Continuous-Time Option Games: Review of Models and Extensions

Part 1: Duopoly under Uncertainty

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Abstract

The theory of *option games* being the combination of two successful theories, namely real options and game theory, has a great potential to applications in many real situations. Although the option games literature is very recent, it has been experimenting a fast growth in the last five years. It considers in the same model, besides the key factors for investment decisions such as uncertainty, flexibility, and timing, the effect of competition with the possible strategies for each firm.

This paper reviews a selected literature on continuous-time models of option games and provides some new insights and extensions. This review is divided into two parts or two papers. In this paper we analyze models of duopoly under uncertainty – both symmetrical and asymmetrical. First we present a brief survey of option games with a summary of possible equilibriums in duopoly - like Cournot and Stackelberg, and types of demand function as well as the effects of this uncertainty on these functions. We discuss concepts like the preemption, non-binding collusion, main and secondary perfect-Nash equilibriums in pure strategies, first mover advantage, situations that mixed strategies are necessary, probability of simultaneous exercise as mistake, effect of the competitive advantage, etc. We show that there are two equivalent ways to calculate both leader and follower values, and two ways to calculate the follower threshold. We also extend the asymmetrical duopoly under uncertainty model analyzed by Joaquin & Buttler by considering issues like *mixed strategies in asymmetric duopoly* and the value of *option to become a leader*. In a second paper will be discussed important option games models like oligopoly under uncertainty, war of attrition and other models of positive externalities, models with either incomplete or asymmetric information, the current option-games models limitations, and suggestions for future research.

Keywords: option games, option exercise games, real options, stochastic game theory, duopoly under uncertainty, preemption, non-binding collusion, mixed strategies in asymmetric duopoly.

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1 - Introduction

The theory of corporate finance in general and especially *capital budgeting*, has been experimenting a fast and rich development in the last 30 years. Most of these new theoretical tools and practical insights come from the development of two theories. First, the option pricing theory and contingent claims approach developed by the seminal papers of Black & Scholes (1973) and Merton (1973). Second, by applications of game theoretic concepts to corporate finance problems starting in 70's mainly *asymmetric information* between *agent* and *principal* with *signalling games* in papers like Ross (1977), Leland & Pyle (1977), Bhattacharya (1979), and Myers & Majluf (1984).

In capital budgeting applications like investment in projects (real assets), early applications of options pricing started a revolution with Myers (1977) coining the term "*real options*". Models development started with Tourinho (1979), Kester (1984), Brennan & Schwartz (1985), McDonald & Siegel (1986), and Paddock & Siegel & Smith (1988), just to mention a few of the best known cases.

The real options allowed a proactive approach to investment decisions in conditions of uncertainty, highlighting the value of the managerial flexibility under uncertainty. Before real options, uncertainty was only a matter of appropriate discounting through an adequate risk-premium, a limited view on the manager's role facing uncertainty in projects and real assets in general. With real options the bases for a modern theory of *investment under uncertainty* were established. However, the problem of investment under uncertainty *and under competition* was demanding a more rigorous framework. The earlier attempts to model competition considered exogenously either an *estimated* entry or a *random* entry of competitors[1], not an endogenous *rational entry* of competitors[2]. Game theoretic concepts for models with strategic interaction between firms were not addressed in real options models until the beginning of 90's.

The practical and theoretical demand of real options models considering also a rational strategic interaction between the "players" - the option exercise of one player changing the real options values

[1] Before option games models, the best known attempts that appeared to model the competition effect were: Kester (1984) considering a *finite time to expiration* of a real option, with this time being function of an estimated competitors entry destroying the firm's real option; Trigeorgis (1986, 1991), the competitor preemption effect is modeled as "*additional dividends*" that are lost by the (non-exercised) real option owner; and also in Trigeorgis (1986, 1991), the competitor entry is modeled with a *Poisson process*, with the competitor's random arrival causing a jump-down in the project value.
[2] Rational entry of competitors includes cases with mixed strategies and Bayesian strategies (incomplete information), because the probabilities in this context are calculated from players' strategies with mutual interaction rationality.

of the others players, led to the birth of *continuous-time option games*[3] literature in the beginning of 90's with the Smets' dissertation (1993, after a working paper in 1991)[4]. Discrete-time option games started with Smit & Ankum (1993), providing an intuitive approach for important option games models, with additional insights in Smit & Trigeorgis (1993)[5].

Game theory is a well-established tool in industrial organization and modeling of imperfect competition. However, standard game theory alone ignores the advances of finance theory on risk-return and on the managerial flexibility value under uncertainty. Game theory and options pricing are *complementary* approaches, providing together a framework with rich potential of applications. Both theories won the Nobel Prize in Economics, game theory in 1994 with Nash, Selten, Harsanyi, and options pricing theory in 1997 with Scholes and Merton (with references to project applications - real options, by the Sweden Academy communication).

About the combination of option pricing and game theory, Ziegler (1999, p.133) wrote: "… game theory analysis of options in effect replaces the maximization of *expected utility* encountered in classical game theory models with the maximization of the value of an *option* … option-pricing approach has the advantage that it automatically takes the time value of money and the price of *risk* into account". He also highlights the "link between markets and organizations" with options setting payoffs using market criteria and game theory taking the structure of organizations into account.

The first (real) option games *textbook* appeared with Huisman (2001), focusing important theoretical models of option games in continuous-time mainly for technology applications. Before, Grenadier (2000) edited a good selection of option games papers. A nice new addition is the forthcoming textbook of Smit & Trigeorgis (2003) focusing mainly discrete-time option games models, in a light and thorough approach, with many practical examples.

Although many new papers have been written in the last years, the combination of real options with game theory has a large potential of new models considering the vast individual literature on game theory and real options. The aim of this paper is to summarize a selected representative literature on continuous-time option games, presenting typical tools and the key concepts used in these models.

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[3] The term "option games" first appeared in Lambrecht & Perraudin (1994).

[4] This model was summarized in the first real options textbook, the outstanding work of Dixit & Pindyck (1994, chapter 9). These authors took the *attractive risk* to include in their book the *just born* option games model of Smets (1993) that was starting a new literature. The price was two minor misunderstandings pointed out by Huisman & Kort (1999).

[5] The latter is well summarized in the very good real options textbook of Trigeorgis (1996, chapter 9).

In the next section of this paper are presented a short review on the main equilibrium possibilities, game theoretic concepts and models, types of demand curves and how uncertainty is inserted into the demand curve. In the third section it is presented the symmetrical duopoly model, showing two equivalent ways to calculate either the leader or the follower value, and with discussion on mixed strategies equilibrium and the possibility of non-binding collusion equilibrium. In third section it is discussed the asymmetrical duopoly model, extending the model of Joaquin & Buttler (2000) with a mixed strategies mixed strategy theorem for asymmetric duopoly under uncertainty. In the fourth section some conclusions and suggestions are presented. The appendixes provide some classical option games proofs omitted in the main text.

2 - Equilibrium Possibilities and Stochastic Demand Curve

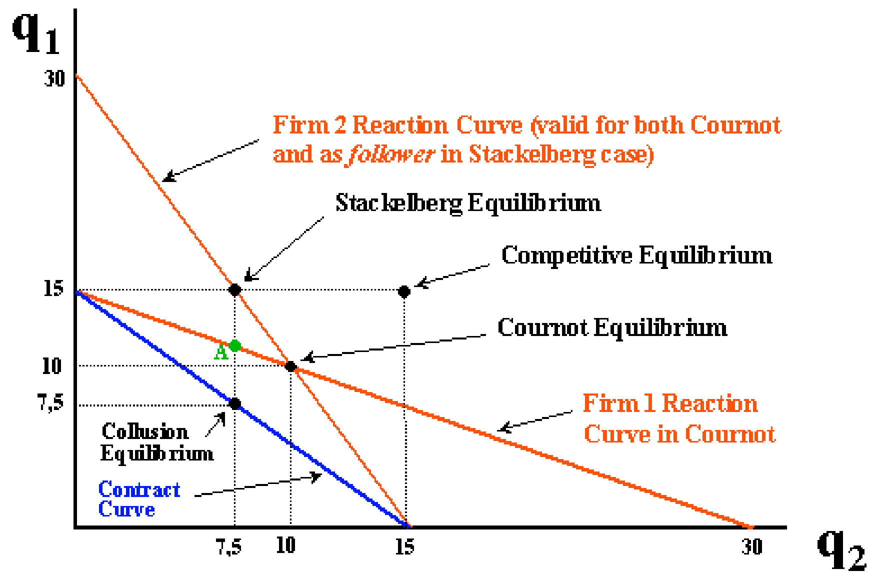
The most important concept in game theory is the ***Nash equilibrium***. This equilibrium occurs when the players (firms, competitors) have no incentive to deviate (to change its strategy) unilaterally. So each firm is doing the best given the other firms strategies. Another important concept is the ***perfect equilibrium*** *in subgames*, used in dynamic games, based in the principle of *sequential rationality*, and closely related to the *backward induction* approach. A profile of strategies is a subgame perfect Nash equilibrium if it induces Nash equilibrium in *every* subgame of this game.

In practice, game theory insights are more relevant with two or few players. Let us present the ***duopoly*** example in order to illustrate some additional classical equilibrium possibilities. Two firms share a certain geographical market of a specific product. Consider a (deterministic) linear ***inverse demand curve*** given by the equation below.

$$\mathbf{p = 30 - Q_T} \qquad \textbf{(1)}$$

Where p is the price of the product, Q_T is the industry production (= $q_1 + q_2$ for the duopoly case). For sake of simplicity, assume that the variable cost is zero – alternatively, consider p as the operational profit *margin*. Let us plot the ***reaction curves*** of the two firms. Reaction curves are the best response functions[6] for the players given the other firms strategies, so that the profit is maximized for each possible strategy of the other firm. Figure 1 shows these curves as well as the equilibrium possibilities for two firms in quantity competition.

[6] In the more general case, best response *correspondences*.



Equilibrium	Profit		Margin p
	Firma 1	Firma 2	
Collusion	112.5	112.5	15
Stackelberg	112.5	56.25	7.5*
Cournot	100	100	10
Competitive	0	0	0

*OBS: (1) Price after the follower entry. Before the follower entry, the leader margin is p = 30 - 15 = 15 = collusion one.
(2) Note that if the game continues, Stackelberg is not Nash equilibrium because firm 1 has incentive to deviate (see point A in green), etc., leading to Cournot equilibrium in long-run.

Figure 1 - Duopoly Equilibrium Possibilities and Quantity Reaction Curves

The red lines are the reaction curves (best response), so that the crossing point is the simultaneous best response for the two firms' game. It is Nash equilibrium because there is no incentive to deviate (no improvement in profit with unilateral change of strategy). This crossing point is also the classical ***Cournot Equilibrium***. The Cournot equilibrium (q_1 = 10 and q_2 = 10, in the figure), also called *quantity* competition equilibrium, is the most likely outcome because it is Nash equilibrium and due to the reasons given below. Hence, in Cournot-Nash equilibrium the firms choose quantity and an equilibrium price clears the market according the inverse demand curve.

