

Optimal Investment Management of Harbour Infrastructures. A Real Options Viewpoint *

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Abstract

Most real problems need to be modeled using multiple state variables, combine multiple Real Options (very often american-style ones) and have complex cash flow functions. In this paper we present a new Scenarios-Monte Carlo method to approach this kind of high-dimensional Real Options problems. The method is based on scenarios spaces built at each exercise date so that the payoff function can be modified at each scenario depending on the optionality. Then scenarios are related in order to calculate the expected value of continuation. The main contribution of the algorithm is that it allows us to price american-style real options while solving decision problems of optimal investment policy.

Keywords: High-Dimensional Real Options, Monte Carlo Simulation, Harbour Infrastructures.

1 Introduction

Literature concerning valuation of projects involving real options mostly focuses on problems where the cash flow function depends on few underlyings or can even be directly modeled through a single stochastic process. Then the valuation model is either a partial differential equation or a binomial or multinomial tree.

We could mention many references, from already classical Dixit and Pindyck's (1994) or Trigeorgis' (1996) books, till papers as Hubalek, F., Schachermayer, W. (1999) or Schwartz, E.S., Zozaya-Gorostiza, C. (2000), without pretending to be comprehensive, but just to illustrate a variety of points of view inside Real Options field.

However most real problems need to be modeled using multiple state variables, combine multiple Real Options (very often american-style ones) and have complex cash flow functions. So we are dealing with what we have called **High-Dimensional Real Options**. For them, the valuation models mentioned above suffer the so called *curse of dimensionality*, and standard Monte Carlo methods do not allow neither to calculate the expected value of continuation nor to modify the payoff function.

In the field of financial derivatives, pricing high-dimensional american options remains an opened problem deeply studied. Until recently the valuation of american options was widely considered out of the scope of Monte Carlo method. But these opinions soon vanished in front of evidence.

Nowadays, Tilley's (1993) bundling algorithm (which first show the potential of simulation applied to american options on a single underlying asset), Barraquand and Martineau's (1995) payoff stratification algorithm and Brodie and Glasserman's (1997a) and (1997b) simulated tree or stochastic mesh algorithms could be considered as classical works.

Recently, Fu, Laprise, Madan, Su, and Wu (2000) review, compare and generalise these and other simulation approaches. Just for mentioning one of the latest contributions, we quote Longstaff and Schwartz's (2001) Least Square Monte Carlo algorithm.

Following the main guidelines developed by these authors, but focusing on high-dimensional real options, we develop a **New Scenarios-Monte Carlo Algorithm**. It is based on scenarios spaces built at each exercise date so that the payoff function can be modified at each scenario depending on the optionality. Then scenarios are related in order to calculate the expected value of continuation.

The main contribution of the algorithm is that it allows us to price american-style real options while solving decision problems of optimal investment policy. As we will see later on, this versatility adds practically no additional computational cost due to the way intermediate information is stored.

The paper has been organised as follows: in section 2 we begin with a description of the scenarios-Monte Carlo algorithm, whose comprehensive mathematical formulation is developed in Appendix A. The real investment valuation problem

we have considered for applying the algorithm is described in section 3. Section 4 contains the application itself. First, in 4.1, we introduce the so called stochastic Net Present Value, needed for solving the valuation problem in 4.2. We finish the section with a subsection 4.3 devoted to software implementation and computational times. Finally, in section 5 we use the information stored when applying the algorithm in 4.2, to solve the decision problem involved. After describing in 5.1 how to state the problem, we present the numerical results of the resolution in 5.2. As usual, we include a last section 6 of conclusions.

2 Scenarios-Monte Carlo Algorithm

The algorithm has two main stages, each one included in one subsection. Subsection 2.1 consists of the generation of the scenarios spaces. Such spaces describe quantitatively situations the project manager would face, each one with its corresponding probability. Each scenario plays a similar role as a node of a valuation tree does. For a comprehensive mathematical formulation see A.1 and A.2. Subsection 2.2 gives us the relation between consecutive scenarios spaces, which will allow us to calculate expected continuation values. Again, for a comprehensive mathematical formulation see A.3. Finally, subsection 2.3 summarises the steps of the scenarios-Monte Carlo algorithm.

2.1 Generation of the Scenarios Spaces

For generating the scenarios spaces, let us consider m state variables denoted by $N_i, i = 1 \dots m$. Each variable has its own dynamic modeled through a stochastic process (the processes can be either independent or correlated). The parameters used in these processes have been adjusted to accomplish risk-neutral valuation.

At a given date τ , we get M samples of each state variable N_i , using Monte Carlo simulation. We build the histogram associated to these M samples, which gives us a probability distribution of the state variable N_i at date τ . So we are considering m different histograms, one for each variable.

From them, we build m -dimensional arrays by making the Cartesian product of the m sets of representative values of the classes of each histogram. The probability of each array is the product of the probabilities of its components. So we get a m -dimensional scenarios space at date τ , which we will denote ϵ_τ , as well as a complete probability distribution on it, denoted by \mathcal{Q}_τ .

