 4.3. This enables us to consider also the situation $0 < \beta_1^{(\rho,\gamma)} \leq 1$ in which the partial spanning model gives a finite trigger and value but the classic models do not. Writing $c = \gamma(1-\rho^2)$, for small c,

$$p^{(\rho,\gamma)}(v) \approx \sup_{\tau<\infty} -\frac{1}{c} \ln\left(1 - c\mathbb{E}^{\mathbb{Q}^0} e^{-r(\tau-t)}(V_\tau - Ke^{r(\tau-t)})^+\right) \approx \sup_{\tau<\infty} \mathbb{E}^{\mathbb{Q}^0} e^{-r(\tau-t)}(V_\tau - Ke^{r(\tau-t)})^+.$$

We can interpret c small to be either $\gamma \to 0$ or $\rho \to 1$. Consider $\rho \to 1$ first. Then, $W = B$ and in fact (see Appendix 7.3) $\mathbb{Q}^0 = \mathbb{Q}$ so

$$\lim_{\rho\to 1} p^{(\rho,\gamma)}(v) = p^{(1,\gamma)}(v) = \sup_{\tau<\infty} \mathbb{E}^{\mathbb{Q}} e^{-r(\tau-t)}(V_\tau - Ke^{r(\tau-t)})^+$$

When $\beta_1^{(\rho,\gamma)} > 1$, (or from (A)(i) this is equivalent to $\beta_1^{(1)} > 1$), the solution to the above optimal stopping problem is $p^{(1)}(v)$, the complete model value given in (3). When $0 < \beta_1^{(\rho,\gamma)} \leq 1$, (or again from (A)(i) we can replace with $\beta_1^{(1)}$) the above problem has infinite value.

Now consider $\gamma \to 0$. In this case we simply obtain

$$\lim_{\gamma\to 0} p^{(\rho,\gamma)}(v) = p^{(\rho,0)}(v) = \sup_{\tau<\infty} \mathbb{E}^{\mathbb{Q}^0} e^{-r(\tau-t)}(V_\tau - Ke^{r(\tau-t)})^+ \tag{40}$$

since $\mathbb{Q}^0$ is independent of γ. When $\gamma \to 0$ in the exponential utility function, we are effectively taking $U(x) = x$, a linear utility function.[17] Although linear utility itself is risk

[17]Note that although we specify $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$, utilities are only defined up to a positive constant. It is equivalent to consider $U(x) = -e^{-\gamma x}$ and hence $\lim_{\gamma\to 0} U(x) = x$.

neutral, when we take $\gamma \to 0$, we do not obtain a risk neutral valuation, as seen above where the expectation is taken under the minimal martingale measure, $\mathbb{Q}^0$. This turns out to correspond to how risk is quantified in the McDonald and Siegel (1986) model of Section 3. That is, we can show that the limiting representation obtained above is equivalent to the McDonald and Siegel valuation. For $0 < \beta_1^{(\rho,\gamma)} \leq 1$, or from (B)(i), equivalently, $0 < \beta_1^{(\rho)} \leq 1$, (40) is infinite. Consider the case $\beta_1^{(\rho,\gamma)} > 1$, or equivalently $\beta_1^{(\rho)} > 1$. Begin with writing $S_u = V_u e^{-r(u-t)}$ and from (40), for $u \geq t$

$$\hat{p}^{(\rho,0)}(S_u) = \sup_{u \leq \tau} \mathbb{E}^{\mathbb{Q}^0}[(S_\tau - K)^+ | S_u = s].$$

Under $\mathbb{Q}^0$, S follows (54). We now derive a Bellman equation in a similar fashion to that in Appendix 7.3. In the continuation region, $\hat{p}^{(\rho,0)}(S_u) > (S_u - K)^+$ and

$$0 = \hat{p}_s^{(\rho,0)} s\eta(\xi - \lambda\rho) + \frac{1}{2}\hat{p}_{ss}^{(\rho,0)} s^2\eta^2$$

Since $p^{(\rho,0)}(v) = \hat{p}^{(\rho,0)}(s)$, we can rewrite the Bellman equation as

$$0 = p_v^{(\rho,0)} v\eta(\xi - \lambda\rho) + \frac{1}{2}p_{vv}^{(\rho,0)} v^2\eta^2.$$

This is exactly the Bellman equation (19) in the McDonald and Siegel (1986) model. □

4.6 A Larger Range of Parameter Values Leading to Investment

We have seen that the valuation under the partial spanning model depends additionally on risk aversion, in comparison to the classic models. The previous section showed that in the limit as risk aversion tends to zero, the valuation coincides with the McDonald and Siegel (1986) valuation of Section 3. We also saw that in the limit as correlation became perfect, the value of the option to invest under the perfect spanning model collapses to the complete market value. In this section we highlight a crucial difference between the partial spanning model and the earlier classic models. This observation has implications for optimal investment and valuation of investment options.

Recall, under the partial spanning model, the firm invests in the case where $\beta_1^{(\rho,\gamma)} > 0$, or equivalently only when $\xi < \lambda\rho + \frac{\eta}{2}$. As described in Section 4.2, outside this case, the firm should postpone investment indefinitely. However, the corresponding critical value in the complete and McDonald and Siegel (1986) models is different.

Fix r, λ and η. We begin with some definitions. Let $\xi^* = \xi^*(\rho,\gamma)$ be the largest[18] value of the project's Sharpe ratio in the partial spanning model, given values of ρ and γ, for which there is a finite investment trigger, and for which the value of the option to invest is finite.[19] Then (see the discussion after (31)) $\xi^* = \lambda\rho + \frac{\eta}{2}$. Let $\xi^*_{DP} = \xi^*_{DP}(1,\gamma)$ be the largest value of the project's Sharpe ratio in the perfect spanning model (see Dixit and Pindyck (1994)) for which there is a finite investment trigger, and for which the value of the option to invest is

[18] In fact ξ^* is the supremum of those values for which there is a finite investment trigger and the option value is finite. When $\xi = \xi^*$ these quantities are infinite.

[19] Note that $\xi^*(\rho,\gamma)$, $\xi^*_{DP}(1,\gamma)$ and $\xi^*_{MS}(\rho,0)$ do not depend on γ.

finite. Then (see Section 3) $\xi^*_{DP} = \lambda$. Finally, let $\xi^*_{MS} = \xi^*_{MS}(\rho, 0)$ be the largest value of the project's Sharpe ratio in the McDonald and Siegel (1986) model given values of ρ for which there is a finite investment trigger, and for which the value of the option to invest is finite. Then (see Section 3) $\xi^*_{MS} = \lambda\rho$. The following theorem is now immediate.

Theorem 4.5
(i) $\xi^*(\rho, \gamma)$ *does not tend to* $\xi^*_{DP}(1, \gamma)$ *as* $\rho \to 1$*;*
(ii) $\xi^*(\rho, \gamma)$ *does not tend to* $\xi^*_{MS}(\rho, 0)$ *as* $\gamma \to 0$*.*

The above theorem observes that the criteria for investment to take place at some time in the future are different in the partial spanning model compared with the complete and McDonald and Siegel models. In the partial spanning model there is a larger critical value for the Sharpe ratio, leading to more situations where the firm should invest. The condition under the partial spanning model, $\xi < \xi^*$ will be satisfied more often than the analogous condition $\xi < \xi^*_{MS}$ under the McDonald and Siegel model, so all other things equal, the firm will be more likely to invest and will invest sooner under the partial spanning model. It is more difficult to compare the conditions for the partial spanning model and complete model since the former applies for $|\rho| < 1$ and the complete model for $\rho^2 = 1$. However, if ρ is near to one, $\xi < \xi^*$ will be satisfied more often than the analogous complete model condition $\xi < \xi^*_{DP}$ and the firm will be more likely to invest.

