Option Contracts in Supply Chains[*]

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Abstract

This article investigates the pricing of options when the demand curve is downward sloping. Our specific application arises in a supply chain setting, where a manufacturer offers the retailer the right to reorder items at a fixed price and/or the right to return unsold goods for a predetermined salvage value. We show that the introduction of option contracts causes the wholesale price to increase and the volatility of the retail price to decrease. Conditions are derived under which the manufacturer is always better off by introducing options. In general, options are not zero sum games. In some cases the retailer also benefits while in other cases the retailer is worse off. If the uncertainty in the demand curve is sufficiently high, the introduction of option contracts alters the equilibrium prices in a way that hurts the retailer. Finally, we demonstrate that if either the manufacturer or the retailer wants to hedge the risk, contracts that pay out according to the square of the price of a traded security are required.

Many studies have been performed in industries such as apparel, sporting goods and toys, where there are long lead times, short selling seasons and high demand uncertainties. For these problems one avenue of research is concerned with the design of contracting relationships that provide retailers with flexibility in responding to unanticipated demand and prices over the sales season. This paper examines such contracting arrangements in a supply chain setting consisting of an upstream party (which we refer to as the manufacturer) whose only access to the product market is via a single downstream party (which we refer to as the retailer). To manage the risk of inventories associated with uncertain demand, it is fairly common for the manufacturer to provide the retailer with an array of products, including *reordering* contracts, or call options, that allow the retailer to purchase additional goods at a predetermined time for a fixed price, and *return* contracts, or put options, that allow the retailer to return unsold goods at a predetermined salvage price. By purchasing inventory, together with a portfolio of these supply chain call and put options, the retailer has more choices that allows a strategy to be put into place to best meets its interests. The manufacturer's goal is to design the terms of the reordering and return option contracts and establish their prices, together with the wholesale price, so as to induce the retailer to take optimal actions that best serve the manufacturer's interests.[1]

By introducing reorder and return option contracts, the manufacturer alters the retailer's sequence of decisions. This in turn has a feedback effect in that the equilibrium wholesale and retail prices are affected. In this paper we are particularly interested in how these prices adjust after the introduction of supply chain options. We are also interested in establishing the pricing mechanism that the manufacturer uses for the supply chain options. Indeed, our problem environment is set up so that we can closely examine the pricing of option contracts in a downward sloping demand curve environment. Moreover, since we assume that there are sufficient financial products that span all uncertainty, we are able to unambiguously value the benefit of the supply chain options without explicitly incorporating risk aversion factors.

The usual approach in pricing real options follows the Black-Scholes (1973) and Merton (1974) paradigm, in which contracts are replicated by dynamic self financing trading schemes in the underlying asset and in riskless bonds. In this approach, derivative contracts are redundant and do not affect prices of assets in the marketplace. In order to investigate how derivative contracts might impact prices, it is necessary to move away from the typical partial equilibrium arbitrage free paradigm and to allow for the possibility that these claims have feedback effects that may alter equilibrium prices of the underlying assets.

In the financial options literature, there is a large number of studies that have investi-

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[1] The existence of multiple decision makers with different ownership interests results in departures from first-best solutions and creates strong incentives for parties to enter into contracts that enhance system-wide performance and improve channel coordination.

gated how listed option contracts could alter the dynamics of asset prices. The popular view that derivatives were beneficial because they expanded the investment opportunity set, allowing traders to more precisely mold returns in accordance with their beliefs and preferences, lacked rigor, because it ignored potentially harmful feedback effects. Theoretical models that explore these feedback effects draw different conclusions. Detemple and Selden (1988) as well as Grossman (1988) identify different sets of conditions whereby the introduction of options leads to a reduction in volatility of prices, whereas Stein (1989) identifies conditions that lead to increased volatility. The empirical evidence, provides modest support for options to lead to reductions in volatility.[2] The empirical literature also concludes that the listing of call (put) options is associated with positive (negative) excess abnormal returns on the underlying asset, while the simultaneous listing of both has little effect.[3] The results of our analysis in the real options setting echo some of the results from this literature in the sense that we find the introduction of supply chain options is accompanied by price effects, both at the wholesale and at the retail levels. In particular, we show that retail prices are less volatile then they would be if no options were utilized. We also identify conditions under which retail prices may increase, or decrease. Finally, with regard to wholesale prices, we show that they will either stay the same or increase.