2.2 Relation between Consecutive Scenarios Spaces. Expected Continuation Value

In order to be able to calculate expected continuation values, we need to relate consecutive scenarios spaces. Let us consider at date τ a scenario e_τ belonging to the scenarios space ϵ_τ . Taking this scenario as initial value, we get M samples S_i at $\tau + 1$ of m -dimensional arrays associated to e_τ . For each S_i , we find the

scenario $e_{\tau+1}(i)$ belonging to $\epsilon_{\tau+1}$ which is closest to S_i using the Euclidean norm.

We consider then that $e_{\tau+1}(i)$ is related to e_τ . Figure 1 illustrates this procedure.

So e_τ is related M times with scenarios at $\tau + 1$. The transition probability from e_τ to one of these $\tau + 1$ -scenarios is given by the frequency the $\tau + 1$ -scenario appears. Figure 2 outlines this procedure.

The expected continuation value CV at e_τ is then calculated as usual, using the values of the payoff function (denoted by Pf) at each related scenario at $\tau + 1$, and the corresponding risk-neutral transition probabilities p_i , $i = 1, \dots, l$:

$$CV(e_\tau) = (p_1 \times Pf(e_{\tau+1}^1) + \dots + p_l \times Pf(e_{\tau+1}^l)) \times (1 + r)^{-1} \quad (1)$$

where r denotes the riskless discount rate.

2.3 Valuation Algorithm Steps

Let us consider a valuation problem involving m state variables and n possible exercise dates denoted by τ , $\tau = 0, \dots, n$ ($\tau = 0$ represents the actual date). We will denote by Ef the function which gives as the value of the investment if we decide to invest. Depending on the optionality of the problem at each date τ and at each scenario e_τ belonging to the scenarios space ϵ_τ , Ef can be as simple as in the case of an american put option on a single non-dividend stock, or much more complex as in the problem modeled in 4.2.

On the other hand, we denote by Pf , as in 2.2, the payoff function of the valuation problem. When defining Pf , the function Ef will be involved, building

complicated payoff functions, as will also see later on. As usual in valuation trees, the algorithm works backwards. The steps are the following:

- We generate the scenarios space at maturity $\tau = n$ as described in 2.1.
 - At each scenario e_n , we calculate Ef considering the optionality involved.
 - At each scenario e_n , we calculate $Pf(e_n)$ (when solving the real problem in section 4, Pf will be the maximum between Ef and 0).
- We generate the scenarios space at date τ , $0 \leq \tau < n$, as described in 2.1.
 - At each scenario e_τ , we calculate Ef considering the optionality involved, and the expected continuation value $CV(e_\tau)$ as described in 2.2.
 - At each scenario e_τ , we calculate $Pf(e_\tau)$ (when solving the real problem in section 4, Pf will be the maximum between Ef and $CV(e_\tau)$).
- The final value is given by $Pf(e_0)$ (let us note that the scenarios space at date $\tau = 0$ has a unique scenario e_0).

This algorithm has been tested for an american put option on a single non-dividend stock with the following characteristics: one month to expiry, stock price $S = 40$, strike price $K = 45$, riskless discount rate 5%, volatility 30% and exercise dates τ , $\tau = 0, \dots, 20$. The algorithm uses then the well known

functions $Ef(e_\tau) = K - S_{e_\tau}$ and $Pf(e_\tau) = \max(Ef(e_\tau), 0)$ at maturity or $Pf(e_\tau) = \max(Ef(e_\tau), CV(e_\tau))$ at an intermediate date. The results show that the price given by the algorithm differs from a binomial tree in 0.01 units, and from the Barone-Adesi and Whaley Approximation in 0.02 units.

3 Real Problem Description

In this section we describe the investment project whose valuation illustrates the use of the scenarios-Monte Carlo algorithm proposed. First we introduce the state variables involved in the statement of the valuation problem by studying the goods flow. Next we develop the real options that must be taken into account in order to properly value the whole investment project.

3.1 Goods Flow: State Variables

We consider a harbour that already operates and has an approximate traffic volume of four million tons of about thirty different goods, mainly from iron and steel industry. The three operative docks that support this traffic will be called North dock, Central dock and South dock. It also operates a jetty, specialised in unloading mineral.

Due to the continuous increase in services demand during the last years, harbour agents have developed an extension project that involves an estimate investment of 102 million euros. It consists of three stages. Each one means construct-

ing and bringing into operation a new dock. We will refer to these three new infrastructures as New North dock, West dock and New South dock. New North dock is already being constructed, so the scope of our valuation problem are West and New South docks.

As the arrows show in Figure 3, traffic of goods is distributed among the already operative structures and the new ones. When old Central dock saturates, the extra goods flow over its capacity is sent to West dock. In such redistribution, ten state variables are involved (whether the jetty remains operative or not, is a decision strictly political, so we will not consider its traffic in our model).

New South dock would be specialised in containers. If operative, it will manage the extra containers from Central dock, otherwise they will be sent to West dock. New South dock will also manage the extra containers traffic from neighbour harbour of Valencia. This surplus will never be sent to West dock. For modelling these traffics we need three state variables.

So the cash flow function of the valuation model involves 13 state variables and, of course, cannot be described by a single standard stochastic process. Each of these variables will be modeled through their own stochastic process, whose parameters have been adjusted using data time series and managers' prospects.