This feature clearly distinguishes the partial spanning model from the classic models. In Proposition 4.4, we saw that as $\rho \to 1$, the value and investment trigger level approach the complete model value and trigger. However, for values of correlation close to one, there is a range of values for the project's Sharpe ratio ξ for which the value and investment trigger for the partially spanned model are finite, but the complete model solution is degenerate. That is, the complete model recommends the firm waits indefinitely, whilst our model gives a trigger level that the project value must reach for investment to take place. Similarly, we saw that as $\gamma \to 0$, the value and investment trigger level approach the McDonald and Siegel model value and trigger. However, for low risk aversion, there is again a range of values for the Sharpe ratio ξ for which the value and investment trigger for the partially spanned model are finite, but the McDonald and Siegel model solution is degenerate. That is, their model recommends the firm waits indefinitely, whilst our model gives a trigger level that the project value has to reach for investment to take place.

We can now ask how these new features might impact on a firm's investment decision if the firm were using the classic complete model. Suppose the firm decides that it has a traded asset which has correlation quite close to one with the project. Then, in some sense it is reasonable to approximate the value of the option using the well known complete (Dixit and Pindyck (1994)) model. However, our discussion (and the result of Theorem 4.5) shows that if the complete model concludes the firm should postpone investment, then it not possible to conclude that it is optimal to postpone investment in the partial spanning model. Rather, it may in fact be optimal to invest at some finite trigger level. Thus a widely held belief that a complete model is a good approximation tool in an "almost complete" situation is wrong. Using the complete model can lead the firm to an incorrect conclusion concerning investment timing.

A similar question can be asked for a firm using the McDonald and Siegel (1986) model. If the firm believes risk aversion should not play a major role, then it can approximate the option value via the McDonald and Siegel model.[20] Again, our results show that if the McDonald and Siegel (1986) model concludes investment should be postponed indefinitely then it is not possible to conclude the same in the partial spanning model. Again in this case, the CAPM style model is not a good approximation if risk aversion is non-negligible and can lead to the wrong conclusion on investment timing.

We can also compare the two classic models to each other. The complete model criteria of $\xi < \xi^*_{DP}$ is more easily satisfied than the corresponding criteria $\xi < \xi^*_{MS}$ in the McDonald and Siegel model (at least in the natural case where the Sharpe ratios and correlation is positive), so a firm following the complete model is more likely to invest and will invest sooner than one following the McDonald and Siegel model. We observe this later in Figure 3.

We will return to observe the implications of Theorem 4.5 in the numerical examples of Section 5.

5 Implications of Partial Spanning for Valuation of the Option to Invest and Optimal Investment

In this section we examine more closely the value of the option to invest obtained under the partial spanning model of Section 4, and the associated investment trigger level. We will illustrate a striking feature of the model represented in Theorem 4.5, that there is are additional situations where the partial spanning model can be solved but the complete and McDonald and Siegel (1986) models are degenerate. This observation has important implications for valuation and investment which we outline in this section. Additionally, via both comparative statics and numerical examples, we observe the effects of imperfect correlation and risk aversion on the value of the option to invest and trigger level.

We first observe the effect of imperfect correlation on the value of the option to invest and the investment timing decision. Recall from (31) that $\xi - \lambda\rho$ is a crucial quantity in determining the value of the option to invest. It appears in the root $\beta_1^{(\rho,\gamma)}$ which in turn appears in the value, (35). Set $\alpha = \xi - \lambda\rho$. In our analysis of the effect of correlation on the value of the option, we will fix the value of α. This can be interpreted in terms of the below equilibrium return shortfall identified by McDonald and Siegel (1986). Write $\delta = -\eta\alpha$ where $\delta = \hat{\nu} - \nu$. Fixing α is equivalent to fixing the value of δ. This is consistent with assuming that δ is a fundamental parameter, in agreement with the discussion in Dixit and Pindyck (1994)[Chapter 5].

Consider first the partial spanning asset model of Section 4, and assume $\beta_1^{(\rho,\gamma)} > 1$, or equivalently $\xi < \lambda\rho = \xi^*_{MS}$. In this case, even in the limit of perfect correlation, there is a finite investment trigger level. Figure 1 corresponds to this case. Note that on all figures, the displayed values and triggers represent discounted values. We have used the same parameter values as the example in Section 5.4 of Dixit and Pindyck (1994) (and Pindyck (1991)) for comparison purposes. Parameters for all graphs are given in the captions.

The top graph displays the investment trigger $\tilde{V}^{(\rho,\gamma)}$ obtained from (32). Each value of $\tilde{V}^{(\rho,\gamma)}$ corresponds to the level of S where the two sides of (32) are equal. This is where the

[20]Note that the McDonald and Siegel (1986) model allows for an aversion to risk in the sense of CAPM, rather than in the sense of a concave utility function.