In our framework, the benefit of the option program to the manufacturer is influenced by the uncertainty in the demand curve and the cost of fulfilling expedited orders that arise from options being exercised late in the season. The manufacturer will only introduce options if they increase profits. Expanding the investment opportunity set, however, may not necessarily improve the condition of the retailer. Indeed, we show that when the demand curve is very uncertain, then the retailer is worse off when options are introduced. This result may, at first glance, be a bit surprising, since one might surmise that when uncertainty is high, the retailer will be more inclined to use option contracts. Of course, the manufacturer recognizes that this is the case, and adjusts the wholesale and option prices accordingly. The option contracts are not zero sum games between the manufacturer and the retailer, and, as we shall see, there are cases where both parties benefit. The pricing of the supply chain options is of particular interest. The reorder options are shown to be less valuable than otherwise identical cash settled contracts based on the equilibrium retail price. This result is tied to the fact that in a downward sloping curve environment, options that appear to be "in-the-money" at expiration, may rationally go unexercised.

The equilibrium wholesale price can be affected by the introduction of reorder options. However, once introduced, we show that the wholesale price will not readjust again if the manufacturer chooses to introduces return options, provided the range of strike prices are curtailed.

[2]Examples of studies include Damodaran and Lim (1991) and the references in Damodaran and Subrahmanyam (1992).

[3]Surprisingly, Conrad (1989) finds that the abnormal returns are generated around the listing date, rather than the earlier announcement date.

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Finally, we show that the net present value of profits for the manufacturer and retailer can be related to the value of a highly nonlinear derivative claim. In particular, their values are fully determined by the price of an option that has payouts based on the *square* of the price of a traded security. While Jarrow and van Deventer (1996) have used these contracts, often referred to as power options, in pricing demand deposit accounts, this economic justification, in a real option setting, is, as far as we know, new.

The paper proceeds as follows. In section 1 we review related literature from supply chain management. In section 2 we list the basic assumptions and establish the decision and pricing problems. We formulate the problem for the case where the manufacturer is considering the use of reordering options. Later on we show how the solutions to this problem can be used to solve for the returns option policies and their prices. In section 3 we establish the benchmark case, where the manufacturer does not provide the retailer with any option contracts. This situation is of some interest in its own right. Specifically, after purchasing inventory, the retailer retains the option to withhold some product from the market. This *retention* option is shown to be valuable, especially when there is large uncertainty in the demand curve. In section 4 we consider the case when the retailer is allowed to purchase reorder options with inventory purchases. We closely investigate how the equilibrium wholesale price and retail prices are affected by the introduction of such contracts. In section 5 we isolate conditions under which both parties benefit from the option program. We also identify cases where the option program benefits the manufacturer, but not the retailer. The value of the option program is also related to the value of a derivative instrument that has payouts linked to the square of the price of a traded security. Section 6 examines the consequences of a returns policy and shows that the equilibrium prices can be obtained from the solution to the earlier gaming problems. Section 7 extends the results to cases where the uncertainty in the linear demand curve follows a more general distribution than that prescribed earlier. We illustrate that our above results hold true when uncertainty in the linear demand curve follows a continuous distribution.

1   Literature Review

Building large inventories can be expensive if large demands are not realized. Of course, not building inventory can lead to shortages with large opportunity costs. These trade offs need to be carefully assessed. One way for the retailer to manage this risk is to transfer it to the manufacturer who may be better equipped to absorb it. Contracts are designed not only to meet the risk aversion needs of the parties in the supply chain, but also to help coordinate the channel. Examples of contracting relationships include Eppen and Iyer (1997), who consider back up agreements, Serel, Dada and Moskowitz (2001), who consider capacity guarantees, and Brown and Lee (1997), who consider pay-to-delay capacity reservation contracts. Barnes-

Schuster, Bassok and Anupindi (2000), show that all of these contracts can be viewed as special cases of option contracts that permit expedited orders. For excellent reviews of this literature see Anupindi and Bassok (1999), Lariviere (1999) and Tsay Nahmias and Aggrawal (1999). Within this literature, an important consideration is channel coordination. Oftentimes this can be accomplished through non linear pricing as in Iyer and Bergen (1997). In other cases, coordinating the supply chain is accomplished by modifying the contract between wholesalers and retailers. For example, Barnes-Schuster, Bassok, and Anupindi (2000), show that channel coordination could be achieved by modifying the nature of the strike price in their contracting arrangements. Other examples of contracting mechanisms that induce channel coordination include Pasternak (1985), who considers product returns, and Taylor (2001), who considers combination of returns and retailer price protection. In these studies the typical methodology employed builds off the classical newsvendor model where inventory levels are established before demand is realized.

Our study is related to the above papers in that we consider contracting arrangements in a supply chain in a newsvendor environment. Our approach, however differs in that we do not take the retail price and the demand distribution as given; rather, we assume demand distributions are influenced by pricing decisions. In particular, we assume the retailer is faced with an uncertain downward sloping demand curve, and hence has to establish a retail price conditional on previous inventory decisions. In this regard our analysis relates to an extended single agent newsvendor problem discussed by Petruzzi and Dada (1999), who permit stocking policies and prices to be determined simultaneously. It also relates to Ritchken and Tapiero (1986) who investigate option contracts for purchasing decisions, in which price and quantity uncertainty are correlated. Both these studies deal with single agents, whereas our analysis deals with two agents in the presence of an uncertain linear downward sloping demand curve.