3.2 Real Options: Control Variables

As usually happens with huge investment projects that are developed during a quite long period of time, the extension project described before involves certain optionality. That means that the project manager has the possibility of deciding how to achieve his goals choosing among different ways. Such ways often concern investment dates but are not the unique ones as we will see later on. It is very important to properly modelize such optionality as, the more flexible the project is, the more valuable it becomes.

On the other hand, once the project is valuated, it remains opened the question of which is the best sequence of decisions to be taken. We will see how the information provided by the scenarios Monte Carlo method when solving the valuation, can be used to answer such project manager's question.

We will modelize the optionality through the so-called control variables. Each one of the different values these variables take, represents each one of the possible decisions. Let us consider what the options included in the project are, as well as their corresponding control variables:

- West dock has six possible investment dates (measured in years). Its control variable will be denoted by τ_j^W , $j = 0, \dots, 5$, where $j = 0$ represents the actual time. On the other hand, West dock could also not be built. This fact will be denoted by τ_{-1}^W .
- New South dock has also six possible investment dates (measured in years).

Its control variable will be denoted by τ_j^{NS} , $j = 0, \dots, 5$, where $j = 0$ represents the actual time. On the other hand, New South dock could also not be built. This fact will be denoted by τ_{-1}^{NS} .

- New South dock could have a mixed financing public-private, i. e. the project manager can decide the percentage of public (and so of private) financing. Its control variable will be denoted by F , where F represents the four possible percentages considered of public financing. Then we have that $F \in \{100\%, 75\%, 50\%, 25\%\}$.

Each combination of values of these control variables means, on one hand, a certain possible decision to be taken by the project manager. On the other hand, determines the structure of the Ef function that will be used in section 4. We will get all combinations of control variables which must be considered, using two different methods depending on whether we are solving the valuation model, or the decision model. Both methods generate exactly the same combinations, but in a different order as we will see in section 5. This fact will be essential to solve the decision problem using the information stored during the valuation process.

For the valuation model, at each exercise date τ we have to introduce a set C_τ of combinations of control variables that are admissible at the date considered. For example, the optionality does not include the possibility of selling an investment that already has been done before date τ . Such combinations will not be taken into account as we will see in 4.2. For the decision model, we make the standard

Cartesian product of the set control variables values and get the grid of 174 points used in 5. In both cases we exclude the possibility of no investment at all, i. e. τ_{-1}^W and τ_{-1}^{NS} simultaneously.

To denote any of these combinations we will use what we call a control array.

The standard control array is then:

$$(\tau_j^W, \tau_k^{NS}, F) \quad j, k = 0, \dots, 5; \quad F \in \{100\%, 75\%, 50\%, 25\%\}. \quad (2)$$

Let us note that if τ_{-1}^{NS} is a component of the control array then F will not appear as the percentages modelized by F are referred only to New South dock.

4 Valuation of the Investment Project as an American Style Real Option

In this section we apply the scenarios-Monte Carlo method to the real valuation problem described. We will follow the steps summarised in 2.3. In order to do so, first we have to build in subsection 4.1 the function Ef included in the payoff function. We will see in 4.2 how the number of control variables combinations introduced in 3.2 increases only the complexity of the evaluation of the Ef function, while the structure of the valuation algorithm itself remains the same.

4.1 Stochastic Net Present Value

Let us consider an investment project with a generic number m of state variables, a date τ , a scenario e_τ belonging to the scenarios space built at τ and the set of admissible combinations of the control variables C_τ introduced in 3.2. For a given scenario e_τ , which determine the initial conditions for the m state variables involved, and for each control array $(\tau_j^W, \tau_k^{NS}, F) \in C_\tau$ we can calculate a Net Present Value (NPV) associated.

Let us note that, as the state variables are stochastic processes, there is not a single value for each one of these NPV but a wide enough samples of them, obtained using Monte Carlo simulation. As usual, we are able to build from each of these samples, its corresponding discrete probability distributions and then get its associated expected value. Each one of these expected values will be denoted by:

$$E \left[NPV \left(e_\tau, \tau_j^W, \tau_k^{NS}, F \right) \right] \quad (3)$$

We define the function Ef introduced en 2.3 as follows:

$$Ef(e_\tau) = \max_{(\tau_j^W, \tau_k^{NS}, F) \in C_\tau} \left\{ E \left[NPV \left(e_\tau, \tau_j^W, \tau_k^{NS}, F \right) \right] \right\} \quad (4)$$

Once we have defined the Ef function, the payoff function Pf is then evaluated following the steps in 2.3.

4.2 Real Problem Valuation

Let us consider the whole real investment problem described in section 3.1 including all the real options modeled in section 3.2. We will follow the steps of the scenarios-Monte Carlo algorithm of section 2.3 and use the notation introduced in all previous sections to approach the valuation.

Let us remember that we are working with a total of 13 state variables and 178 arrays of combinations of control variables. For calculating the expected NPV as explained in 4.1 we will consider that the life of the project is $T = 25$ years (see A.1). The valuation will be made as an American style option with six possible exercise dates denoted by τ , $\tau = 0, 1, \dots, 5$ (where $\tau = 0$ denotes the actual date). Next we describe the steps of the valuation.