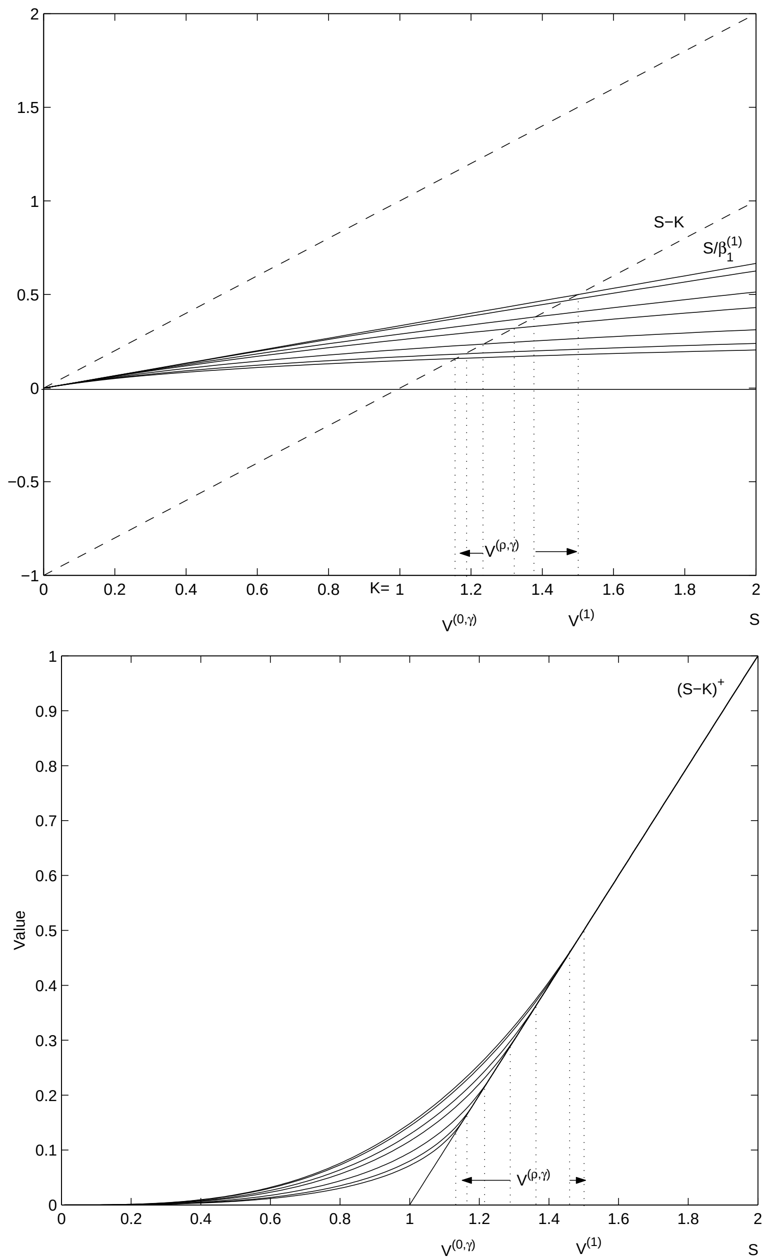

Figure 1: *Case* $\beta_1^{(\rho,\gamma)} > 1$. *The top figure shows the investment trigger* $\tilde{V}^{(\rho,\gamma)}$ *from (32) for a range of correlations. The solid lines correspond to the RHS of (32) with, from top to bottom,* $\rho = 1, 0.99, 0.95, 0.9, 0.75, 0.5, 0$. *When* $\rho^2 = 1$, *the investment trigger* $\tilde{V}^{(\rho,\gamma)} = \tilde{V}^{(1)} = 1.5$. *For* $|\rho| < 1$, $\tilde{V}^{(\rho,\gamma)} < \tilde{V}^{(1)}$. *The lower figure shows the value of the option to invest obtained from (35) for a range of correlations. The highest value corresponds to* $\rho^2 = 1$, *and* $\tilde{V}^{(1)} = 1.5$ *in this case. For* $|\rho| < 1$, *option values lie beneath the perfect correlation value, and correspond to values* $\rho = 0.99, 0.95, 0.9, 0.75, 0.5, 0.0$ *from highest to lowest. Observe* $\tilde{V}^{(\rho,\gamma)} < \tilde{V}^{(1)}$ *for* $|\rho| < 1$. *In both figures, parameters are* $K = 1, \eta = 0.2, \lambda = 0.3, r = 0.04, \alpha = \xi - \lambda\rho = -0.2, \beta_1^{(\rho,\gamma)} = 1 - 2\alpha/\eta = 3 = \beta_1^{(1)}$, $\gamma = 10$.

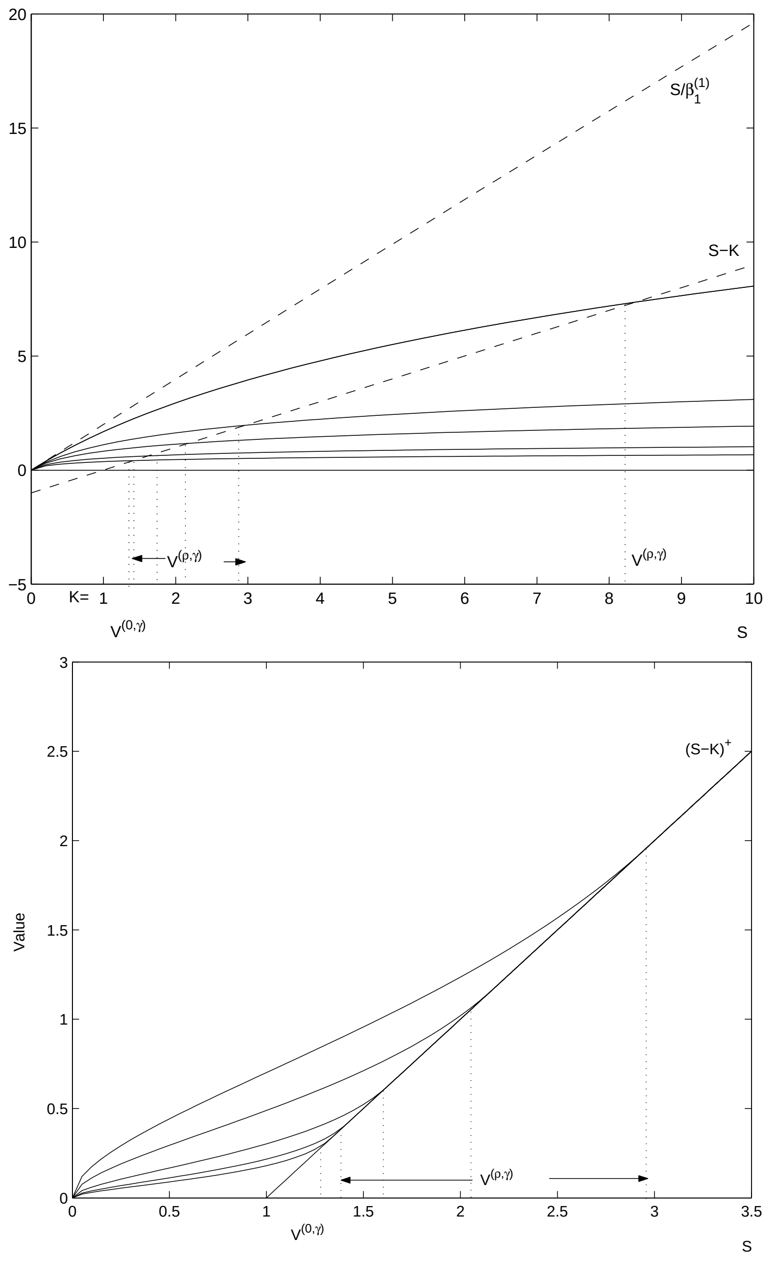

Figure 2: *Case* $0 < \beta_1^{(\rho,\gamma)} \leq 1$. *The top figure shows the investment trigger* $\tilde{V}^{(\rho,\gamma)}$ *from (32) for a range of correlations. The top dashed line corresponds to the complete model with* $\rho^2 = 1$. *Since* $\beta_1^{(1)} \leq 1$, $\tilde{V}^{(1)} = \infty$ *shown on the graph as the line never crosses the lower dashed line* $S - K$. *The solid lines correspond to the RHS of (32) with, from top to bottom,* $\rho = 0.99, 0.95, 0.9, 0.75, 0.5, 0$. *For* $|\rho| < 1$, $\tilde{V}^{(\rho,\gamma)} < \tilde{V}^{(1)} = \infty$. *The lower figure shows the value of the option to invest from (35) for a range of correlation values. Option values correspond to values* $\rho = 0.95, 0.9, 0.75, 0.5, 0.0$ *from highest to lowest. Again,* $\tilde{V}^{(\rho,\gamma)} < \tilde{V}^1 = \infty$. *Parameters are* $K = 1, \eta = 0.2, \lambda = 0.3, r = 0.04, \alpha = \xi - \lambda\rho = 0.05, \beta_1^{(\rho,\gamma)} = 1 - 2\alpha/\eta = 0.5 = \beta_1^{(1)}$, $\gamma = 10$.