Several studies have investigated contingent claim pricing in the presence of a downward sloping demand curve. Triantis and Hodder (1990) examine the pricing of complex options that arise in a flexible production system which allows the firm to switch its output mix over time, in the presence of demand curves which are stochastic and downward sloping. Pindyck (1989) and He and Pindyck (1989) also incorporate downward sloping demand curves to examine capacity choice decisions in a real options framework. Our study differs from these in that we have a specific supply chain in which the optimal contracts can be designed and priced through the appropriate use of principal-agent theory.

2 The Basic Model

We consider a simple supply chain consisting of a single manufacturer who produces a product and sells it at a wholesale price to a retailer. At date 0, the future date 1 demand curve

is uncertain. Without any risk management contracts, the retailer is faced with bearing this risk. The manufacturer assists in risk sharing by allowing the retailer to supplement inventory decisions with option purchases. Each option contract provides the retailer with the right to purchase an additional unit at date 1, after uncertainty is revealed, for a predetermined price of X dollars.

Let I and U be the number of goods and options purchased at date 0 by the retailer. The price of each good is S_0 and the price of each option is C_0. The retailer's problem is to determine the optimal mix of inventory and options, given their prices. The manufacturer's problem is to establish the optimal prices of both the product and options that induces the retailer to take rational decisions that further the interests of the manufacturer. The manufacturer is particularly keen to access the value to the firm of issuing option contracts and to assess the role of the strike price of the contract.

We assume the demand in period 1 is linear. Denoting period one prices by S_1, the inverse demand curve is:

$$S_1 = \alpha - \delta Q \tag{1}$$

where Q is the total amount of inventory released into the market. Here α is stochastic, and if $\delta > 0$, α/δ can be viewed as the maximum potential size of the market.

At date 1 the uncertainty in the market size factor, α, is resolved, and the retailer responds by establishing how many units of inventory to release into the market and how many options to exercise. Let $0 \leq q \leq I$ be the number of units of inventory released and let $0 \leq v \leq U$ be the number of options exercised. The total number of items released into the market place is $Q = q + v$. We assume that any excess inventory that is held back has no salvage value. [4] Let R_1 be the net cash flow that the retailer makes in period 1. Then

$$R_1(q, v | I, U, \alpha = a) = (q + v)(a - \delta(q + v)) - vX \tag{2}$$

In period 1, the retailer chooses q and v to maximize equation (2) subject to the inventory and option constraints. If the exercise price X is too high, then the retailer, who has exclusive rights to the market, has incentives to renegotiate the price. To ensure that the option contract is credible, the manufacturer must therefore set the exercise price no higher than the marginal cost of expedited production in period 1. [5]

Now, consider the retailer's decision at date 0. Let R_0 be the net present value associated with purchasing I units of inventory and U options at date 0 and optimally managing the project in period 1. Then

$$R_0(I, U) = -IS_0 - UC_0 + E[R_1 D_1] \tag{3}$$

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[4] This is consistent with a downward sloping demand curve.

[5] This assumes that the manufacturer builds to order and is faced with a linear cost structure. Note that if the exercise price is set too low, then the retailer is exposed to the risk of the manufacturer defaulting.

where D_1 is the state dependent stochastic discount rate, commonly referred to as the pricing kernel.

We assume uncertainty in the demand curve is represented by a Bernoulli process. In particular:

$$\alpha = \begin{cases} a_H & \text{with probability} \quad p \\ a_L & \text{with probability} \quad 1-p \end{cases} \tag{4}$$

We also assume that there exists a traded security that pays out a_H dollars in the high (H) state, and a_L dollars in the low (L) state. The price of this security at date 0 is A_0. In addition, a riskless bond exists that pays out \$1 in period 1. Its current price is $B_0 < 1$. The existence of traded securities that spans the uncertainty in the demand curve allows the pricing kernel to be uniquely determined.[6] This assumption allows us to perform the valuations without regard to the specific risk preferences of the retailer and manufacturer. In addition, the valuation can proceed, even if there is not consensus on the value of p.

Let e_H (e_L) be the Arrow Debreu state prices corresponding to a \$1.0 payout only in the high (low) state and \$0 payout otherwise. Clearly:

$$\begin{aligned} A_0 &= a_H e_H + a_L e_L \\ B_0 &= e_H + e_L. \end{aligned}$$

Given the state prices we have:

$$R_0(I,U) = -IS_0 - UC_0 + e_H R_1(q_H^*, v_H^*|I,U,a_H) + e_L A R_1(q_L^*, v_L^*|I,U,a_L), \tag{5}$$

where $R_1(q_H^*, v_H^*|I,U,a_H)$ and $R_1(q_L^*, v_L^*|I,u,a_L)$ are the maximum values of R_1 in equation (2) with $\alpha = a_H$ and $\alpha = a_L$ respectively.