- We generate the scenarios space at maturity $\tau = 5$
 - At each scenario e_5 , we calculate the following expected values considering all the admissible optionality at $\tau = 5$ (i.e. the elements of C_5):
 1. West dock will never be built and New South dock will be built at $\tau = 5$ with four different percentages of investment:

$$\left\{ E \left[NPV \left(e_5, \tau_{-1}^W, \tau_5^{NS}, F \right) \right]; F \in A \right\} \quad (5)$$

2. West dock will be built at date $\tau = 5$ and New South dock will

never be built:

$$E \left[NPV \left(e_5, \tau_5^W, \tau_{-1}^{NS} \right) \right] \quad (6)$$

3. West and New South dock are both built at date $\tau = 5$, the latter with four possible percentages of investment:

$$\left\{ E \left[NPV \left(e_5, \tau_5^W, \tau_5^{NS}, F \right) \right]; F \in A \right\} \quad (7)$$

- Following equation 4, the $Ef(e_5)$ is the maximum of these expected values.
- The payoff function at each e_5 is then $Pf(e_5) = \max(Ef(e_5), 0)$

- We generate the scenarios space at time $\tau = t$, $0 \leq t < 5$

- At each scenario e_t , we calculate the following expected values considering all the admissible optionality at $\tau = t$ (i.e. the elements of C_t):

1. West dock will never be built and New South dock will be build at $\tau = t$ with four different percentages of investment:

$$\left\{ E \left[NPV \left(e_t, \tau_{-1}^W, \tau_t^{NS}, F \right) \right]; F \in A \right\} \quad (8)$$

2. West dock will be built at date $\tau = t$ and New South dock will never be built:

$$E \left[NPV \left(e_t, \tau_t^W, \tau_{-1}^{NS} \right) \right] \quad (9)$$

3. West dock is built at date $\tau = t$ and New South dock will be built afterwards, at any of the possible dates between $\tau = t$ and $\tau = 5$ and with any of its possible percentages of investment:

$$\left\{ E \left[NPV \left(e_t, \tau_t^W, \tau_k^{NS}, F \right) \right] , t \leq k; F \in A \right\} \quad (10)$$

4. New South dock is built at date $\tau = t$ with any of its possible percentages of investment and West dock will be built afterwards, at any of the possible dates between $\tau = t$ and $\tau = 5$:

$$\left\{ E \left[NPV \left(e_t, \tau_j^W, \tau_t^{NS}, F \right) \right] , t \leq j; F \in A \right\} \quad (11)$$

- Following equation 4, the $Ef(e_t)$ is the maximum of these expected values.
 - We calculate the expected continuation value $CV(e_t)$
 - The payoff function at each e_t is then $Pf(e_t) = \max(Ef(e_t), CV(e_t))$.
- The value of the investment project is the value of the payoff function at e_0 , the unique scenario belonging to the scenarios space at date $\tau = 0$:

$$\text{Value of the Project} = Pf(e_0)$$

4.3 Software Designed for the Real Problem Valuation

From the point of view of software design, the main advantage of the algorithm is that is especially adequate for parallel processing. As the number of variables and

the complexity of the payoff functions is so high, this feature becomes essential. The program designed has been divided into two parts: the first one concerning scenarios spaces and probability distributions (implementing the process described in 2.1), and the second one approaching the valuation itself, which include, on one hand the calculation of the transition probabilities (implementing 2.2), and on the other hand evaluating the payoff function (involving equations from 5 to 11).

This software has been designed to be run in a Gray-Silicon Graphics Origin 2000 computer with 64 processors (MIPS R1200 at 300 MHz), 16 GBytes central memory and 390 GBytes on disk. The operative system used is IRIX 6.5.5. This computer belongs to the University of Valencia and is available for members of research programs developed by the departments and institutes inside the University.

Let us describe in detail the software structure of the two parts as well as the computational cost involved. For generating the scenarios spaces we use 5 processors, one for each exercise date. The estimate computing time is 4'. For each scenarios space we consider a number of scenarios that conform at least 95% of the probability distribution. The files containing the scenarios as well as its corresponding probabilities are stored in order to be used in the second part of the program.

The second part of the program is design to make the valuation itself, so it

will work backwards and need sequential processing. However, at each exercise date we can make use of parallel processing to reduce computational times. We will use one processor for each type of expected value and the extra processors for the expected continuation values.

At maturity, we need 9 processors to calculate equations from 5 to 7. The computing time is around $5'$. At any of the intermediate dates $0 < \tau = t < 5$, we need to calculate $5 + 2 \times (4 \times (6 - t))$ expected values. That means a number of processors between 21 ($t = 4$) and 45 ($t = 1$). So we will use between 43 and 19 processors to reduce with parallel processing the computational cost of the expected continuation values.

For each one of these intermediate dates we estimate a computing time of $3'$ at $t = 4$, $3'$ at $t = 3$, $5'$ at $t = 2$ and $10'$ at $t = 1$. At $\tau = 0$, as there is only one scenario, we can consider that the computational cost is due only to expected continuation values and so we can make use of all processors available. We will assign no computational cost.