curve

$$f(S) = \frac{1}{\gamma(1-\rho^2)} \ln \left[1 + \frac{\gamma S(1-\rho^2)}{\beta_1^{(\rho,\gamma)}} \right]$$

intersects the line $S - K$ represented by the dashed line. The top line (labelled $S/\beta_1^{(1)}$) corresponds to $\rho^2 = 1$, and gives $\tilde{V}^{(1)} = \frac{\beta^{(1)}}{1-\beta^{(1)}} K = 1.5$.[21] The remaining curves all correspond to $|\rho| < 1$. This leads to lower investment triggers ($\tilde{V}^{(\rho,\gamma)}$) as $|\rho|$ is decreased, hence the less close the spanning asset is to the real asset, the sooner the firm should invest. The discounted investment cost is $K = 1$, and it can be seen that even when the correlation between the project value and partial spanning asset is zero, the investment trigger is in excess of one. That is, even when there is no correlated asset at all, the option to invest still has value and investment should only take place when S rises above $\tilde{V}^{(0,\gamma)} > K$.

The lower panel of Figure 1 gives the value of the option to invest. The curved lines represent option value, calculated via (35) for a range of correlation values. The highest curve corresponds to perfect correlation, where option value $p^{(1)}(v)$ is given in (3). In this case when $S = \tilde{V}^{(1)} = 1.5$, the firm should invest. As correlation is reduced, the option value curve shifts downwards and, corresponding to the top panel, the investment trigger falls.

Now consider the case $0 < \beta_1^{(\rho,\gamma)} \leq 1$ (or $\xi^*_{MS} \leq \xi < \xi^*$) in the partial spanning asset model. In the limiting case of perfect correlation, our model coincides with the complete model of Section 3 and the firm should postpone investment. The investment trigger and option value in this case are infinite. However, for $-1 < \rho < 1$ our partial spanning model gives a finite investment trigger and option value. Figure 2 illustrates this behavior. Figure 2 is the analog of Figure 1 with $0 < \beta_1^{(\rho,\gamma)} \leq 1$ and illustrates the investment trigger levels, $\tilde{V}^{(\rho,\gamma)}$, for various values of correlation in the top panel. The perfect correlation case is represented by the line labelled $S/\beta_1^{(1)}$, which now never crosses the dotted line $S - K$. This is the complete model whereby the firm should postpone investment. Once correlation is less than perfect however, the curves corresponding to

$$f(S) = \frac{1}{\gamma(1-\rho^2)} \ln \left[1 + \frac{\gamma S(1-\rho^2)}{\beta_1^{(\rho,\gamma)}} \right]$$

cross $S - K$ at a finite trigger, $\tilde{V}^{(\rho,\gamma)}$, marked on the figure. For instance, when correlation drops from 1 to 0.99, the investment trigger goes from being infinite to having finite value 8.36. Hence in the case $0 < \beta_1^{(\rho,\gamma)} \leq 1$ (or $\xi^*_{MS} \leq \xi < \xi^*$) the assumption of perfect correlation gives the firm the signal to postpone investment indefinitely, whilst as soon as we (more realistically) assume less than perfect correlation, the firm should invest at a certain trigger level. Again, as correlation decreases, $\tilde{V}^{(\rho,\gamma)}$ decreases. Even when correlation is zero and there is no correlated asset, the option to invest still has value and the trigger of 1.33 is still greater than $K = 1$.

The lower panel of Figure 2 gives the corresponding option values for a range of correlations. Recall if correlation is perfect, for these values for $\beta_1^{(\rho,\gamma)}$, the investment trigger is infinite and the option value is also infinite. The firm should postpone investment in this

[21] In fact, Dixit and Pindyck (1994) obtain an investment trigger of 2 due to their treatment of investment costs. Our assumption of growing costs results in a lowered investment trigger compared with Dixit and Pindyck (1994).

case. However, for less than perfect correlation, the option value is finite and there is a finite investment trigger $\tilde{V}^{(\rho,\gamma)}$, above which the firm should invest. The highest option value in the lower panel of Figure 2 corresponds to $\rho = 0.95$ (taking higher values swamped the remaining lines on the graph). We see again that even when there is no correlated asset, the investment trigger is still greater than K. One final observation from the lower panel of Figure 2 is that the option value is not convex in the project value. We will return to this point later in this section, following a discussion of the effect of risk aversion.

Displayed in Figure 3 are option values for various values of risk aversion, γ. Both appear to be very similar to the lower panels of Figures 1 and 2. This is because correlation and risk aversion have similar effects on the option value and investment. For this reason, we do not repeat graphs for the investment triggers with varying risk aversion. In addition, the trigger information is contained in the value graphs as the level where the function smooth pastes to the option payoff.

The upper panel treats the case $\beta_1^{(\rho,\gamma)} > 1$ (or $\xi < \lambda\rho = \xi^*_{MS}$) whilst the lower panel considers $0 < \beta_1^{(\rho,\gamma)} \leq 1$ (or $\xi^*_{MS} \leq \xi < \xi^*$). Correlation is held fixed at 0.9 for both panels (apart from one exception noted below), other parameter values are given in the caption. In the top panel, the highest curve corresponds to the McDonald and Siegel valuation, which is the special case $\gamma = 0$. The line underneath this corresponds to taking $\rho = 1$, for comparison. The complete market valuation in this case lies below the McDonald and Siegel valuation. They differ only in the value of α used, since otherwise, as we observed earlier, taking $\gamma = 0$ or $\rho = 1$ impacts in the same way on the value. In our example, the complete model uses a more negative α and therefore has a lower value than the McDonald and Siegel value. We also observed this feature earlier in Section 4.6 in the context of the critical values of the Sharpe ratio ξ. Our observation is also confirmed by the earlier discovery that the McDonald and Siegel (1986) valuation corresponds to the valuation under a new measure, see Proposition 4.4. This ordering would reverse for instance if the Sharpe ratio of the traded partial spanning asset P were negative. It would also reverse if we were to compare the McDonald and Siegel valuation with $-1 < \rho < 0$ with perfect negative correlation. Other curves for increasing γ show that increasing risk aversion reduces the option value and the investment trigger $\tilde{V}^{(\rho,\gamma)}$.

The lower panel of Figure 3 treats the second case, $0 < \beta_1^{(\rho,\gamma)} \leq 1$ (or $\xi^*_{MS} \leq \xi < \xi^*$). When $\gamma = 0$, the McDonald and Siegel option value and investment trigger are infinite. On the graph are values for risk aversions $\gamma = 1, 5, 10, 20$, from highest to lowest on the graph.

We have seen the effect of correlation and risk aversion via the figures. It is also possible, although tedious, to obtain the following comparative statics result.

Proposition 5.1 *Under the partial spanning model of Section 4, for fixed α, the investment trigger $\tilde{V}^{(\rho,\gamma)}$ and value of the option to invest $p^{(\rho,\gamma)}(v)$ are*
(i) increasing in $|\rho|$; and
(ii) decreasing in γ.