The retailer's time 0 optimization problem is given by

$$Max_{I \geq 0, U \geq 0} R_0(I,U) \tag{6}$$

The objective of the manufacturer is to maximize value by appropriately determining the wholesale price, S_0, and the charge for each option, C_0. Let M_0 be the net present value associated with this specific project. Then.

$$M_0 = IS_0 + UC_0 + e_H[Xv_H^*] + e_L[Xv_L^*] - K_0(I) - e_H K_H(v_H^*) - e_L K_L(v_L^*). \tag{7}$$

In this equation, $K_0(I)$ represents the total cost of making up I units for delivery at date 0, and $K_H(v_H^*)$ ($K_L(v_L^*)$) represents the cost of expediting an additional v_H^* (v_L^*) units on date 1 in the high (low) state. For simplicity, we shall assume that all costs are linear, i.e., $K_0(q) = K_0 q$,

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[6]For an in depth discussion on this point see Duffie (1996).

$K_H(q) = K_H q$ and $K_L(q) = K_L q$, where K_0, K_L, K_H are known constant marginal production costs. To ensure that the manufacturer prefers committed orders to expedited orders, we require $K_j B_0 > K_0$ for $j = L, H$. Further, to ensure that the manufacturer does not find it advantageous to build inventories as a contingency we require $K_0 \geq K_j e_j$, for $j = L, H$. Finallly, we assume that $K_0 < A_0$, since A_0 represents the value at date 0 of a unit of product sold at the highest possible retail price in date 1.

The manufacturer's problem is to establish the wholesale price, S_0, the reorder option price, C_0, and the appropriate strike, X, such that:

$$Max_{S_0, C_0, X} M_0(S_0, C_0, X) \tag{8}$$

given the fact that the retailer responds optimally for each action. This formulation is a standard principal-agent problem, or Stackelberg game, with the manufacturer being the principal.

In what follows we often find it helpful to represent the uncertainty in the demand curve in terms of volatility. Let μ_A and σ_A^2 represent the mean and variance of the intercept term of the demand curve under the risk neutral measure.[7] Then

$$\mu_A = \frac{A_0}{B_0}$$
$$\sigma_A^2 = \frac{e_H e_L}{B_0^2}(a_H - a_L)^2,$$

and we have:

$$a_H = \frac{A_0}{B_0} + \sqrt{\rho}\sigma_A$$
$$a_L = \frac{A_0}{B_0} - \frac{1}{\sqrt{\rho}}\sigma_A$$

where $\rho = \frac{e_L}{e_H}$.

3 Pricing with no Supply Chain Option Contracts

We begin by considering how the retailer will respond to a given price set by the manufacturer.

Lemma 1

If the manufacturer does not provide options, the optimal quantity of inventory the retailer

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[7]Under this measure the expected growth rate of all traded securities equals the risk free rate. In this model it is determined by the two risk-neutral probabilities $q_j = e_j/B_0, j = L, H$.

orders at time 0 is given by:

$$I^* = \begin{cases} \frac{a_H e_H - S_0}{2\delta e_H} & \text{if} \quad 0 < S_0 \le (a_H - a_L)e_H \\ \frac{A_0 - S_0}{2\delta B_0} & \text{if} \quad (a_H - a_L)e_H \le S_0 \le A_0 \\ 0 & \text{if} \quad S_0 > A_0. \end{cases}$$

Proof: See Appendix

Lemma 1 implies that the quantity ordered by the retailer decreases as the wholesale price increases, first at a low rate up to a critical point, and then at a faster rate. Given the response function, the manufacturer can establish the optimal pricing policy. The results are summarized below.

Proposition 1

(i) With no options, the manufacturer's optimal pricing policy is:

$$S_0^* = \begin{cases} \frac{A_0 + K_0}{2} & \text{if} \quad \sigma_A \le \eta(K_0) \\ \frac{a_H e_H + K_0}{2} & \text{if} \quad \sigma_A > \eta(K_0) \end{cases} \tag{9}$$

where

$$\eta(K_0) = \frac{\sqrt{\rho}}{1 + \rho + \sqrt{1 + \rho}} \left[(1 + \rho)\frac{K_0}{B_0} + \sqrt{1 + \rho}\frac{A_0}{B_0} \right]$$

$$\rho = \frac{e_L}{e_H}.$$

(ii) The retailer's optimal ordering and selling response is

$$\begin{array}{llll} I^* = \frac{A_0 - K_0}{4\delta B_0}, & q_L^* = I^*, & q_H^* = I^* & \text{if} \quad \sigma_A \le \eta(K_0) \\ I^* = \frac{e_H a_H - K_0}{4\delta e_H}, & q_L^* = \frac{a_L}{2\delta}, & q_H^* = I^* & \text{if} \quad \sigma_A > \eta(K_0) \end{array}$$