The figure also shows the ***Stackelberg Equilibrium***. In this case there is a leader that enters first in the market fixing a maximizing monopolistic quantity (q_1 = 15 and q_2 = 0, in the example), and when firm 2 enters as follower, firm 2 observes the leader production and set its follower production according its reaction curve. Hence, both the production (q_1 = 15 and q_2 = 7.5) and the profit are higher for the leader than for the follower, the named *Stackelberg's* ***first-move advantage***. It is possible if we assume an immutable capacity commitment by the leader. However, if the game continues it is not Nash equilibrium because there is an incentive for the leader to reduce its production in order to maximize profit (see point A in the figure). After this, the follower will react in the same way, until reaching the Cournot-Nash equilibrium. Fudenberg & Tirole (1991, pp.74-76) point out that there is a problem of ***time consistency*** with the Stackelberg equilibrium, because the leader quantity in Stackelberg is not a best response for the follower production. In short, also in timing games with one firm entering before, the Cournot outcome is the most likely equilibrium.

The other possibility is the ***Collusion Equilibrium*** ($q_1 = q_2 = 7.5$). It maximizes the *joint* profit of the firms (the *contract curve* showed in blue) sharing the resulting profit. However, without a binding contract, it is not Nash equilibrium because there are incentives to deviate for both firms. In addition, collusion contracts are in general illegal or unethical. In the next section we will see a case that *non-binding* (or *tacit*) collusion can be Nash-equilibrium depending of the industry conditions.

The figure also shows the theoretical case of ***Perfect Competitive Equilibrium***. In this case both firms rise production until reach zero profit margin. For the two firms' case, it is not Nash equilibrium as the reader can easily check out.

Why not a *price* competition, the named ***Bertrand Equilibrium***? The classical paper by Kreps & Scheinkman (1983) makes the comparison of Bertrand x Cournot competition[7]: "Under mild assumptions about demand the unique equilibrium outcome is the Cournot outcome". In their paper with two-stage game, firms choose capacity (Cournot) in the first stage, and then follow a Bertrand competition in prices. The resulting outcome is the standard Cournot outcome!

Let us discuss a little bit the inverse demand curve function. A generic equation is given below.

$$\mathbf{P = Y(t) . D(Q_T)} \qquad \textbf{(2)}$$

Where P is the price, $D(Q_T)$ is the (inverse) demand function, QT is total industry output, and Y is a multiplicative stochastic demand shock, so that $Y(t = 0) = 1$ and $Y(t > 0)$ is uncertain given by a stochastic process. Some possibilities for a deterministic ($Y = 1$ forever) demand function are:

- Linear Demand: $\mathbf{P = a - b\, Q_T}$, with $a > 0$, $b > 0$, and $a > b\, Q_T$ **(3a)**
- Iso-Elastic Demand: $\mathbf{P = 1 / (Q_T + W)}$ **(3b)**

[7] See also Varian (1992, pp.301-302) for a discussion on this point.

- Exponential Demand: $\mathbf{P = a \, . \, exp[- \varepsilon \, Q_T]}$ **(3c)**

The *linear demand* function is the most simple and popular one. The problem is that when QT rises until $Q_T = a/b$, the price goes to zero (and is negative if Q_T is higher). So, this function can be approximately representative for one industry only for a certain range of Q_T.

The *iso-elastic demand* in its simpler format is $P = 1 / Q_T$. However, this simpler format for the iso-elastic demand has a drawback. The total revenue = P.Q is constant, so that in presence of variable operational costs the industry (monopoly or oligopoly in collusion) will find optimal to reduce the production to near zero (with prices going to infinite) in order to reduce the operational cost without decreasing the revenue! Of course this cause mathematical problems. We adopt the solution analyzed by Agliari & Puu (2002), using the *modified* iso-elastic demand curve $P = 1 / (Q + W)$, so that the hyperbola is translated to a new position intersecting the ordinate-axis. With this trick the maximum price is 1/W when production tends to zero.

The *exponential demand* function has the good properties of non-zero or negative prices and finite prices when Q_T tends to zero. In addition, the parameter ε can be interpreted as a kind of *elasticity of demand*. So, we think that this function is a good alternative for demand function.

Now, let us insert the stochastic multiplicative factor Y(t). We assume here that Y follows a geometric Brownian motion (GBM) given by:

$$\mathbf{dY / Y = \alpha \, dt + \sigma \, dz} \qquad \textbf{(4)}$$

Where α is the (real) drift[8], σ is the volatility, and dz is the Wiener increment[9]. Let us assume that the process starts with Y(t=0) = 1. Of course others stochastic processes are possible.

In order to ease the reader feeling on the stochastic demand, let us present a numerical example with the following linear demand function: $\mathbf{P = Y \, (500 - 25 \, Q_T)}$. Figure 2 shows the initial demand curve (Y = 1) and a future demand curve after stochastic shocks that increase the demand up to Y = 2.

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[8] For the *risk-neutral* GBM, just replaces α by the *risk-neutral drift* = $r - \delta$, where r is the risk-free discount rate and δ is the dividend yield. The risk-neutral drift is also equal to $\alpha - \pi$, where π is the risk-premium.

[9] A non-biased stochastic increment given by $dz = dt^{0.5} \, . \, N(0, 1)$, where N(0, 1) is drawn from the standard Normal distribution.

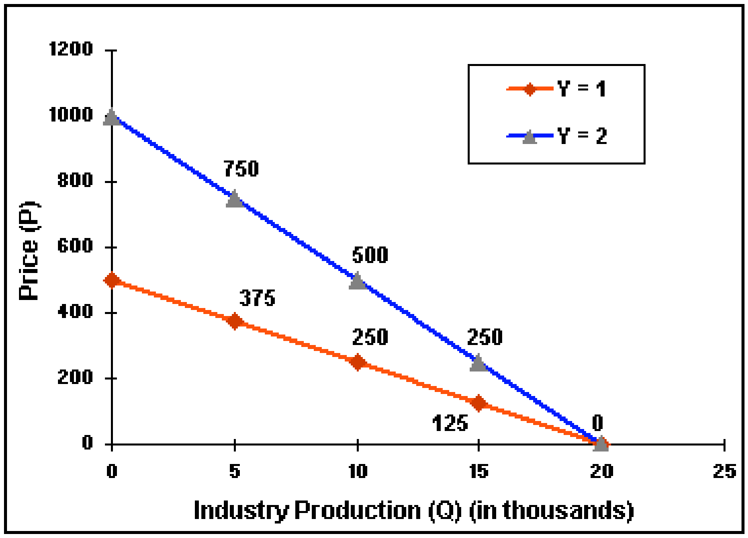

Figure 2 - Stochastic Linear Demand Curve

The figure shows that if the industry output is 10 thousands units (per annum) the product price is initially $250 per 1,000 units. If the stochastic variable doubles the value, the same happen with the price (rises to $500/1,000 units).

3 - Symmetrical Duopoly under Uncertainty

The Symmetrical Duopoly under Uncertainty Model was the first known option games model due to Smets (1993). In addition to the historical relevance, this is model has a great theoretical importance - foundations of stochastic timing-games of preemption in continuous-time - and also practical significance for duopoly when there is *no* competitive advantage for one firm. This section summarizes the in-depth analysis performed by Huisman & Kort (1999). They extended the previous literature (Smets, Dixit & Pindyck) in many ways, especially by allowing *mixed-strategies* equilibrium and analyzing the possibility of *non-binding collusion* equilibriums. In addition, they presented many propositions that establish the conditions for the occurrence of different equilibriums, in a rigorous approach. For sake of space we present only selected results[10]. The explanation here presents only some minor modifications on original notation. However, we include some issues not discussed in the paper, e.g., the *two ways to calculate the value of the follower and the leader* and some additional explanations not addressed in the paper.

This duopoly model is symmetrical in the sense that the players are homogeneous firms (equal firms), so that this suggests symmetrical strategies.

[10] For additional results, discussions, and charts, see the webpage at: http://www.puc-rio.br/marco.ind/duopoly2.html

The first difference of Huisman & Kort model - when compared with Smets or Dixit & Pindyck - is that the two firms are already active in the market producing one unit each, and they are considering the exercise of (a definitive) one *perpetual option to expand* the production. This means that the "entry" (read *expansion*) of a firm affects the current profit flow of the other firm – reducing the profit flow because this model consider ***negative externalities***. The investment to expand the production (exercise price of the real option) is the same for both firms and denoted by I.

The firms face a (inverse) demand curve expressed by the *profit flow* P(t) for firm i given by:

$$\mathbf{P = Y(t) . D(N_i, N_j)} \qquad \textbf{(5)}$$

Where Y(t) is the stochastic demand shock following a GBM and $D(N_i, N_j)$ is a deterministic demand parameter *for firm i*, which depends of the status of firms i and j. The possible values of D(Ni, Nj) are:

- D(0, 0) means that both firms not invested yet (but there is a profit flow Y D(0, 0) because the firms are already active in the market);
- D(1, 0) means that firm i invested and is the "leader" because the firm j has not invested yet;
- D(0, 1) means that firm i is the "follower" because only the other firm (j) has invested becoming the leader; and
- D(1, 1) means that both firms invested in the market (simultaneous investment).

These factors $D(N_i, N_j)$ are adjusted to the production level. The operational cost is zero or it is also included in D, so that P is interpreted as *profit flow* instead price. For example, the profit flow of the leader is Y.D(1, 0), and this profit flow is higher than Y.D(0, 0). That is, the higher production level when investing and assuming the leader role, is already included in the value D(1, 0). The change of status from D(0, 0) to D(1, 0) demands the sunk investment of I. In addition, due to the symmetry of the problem, when one firm is profiting Y D(1, 0), the other firm is profiting Y D(0, 1), etc. For the called "***new market model***" – the original case of Smets and in Dixit & Pindyck where firms are not active in the market in t = 0, we have **D(0, 0) = D(0, 1) = 0**.

Huisman & Kort assume that firms are risk-neutral so that the discounting is performed with risk-free interest rate. However, it is easy to extend the model to risk-averse firms in *contingent claims* approach by supposing that the stochastic process drift α is a risk-neutral drift.

The deterministic demand parameters have the additional constraint of *negative externality* (the option exercise of one firm reduces the value of the other firm) given by the inequality:

$$\mathbf{D(1, 0) > D(1, 1) > D(0, 0) > D(0, 1)} \qquad \textbf{(5a)}$$

Other model assumption is the *first mover advantage*, given by the inequality:

$$\mathbf{D(1, 0) - D(0, 0) > D(1, 1) - D(0, 1)} \qquad \textbf{(5b)}$$

This inequality says that the gain when becoming leader is higher than the gain when becoming follower (for the same Y, and considering the same investment I).

Before any options exercise (when Y is below the *leader threshold* Y_L), the value of each firm is the current cash-flow profit in *perpetuity*, $\mathbf{Y\ D(0, 0) / (r - \alpha)}$, with $r > \alpha$, *plus the option* to exercise the investment option (as leader or as follower, with 50% each - as we will see later) net of the competitive losses due to the negative externality from the rival entry.

As standard in *timing games*, the solution is performed *backwards*. This means that first we need to estimate the value of the follower (given that the leader entered before), and then the leader value given that the leader knows that the optimal follower entry can happen in the future. Here it is considered that any firm can become the leader (the roles are *not* exogenously assigned). Let us see two ways to estimate the follower value. First, by using the traditional contingent claims steps (Itô's Lemma, etc.) the value of the follower F is given by the ordinary differential equation (ODE) below.

$$\mathbf{0.5\,\sigma^2\,Y^2\,F_{YY} + \alpha\,Y\,F_Y - r\,F + Y\,D(0, 1) = 0}$$

Where the first three terms corresponds to the *homogeneous* part of the ODE, and the last term in the right side is the non-homogeneous term also called the "cash-flow" because it is the profit flow for the follower before the option exercise. Note that this cash-flow term doesn't exist in the *new market model* of Dixit & Pindyck. In Huisman & Kort the follower earns cash flow even before the option (to expand) is exercised. The ODE solution is a *homogeneous solution* of type $\mathbf{A\,Y^{\beta1} + B\,Y^{\beta2}}$, plus a *particular solution*, where $\beta_{1,2}$ are the roots from the equation $\mathbf{0.5\,\sigma^2\,\beta^2 + (\alpha - 0.5\,\sigma^2)\,\beta - r = 0}$. Economic condition makes the constant B in the *negative* root (β_2) term equal zero[11].