Proof: The calculations are available from the author upon request. □

Observe from the lower panels of Figures 2 and 3, that the option value is no longer everywhere convex in the project value when $0 < \beta^{(\rho,\gamma)} < 1$, or $\xi^*_{MS} < \xi < \xi^*$.[22] We can

[22] However, if we recall the case of perfect correlation, where the option value is zero when $V = 0$ and infinite for any positive V, then, in fact, this is not convex either.

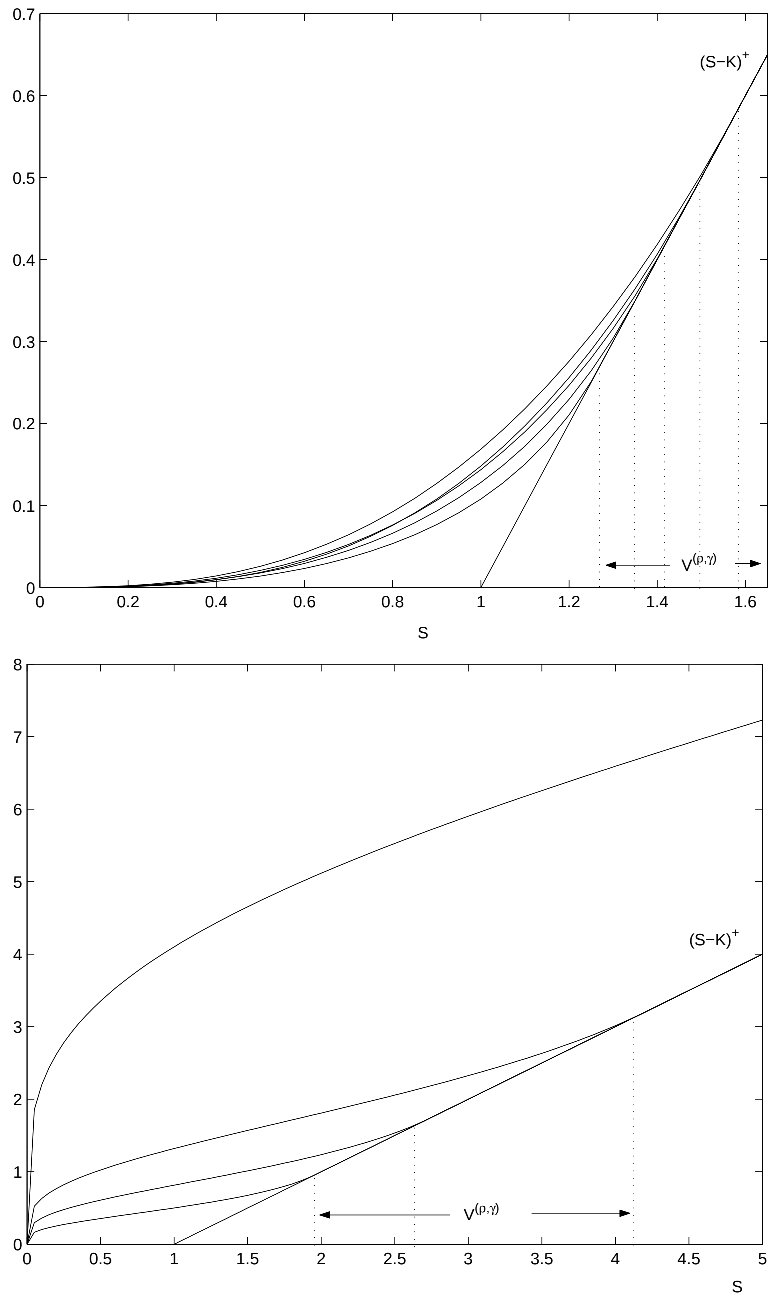

Figure 3: *Both figures show the value of the option to invest for a range of* γ *against the discounted project value for a fixed correlation* $\rho = 0.9$. *In the top panel,* $\beta_1^{(\rho,\gamma)} > 1$. *The highest value corresponds to the McDonald and Siegel valuation* ($\gamma = 0$). *Lower values correspond to increasing* γ *to values 5, 10 and 20. For the top panel,* $\alpha = -0.17, \beta_1^{(\rho,\gamma)} = 1 - 2\alpha/\eta = 2.7$. *In the lower panel,* $0 < \beta_1^{(\rho,\gamma)} \leq 1$. *Option values correspond to* $\gamma = 1, 5, 10, 20$, *from highest to lowest on the graph. When* $\gamma = 0$, *the McDonald and Siegel trigger* $\tilde{V}^{(\rho,0)} = \infty$. *For the lower panel,* $\alpha = 0.08, \beta_1^{(\rho,\gamma)} = 1 - 2\alpha/\eta = 0.2$. *Parameters common to both panels are* $K = 1, \eta = 0.2, \lambda = 0.3, r = 0.04$.

show

Proposition 5.2 *Assume the value of the option to invest,* $p^{(\rho,\gamma)}(v)$*, is given in (35).*
(i) If $\beta_1^{(\rho,\gamma)} \geq 1$*, or equivalently* $\xi \leq \xi^*_{MS}$*,* $\frac{\partial^2}{\partial v^2} p^{(\rho,\gamma)}(v) > 0$ *and the value of the option is convex in* v*.*
(ii) If $0 < \beta_1^{(\rho,\gamma)} < 1$*, or equivalently* $\xi^*_{MS} < \xi < \xi^*$*, the value of the option may be convex or concave depending on the value of* v*.*

Proof: We can prove this result either by straightforward differentiation or by using the representation in (51). See Appendix 7.4 for details. □

The result of the proposition confirms what we have seen in the figures. There are two effects operating here. Firstly, a convexity effect from optionality. In a complete market, option values are convex in the underlying asset, in this case, project value. This convexity effect carries over to our incomplete market, and dominates in the case $\beta_1^{(\rho,\gamma)} \geq 1$. That is, in the case $\xi \leq \xi^*_{MS}$, the option value is convex regardless of risk aversion, and model parameters. However, in other situations, when $\beta_1^{(\rho,\gamma)} < 1$ or $\xi^*_{MS} < \xi < \xi^*$, there is a second effect, that of risk aversion, influencing the value in the partial spanning model. Risk aversion and incompleteness mean that the additional option value from an extra dollar of project value diminishes as project value increases. In the case $\xi^*_{MS} < \xi < \xi^*$, this becomes important and the effect of risk aversion can outweigh the optionality effect, resulting in concavity in the project value.

6 Conclusion

This paper has formulated and analyzed a new model for corporate investment decisions. It recognizes that private risks exist, and introduces these via a partial spanning asset, one which has less than perfect correlation with the underlying project. In doing so, the paper extends the well known complete real options models including those of Amram and Kulatilaka (1999), Brennan and Schwartz (1985), Dixit and Pindyck (1994), Mason and Merton (1985), Pindyck (1991) and Trigeorgis and Mason (1987) to incompleteness, as suggested by Pinches (1998). Reassuringly, these models, as well as those based on CAPM such as McDonald and Siegel (1986), are recovered as limiting cases of the partial spanning model.