Proof: See Appendix

The proposition tells us that the manufacturer chooses a higher price if the variance is lower than a threshold value, $\eta^2(K_0)$, and a lower price otherwise. To gain insight on this result, first consider the case where there is no demand uncertainty, namely $\sigma_A = 0$. In this case, it is easy to show that the equilibrium wholesale price is $S_0 = \frac{A_0 + K_0}{2}$ and the optimal order quantity is $I = \frac{A_0 - K_0}{4\delta B_0}$. If $\sigma_A \le \eta(K_0)$, this wholesale price and retailer response remains optimal. Specifically, the retailer will ensure enough inventory is purchased so that in the low state the optimal quantity is released, while in the high state there is only minor regret.

As the volatility expands, the consequences are no longer minor. Eventually, the retailer will want to order more than is required for the lower state, but not enough to cover the optimal amount in the high state. If the low state occurs, the retailer has to establish the amount to be

released into the market. By retaining some of the units, the retailer ensures a higher per unit cost. However, since the units are costly, the retailer will take this into account when the he makes the ordering decision in period 0, and will be less likely to order too much. The retailer's ability to control how much inventory is released into the market is called a *retention* option. As the volatility of the demand curve expands, the value of this retention option also expands.

The ability of the retailer to control the number of items that are released for sale is valuable, and is recognized by the manufacturer. Indeed, it can be shown that if the retailer was committed to releasing all inventory that was ordered in period 0, then the manufacturer would set the equilibrium price at $\frac{A_0+K_0}{2}$ regardless of the variance.

The manufacturer, recognizes the importance of the retailer's retention option, and induces the retailer to purchase more inventory, by offering a lower price when the variance is large ($\sigma_A > \eta(K_0)$). The retailer responds to the lower price by purchasing more units than necessary for the lower state, but not quite enough for the high state. Since the gap between states is sufficiently large, and since the per unit cost is low, the retailer is prepared to withhold some inventory in the low state, rather than releasing it into the market.

For the low variance scenarios, the retailer has little incentive to build excess inventory, beyond what is optimal for the lower state, since the additional revenues captured by having such inventory available, if the high state occurs, does not offset the additional costs of purchasing inventory at date 0, which may go unused. The manufacturer, of course, recognizes that the retention option is not worth much, and hence charges a price, based on the premise that the retailer will rationally commit to selling all inventory purchased.

The table below summarizes the wholesale and retail prices in periods 0 and 1 respectively, together with the state dependent quantities that are released into the market place.

Variance	Wholesale Price S_0	Order Quantity I_0	Amount Released q_L	q_H	Retail Price S_L	S_H
$\sigma_A \le \eta(K_0)$	$\frac{A_0+K_0}{2}$	$\frac{A_0-K_0}{4\delta B_0}$	$\frac{A_0-K_0}{4\delta B_0}$	$\frac{A_0-K_0}{4\delta B_0}$	$a_L - \frac{A_0-K_0}{4B_0}$	$a_H - \frac{A_0-K_0}{4B_0}$
$\sigma_A > \eta(K_0)$	$\frac{a_H e_H + K_0}{2}$	$\frac{e_H a_H - K_0}{4 \delta e_H}$	$\frac{a_L}{2\delta}$	$\frac{e_H a_H - K_0}{4\delta e_H}$	$\frac{a_L}{2}$	$\frac{3a_H}{4} + \frac{K_0}{4e_H}$

4 Pricing with Supply Chain Option Contracts

We now reconsider the above problem, but this time we also assume that the manufacturer offers a reorder option, in which each contract provides the retailer with the option of purchasing one extra unit at a predetermined price of X. Assume the unit cost of this option is C_0 and the wholesale price for the product is S_0. In period 1, the retailer maximizes the following function:

$$R_1(q, v | I, U, a) = (q + v)(a - \delta(q + v)) - Xv.$$

Lemma 2

The optimal policy for the retailer in period 1 is to release all inventory into the market before any options are exercised. That is, if $v^* > 0$, then $q^* = I$. Equivalently,

$$v^*(I - q^*) = 0$$

Proof: See Appendix

Lemma 3

At date 1, the retailer's optimal policy has the following form:

(i) If $I > \frac{a}{2\delta}$ then the optimal number of units to sell is $q^* = \frac{a}{2\delta}$ with $v^* = 0$.

(ii) If $\frac{a-X}{2\delta} < I < \frac{a}{2\delta}$ then it is optimal for the retailer to sell all inventory, but not to exercise any options.

(iii) If $0 < I < \frac{a-X}{2\delta}$ then it is optimal to sell all inventory and to exercise $v^* = Min[U, \frac{a-X}{2\delta} - I]$ options.