[11] Because economic logic tells that when demand Y tends to zero, the follower value needs to go to zero as well.

So the solution for the follower's ODE is given by:

$$\mathbf{F(Y) = A\,Y^{\beta 1} + Y\,D(0,1)/(r-\alpha)} \qquad \text{if} \quad \mathbf{Y \leq Y_F} \qquad \mathbf{(6a)}$$

$$\mathbf{F(Y) = Y\,D(1,1)/(r-\alpha) - I} \qquad \text{if} \quad \mathbf{Y \geq Y_F} \qquad \mathbf{(6b)}$$

Where Y_F is the follower's optimal investment threshold. We need to find out two unknowns, the constant A and the threshold Y_M. For this, we need to apply the boundary conditions *value matching* and the *smooth pasting*:

$$\mathbf{F(Y = Y_F) = Y_F\,D(1,1)/\ (r-\alpha) - I} \qquad \mathbf{(7a)}$$

$$\mathbf{F_Y(Y = Y_F) = D(1,1)\ /\ (r-\alpha)} \qquad \mathbf{(7b)}$$

With subscript Y in 7b denoting first derivative in relation to Y. By deriving the equation 6a at Y = Y_F and equalizing to equation 7b, and by equalizing eq.6a at Y = Y_F to equation 7a, we get two equations with two unknowns (A and Y_F). With some algebra we get the following values:

$$\mathbf{A} = \frac{Y_F^{1-\beta_1}}{\beta_1}\ \frac{D(1,1) - D(0,1)}{(r-\alpha)} \qquad \mathbf{(8a)}$$

$$Y_F = \frac{\beta_1}{\beta_1 - 1}\ \frac{(r-\alpha)\ I}{D(1,1) - D(0,1)} \qquad \mathbf{(8b)}$$

By substituting the constant A into equation 6a, we finally find the follower value.

Next, the second way to find both the follower value and the threshold, based in the concept of *first hitting time* and *expected discount factor*. Let T* be the first time that the stochastic variable hits a (superior) level Y* (here Y* = Y_F). The follower value here has two components. First the profit flow before exercising the option (remember that here both firms are active in the market even before the option exercise), from t = 0 until t = T* (or T_F). Second, the profit flow after the option exercise at Y_F, net of investment I, from t = T* = T_F until infinite. See the integrals below[12]:

[12] Note that the *follower value* assumes that the other firm entered as leader at t = 0. If instead we want the *value of the firm* planning to be follower, assuming that the other firm will enter as leader at $T_L > 0$, three integrals are necessary: one from 0 to T_L, with D(0, 0); other from T_F to T_L, with D(0, 1), and the last from T_F to infinite, with D(1, 1).

$$F(Y) = E\left[\int_0^{T^*} e^{-rt}\, Y(t)\, D(0,1)\, dt\right] + E\left[\int_{T^*}^{+\infty} e^{-rt}\, Y(t)\, D(1,1)\, dt\right] - I$$

Expected Profit Before Exercise — Expected Profit After Option Exercise

Note that for the *new market model* D(0, 1) = 0 and the first integral is zero (Dixit & Pindyck case). At t = T* the firm gets a profit flow in perpetuity, with present value at T* of $\mathbf{Y_F\, D(1, 1)/(r - \alpha)}$. By carrying that in present value, from T* to current instant (t = 0), we obtain:

$$F(Y) = D(0,1)\; E\left[\int_0^{T^*} e^{-rt}\, Y(t)\, dt\right] + E\left[e^{-r T^*}\right] \frac{Y_F\, D(1,1)}{r - \alpha} - I$$

The ***stochastic discount factor*** depends only on the stochastic process parameters and the discount rate r, and is given by the very simple equation below:

$$E\left[e^{-r T^*}\right] = \left(\frac{Y}{Y_F}\right)^{\beta_1} \qquad \textbf{(9)}$$

For the proof, see **Appendix A**. The value of the first expectation in the F(Y) equation is given by:

$$E\left[\int_0^{T^*} e^{-rt}\, Y(t)\, dt\right] = \frac{Y}{r - \alpha}\left[1 - \left(\frac{Y}{Y_F}\right)^{\beta_1 - 1}\right] \qquad \textbf{(10)}$$

For the proof of this expectation, see **Appendix B**[13]. By substituting the expectations results, eqs. 9 and 10, into the last equation of the follower value F(Y), the reader can verify that we find out the same result for the follower value encountered with the first method (eqs. 7 and 8a). For the more common case of *new market model*, this second method is easier because the equation of expected discount factor is easy to remember and the first integral is zero.

However, how to calculate the *threshold* value with the second method? Let us present a simple method using the *standard optimization* approach for the investment decision (for details of this method, see Dixit & Pindyck & S⌀dal, 1999).

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[13] Although these proofs are presented in Dixit & Pindyck (chapter 9, appendix), we show some intermediate steps not presented in that textbook.

Let the *net present value* from the option exercise be $\mathbf{NPV_F = V(Y) - I}$. The maximization of the project is a trade-off between waiting for a higher value of the project V(Y) and the discount factor (higher as earlier we exercise the option). If we wait for a too high value of V(Y), we can wait too long time and the discount factor can be too small. So, in this cost-benefit balance of waiting policy, there is one optimal value for Y that maximizes the present value of the expected payoff from the option exercise, namely the function G(Y) below:

$$G(Y) = \underset{Y}{\text{Max}}\ E\,[\,e^{-r\,t}]\,.\,[V(Y) - I] \tag{11}$$

The benefit from exercising the option here is $\mathbf{V(Y) = Y\,[D(1,1) - D(0,1)]\,/\,(r - \alpha)}$. Denote the expected discounting factor by $R(Y_0, Y_F)$, that is (for notational simplicity make $Y_0 = Y$):

$$\mathbf{R(Y, Y_F) = E[\exp(-r\,T^*)] = (Y/Y_F)^{\beta 1}} \tag{12}$$

So, the maximization problem becomes:

$$\mathbf{G(Y) = Max\ \ R(Y, Y_F)\,.\,(\{\,Y\,[D(1, 1) - D(0, 1)]\,/\,(r - \alpha)\} - I\,)} \tag{13}$$

The *first order condition* to maximize the above equation determines that we take the partial derivative of G in relation to the stochastic control variable Y, equaling it to zero. We obtain:

$$\mathbf{R(Y, Y_F).[D(1, 1) - D(0, 1)]/(r - \alpha) + R_{Y_F}(Y, Y_F).Y_F\,[D(1, 1) - D(0, 1)]/(r - \alpha) = R_{Y_F}(Y,Y_F)\,.\,I}$$

The value of derivative of the expected discount factor in relation to Y_F, is simply:

$$\mathbf{R_{Y_F}(Y, Y_F) = -\,\beta_1\,Y^{\beta 1}\,/\,[Y_F^{(\beta 1 + 1)}]} \tag{14}$$

Substituting this and eq.12 into the equation of the first order condition, and after *few* algebra steps, it is easy to conclude that the resulting threshold equation is the same of equation 8b, obtained with the first method. This second method is even better (simpler) for the case of the new market model.

For the leader value L(Y), we can also apply either of these two methods. For example, the leader value is given by the sum of integrals net of the investment cost I:

$$L(Y) = E\left[\int_0^{T^*} e^{-rt}\,Y(t)\,D(1,0)\,dt\right] + E\left[\int_{T^*}^{+\infty} e^{-rt}\,Y(t)\,D(1,1)\,dt\right] - I$$

Expected Profit in Monopoly Phase — Expected Profit in Duopoly Phase

That is, entering as leader the firm experiments a phase as monopolistic with profit flow Y D(1, 0), and when the follower enters (at $t = T^* = T_F$) the profit flow drops to Y D(1, 1). We can follow with the similar steps used for the follower case, in order to calculate the leader value. However, perhaps it is easier another method, namely the differential equation approach for the value of the leader *during the monopolistic phase*, denoted by **V(Y) = L(Y) + I**. This value V(Y) needs to match the value of simultaneous investment (follower value) at the boundary point $Y = Y_F$. The differential equation of V(Y) is given by:

$$0.5\,\sigma^2\,Y^2\,V_{YY} + \alpha\,Y\,V_Y - r\,V + Y\,D(1,0) = 0$$

The last term (non-homogeneous part) is the "cash flow", represented by the profit flow during this *monopolistic phase*. Again, the ODE solution is given by a general solution from the homogeneous part plus the particular solution related to the cash flow.

$$V(Y) = B\,Y^{\beta_1} + \frac{Y\,D(1,0)}{r - \alpha} \qquad \textbf{(15)}$$

The constant B is the parameter that remains to be calculated, requiring only one boundary condition for that. The biggest difference compared with the constant A (eq.8a) from the follower value, is that the constant B is negative, reflecting in the (expected) leader value, the losses due to the possible future follower investment exercise. This is mathematically showed below. The relevant boundary condition is the value-matching at the point that the follower entry (at Y_F). The smooth-pasting condition is not applicable here because this point is not an optimal control of the leader, it is derived of one optimization problem but from the *other* player. This boundary condition is:

$$V(Y_F) = Y_F\,D(1,1)\,/\,(r - \alpha)$$

The leader value during the monopolist phase is equal to the simultaneous investment value at Y_F. Equaling the two last equations, we get the value of the constant B in function of Y_F.

$$B = \frac{Y_F}{Y_F^{\beta_1}} \frac{[D(1,1) - D(1,0)]}{(r - \alpha)} \tag{16}$$

Note that the constant value is negative because D(1, 1) < D(1, 0). This means that the effect of the follower entering is to decrease the leader value, as expected by the economic intuition in this duopoly. The negative value of the constant means that the leader value function is concave.

The leader value in the monopoly phase V(Y) is obtained by substituting this constant (eq.16) into the leader equation (eq.15).

With V(Y), we can find out the *value of becoming a leader*, **L = V – I**. Hence, the value of becoming leader **if $Y < Y_F$** is given by:

$$L(Y) = \frac{Y\, D(1,0)}{r - \alpha} + \left(\frac{Y}{Y_F}\right)^{\beta_1} \frac{Y_F [D(1,1) - D(1,0)]}{r - \alpha} - I \tag{17}$$

The reader can verify that this value is the same that could be obtained by using the (second) method based on the value of the integral expectations. **If $Y \geq Y_F$**, the value of becoming leader is equal to the value of becoming follower, which is equal to the ***value of simultaneous investment*** **S(Y)**:

$$L(Y) = S(Y) = \frac{Y\, D(1,1)}{r - \alpha} - I \tag{18}$$

The value of simultaneous investment is also important because it is necessary to answer fundamental questions like: (a) what if I deviate from the follower waiting strategy by investing? and (b) I want to become a leader, but what if the rival has the same idea at the same time and both invest simultaneously? In other words, it is necessary to check if a follower strategy is a Nash-equilibrium and the expected value (or expected losses) if by "mistake" both invest at the same time (this will be important for the calculus of mixed strategy equilibriums).