Our main conclusion is that the classic models are overstating the worth of the option to invest, and recommending a firm waits too long to invest. The classic models may be leading to underinvestment on the part of firms. Our more realistic model incorporating a partial spanning asset and risk aversion recommends firms invest earlier, as the option is not as valuable as the classic models claim, due to private risks. Moreover, we discover that approximating investment decisions with the complete model when a highly correlated spanning asset can be found, can lead to the wrong decision. In particular, if the complete model concludes the firm should postpone investment indefinitely, then it not possible to conclude that it is optimal to postpone investment in the partial spanning model. It may in fact be optimal to invest at some finite trigger level. Thus a widely held belief that a complete model is a good approximation tool in an "almost complete" situation is incorrect. It is also the case that the CAPM style model of McDonald and Siegel (1986) is not a always a good approximation if risk aversion is non-negligible, and again, can lead to the wrong conclusion concerning the timing of investment.

Various extensions of the partial spanning model presented could be undertaken. Firstly, models other than lognormal could be assumed for the project value, see Footnote 7. For instance, mean reverting models are increasingly popular, see Bhattacharya (1978), Dixit and Pindyck (1994)[Chapter 5.5] and Schwartz (1997). Although we have analyzed options to invest in this paper, the same framework could be used to examine abandonment decisions as well. The conclusion of such an extension would be that the firm should abandon earlier (compared to a complete or CAPM style model, as treated in Myers and Majd (1990) and McDonald and Siegel (1986) respectively) when the correlation between the project value and traded asset is less than perfect. The constant abandonment trigger level would rise as correlation decreased away from one. Investment (and similarly, abandonment) decisions with a finite horizon can also be modeled via a partial spanning approach. Of course, the option value and trigger level cannot be found in closed form in this case, making general conclusions difficult to obtain.

Empirical testing of real options models is beginning, see Berger et al (1996) and Moel and Tufano (2002). This paper raises many questions which could potentially be empirically tested. For instance, what are typical correlations between projects and the traded assets that firms have access to ? Are firms investing earlier than classic real options models recommend ? Can we determine if firms are investing when the project Sharpe ratio is higher than the critical value suggested in the classic models ? Much work remains to be done in this area.

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7 Appendix

7.1 Time Consistency of Utility Functions

In Merton (1969) style investment problems, utility of terminal wealth is maximized and the optimal investment strategy over this fixed time period is obtained. Inherently, the utility function is being applied to value an amount of wealth at a fixed, terminal date T', and in this type of problem, it is not necessary to consider how the investor values wealth at earlier dates $T < T'$. Denote the utility function used to value wealth at T' by $U_{T'}(x)$.

However, in the perpetual problem of this paper, we need to describe how the firm values the amount $X_\tau + (V_\tau - Ke^{r(\tau-t)})^+$ at an intermediate date $\tau < \infty$. To do this, we need to think about time consistency properties of utility functions. This is a new approach, although alternative ways of treating infinite horizon portfolio problems exist, such as those maximizing the growth rate of a portfolio, see Hakansson (1979).

We temporarily forget about the option and infinite horizon complications and concentrate on a finite horizon $T' \geq t$ where t is fixed. Assume we aim to maximize expected utility of wealth, given wealth follows (24). Consider also an intermediate date $t \leq T \leq T'$. At T', we assume utility of wealth is described by the exponential utility function

$$U_{T'}(x) = -\frac{A_{T'}}{\gamma_{T'}} e^{-\gamma_{T'} x}$$

where $A_{T'}$ is some constant and the constant absolute risk aversion $\gamma_{T'}$ reflects risk aversion at date T'.

To decide how to value wealth at the intermediate time T, consider choosing *any* investment strategy over $[t, T]$ and the optimal strategy between T and T'. This optimal strategy is the Merton (1969) strategy given by

$$\theta_M^* = \frac{\lambda e^{-r(T'-T)}}{\gamma_{T'} \sigma}$$

We can write (using the Merton (1969) solution)

$$\sup_{\theta_u, t \leq u \leq T'} \mathbb{E} U_{T'}(X_{T'}) = \sup_{\theta_u, t \leq u \leq T} \mathbb{E} \left[-\frac{A_{T'}}{\gamma_{T'}} e^{-\gamma_{T'} e^{r(T'-T)} X_T} e^{-\frac{1}{2}\lambda^2 (T'-T)} \right] \tag{41}$$

The right hand side is now an optimization problem over the sub-horizon $[t, T]$. To value consistently with T' cashflows, we would like the utility function valuing cashflows at T to take the form

$$U_T(x) = -\frac{A_T}{\gamma_T} e^{-\gamma_T x} \tag{42}$$

where A_T is constant and γ_T reflects risk aversion for time T. From (41) we must have

$$-\frac{A_{T'}}{\gamma_{T'}} e^{-\gamma_{T'} e^{r(T'-T)} x} e^{-\frac{1}{2}\lambda^2 (T'-T)} = -\frac{A_T}{\gamma_T} e^{-\gamma_T x}$$

and so require

$$\gamma_{T'} e^{rT'} = \gamma_T e^{rT} = \gamma_t e^{rt} \tag{43}$$

and

$$\frac{A_{T'}}{\gamma_{T'}} e^{-\frac{1}{2}\lambda^2 T'} = \frac{A_T}{\gamma_T} e^{-\frac{1}{2}\lambda^2 T} = \frac{A_t}{\gamma_t} e^{-\frac{1}{2}\lambda^2 t} \tag{44}$$

where in both (43) and (44), A_t is a constant and γ_t is the constant absolute risk aversion parameter for today, t. From here on, and throughout the main text of the paper, we denote $\gamma_t = \gamma$ and $A_t = A$ for ease of notation.

Using (43) in (44) gives

$$A_T = Ae^{(\frac{1}{2}\lambda^2 - r)(T-t)}. \tag{45}$$

Now combining (42), (45) and (43) gives the time consistent utility function for cashflows at T as

$$U_T(x) = -\frac{A}{\gamma}e^{-\gamma e^{-r(T-t)}x}e^{\frac{1}{2}\lambda^2(T-t)}.$$

We see that the terminal date T' has disappeared in the time consistent utility used at $T \leq T'$. For this reason, this is also the appropriate utility to use when the horizon T' is infinite.

Returning to our perpetual problem, the firm receives $(V_\tau - Ke^{r(\tau-t)})^+$ at τ and has generated X_τ from hedging. The appropriate time-τ utility function to use is

$$U_\tau(x) = -\frac{A}{\gamma}e^{-\gamma e^{-r(\tau-t)}x}e^{\frac{1}{2}\lambda^2(\tau-t)} \tag{46}$$

where A is a constant and γ is the constant absolute risk aversion of the firm today.