Proof: See Appendix

Figure 1 and Figure 2 show the two possible sets of regions of (I, U) over which the optimal responses by the retailer in period 1 can be identified.

Insert Figures 1 and 2 Here

Having characterized the retailer's policy in period 1, we now can turn attention to the retailer's optimization problem in period 0.

Lemma 4

If $C_0 > Max[0, S_0 - XB_0]$ then either

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1. $U^* = 0$, or

2. $I^* \geq \frac{a_L - X}{2\delta}$ and $U^* \leq \frac{a_H - X}{2\delta} - I^*$

Proof: See Appendix

Lemma 4 provides us the region over which it might be optimal to hold a positive number of options. We now investigate the conditions that result in the retailer holding option positions.

Lemma 5

The solution to the retailer's optimization problem at date 0 that involves options is:

$$I_1^* = \frac{a_L - X}{2\delta} + \frac{C_0 - S_0 + XB_0}{2\delta e_L} \tag{10}$$

$$U_1^* = \frac{a_H - X}{2\delta} - \frac{C_0}{2\delta e_H} - I_1^* \tag{11}$$

if

$$C_0 > S_0 - Xe_H - a_L e_L \tag{12}$$

and if either

$$a_L > a_H - X \tag{13}$$

$$Max[0, S_0 - X B_0] \leq C_0 \leq [S_0 - XB_0 + e_L(a_H - a_L)]\frac{e_H}{B_0} \tag{14}$$

or

$$a_L < a_H - X \tag{15}$$

$$Max[0, S_0 - XB_0] \leq C_0 \leq Min[S_0 - Xe_H, [S_0 - XB_0 + e_L(a_H - a_L)]\frac{e_H}{B_0}] \tag{16}$$

In all other cases where $C_0 \geq Max[0, S_0 - XB_0]$, the optimal solution contains no options.[8]

Proof: See Appendix

If $a_L > a_H - X$, and if the call is priced in the interval given by equation (14) then the retailer will hold a positive position in reorder options, and the optimal response should be in region R3 of Figure 1. Similarly, if $a_H \leq a_H - X$, and if equation (16) holds, then the retailer's optimal response should be in R3 of Figure 2. If neither of these conditions apply, then the optimal solution will not contain any options.

Once the retailer's response has been fully characterized, the optimal pricing policy by the manufacturer can now be established.

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[8]We restrict attention to this case, since if it does not hold, then it can easily be shown that inventory building is never optimal, and the retailer will only hold onto option positions.

Proposition 2

(i) If the manufacturer is going to use options, then the wholesale price and option price will be set at:

$$S_0^* = \frac{A_0 + K_0}{2} \tag{17}$$

$$C_0^* = \frac{(a_H + K_H - 2X)e_H}{2}, \tag{18}$$

where the strike price is curtailed as:

$$\frac{a_L e_L + K_0 - K_H e_H}{2e_L} \leq X \leq \frac{a_H + K_H}{2}. \tag{19}$$

if and only if the cost of expedited production, K_H, is constrained relative to the cost of regular production, K_0, as:

$$\widetilde{K}_L \leq K_H \leq \widetilde{K}_H \tag{20}$$

where

$$\widetilde{K}_L = Max \left[\frac{K_0}{B_0}, \frac{K_0 - a_L e_L}{e_H} \right]$$

$$\widetilde{K}_H = Min \left[\frac{K_0}{e_H}, \overline{K}_H \right]$$

and

$$\overline{K}_H = \frac{K_0 + (a_H - a_L)e_L}{B_0}$$

(ii) The retailer's optimal response is given by:

$$I^* = \frac{a_L}{4\delta} + \frac{K_H e_H - K_0}{4\delta e_L} \tag{21}$$

$$U^* = \frac{1}{4\delta}(a_H - a_L) - \frac{1}{4\delta e_L}(K_H B_0 - K_0) \tag{22}$$

$$q_L = q_H = I^* \tag{23}$$

$$v_L = 0, \quad v_H = U^*. \tag{24}$$

In all other cases where $C_0 \geq Max[0, S_0 - XB_0]$, the manufacturer will not find it optimal to use options.

Proof: See Appendix

Notice that if options are used the structure for the equilibrium wholesale price simplifies in that it no longer depends on the magnitude of the variance.