Note that the simultaneous investment can be the optimal policy for both if the state of demand is so high that $Y \geq Y_F$, but a mistake in case of $Y < Y_F$.

The remaining issue is the ***leader threshold***. Without the preemption menace, the firm will invest optimally at the *monopolistic threshold* Y_M. However, due to the menace of preemption, firms cannot wait so long to invest. If one firm wait until $Y = Y_M$, the other firm can invest at YM - ε, but the first firm could preempt the rival by investing before when Y = YM - 2 ε, etc. This process stops when one firm has no more incentive to preempt the rival.

Firm 1 has incentive to become leader if $L_1 > F_1$ and, most important for firm 1 decision, firm 1 knows that firm 2 has also the incentive to become a leader if $L_2 > F_2$. So, the firm 1 strategy to become a leader is to invest when $L_2 = F_2$. However, due the symmetry of the problem, $L_1 = L_2$ and $F_1 = F_2$ for all Y, so that the leader threshold is defined as the Y in the interval $0 < Y < Y_F$ so that the values of the leader and the follower are equal, i.e.,:

$$\mathbf{Y_L := \{ 0 < Y < Y_F \mid L(Y) = F(Y) \}} \quad \textbf{(19)}$$

There is a proposition in Huisman & Kort showing that this leader threshold Y_L is *unique*. Due to the symmetry, this threshold is valid for both firms.

Figure 3 below shows the leader and follower values and the optimal entry as leader and as follower. The numerical inputs are the same as the Huisman & Kort paper's, that is, our ***base case*** has the inputs: α = 5%, σ = 20%, r = 10%, Y(t = 0) = 1, I = 20 (for each firm), and the deterministic stochastic factors are D(0, 1) = 1, D(0, 0) = 2, D(1, 1) = 2.5, D(1, 0) = 4.

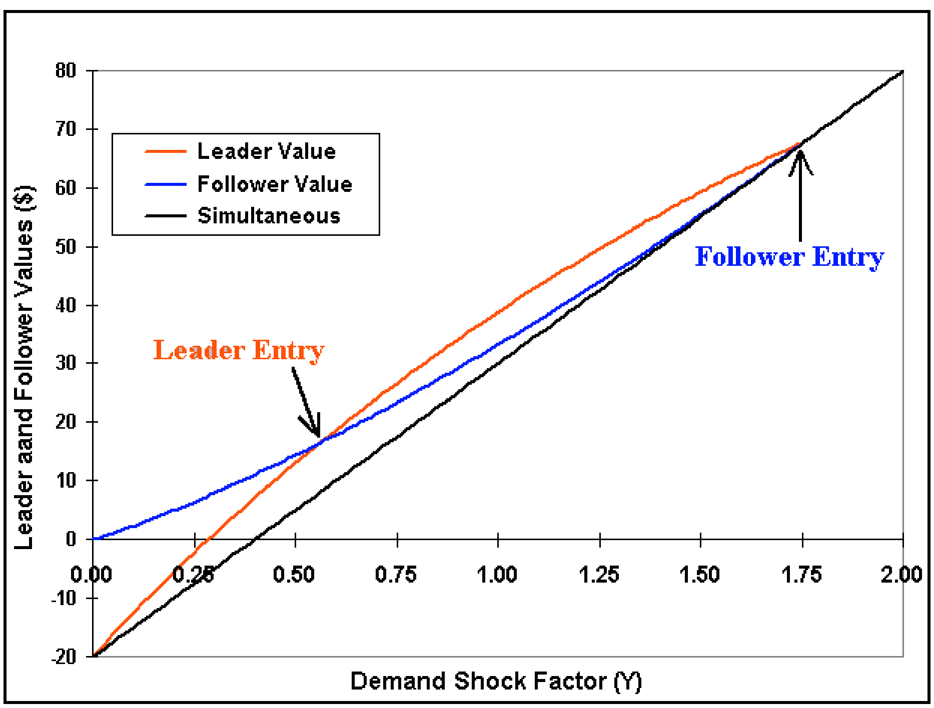

Figure 3 - Leader and Follower Values and Entry Thresholds

Another interesting extension of Huisman & Kort is to consider the possibility of collusion equilibriums *without a binding contract* between the firms (without communication). They analyze if there are situations that a *tacit* wait and see policy is equilibrium. Firms could calculate if it is the best strategy or not the wait and see until one level $Y = Y_C$, when both invest simultaneously - or one firm invest and the other one invest immediately after that.

Collusion will only be Nash equilibrium if there is *no unilateral incentive to deviate*. Deviation means to earn the leader payoff with the other firm choosing optimally to invest much later as the follower. Denoting the ***collusion value*** of each firm by $\mathbf{C(Y, Y_C)}$, collusion will be Perfect-Nash equilibrium **only if** $\mathbf{C(Y, Y_C) \geq L(Y)}$ at least for all Y in the **interval** $\mathbf{(0, Y_F)}$. If this occurs, there are *infinite* collusion Nash-equilibriums possibilities. From these equilibriums the Pareto optimal one is to invest at the collusion threshold Y_C given by:

$$\mathbf{Y_C = \frac{\beta_1}{\beta_1 - 1} \frac{(r - \alpha)\ I}{D(1, 1) - D(0, 0)}} \quad \mathbf{(20)}$$

Figure 4 below presents the leader value, the follower value, and the *collusion value* as functions of the stochastic demand shock Y. The used parameters were the same of the previous chart. Note that the collusion value is always higher than the leader value, so that the collusion equilibrium is feasible (firms in collusion have no incentive to cheating). For the base-case inputs, the collusion value curve smooth-pasting on the simultaneous value line at $Y_C = 5.29$, a very high value that is out of the chart.

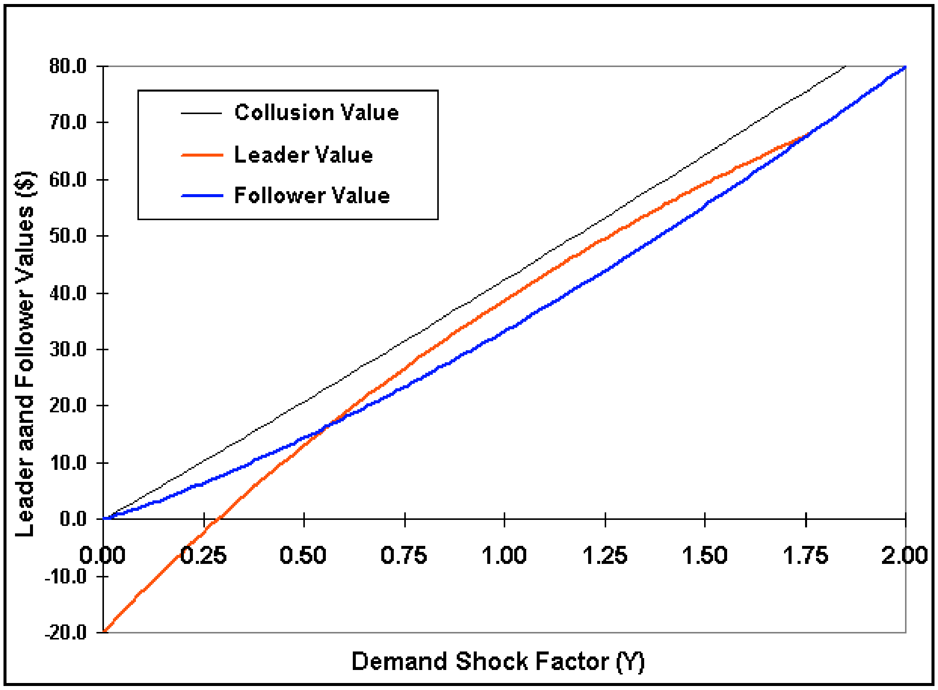

Figure 4 - Non-Binding Collusion Strategy as Nash-Equilibrium

If the *first-mover advantage* is *sufficiently large*, the collusion can be destroyed by the preemption. Figure 5 illustrate this point, when we rise this first mover advantage by setting D(1, 0) = 5.

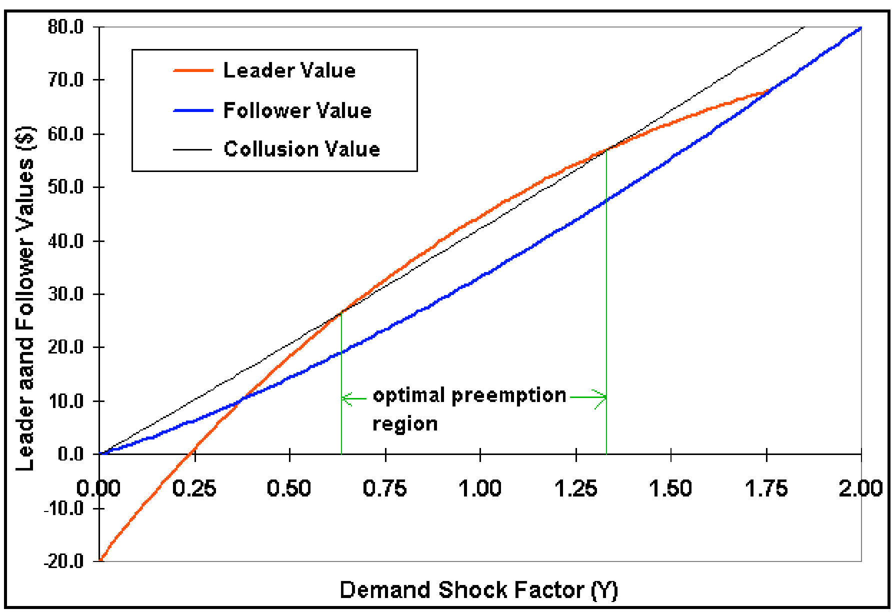

Figure 5 - Collusion Strategy Destroyed by Preemption

In the figure above there is a region where preemption is optimal due to the higher value for the leader role when comparing with the collusion one. So, in this case the firms in collusion have incentive to cheating and collusion here is not Perfect-Nash equilibrium.

Now, imagine that the initial state of the demand is between Y_L and Y_F. Both firms have incentive to become leader because L > F (see the previous figures to clarify this point). There is no logic to imagine that, without any communication, the other firm will let the rival to become leader so that the probability of simultaneous investment is zero. Both firms will wish the higher leader payoff, but both firms fear the possibility of becoming worse in case of simultaneous investment (a "mistake"). The payoff from simultaneous exercise is lower even than the follower payoff. Assuming non-communication between the players, the only rational way to treat this problem in the game theory context is by allowing ***mixed strategies***[14].

With mixed strategies, firms will rationally calculate the optimal probability to exercise the investment option aiming the leader payoff, but considering the positive probability of simultaneous investment. Firms will play a *simultaneous game* (with possibly infinite rounds) where the firm i can choose invest with probability $\mathbf{p_i}$ and not invest with probability $\mathbf{1 - p_i}$; players i = 1 or 2.

The analysis of mixed strategies in continuous-time preemption games, must be performed carefully and using special tools. The passage from discrete-time to continuous-time presents problems when using traditional limit considerations. Fudenberg & Tirole (1985) reported that the usual methods present "loss of information" in this passage, with the continuous case not representing the limit of the discrete case. In addition, with traditional tools many strategies converge with probability 1 to be played at the instant t = 0, a non-consistent result.