The firm's investment problem is to solve

$$G(x,v) = \sup_{t\leq\tau}\sup_{\theta_u, t\leq u\leq\tau}\mathbb{E}_t\left[U_\tau\left(X_\tau + (V_\tau - Ke^{r(\tau-t)})^+\right)|X_t = x, V_t = v\right]$$

for the optimal investment time τ and hedge θ in the partial spanning asset P. Using the time consistent utility gives

$$G(x,v) = \sup_{t\leq\tau}\sup_{\theta_u, t\leq u\leq\tau}\mathbb{E}_t\left[-\frac{A}{\gamma}e^{\frac{1}{2}\lambda^2(\tau-t)}e^{-\gamma e^{-r(\tau-t)}(X_\tau+(V_\tau-Ke^{r(\tau-t)})^+)}|X_t = x, V_t = v\right] \tag{47}$$

7.2 Derivation of the Bellman equation (25)

In this section, we develop a Bellman equation for the investment problem in (47). Define for $u \geq t$,

$$H(y,s) = \sup_{u\leq\tau}\sup_{\theta_r, u\leq r\leq\tau}\mathbb{E}_u\left[-\frac{A}{\gamma}e^{\frac{1}{2}\lambda^2(\tau-u)}e^{-\gamma(Y_\tau+(S_\tau-K)^+)}|Y_u = y, S_u = s\right]$$

where $Y_u = X_u e^{-r(u-t)}$ and $S_u = V_u e^{-r(u-t)}$ are the discounted wealth and project cashflows respectively. Now

$$\begin{aligned} H(Y_u, S_u) \quad = \quad & \text{Max}\left\{-\frac{A}{\gamma}e^{-\gamma(Y_u+(S_u-K)^+)} \quad ; \right. \\ & \left. \sup_{\tau\geq u+du}\sup_{\theta}\mathbb{E}_u\left[\mathbb{E}_{u+du}\left(-\frac{A}{\gamma}e^{\frac{1}{2}\lambda^2(\tau-(u+du)+du)}e^{-\gamma(Y_\tau+(S_\tau-K)^+)}\right)\right]\right\} \end{aligned} \tag{48}$$

Concentrating on the second term above, we see it can be written as

$$\begin{aligned}
&\sup_{\theta_r,\, r\le u+du} \mathbb{E}_u\left[\sup_{\tau\ge u+du}\sup_{\theta_r,\, r\ge u+du} e^{\frac{1}{2}\lambda^2 du}\mathbb{E}_{u+du}\left(-\frac{A}{\gamma}e^{\frac{1}{2}\lambda^2(\tau-(u+du))}e^{-\gamma(Y_\tau+(S_\tau-K)^+)}\right)\right] \\
=& \sup_{\theta_r,\, r\le u+du} \mathbb{E}_u[e^{\frac{1}{2}\lambda^2 du}H(Y_{u+du},S_{u+du})] \\
=& \sup_{\theta_r,\, r\le u+du} \mathbb{E}_u[e^{\frac{1}{2}\lambda^2 du}(H(Y_u,S_u)+dH(Y_u,S_u))] \\
=& H(Y_u,S_u)\left(1+\frac{1}{2}\lambda^2 du\right)+\sup_{\theta_r,\, r\le u+du}\mathbb{E}_u dH(Y_u,S_u)
\end{aligned}$$

Using this expression in (48) gives

$$0=\text{Max}\left\{-\frac{A}{\gamma}e^{-\gamma(Y_u+(S_u-K)^+)}-H(Y_u,S_u)\,;\ \frac{1}{2}\lambda^2 H(Y_u,S_u)du+\sup_{\theta_r,\, r\le u+du}\mathbb{E}_u dH(Y_u,S_u)\right\}$$

In the continuation region $H(y,s)>-\frac{A}{\gamma}e^{-\gamma(y+(s-K)^+)}$ and

$$\mathcal{L}H=0 \tag{49}$$

where

$$\mathcal{L}H=\frac{1}{2}\lambda^2 H+\xi\eta s H_s+\frac{1}{2}\eta^2 s^2 H_{ss}+\sup_{\tilde{\theta}}\left\{\tilde{\theta}\lambda\sigma H_y+\frac{1}{2}\tilde{\theta}^2\sigma^2 H_{yy}+\tilde{\theta}\sigma\rho\eta s H_{ys}\right\}$$

and $Y_u=X_u e^{-r(u-t)}, S_u=V_u e^{-r(u-t)}, \tilde{\theta}_u=\theta_u e^{-r(u-t)}, \tilde{P}_u=P_u e^{-r(u-t)}$ giving dynamics $dY_u=\tilde{\theta}_u\frac{d\tilde{P}}{\tilde{P}}$, $\frac{dS}{S}=\eta(\xi du+dW)$, $\frac{d\tilde{P}}{\tilde{P}}=\sigma(\lambda du+dB)$.

In the stopping region $H(y,s)=-\frac{A}{\gamma}e^{-\gamma(y+(s-K)^+)}$ and

$$\mathcal{L}H\le 0.$$

The stopping region is characterized by the investment time τ^* given by

$$\tau^*=\inf\left\{u\ge t: S_u=\tilde{V}^{(\rho,\gamma)}\right\}=\inf\left\{u\ge t: V_u=\tilde{V}^{(\rho,\gamma)}e^{r(u-t)}\right\}$$

That is, when the discounted project value reaches the constant level $\tilde{V}^{(\rho,\gamma)}$, investment optimally takes place. Optimizing over $\tilde{\theta}$ gives

$$\tilde{\theta}^*_u=\frac{-\lambda H_y-H_{ys}\rho s\eta}{H_{yy}\sigma}.$$

and substituting this back into (49) gives

$$0=\frac{1}{2}\lambda^2 H+\xi\eta s H_s+\frac{1}{2}\eta^2 s^2 H_{ss}-\frac{1}{2}\frac{(\lambda H_y+\rho\eta s H_{ys})^2}{H_{yy}} \tag{50}$$

in the continuation region.

Finally, observe from (47),

$$\begin{aligned}
G(x,v) &= \sup_{t\leq\tau}\sup_{\theta_u,t\leq u\leq\tau}\mathbb{E}_t\left[-\frac{A}{\gamma}e^{\frac{1}{2}\lambda^2(\tau-t)}e^{-\gamma e^{-r(\tau-t)}(X_\tau+(V_\tau-Ke^{r(\tau-t)})^+)}|X_t=x,V_t=v\right]\\
&= \sup_{t\leq\tau}\sup_{\theta_u,t\leq u\leq\tau}\mathbb{E}_t\left[-\frac{A}{\gamma}e^{\frac{1}{2}\lambda^2(\tau-t)}e^{-\gamma(Y_\tau+(S_\tau-K)^+)}|X_t=x,V_t=v\right]\\
&= \sup_{t\leq\tau}\sup_{\theta_u,t\leq u\leq\tau}\mathbb{E}_t\left[-\frac{A}{\gamma}e^{\frac{1}{2}\lambda^2(\tau-t)}e^{-\gamma(Y_\tau+(S_\tau-K)^+)}|Y_t=x,S_t=v\right]\\
&= H(x,v)
\end{aligned}$$

We can rewrite (50) and the continuation region as $G(x,v) > -\frac{A}{\gamma}e^{-\gamma(x+(v-K)^+)}$,

$$0=\frac{1}{2}\lambda^2 G+\xi\eta vG_v+\frac{1}{2}\eta^2v^2G_{vv}-\frac{1}{2}\frac{(\lambda G_x+\rho\eta vG_{xv})^2}{G_{xx}}$$

with

$$\begin{aligned}
G(x,0) &= -\frac{A}{\gamma}e^{-\gamma x}\\
G(x,\tilde{V}^{(\rho,\gamma)}) &= -\frac{A}{\gamma}e^{-\gamma(x+(\tilde{V}^{(\rho,\gamma)}-K)^+)}\\
G_v(x,\tilde{V}^{(\rho,\gamma)}) &= AI_{\{\tilde{V}^{(\rho,\gamma)}>K\}}e^{-\gamma(x+(\tilde{V}^{(\rho,\gamma)}-K)^+)}
\end{aligned}$$

7.3 Proof of Proposition 4.3

A similar representation in terms of the minimal martingale measure is shown in the case of finite time American options in Musiela and Zariphopoulou (2003) and in the European case by Henderson (2002). We extend the representation here to the perpetual American option under consideration.