Therefore, the whole valuation process involves 64 processors and has an estimate total computing time of $30'$. At the same time, the files containing the expected values at each scenario and for each control combination are stored in order to be used for solving the decision problem. The computational cost of such resolution will be of $1'$, as we will see in 5.

5 Optimal Investment Policy Decision Problem

Once we have finished the valuation of the project, in this section we will solve the decision problem of optimal investment policy that the project manager would face. As we described in 3.2, all possible combinations of the control variables build a grid of 178 points. Each combination is given by a control array and represents a decision that could be taken by the project manager. We have divided this section into two subsections. First we will describe how to state the decision problem so that we use the information already stored during the valuation process. In the second subsection we present the numerical results obtained.

5.1 Statement of the Decision Problem

As we have said before, we have built a grid containing all possible 178 decisions. We will denote the set of elements of this grid by \mathcal{G} . In order to be able to decide which one is the best according to the project manager's criteria concerning returns and risk, we have to assign to each point a measure of these two parameters: return and risk. The procedure we use is the following:

- Let us consider a given control array $(\tau_j^W, \tau_k^{NS}, F)$ which represents a generic point of the grid. Then we find the date τ of the valuation process described in 4.2 for which the control array considered belongs to the set of admissible arrays, i. e. the date τ for which $(\tau_j^W, \tau_k^{NS}, F) \in C_\tau$ Let us note that each

control array belongs to a unique C_τ .

- Once we have found the suitable date τ , we identify the equation between 5 and 7 if τ is maturity, or between 8 and 11 otherwise, involving the control array considered. So we have found an expected NPV which has been calculated at each scenario e_τ of the scenarios space built at date τ , considering that the given control array $(\tau_j^W, \tau_k^{NS}, F)$ has fixed values.
- Let us consider the set of all these expected NPVs when e_τ varies along the whole scenarios space ϵ_τ built at date τ :

$$\left\{ E \left[NPV \left(e_\tau, \tau_j^W, \tau_t^{NS}, F \right) \right] \right\}_{e_\tau \in \epsilon_\tau} \quad (12)$$

We will denote this set by $\mathcal{G}(\tau_j^W, \tau_t^{NS}, F)$. This way we have assigned to each point of the grid represented by a control array, a space $\mathcal{G}(\tau_j^W, \tau_t^{NS}, F)$ of expected NPVs, parallel to that of scenarios at date τ . Let us note that the same scenarios space ϵ_τ will be used to build such a parallel space for all control arrays belonging to the same C_τ .

- As described in 2.1, the scenarios space ϵ_τ has a probability distribution \mathcal{Q}_τ on it, so that each scenario $e_\tau \in \epsilon_\tau$ occurs with a certain probability p_{e_τ} . We can consider then that the expected NPV belonging to the space $\mathcal{G}(\tau_j^W, \tau_t^{NS}, F)$ and associated to e_τ also occurs with probability p_{e_τ} . This way \mathcal{Q}_τ becomes a probability distribution on the space $\mathcal{G}(\tau_j^W, \tau_t^{NS}, F)$. Figure 4 represents $\mathcal{G}(\tau_{-1}^W, \tau_1^{NS}, 100)$ versus $\mathcal{Q}_{\tau=1}$.

- As we explained in section 4, each element of $\mathcal{G}(\tau_j^W, \tau_t^{NS}, F)$ gives a measure of the returns of the project for a given scenario e_τ and a given control array. Then we use the probability distribution \mathcal{Q}_τ on $\mathcal{G}(\tau_j^W, \tau_t^{NS}, F)$ to get its corresponding expected value, and so get a return measure associated only to the given control array $(\tau_j^W, \tau_t^{NS}, F)$. We will denote such return measure by $Rtn(\tau_j^W, \tau_t^{NS}, F)$
- Using again \mathcal{Q}_τ , we can consider as a risk measure associated to the control array $(\tau_j^W, \tau_t^{NS}, F)$, the standard deviation of \mathcal{Q}_τ on $\mathcal{G}(\tau_j^W, \tau_t^{NS}, F)$. We will denote such risk measure by $Rsk(\tau_j^W, \tau_t^{NS}, F)$.

The decision problem can then be stated as follows:

$$\begin{aligned}
& \max_{(\tau_j^W, \tau_k^{NS}, F) \in \mathcal{G}} Rtn(\tau_j^W, \tau_k^{NS}, F) \\
& \text{subject to} \\
& Rsk(\tau_j^W, \tau_k^{NS}, F) \leq risk^*
\end{aligned} \tag{13}$$

where $risk^*$ denotes the admissible risk level considered by the project manager.

5.2 Numerical Resolution of the Decision Problem

As we have described in 5.1, the numerical resolution of the decision problem stated, consist of calculating expected values and standard deviations of probability distributions already stored from the valuation process. For each point of the grid of control variables combinations, we get a measure of return and risk.

The whole process of assigning values to points of the grid will need just one processor and an estimate computing time of 1 minute.

The following Figure 5 shows the surface which contains the optimal solution given by $Rtn(\tau_j^W, \tau_k^{NS}, F = 25)$, with $j = -1, 0, 1, \dots, 5$ and $k = -1, 0, 1, \dots, 5$.