In order to determine the symmetrical mixed strategies, Huisman & Kort used the same tool applied in Fudenberg & Tirole (1985): they specify "probabilities" named of "*atoms*" $\mathbf{p(\tau)}$ that, if positive, indicate *cumulative probability of exercise* $G_i(\tau)$ equal to 1. So, τ is defined as the first time that some player will exercise the option to invest given that nobody exercise the option before. This kind of resource is taken from *optimal control literature* (e.g., see Birge & Louveaux, 1997, p.289).

The fundamental idea is that this ***control doesn't take time***. Using this analogy, the control here is the result of a simultaneous game that can be repeated. This is like an instantaneous automatic optimizer. In this way, a simultaneous game with two players *even if repeated infinite times*, is played

[14] When communication is allowed - as suggested in Joaquin & Buttler (2000) - there exists the alternative of a *bargain-game* (sharing the surplus L – F) with agreement for only one firm to become leader avoiding the simultaneous investment "mistake". However, in many situations this kind of agreement is illegal or contrary to accepted convention.

instantaneously (no time consuming). This approach determines the probabilities from the mixed strategies and can be proved that this is the true limit of the equivalent game in discrete-time.

The simultaneous game, which can be repeated infinite times, at the instant that one (or both) player will exercise the option (that is, at τ), is showed in strategic form below, together with the option exercise probabilities.

		Firm 2: $p_2(\tau)$	Firm 2: $1 - p_2(\tau)$
Firm 1	$p_1(\tau)$	S(Y(t)) , S(Y(t))	L(Y(t)) , F(Y(t))
Firm 1	$1 - p_1(\tau)$	F(Y(t)) , L(Y(t))	Repeat the game

Figure 6 - Simultaneous Game at τ

The firm 1 payoff, V_1 (not yet optimized) is given by:

$$\mathbf{V_1 = p_1\, p_2\, S + p_1\, (1 - p_2)\, L + (1 - p_1)\, p_2\, F + (1 - p_1)\, (1 - p_2)\, V_1} \qquad \textbf{(21)}$$

The last term means that, in case of repetition we get the payoff V_1 because due to the definition of τ when this game is played (at τ) is certain that some player (or both) will exercise the option to invest. Alternatively, an equivalent (perhaps more intuitive) way to obtain V1 is given below:

$$\mathbf{V_1 = [\, p_1\, p_2\, S + p_1\, (1 - p_2)\, L + (1 - p_1)\, p_2\, F\,]\, .\, [1 + (1 - p_1)\, (1 - p_2) + (1 - p_1)^2\, (1 - p_2)^2 + ...\,]}$$

The term between the first term in brackets is the expected payoff of one round in case of definition (investment of one or two players). The second term in brackets multiplies that expected payoff and consider the case of definition in the first round (multiplying by one), in the second round [multiplying by $(1 - p_1)$ $(1 - p_2)$], and etc., until infinite. The second term in brackets is just the infinite sum of a convergent geometric progression, so that it is easily calculated. Hence, the equation for the (non-optimized) payoff from the simultaneous game for the firm i, V_i, i = 1 or 2, is:

$$\mathbf{V_i = \frac{p_i\, p_j\, S + p_i\, (1 - p_j)\, L + (1 - p_i)\, p_j\, F}{1 - [(1 - p_i)\, (1 - p_j)]}} \qquad \textbf{(22)}$$

Next, players need to set the *optimal* probability to exercise the option, that is, the probability that maximizes the expected payoff V_i. The *first order condition* for this optimization problem for the firm 1 is $\partial V_1 / \partial p_1 = 0$, given that the rival is planning to exercise the option with probability p_2. The second order condition indicates a maximization problem. Note also that, due to the symmetry of the problem, the optimal probabilities must be equal. That is, $\mathbf{p_1 = p_2 = p^*}$. This permits a further simplification. Using the first order condition and the symmetry insight, after some algebra we get the following simple equation for the optimal probability of investing in mixed strategies:

$$p^* = \frac{L - F}{L - S} \qquad \textbf{(23)}$$

Figure 7 gives a *geometric interpretation* for the mixed strategy optimal probability of exercise p*.

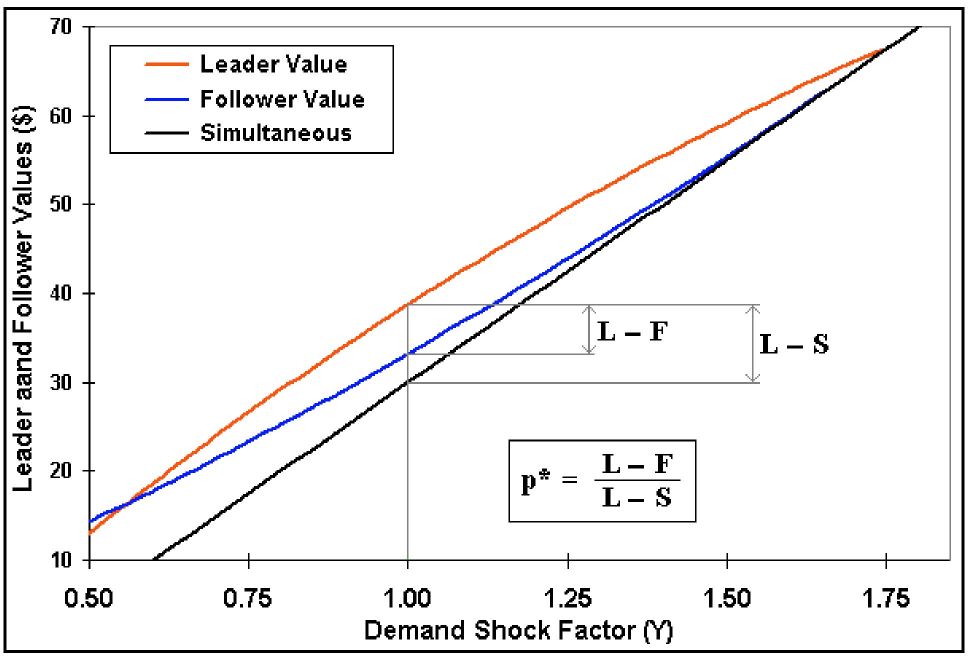

Figure 7 - Geometric Interpretation of Mixed Strategy Exercise Probability

In the figure above, it is easy to see that when L = F and L > S, we have p* = 0. When L tends to both F and S, we have p* = 1. In the mixed strategy equilibrium, the probability p* can be interpreted as equal the ratio of the possible benefit from preemption (L – F) to the range of total possible variation (L – S). Note that L – S = [(L – F) + (F – S)].

Let us calculate the *probability of only one of the firms to exercise* the option and *the probability of simultaneous exercise* in this game. For that, look again the Figure 6 with the strategic-form of the simultaneous game and, using a similar reasoning from the payoff, we can write that the probability **pr(one = i)** that only the firm i investing in one round from the simultaneous-game at τ is given by:

$$\mathbf{pr(one = i) \;=\; p(\tau)\,(1 - p(\tau)) \;+\; (1 - p(\tau))\,.\,(1 - p(\tau))\,.\,pr(one = i)}$$

So, the probability of pr(one = i) is given by:

$$\mathbf{pr(one = i) = \frac{1 - p(\tau)}{2 - p(\tau)}} \qquad \mathbf{(24)}$$

Of course, the probability pr(one = j) of only the firm j investing in this simultaneous-game at τ is exactly the same due to the symmetry. Now, the *probability of simultaneous exercise* (or probability of "mistake", if $Y < Y_F$), denoted by **pr(two)**, using a similar reasoning is given by:

$$\mathbf{pr(two) \;=\; \frac{p(\tau)}{2 - p(\tau)}} \qquad \mathbf{(25)}$$

Note that the three probabilities sum one, that is, **p(one = i) + pr(one = j) + pr(two) = 1**. This is because there is no possibility of non-exercise in this simultaneous-game due to the definition of τ. We can use the previous optimal value that we found for the probability $p(\tau)$ (eq.23) to estimate the probabilities of each firm investing and the probability of simultaneous exercise. Example, at Y_L we know that L = F and L > S. So, $p(\tau) = 0$. In this case pr(two) = 0 and pr(one = i) = pr(one = j) = 1/2. Note also that the pr(two) is consistent in the limits when Y tends to Y_F or Y_L, respectively 1 and 0.

Therefore, when the market births with $Y < Y_L$ there are 50% chances each to became leader and zero probability of "mistake" when Y reach Y_L. This conclusion from Dixit & Pindyck with trivial mixed strategies outcome is correct only in this case.

4 - <u>Asymmetrical Duopoly under Uncertainty</u>

The asymmetrical duopoly under uncertainty model is a more realistic hypothesis in most industries. Here firms are non-homogenous because, for the same investment, one firm has *lower operational cost* than the other. This means that one firm has ***competitive advantage*** over the rival.

Here we summarize and extend the known model of Joaquin & Buttler (2000). Following them, we assume a *linear inverse demand* function, with quantities determined by a Cournot competition. Both firms are based in the same country and both are considering the investment in the same foreign country. The *demand function is deterministic* and fixed with time. However, the ***exchange rate X(t) is uncertain*** and evolves as a stochastic process modeled as a GBM. Mathematically it is equivalent to consider a multiplicative stochastic demand shock Y(t). The deterministic linear demand function is:

$$\mathbf{P = a - b\,Q_T}, \text{ with } a > 0,\ b > 0, \text{ and } a > b\,Q_T \qquad \mathbf{(26)}$$

Where $P(Q_T)$ is the price of the product *in foreign currency*, which is function of the total output in this market Q_T. For the price in *domestic currency*, just multiply by the exchange rate X(t).

In case of investment option exercise, there exists a *variable operational cost* $\mathbf{c_i}$ for firm i where i can be "**l**" or "**h**", for **low-cost** and **high-cost** firms, respectively. The competitive advantage of low-cost firm is expressed by $c_l < c_h$. The function *profit flow* $\pi_i(Q_i)$ for the firm i *in foreign currency* is:

$$\boldsymbol{\pi_i(Q_i) = Q_i\,[\,a - b\,Q_T - c_i\,]} \qquad \mathbf{(27)}$$

Using contingent claims, and with *the dividend yield* $\delta > \mathbf{0}$ being interpreted as a *foreign currency yield*, the *present value* of a *perpetual* profit flow is given by dividing the equation 27 by δ.

The optimal monopolistic profit flow and the equilibrium profit flows from the Cournot duopoly are:

$$\pi_{M_i} = \frac{(a - c_i)^2}{4\,b} \qquad \mathbf{(28)}$$

$$\pi_l = \frac{(a - 2c_l + c_h)^2}{9b} \quad \Big| \quad \pi_h = \frac{(a - 2c_h + c_l)^2}{9b} \qquad \mathbf{(29)}$$

We'll put directly the results[15], which can be obtained with either of the two methods presented before, being here a "new market model" (which ease the calculus). The ***follower value*** for the high-cost firm $F_h(X)$, exercising the option as follower at the threshold X^*_{Fh} is given by the equation below (for the less probable low-cost firm as follower, just switch "l" and "h").