First, write the value in (35) as

$$p^{(\rho,\gamma)}(v)=-\frac{1}{\gamma(1-\rho^2)}\ln\Gamma(v). \tag{51}$$

We now propose a form for $\Gamma(v)$ and verify it satisfies the pde (27) and associated boundary conditions. For the proposition to be true, we must have that

$$\Gamma(v)=\inf_{\tau<\infty}\mathbb{E}^{\mathbb{Q}^0}(e^{-\gamma(1-\rho^2)e^{-r(\tau-t)}(V_\tau-Ke^{r(\tau-t)})^+}|V_t=v) \tag{52}$$

where $\mathbb{Q}^0$ defined via

$$\frac{d\mathbb{Q}^0}{d\mathbb{P}}=\exp\left(-\lambda B_T-\frac{1}{2}\lambda^2T\right) \tag{53}$$

is the minimal martingale measure under which the traded asset P is a martingale. Specifically, under $\mathbb{Q}^0$, the traded asset follows

$$\frac{dP}{P}=rdt+\sigma dB^0$$

where $B^0 = B + \lambda t$ is a $\mathbb{Q}^0$ Brownian motion. Writing the original $\mathbb{P}$ Brownian motions W and B as $dW = \rho dB + \sqrt{1-\rho^2}dZ$ where Z is independent of B, the project value follows

$$\frac{dV}{V} = \nu dt + \eta(\rho dB + \sqrt{1-\rho^2}dZ)$$

under $\mathbb{P}$. Substituting for the $\mathbb{Q}^0$ Brownian motion and leaving the drift of the independent Brownian motion Z unchanged, gives

$$\frac{dV}{V} = (\nu - \lambda\eta\rho)dt + \eta(\rho dB^0 + \sqrt{1-\rho^2}dZ)$$

under the minimal martingale measure $\mathbb{Q}^0$. Finally, discounting gives S follows

$$\frac{dS}{S} = (\nu - r - \lambda\eta\rho)dt + \eta(\rho dB^0 + \sqrt{1-\rho^2}dZ) \tag{54}$$

under $\mathbb{Q}^0$.

Returning to (52), we see boundary condition (28) is satisfied. Rewrite (52) for $u \geq t$ as

$$\hat{\Gamma}(s) = \inf_{u \leq \tau} \mathbb{E}_u^{\mathbb{Q}^0}(e^{-\gamma(1-\rho^2)(S_\tau - K)^+}|S_u = s)$$

where $S_u = V_u e^{-r(u-t)}$. We develop a Bellman equation for $\hat{\Gamma}(s)$ similarly to that derived in 6.2. Write

$$\hat{\Gamma}(s) = \min\left[e^{-\gamma(1-\rho^2)(S_u-K)^+}; \inf_{u+du\leq\tau} \mathbb{E}_u^{\mathbb{Q}^0}\mathbb{E}_{u+du}^{\mathbb{Q}^0} e^{-\gamma(1-\rho^2)(S_\tau-K)^+}\right]$$

which reduces to

$$0 = \min\left[e^{-\gamma(1-\rho^2)(S_u-K)^+} - \hat{\Gamma}(S_u); \mathbb{E}_u^{\mathbb{Q}^0} d\hat{\Gamma}(S_u)\right]$$

In the continuation region, $\hat{\Gamma}(s) < e^{-\gamma(1-\rho^2)(s-K)^+}$ and

$$0 = \mathbb{E}_u^{\mathbb{Q}^0}\left(\hat{\Gamma}_s dS + \frac{1}{2}\hat{\Gamma}_{ss}(dS)^2\right).$$

Substituting from the dynamics in (54) and taking expectations, we obtain

$$0 = \hat{\Gamma}_s S\eta(\xi - \lambda\rho) + \frac{1}{2}\hat{\Gamma}_{ss}S^2\eta^2$$

Now since $\hat{\Gamma}(s) = \Gamma(v)$, we obtain

$$0 = v\Gamma_v\eta(\xi - \lambda\rho) + \frac{1}{2}\eta^2 v^2 \Gamma_{vv}$$

which is exactly the pde (27). The condition (29) is obtained from the stopping condition.

□

7.4 Proof of Proposition 5.2

The first idea of a proof is simply to differentiate (35) twice in v to obtain

$$\frac{\partial^2}{\partial v^2}\left\{p^{(\rho,\gamma)}(v)\right\} = -\frac{\beta_1^{(\rho,\gamma)}}{cv^2}(1-e^{cp^{(\rho,\gamma)}})\left[\beta_1^{(\rho,\gamma)} - 1 - \beta_1^{(\rho,\gamma)}(1-e^{cp_1^{(\rho,\gamma)}})\right]$$

where $c = \gamma(1-\rho^2)$. The term outside the square brackets is positive for any $\beta_1^{(\rho,\gamma)} > 0$. Now observe that if $\beta_1^{(\rho,\gamma)} \geq 1$, the term inside the square brackets is positive also, and overall the second derivative is greater than zero. However, if $\beta_1^{(\rho,\gamma)} < 1$, the sign of the square bracket term is indeterminate and the second derivative may be of either sign.

A more enlightening approach is to write for functions $f(v), \Gamma(v)$ (where $\Gamma(v)$ is given in (33) and the relationship between option value and $\Gamma(v)$ is given in (51))

$$\{f \circ \Gamma\}'' = f''(\Gamma)(\Gamma')^2 + \Gamma'' f'(\Gamma)$$

Identifying $f(x) = -\frac{1}{c}\ln x$, this is convex and decreasing. The convexity or concavity of $p^{(\rho,\gamma)}(v)$ depends upon the sign of $\{f.\Gamma\}''$ above. The first term $f''(\Gamma)(\Gamma')^2$ is always positive. We need to determine the sign of the second term above.

In the case $\beta_1^{(\rho,\gamma)} \geq 1$, $\Gamma(v)$ is concave and decreasing, and hence the second term above is positive. Overall, $p^{(\rho,\gamma)}(v)$ is convex in v in this case.

Consider now $\beta_1^{(\rho,\gamma)} < 1$. Now $\Gamma(v)$ is convex and decreasing, and hence the second term above is now negative. Whether $p^{(\rho,\gamma)}(v)$ is convex or concave in v depends on which term dominates.

For example, set $\beta_1^{(\rho,\gamma)} = \frac{1}{2}$. Then $\Gamma(v) = 1 - kv^{\frac{1}{2}}$, $(\Gamma')^2 \sim \frac{1}{v}$ and $\Gamma'' \sim \frac{1}{v^{3/2}}$. In this case, for small v, $\frac{1}{v^{3/2}} > \frac{1}{v}$ so the second, negative term dominates and $p^{(\rho,\gamma)}(v)$ is concave. However, for large v, the first term dominates and the option value is convex. □