The control variables combination that gives the maximum expected return is $(\tau_5^W, \tau_1^{NS}, F = 25)$. The associated risk is 41.63%.

6 Conclusions

The main contribution of this work is a new algorithm which allows us to price high dimensional american style real options, solving at the same time the decision problem involved. The latter with no additional computational cost due to the way information is stored during the valuation process.

Using this new algorithm, we have been able to approach an investment on harbour infrastructures capturing all the complexity intrinsic to this type of projects, with multiple state and control variables, complex cash flow functions and multiple exercise dates.

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A Theoretical Description of the Scenarios-Monte Carlo Algorithm

In this appendix we develop a comprehensive mathematical description of the scenarios Monte Carlo algorithm. First we approach the process of generation of scenarios spaces in A.1. Next we complement it with the numerical algorithm suggested for implementing such generation in A.2. Such numerical algorithm is summarised in 2.1 and applied in 4.2. Finally, A.3 contains the theoretical development of how consecutive scenarios spaces are related in order to calculate expected continuation values, also summarised in 2.2 and applied 4.2.

A.1 Generation of the Scenarios Spaces

Let us consider a theoretical investment project whose life is denoted by T . The following set

$$\{\tau_0, \dots, \tau_n\}, \quad 0 \leq \tau_0, \quad \tau_i < \tau_{i+1}, \quad i = 1, \dots, n$$

contains its possible investment dates. Then, for each τ_j , our work time interval will be

$$I_{\tau_j} = [\tau_j, \tau_j + T]$$

Let $N_i(t)$, $i = 1, \dots, m$ be the state variables of the problem, each one evolving according to a stochastic process

$$dN_i(t) = \mu_i(N_i, t)dt + \sigma_i(N_i, t)dW_i \tag{14}$$

where $E[dW_i dW_j] = \rho_{ij}$, $i, j = 1, \dots, m$ denote their correlations.

For a given τ_j , let us consider the following elements:

- Let \mathcal{L}_{τ_j} be the space of possible outcomes at time τ_j of the m state variables, and $\mathcal{Q}_{\tau_j}^c$ a continuous probability distribution on it induced by the dynamics given in (14)
- Let $\{V_{\tau_j}^k\}_{k=1}^{s_j}$ be a finite and disjoint partition of \mathcal{L}_{τ_j} in m -dimensional cubes.
- Let $(N_1^k(\tau_j), \dots, N_m^k(\tau_j))$, $k = 1, \dots, s_j$, be the representative arrays of the cubes $\{V_{\tau_j}^k\}_{k=1}^{s_j}$ (for example, the geometric centre in the case of a finite cube). We denote by ϵ_{τ_j} the set of these representative arrays.

Then we will say that ϵ_{τ_j} is a **scenarios space** at date τ_j , and that each of its elements $(N_1^k(\tau_j), \dots, N_m^k(\tau_j))$ is a **scenario** at time τ_j . Abbreviatedly, we will denote these scenarios by $e_{\tau_j}^k$, $k = 1, \dots, s_j$.

Next we build a complete discrete probability distribution on the scenarios space ϵ_{τ_j} . For each scenario $e_{\tau_j}^k$ belonging to ϵ_{τ_j} , its probability $q_{\tau_j}^k$ will be given by

$$q_{\tau_j}^k = \int_{V_{\tau_j}^k} d\mathcal{Q}_{\tau_j}^c. \quad (15)$$

We will denote by \mathcal{Q}_{τ_j} the discrete probability distribution on ϵ_{τ_j} obtained this way.

A.2 Numerical Algorithm for Scenarios Space Generation

To construct the discrete space of scenarios ϵ_{τ_j} and the discrete probability distribution \mathcal{Q}_{τ_j} , we will proceed as follows:

- At a given τ_j , we get M samples of each state variable N_i , using Monte Carlo simulation of the discretized dynamics equations

$$N_i(t_{j+1}) = N_i(t_j) + \mu_i(N_i(t_j), t_j) \Delta t + \sigma_i(N_i(t_j), t_j) \alpha_i \sqrt{\Delta t} \quad (16)$$

with $i = 1, \dots, m$, $\Delta t = t_{j+1} - t_j$, and α_i a random variable following the normal distribution $N(0, 1)$ and verifying $E[\alpha_i \alpha_j] = \rho_{ij}$, $i, j = 1, \dots, m$.

- For each N_i , we build the histogram associated to this M samples, which give us a probability distribution of the state variable N_i at time τ_j .
- We build a space of m -dimensional arrays $(N_1^k(\tau_j), \dots, N_m^k(\tau_j))$ (scenario at time τ_j) by making the Cartesian product of the representative values of the classes of the m histograms. This space is the **scenarios space** ϵ_{τ_j} at time τ_j .
- The probability of each array is the product of the probabilities of its components. Then, we get a complete probability distribution \mathcal{Q}_{τ_j} on ϵ_{τ_j} .