[15] For the intermediate steps and additional discussions, see: http://www.puc-rio.br/marco.ind/duopoly3.html

$$F_h(X) = \begin{cases} \left[\dfrac{(a - 2c_h + c_l)^2}{9b} \dfrac{X^*_{Fh}}{\delta} - I \right] \left(\dfrac{X}{X^*_{Fh}} \right)^{\beta_1} & \text{if } X < X^*_{Fh} \\ \dfrac{(a - 2c_h + c_l)^2}{9b} \dfrac{X}{\delta} - I & \text{if } X \geq X^*_{Fh} \end{cases} \tag{30}$$

Where β_1 is the positive (> 1) root of the quadratic equation $0.5\ \sigma^2\ \beta^2 + (r - \delta - 0.5\ \sigma^2)\ \beta - r = 0$. The format of the follower value equation when the exchange rate is below the threshold has natural interpretation. The first term, between brackets, is the NPV from the option exercise at X^*_{Fh}. The multiplicative second term is the expected value of the stochastic discounted factor (recall eq.9), from a random time of the follower exercise T^*_{Fh}. The second line in the equation 30 is also the ***value of simultaneous exercise*** for any X (for $X < X^*_{Fh}$, it is the value of a *mistake*).

The ***threshold*** *for the high-cost firm as* ***follower*** is:

$$X^*_{Fh} = \frac{\beta_1}{\beta_1 - 1} \frac{9b\ \delta\ I}{(a - 2c_h + c_l)^2} \tag{31}$$

The ***leader value*** for the low-cost firm $\mathbf{L_l}$ (again, for the less probable high-cost firm as leader, just switch "l" and "h"), **if** $\mathbf{X < X^*_{Fh}}$, is given by the equation below.

$$L_l = \frac{(a - c_l)^2}{4b} \frac{X}{\delta} + \left[\frac{(a - 2c_l + c_h)^2}{9b} - \frac{(a - c_l)^2}{4b} \right] \frac{X^*_{Fh}}{\delta} \left(\frac{X}{X^*_{Fh}} \right)^{\beta_1} - I \tag{32}$$

The format of this equation also permits an intuitive explanation. The first term of the right side is the monopoly profit in perpetuity of the leader firm. The middle term is the expected present value of the competitive losses (the value between brackets is negative), which will occur at the follower entry (decreasing the monopoly profit in perpetuity). The last term is the investment required to become leader. If $\mathbf{X \geq X^*_{Fh}}$ the leader value is equal to the value of *simultaneous* investment.

In order to get the leader value, we found the *same* constant A_i used in Joaquin & Buttler (eq.16.6e) but in a more heuristic format that permits a <u>quick extension to other demand curves</u>, showed below:

$$A_l = \left[\frac{(a - 2c_l + c_h)^2}{9b} - \frac{(a - c_l)^2}{4b} \right] \frac{(X^*_{Fh})^{1 - \beta_1}}{\delta} \tag{33}$$

We can identify between brackets the difference of profit flows format: the profit flow from duopoly less the profit flow from monopoly. In this way it is easy to see that this constant is negative: we know that the profit flow from the duopoly phase is lower than the profit flow from the monopolistic phase. The negative value of the constant means that the leader value function is concave, meaning that the effect of the follower entering is to decrease the leader value, as expected by the intuition.

Figure 8 below shows the leader, follower, and simultaneous investment values for both low-cost and high-cost firms, with the same numerical inputs from the example used by Joaquin & Buttler.

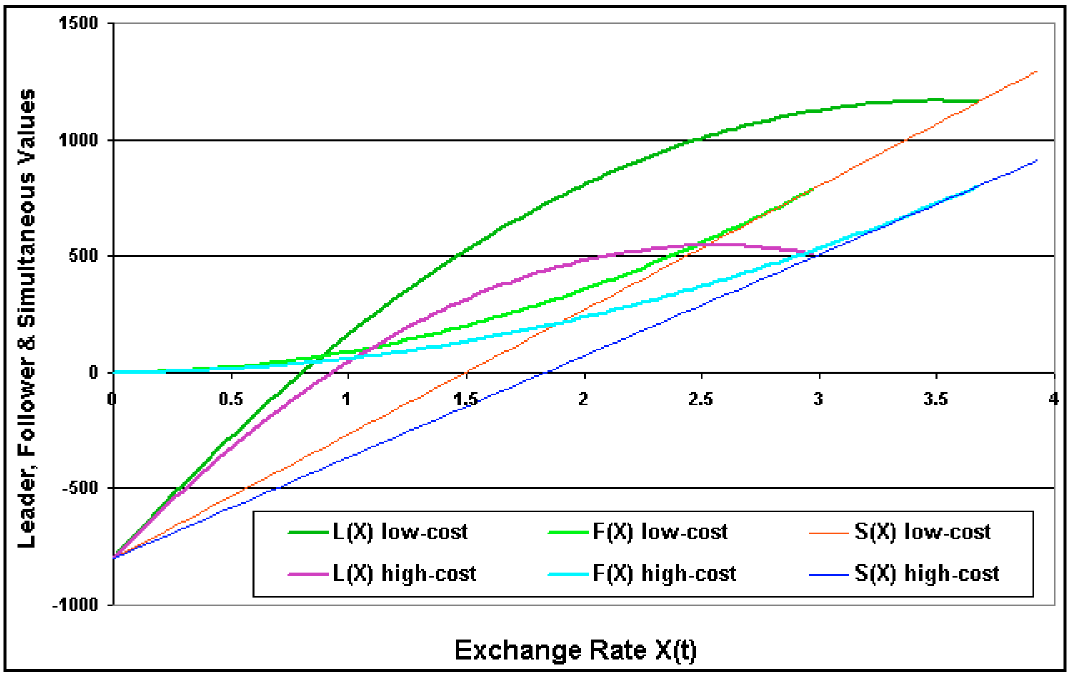

Figure 8 - Leader, Follower, and Simultaneous Values for Low-Cost and High-Cost Firms

How to estimate the leader threshold? This issue needs some additional considerations when compared with the symmetric case of section 3. Without the preemption menace, the low-cost firm will invest optimally at the monopoly threshold X_{Ml}. However, due to the menace of preemption, firms cannot wait so far to invest. If one firm wait until $X = X_M$, *in some cases* the other firm can invest at X_M - ε, etc. This process stops when one firm has no more incentive to preempt the rival. Firm l has incentive to become leader if $L_l > F_l$ but it is not necessary to invest at this point because the low-cost firm knows that firm high-cost has incentive to become leader only if $L_h \geq F_h$. So, **firm low-cost strategy to become leader is to invest when $L_h = F_h$** (or an infinitesimal value before).

Figure 9 shows this issue. It is a zoom from previous one showing only leader and follower values.

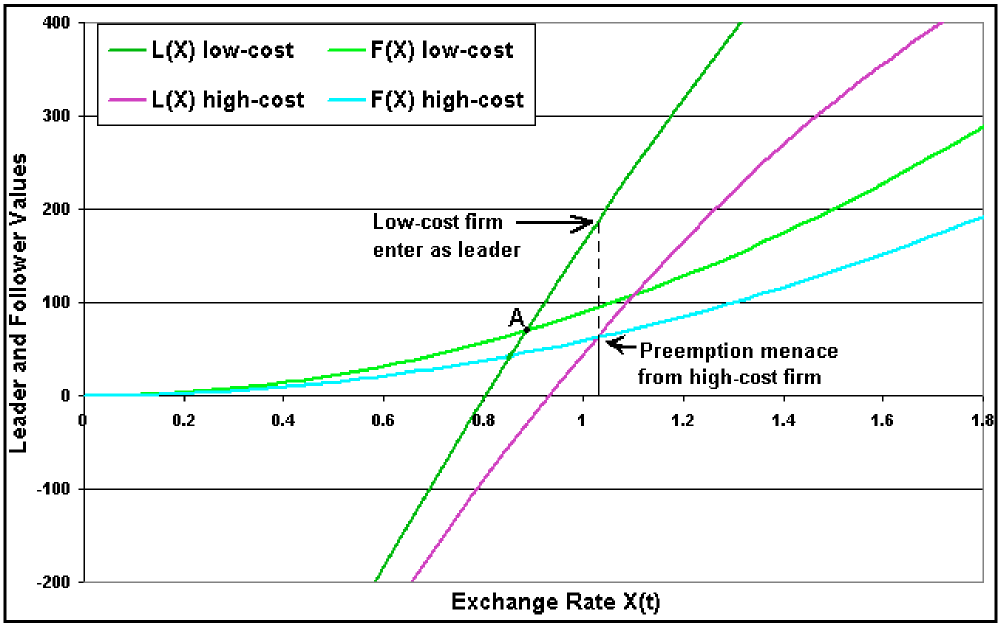

Figure 9 - The Leader Threshold Determined by the Preemption Menace

In this case X_{Ll} is the value of the stochastic exchange rate where $\mathbf{L_h(X_{Ll}) = F_h(X_{Ll})}$ and $\mathbf{X_{Ll} < X_{Fh}}$. In other cases, depending mainly on the difference between the operational costs c_l and c_h, the competitive advantage could be higher disappearing the menace of preemption before the optimal monopolistic exercise at $X = X_{Ml}$. In this case the <u>competitive advantage is so high that the low-cost firm ignores the competition by investing at the monopolistic threshold</u> X_{Ml} - this is an *open-loop strategy*. As example, by rising c_h (from $21 used in Joaquin & Buttler to $23.5) the values of leader and follower for high-cost firm shows that this firm will never want to be the leader. See Figure 10.

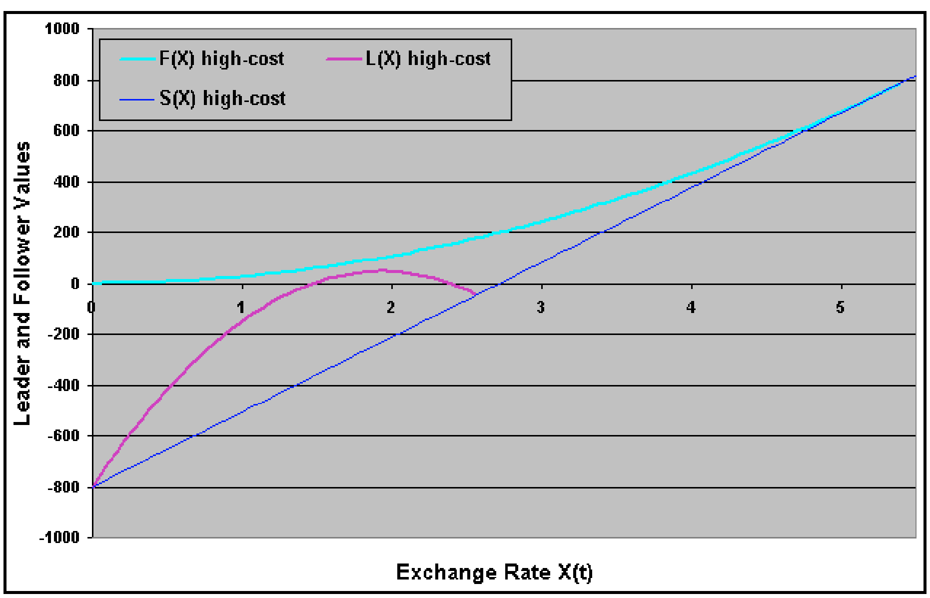

Figure 10 - There Is No Danger of Preemption by High-Cost Firm $F_h > L_h \ \forall \ X$

In the above figure the follower curve (light blue) is always above the leader value (violet) for the high-cost firm. In this case it is always better for the high-cost firm to be the follower, waiting until the exchange rate reach the level X_{Fh} = 5.44 in this case. For the rival low-cost firm, this means that there is no menace of preemption so that the low-cost firm can ignore the competition by investing at the monopolistic threshold X_{Ml}. So, the **leader threshold is the minimum between its monopolistic threshold and the other firm minimum level with incentive to become a leader**. This is essentially the "Result 3" from the Joaquin & Buttler paper.