If the variance is low ($\sigma_A^2 \leq \eta(K_0)^2$), then the introduction of options does not affect the equilibrium wholesale price. The introduction of options, however, does affect the retailer's

response, the quantities released into the market, and hence alters the retail prices in period 1. In particular, for this low volatility case, the retailer, by purchasing options, changes inventory holdings from $\frac{A_0 - K_0}{4\delta B_0}$ to $\frac{a_L}{4\delta} + \frac{K_H e_H - K_0}{4\delta e_L}$. The difference is equal to $\frac{e_H}{4\delta B_0 e_L}[(a_H - a_L)e_L - (K_H B_0 - K_0)] > 0$, due to the condition $K_H < \overline{K}_H$ in Proposition 2. Therefore, with options available, the initial order quantity is reduced. Similarly to the case with no options, all inventory purchased is committed for sale. That is, the option not to release inventory in the low state in period 1 is not valuable. The optimal position is structured so that in the low state no options are exercised, while in the high state all options are exercised. Notice, that as K_H increases, the difference between the order quantities with and without options narrows. This reflects the fact that if the cost to the manufacturer of expediting an order increases, then the benefit of options for the manufacturer decreases, and prices will be set so that the retailer does not find it beneficial to use options.

For the low variance case, the introduction of options leads to a reduction in the order quantity. Further, in the low state, options are not exercised, which means that the total quantity of units released into the market is lower. In the high state, however, all options are exercised, and the total amount released into the market place exceeds the case, for which there were no options. Table 1 shows the resulting equilibrium retail prices.

The volatility of the retail price is proportional to the range. When $\sigma_A^2 \geq \eta(K_0)^2$, the difference in the ranges of retail prices with options and without options simplifies to $\frac{B_0}{4e_L}[\frac{K_H e_H - K_0}{e_H}] - \frac{a_L}{4} < 0$, where the last inequality follows from the fact that $K_H e_H - K_0 \leq 0$. Hence, when $\sigma_A^2 \geq \eta(K_0)^2$, the introduction of options lead to a less volatile retail price. This suggests that if the cost of expediting an order, K_H, can be reduced, it will lead to less volatile prices.

Now consider the volatility of retail prices when $\sigma_A^2 < \eta(K_0)^2$. In this case, the difference in the retail price ranges with and without options simplifies to $\frac{K_H B_0 - K_0 - (a_H - a_L)e_L}{4e_L} < 0$, where the inequality follows from the condition $K_H < \overline{K}_H$ in Proposition 2. Therefore, the introduction of options leads to a reduction in retail price volatility in this case as well.

Proposition 2 states that in the low state, none of the options are exercised. In this state, the retail price, $S_L = \frac{3a_L}{4} + \frac{K_0 - K_H e_H}{4e_L}$. From equation (19), the feasible set of strikes are:

$$\frac{a_L e_L + K_0 - K_H e_H}{2e_L} \leq X \leq \frac{a_H + K_H}{2}.$$

Since the lowest feasible strike price is higher than the equilibrium retail price, S_L, at first glance it appears that the retailer is allowing some in-the-money reorder options to expire worthless! However, this is not the case. By exercising an additional option in the presence of a downward sloping demand curve, the retailer increases the amount of items that are released into the market, and this affects the retail price in an adverse way.

If the variance is high ($\sigma_A^2 > \eta^2$) then the introduction of option contracts does affect the

equilibrium wholesale price. In particular, the introduction of options increases the wholesale price from $\frac{a_H e_H + K_0}{2}$ to $\frac{A_0 + K_0}{2}$. In this case, the retailer's inventory holdings drop from $\frac{a_H e_H - K_0}{4\delta e_H}$ to $\frac{a_L}{4\delta} + \frac{K_H e_H - K_0}{4\delta e_L}$. The option the retailer has of not releasing inventory is now made worthless by the call option. Specifically, the retailer commits to releasing all inventory ordered at date 0, regardless of the state. If the high state occurs then all U^* options are exercised, increasing the amount released to $\frac{a_H - K_H}{4\delta}$.

The table below compares the wholesale and state dependent retail prices, the retailer order quantities, and the quantities released into the market, for the case with and without options.

	Wholesale Price S_0	Order Quantity I_0	Option Price C_0	Order Quantity U_0
No Options $\sigma_A^2 \le \eta^2$ $\sigma_A^2 > \eta^2$	$\frac{A_0+K_0}{2}$ $\frac{a_H e_H + K_0}{2}$	$\frac{A_0-K_0}{4\delta B_0}$ $\frac{e_H a_H - K_0}{4\delta e_H}$	- -	- -
With Options	$\frac{A_0+K_0}{2}$	$\frac{a_L}{4\delta} + \frac{K_H e_H - K_0}{4\delta e_L}$	$\frac{(a_H+K_H-2X)e_H}{2}$	$\frac{a_H-a_L}{4\delta} - \frac{K_H B_0 - K_0}{4\delta e_L}$