A.3 Relation between Consecutive Scenarios Spaces. Expected Continuation Value

Let $e_{\tau_j}^d$ be a given scenario belonging to the scenarios space ϵ_{τ_j} . Let us consider the stochastic processes of each one of its components given by equations 14. In order to get the scenarios belonging to $\epsilon_{\tau_{j+1}}$ which are related with $e_{\tau_j}^d$, as well as its corresponding transition probabilities, we proceed as follows:

- Let us consider the discretized dynamics equations given by 16.
- Using Monte Carlo simulation and $e_{\tau_j}^d$ as initial condition, we get M samples of m -dimensional arrays at date τ_{j+1} . We will denote the set of these samples by:

$$\mathcal{S}_{\tau_{j+1}} = \left\{ \left(N_1^1(\tau_{j+1}), \dots, N_m^1(\tau_{j+1}) \right) \dots \left(N_1^M(\tau_{j+1}), \dots, N_m^M(\tau_{j+1}) \right) \right\} \quad (17)$$

Let us note that $\mathcal{S}_{\tau_{j+1}}$ is a subset of the space $\mathcal{L}_{\tau_{j+1}}$. Abbreviatedly we will denote it as $\mathcal{S}_{\tau_{j+1}} = \left\{ S_{\tau_{j+1}}^1, \dots, S_{\tau_{j+1}}^M \right\}$.

- Let $\{V_{\tau_{j+1}}^k\}_{k=1}^{s_{j+1}}$ be the finite and disjoint partition of $\mathcal{L}_{\tau_{j+1}}$ in m -dimensional cubes introduced in A.1. For each $S_{\tau_{j+1}}^u$, $u = 1, \dots, M$, we can find a cube $V_{\tau_{j+1}}^{k_u}$ such that $S_{\tau_{j+1}}^u \in V_{\tau_{j+1}}^{k_u}$, using the Euclidean norm.
- As we said in A.1, the representative arrays of those cubes are the scenarios $e_{\tau_{j+1}}^{k_u}$, $u = 1, \dots, M$. Then we will say that $S_{\tau_{j+1}}^u$ is related with $e_{\tau_{j+1}}^{k_u}$,

$u = 1, \dots, M$. This way we have built an application $\Psi : \mathcal{S}_{\tau_{j+1}} \rightarrow \epsilon_{\tau_{j+1}}$ between the set $\mathcal{S}_{\tau_{j+1}}$ and the scenarios space $\epsilon_{\tau_{j+1}}$.

- As more than one array $\mathcal{S}_{\tau_{j+1}}^u$ could be related with the same scenario in $\epsilon_{\tau_{j+1}}$, the cardinal of the set $\mathcal{R}_{\tau_{j+1}} = \Psi(\mathcal{S}_{\tau_{j+1}})$ gives us the number of different scenarios belonging to $\epsilon_{\tau_{j+1}}$ that are related with samples in $\mathcal{S}_{\tau_{j+1}}$. Let us denote with l this cardinal. Then, for each $e_{\tau_{j+1}}^v \in \mathcal{R}_{\tau_{j+1}}$ with $v = 1, \dots, l$, the set $\{\Psi^{-1}(e_{\tau_{j+1}}^v)\}$ contains the samples in $\mathcal{S}_{\tau_{j+1}}$ that are related with $e_{\tau_{j+1}}^v$, $v = 1, \dots, l$.

- We define the application $\Lambda : \mathcal{R}_{\tau_{j+1}} \rightarrow [0, 1]$ given by:

$$\Lambda(e_{\tau_{j+1}}^v) = \frac{\text{cardinal}(\Psi^{-1}(e_{\tau_{j+1}}^v))}{\text{cardinal}(\mathcal{S}_{\tau_{j+1}})} \quad (18)$$

with $v = 1, \dots, l$. Let us note that Λ is a probability distribution on $\mathcal{R}_{\tau_{j+1}}$.

- Finally, as all samples in $\mathcal{S}_{\tau_{j+1}}$ come from the same initial condition $e_{\tau_j}^d$, we consider that $e_{\tau_j}^d$ is related with the l scenarios belonging to $\mathcal{R}_{\tau_{j+1}}$, and that the transition probability from $e_{\tau_j}^d$ to $e_{\tau_{j+1}}^v \in \mathcal{R}_{\tau_{j+1}}$, $v = 1, \dots, l$, is given by:

$$\pi(e_{\tau_j}^d \rightarrow e_{\tau_{j+1}}^v) = \Lambda(e_{\tau_{j+1}}^v) \quad (19)$$

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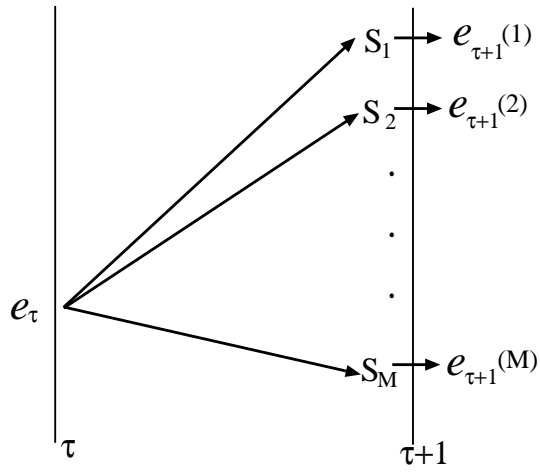