For "new market model" like this case, the *collusion is never Nash-equilibrium* (Huisman & Kort, 1999). With asymmetric firms, this conclusion is strengthened.

Now, imagine that the initial state of the exchange rate is so favorable that ***both firms have incentive to become leader*** because exists an exchange rate region where **L > F** for both firms. However, both firms will be *worse in case of simultaneous exercise* of the option to invest because the value of simultaneous investment is even lower than the follower value, namely **S < F** for both firms. It is obvious that, *without communication* between the firms, ***there is a positive probability of "mistake"*** - simultaneous investment getting a value lower than the follower's one. Figure 11 shows this region.

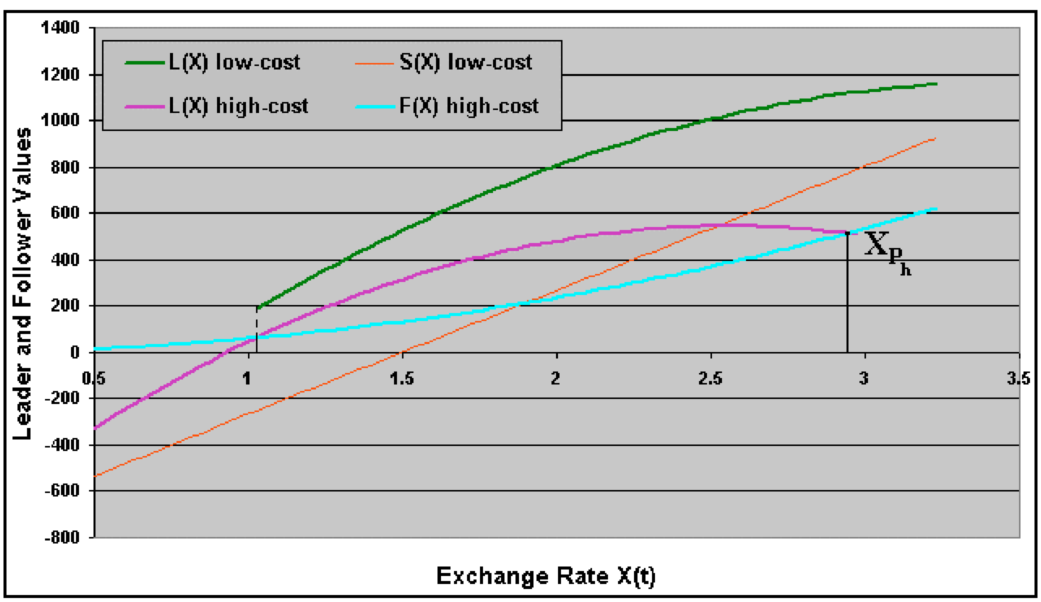

Figure 11 - Preemption Region for the High-Cost Firm: From 1.03 to ~ 2.94

Note that if the high-cost firm has incentives to become leader - the case showed in the figure, the same happen with the low-cost firm. However, we know that the vice-versa is not necessarily valid. Let us denote the region above as ***preemption region for the high-cost firm***. The existence of this region depends on the parameters, specially the difference between the costs, that is, the competitive advantage value. In the base case of Joaquin & Buttler this region exists. This means that the low-cost firm in this region suffers the risk of obtaining the simultaneous exercise value (red line in the above chart) instead the "logical" or "natural" leader value when investing (green curve in the chart).

This risk must be considered when analyzing this game. The high-cost firm threat is credible, because if the high-cost firm enters first, the best for the low-cost firm is to resign with the follower role, unless $X \geq X_{Fh}$. In reality, the less probable strategic outcome, that is, *high-cost firm as leader* and the *low-cost firm as follower* is also a Perfect Nash equilibrium. We need to analyze the ***mixed strategy*** probabilities in order to evaluate the probability or risk of "mistake" and the probability of the less probable high-cost firm emerging as the leader.

Define $\mathbf{X_{Ph}}$ as the ***preemption upper bound*** from the *preemption region for the high-cost firm*, showed in Figure 11 above as approximately equal to 2.94. This upper bound will be used soon in the *mixed strategy theorem*. The lower bound of the region - about 1.03 is the leader threshold (equal for both firms). As in Huisman & Kort model, the desire to become leader (or the probability to invest) is *proportional to the difference between the L and F* values and decreases if the difference between F and S increases. This means that, for this preemption region with positive probability of simultaneous investment, the probability of option exercise for the low-cost firm is strictly higher than the probability of option exercise for the high-cost firm. Even lower, this probability is positive in this region. We need to calculate this risk!

In order to determine the mixed strategies probabilities, we use the Theorem 8.1 partially from the Huisman's textbook (2001, p.204)[16]. He analyzes the asymmetric duopoly case (chapter 8), but it is a little bit different from our case because the asymmetry there occurs with different investments, whereas here the investments are the same and the asymmetry results from the operational costs.

Mixed Strategy Theorem

(a) Consider the scenario with parameters so that exists a non-empty *preemption region for the high-cost firm*. Define the following *probabilities of exercise* $\mathbf{p_l(X)}$ and $\mathbf{p_h(X)}$ for the low-cost and high-cost firms respectively:

[16] In this adaptation of the theorem, a book's typo has been corrected – the equations 8.18 and 8.19 are inverted in the book (at least in the first printing edition).

$$p_l = \frac{L_l - F_l}{L_l - S_l} \quad \Bigg| \quad p_h = \frac{L_h - F_h}{L_h - S_h} \tag{34}$$

Readers of the last section on symmetrical duopoly model will recognize this format and know how to find out results like that using the concept of "atoms" and a maximization process. Now consider the following possible cases of <u>initial value</u> for the stochastic exchange rate X(t = 0):

(a.1) If the current exchange rate X(0) belongs to the *preemption region for the high-cost firm*, then with probability

$$\text{pr(low-cost)} = \frac{p_l\ (1 - p_h)}{p_l + p_h - p_l p_h} \tag{35}$$

low-cost firm invests immediately and high-cost firm invests when X reaches X_{Fh}. With probability

$$\text{pr(high-cost)} = \frac{p_h (1 - p_l)}{p_l + p_h - p_l p_h} \tag{36}$$

high-cost firm invests immediately and low-cost firm invests when X reach X_{Fl}. And, with probability

$$\text{pr(both firms)} = \frac{p_l\, p_h}{p_l + p_h - p_l p_h} \tag{37}$$

both high-cost firm and low-cost firm invest immediately, the named ***probability of mistake***.

(a.2) If the current exchange rate X(0) is lower than X_{Lh} (that is, at the left of the *preemption region for the high-cost firm*) and lower than X_{Ml}, then with probability one low-cost firm invests when X(t) reach the value X_{Ll} = minimum(X_{Lh} , X_{Ml}) and high-cost firm invests when X(t) reach X_{Fh}.

(a.3) If the current exchange rate X(0) is lower than X_{Lh} and higher than X_{Ml}, then with probability one low-cost firm invests immediately and high-cost firm invests when X(t) reach X_{Fh}.

(a.4) If the current exchange rate X(0) is higher than X_{Ph} (that is, at the right of the *preemption region for the high-cost firm*) and lower than X_{Fh}, then with probability one low-cost firm invests immediately and high-cost firm invests when X reach X_{Fh}.

(a.5) If the current exchange rate X(0) is higher or equal than X_{Fh} - that is, a region where is optimal for both firms to invest, then with probability one both firms invest immediately.

(b) If the *preemption region for the high-cost firm* is an empty set (there is no X that L > S for high-cost firm), then:

Low-cost firm will invest with probability one at X_{Ml} as leader (alone) and high-cost firm will invest with probability one only at X_{Fh}, as follower.

Proof: By using the concept of "atoms" we can follow the same steps presented at section 3 for the simultaneous game at τ (when one or two firms invest with probability 1), proving (a.1) with a maximization process to get the equations 34, 35, 36 and 37, so that there are no incentives to deviate. Items (a.2), (a.3) and (a.4) follow because the low-cost firm is better off exercising the option than waiting if $X(t) > X_{Ll}$, and it has no incentive to deviate from the option exercise strategy when $X(t) = X_{Ll}$, whereas for the high-cost firm waiting and see is better than the option exercise for $X(t) < X_{Fh}$. Item (a.5) follows because the option exercise - even simultaneously, is better than wait and see for both firms. Finally, item (b) results from the fact that in this case never is optimal for high-cost to enter as leader, so that there is no menace of preemption and low-cost firm maximizes its profit by entering only at the monopolistic threshold value X_{Ml}.

Figure 12 shows the most probable or main Perfect-Nash equilibrium with low-cost firm entering as leader and high-cost firm as follower.

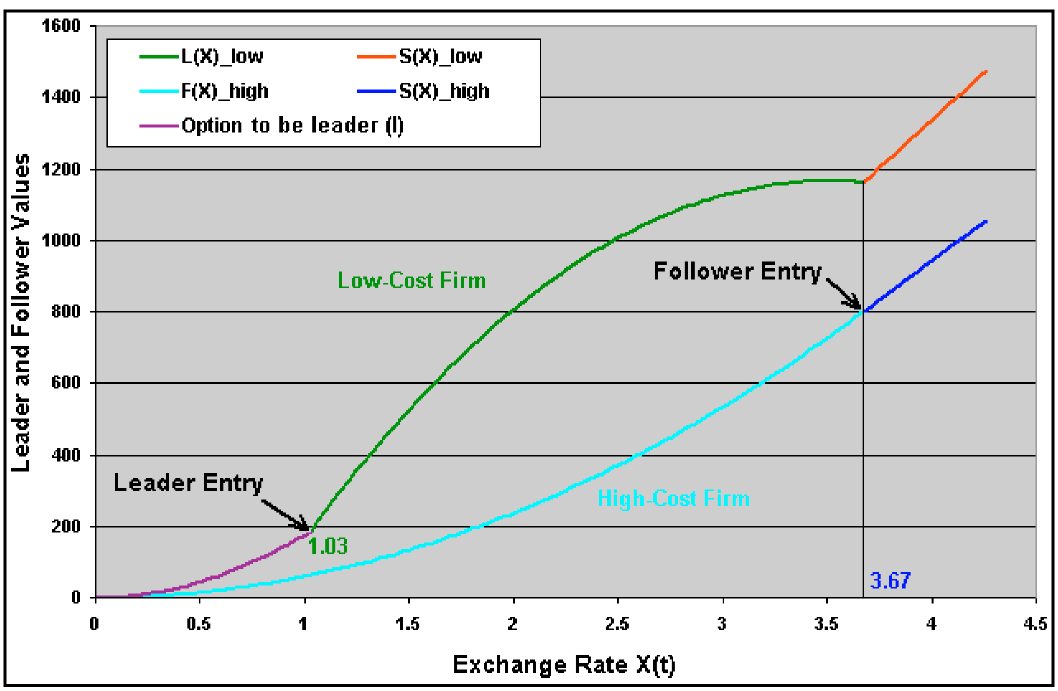

Figure 12 - The Main Perfect Nash Equilibrium in Asymmetric Duopoly

In the figure above appears a new value function: the ***option to become leader***. This value function is calculated with the leader threshold X_{Ll}, by using the *expected value of stochastic discount factor* from the random time to X(t) reaches X_{Ll}, times the leader value function for the low-cost firm.