Figure 1: Relation between consecutive scenarios spaces

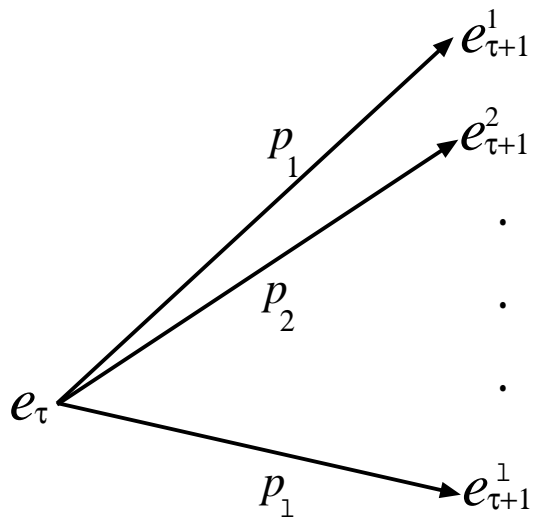


Figure 2: Transition probabilities

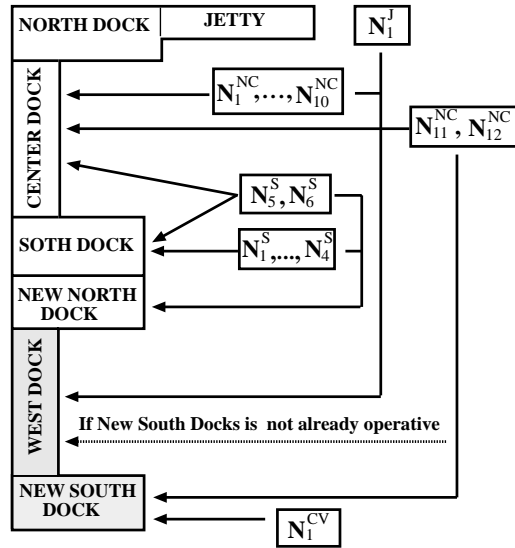


Figure 3: Harbour infrastructure and goods flow. State variables

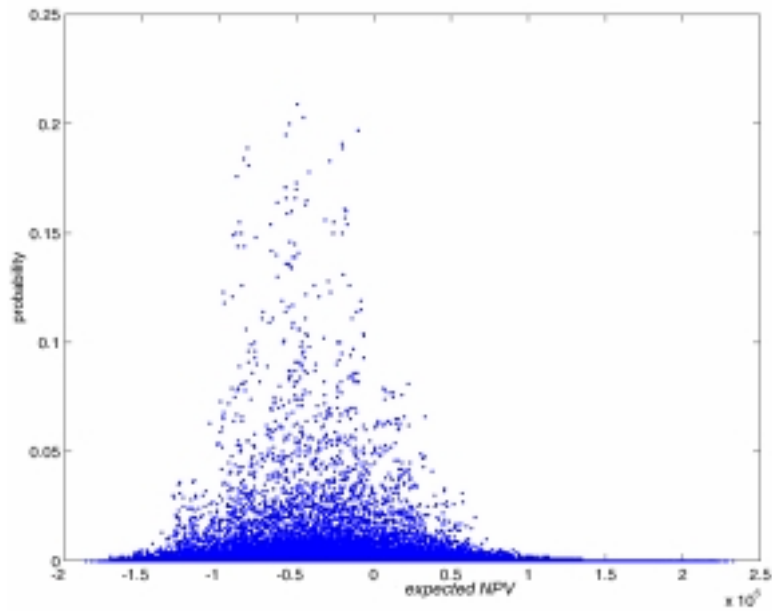


Figure 4: Expected Net Present Values versus probabilities

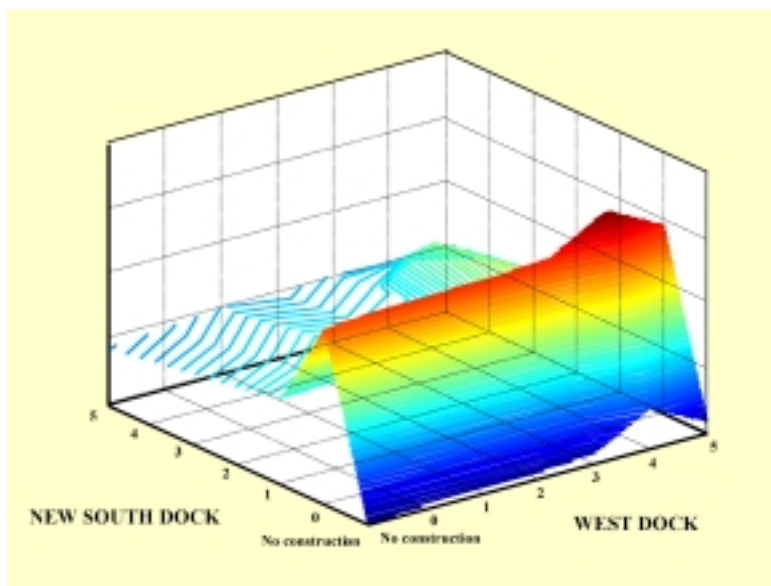


Figure 5: Exercise dates combinations versus returns