Figure 13 below shows the less probable or secondary Perfect Nash equilibrium, with the high-cost firm being the leader and the low-cost firm being the follower.

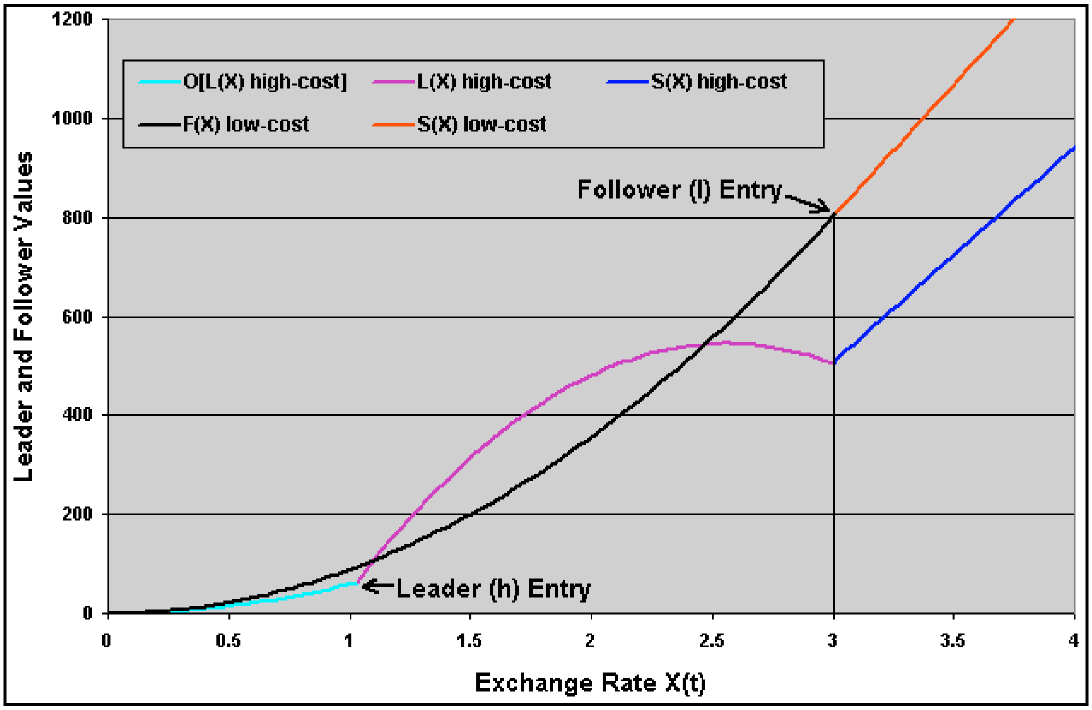

Figure 13 - The Secondary Perfect-Nash Equilibrium in Pure Strategies

Recall that if the game starts with the exchange rate $X < X_L$, the occurrence probability for this equilibrium is zero. However this probability can be strictly positive – although lower than the previous equilibrium probability, if X starts at the called *preemption region for the high-cost firm*.

Recall that, depending on the parameters, there exists the possibility of a simultaneous investment situation when the optimal for both firms is only one firm active in the market. If this simultaneous exercise happens - a mistake with positive probability if the initial exchange rate is in the preemption region for the high-cost firm, the deviation is not feasible because the investment is irreversible.

5 - Conclusions and Part 2 Issues

In this paper we presented a short bibliographical development of option games and the basic related concepts. We discuss the models of symmetric and asymmetric duopoly under uncertainty, being the former of great conceptual relevance and the latter of great practical appeal. Concepts like preemption, non-binding collusion, and mixed strategies are discussed for both cases. We see that in many situations exists a strictly positive probability of "mistake" - simultaneous option exercise when the best for both firms is only one firm exercising the option. This positive probability of mistake can occur even for *asymmetric* duopoly under uncertainty - contrary to popular belief.

Perhaps the major contribution of this paper is to formalize two equivalent ways to calculate the leader value, the follower value and the follower threshold for option exercise. In certain problems one approach can be preferable than the other - creating an *option* to solve option games!

In a next paper (Part 2), we will present the model of oligopoly under uncertainty from Grenadier (2002) with discussion of concepts like the Leahy's "*optimality of myopic behavior*", the change in demand function in order to solve oligopoly as an *artificial perfect competitive market*. In addition, we will review models of positive externalities including war of attrition models with focus in the case discussed in Dias (1997). Part 2 will be completed with comments on others option games models such as asymmetric information models, evolutionary option-games models, bargain option games, etc. We will see that the *option premium* for option games models can be *negative* (Huisman & Kort, 1999), positive but *near zero* when the number of competitors grows (Grenadier, 2002), or even *higher* than the standard real option premium in war of attrition models (Dias, 1997).

APPENDIXES

A) Proof for the First Expectation

In this appendix we prove the expectation $E[\exp(- r\, T^*)] = (Y/Y_F)^{\beta 1}$ by following the appendix in the chapter 9 of Dixit & Pindyck textbook, but including some intermediate steps not showed in that book. To make more general the notation, let the threshold $Y_F = Y^*$.

Suppose an expected discount factor in continuous time, with a risk-free discount rate r:

$$f(Y) = E[\exp(- r\, T^*)]$$

Denoting the first hitting time as T* (first time that Y is equal or larger than Y*), here representing when the option to invest will be optimally exercised, the expected discounted payoff from T* to current date is exactly the current value of the option to invest. Assuming that the current $Y < Y^*$ and by choosing an interval dt sufficiently small that hitting the threshold Y* in the next short time interval dt is an unlike event, the problem restarts from a new level (Y + dY). Therefore we have the dynamic programming-like recursion expression:

$$f(Y) = \exp(-r\,dt)\, E[\, f(Y + dY) \mid Y] = \exp(-r\,dt)\, \{ f(Y) + E[\, df(Y)\,] \}$$

By noting that:

(a) Y follows a geometric Brownian motion with drift α and volatility σ; and
(b) Using the Itô's Lemma for expanding df(Y), and using the subscripts to denote derivatives:

$$df = f_Y(\alpha\, Y\, dt + \sigma\, Y\, dz) + 0.5\, f_{YY}(\sigma^2\, Y\, dt) = f_Y\, \alpha\, Y\, dt + f_Y\, \sigma\, Y\, dz + 0.5\, f_{YY}\, \sigma^2\, Y\, dt$$

Note that we are supposing the infinite time horizon (perpetual option) case so that the variable time is not included in the Itô's Lemma. By substituting df into the previous equation and by noting that E[dz] = 0, and letting $\exp(-r\,dt) \cong 1 - r\,dt$ for a very small dt, we get:

$$f(Y) = (1 - r\,dt)\, \{ f + f_Y\, \alpha\, Y\, dt + 0.5\, f_{YY}\, \sigma^2\, Y\, dt \}$$

With a few algebra (remember dt^2 is zero) the reader can find out the following differential equation:

$$0.5\, \sigma^2\, Y\, f_{YY} + \alpha\, Y\, f_Y - r\, f = 0$$

The general solution of this ODE is:

$$f(Y) = A_1\, Y^{\beta 1} + A_2\, Y^{\beta 2}$$

Where β_1 and β_2 are respectively the positive and the negative roots of the standard quadratic characteristic equation from the differential equation (see an instructive discussion of the characteristic equation in chapter 5, section 2.A, of Dixit & Pindyck's book).

Applying two boundary conditions: as Y approximates to the threshold Y*, T* is probable to be small and the discount factor f(Y) close to 1, so f(Y*) = 1. When Y is close to zero, T* is likely to be large and so the discounted factor close to zero, therefore f(0) = 0 (note: alternatively, is possible to see Y = 0 as an absorbing barrier, so when Y tends to zero, T* tends to infinite or there is no finite time for Y to reach the threshold from Y = 0).

With these results, we can see that $A_1 = (1/Y^*)^{\beta 1}$ and $A_2 = 0$. Therefore the solution for the expected discount factor is:

$$\mathbf{f(Y) = E[exp(- r\, T^*)] = (Y/Y^*)^{\beta 1}}$$

Where Y is the initial value of the stochastic variable (that is at t = 0). The proof is complete.

B) Proof for the Second Expectation

We want to prove the following result of the expectation below (defined by the function g(Y)), being T* the first time that the stochastic process (GBM) of Y reaches the threshold level Y*.

$$g(Y) \;=\; E\left[\int_0^{T^*} Y\, e^{-rt}\, dt\right] = \frac{Y}{r-\alpha}\left\{1 - \left(\frac{Y}{Y^*}\right)^{\beta_1 - 1}\right\}$$

In the expectation of the integral, the main difference - when comparing with the first expectation - is that in the interval between 0 and dt there exists a dividend like profit π given by the value of the integral in this time interval:

$$\pi(Y) \;=\; \int_0^{dt} Y\, e^{-rs}\, ds$$

Calculating the integral, we get:

$$\pi(Y) \;=\; Y\left[\frac{e^{-r\,dt}}{-r} - \frac{e^{-r\,0}}{-r}\right] = \; Y\left[\frac{e^{-r\,dt} - 1}{-r}\right] \cong Y\left[\frac{(1 - r\,dt) - 1}{-r}\right] \;=\; Y\,dt$$

Assuming that currently Y < Y* and by choosing an interval dt sufficiently small that hitting the threshold Y* in the next short time interval dt is an unlike event, the problem restarts from a new level (Y + dY). Therefore we have the dynamic programming-like recursion expression - this time including the profit or dividend-like term π(Y):

$$\mathbf{g(Y) \;=\; \pi(Y) + e^{-r\,dt}\, E\, [\, g(Y + dY) \mid Y]}$$

The term π(Y) was calculated above and the term $e^{-r\,dt}$ E [g(Y + dY) | Y] is calculated using the Itô's Lemma in the same way as calculated in the first expectation (see Appendix A above). Hence,

$$\mathbf{g(Y) \;=\; Y\,dt + (1 - r\,dt)\,\{\, g + g_Y\, \alpha\, Y\, dt + 0.5\, g_{YY}\, \sigma^2\, Y^2\, dt \,\}}$$

With a few algebra it is easy to get an ordinary differential equation (ODE) similar to that showed in the chapter 9 appendix from Dixit & Pindyck book:

$$0.5\,\sigma^2\,Y^2\,g_{YY} + \alpha\,Y\,g_Y - r\,g + Y = 0$$

With the subscripts denoting derivatives. This ODE has a homogeneous part (equal to ODE from Appendix A) and a non-homogeneous part (the last term of the left side in the equation above). The solution of this ODE is the sum of the homogeneous general solution with the particular solution:

$$g(Y) = B_1\,Y^{\beta 1} + B_2\,Y^{\beta 2} + Y \,/\, (r - \alpha)$$

As before, β_1 and β_2 are respectively the positive and the negative roots of the standard quadratic characteristic equation. The value of the constants are calculated by using the boundary conditions for Y = 0 and for Y = Y*. Look the definition of g(Y) for the following reasoning: (a) For g(0) = 0 is necessary that $B_2 = 0$; (b) For g(Y*) = 0 (because T* = 0) we get $B_1 = -(Y^*)^{1-\beta 1} / (r - \alpha)$.

Substituting these constants into the ODE solution, we complete the proof:

$$g(Y) = E\left[\int_0^{T^*} Y\,e^{-rt}\,dt\right] = \frac{Y}{r-\alpha}\left\{1 - \left(\frac{Y}{Y^*}\right)^{\beta_1 - 1}\right\}$$

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