	Amount Released		Retail Price	
	Low State $q_L + v_L$	High State $q_H + v_H$	Low State S_L	High State S_H
No Options $\sigma_A^2 \le \eta^2$ $\sigma_A^2 > \eta^2$	$\frac{A_0-K_0}{4\delta B_0}$ $\frac{a_L}{2\delta}$	$\frac{A_0-K_0}{4\delta B_0}$ $\frac{a_H e_H - K_0}{4\delta e_H}$	$a_L - \frac{A_0-K_0}{4B_0}$ $\frac{a_L}{2}$	$a_H - \frac{A_0-K_0}{4B_0}$ $\frac{3a_H}{4} + \frac{K_0}{4e_H}$
With Options	$\frac{a_L}{4\delta} + \frac{K_H e_H - K_0}{4\delta e_L}$	$\frac{a_H-K_H}{4\delta}$	$\frac{3a_L}{4} + \frac{K_0 - K_H e_H}{4e_L}$	$\frac{3a_H}{4} + \frac{K_H}{4}$

If a cash settled call option existed with a strike price of X on the retail price, then its value would be C_{CASH}, where

$$C_{CASH} = Max(0, \frac{3a_H}{4} + \frac{K_H}{4} - X)\, e_H + Max(0, \frac{3a_L}{4} + \frac{K_0 - K_H e_H}{4e_L} - X)\, e_L \qquad (25)$$

This contract is fundamentally different from the reorder option. With the real option, the only reason the retailer exercises the contract, is to obtain an additional unit of inventory to release into the market. But because the demand curve is downward sloping, releasing an extra unit would result in lowering the retail price. Hence, exercise decisions alter retail prices, and this has a feedback effect into the value of the option. The cash settled option contract is more valuable than the reordering option.

The above observations are summarized below.

Corollary

1. *The introduction of reorder option contracts results in the wholesale price charged by the manufacturer staying the same or increasing.*

2. *The introduction of reorder option contracts results in a less volatile retail price.*

3. *The introduction of reorder option contracts reduces the retailer's order quantity in period 0.*

4. *A cash settled call option contract on the retail price is more valuable than the reorder option with identical terms.*

5. *As the manufacturer's cost of expediting an order in period 1 increases, the option price increases, and the retailer's response reduces towards the case with no options.*

5 The Value of Supply Chain Option Contracts

In this section we analyze the impact of introducing reorder options on the profits of the manufacturer and the retailer.

Proposition 3

(i) The net present value of this project, for the manufacturer, given that no option contracts are used equals

$$M_0^* = \begin{cases} \frac{(A_0 - K_0)^2}{8\delta B_0} & \text{if} \quad \sigma_A^2 \leq \eta^2(K_0) \\ \frac{(a_H e_H - K_0)^2}{8\delta e_H} & \text{if} \quad \sigma_A^2 > \eta^2(K_0), \end{cases} \tag{26}$$

and with option contracts, the net present value is

$$M^* = \frac{a_H^2 e_H + a_L^2 e_L}{8\delta} - \frac{1}{8\delta e_L}[2a_L e_L K_0 - K_0^2 + 2e_H K_0 K_H - e_H^2 K_H^2 + e_H e_L(2a_H K_H - 2a_L K_H - K_H^2)]. \tag{27}$$

(ii) The net present value of this project , for the retailer, given that no option contracts are used equals

$$R_0^* = \begin{cases} \frac{(A_0 - K_0)^2}{16 B_0 \delta} & \text{if} \quad \sigma_A^2 \leq \eta^2(K_0) \\ \frac{a_H^2 e_H^2 + 4a_L^2 e_L e_H - 2K_0 a_H e_H + K_0^2}{16\delta e_H} & \text{if} \quad \sigma_A^2 > \eta^2(K_0) \end{cases} \tag{28}$$

and with option contracts, the net present value is

$$R^* = \frac{M^*}{2}. \tag{29}$$

Proof: See Appendix

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Proposition 4

(i) If $\sigma_A^2 < \eta^2(K_0)$, then the manufacturer will issue options if and only if $\widetilde{K_L} \leq K_H \leq \widetilde{K_H}$.

(ii) If $\sigma_A^2 \geq \eta^2(K_0)$, then the manufacturer will issue options if and only if $\widetilde{K_L} \leq K_H \leq \widetilde{\widetilde{K_H}}$,
where

$$\widetilde{\widetilde{K_H}} = Min[\widetilde{K_H}, K_H^*]$$

and

$$K_H^* = \overline{K}_H - \frac{\rho}{B_0}\sqrt{Max[0, T_1]}$$

and

$$T_1 = (a_H^2 - a_l^2 \rho - 2a_H a_L)e_H^2 + (2a_L - 2a_H + K_0)K_0 e_H.$$

Proof: See Appendix

Proposition 5

(i) If $\sigma_A^2 < \eta^2(K_0)$ and $\widetilde{K_L} \leq K_H \leq \widetilde{K_H}$, then the manufacturer introduces options and the retailer will be better off.

(ii) If $\sigma_A^2 \geq \eta^2(K_0)$ and $\widetilde{K_L} \leq K_H \leq \widetilde{\widetilde{K_H}}$, then the manufacturer introduces options and the retailer will be worse off.

Proof: See Appendix

The value of the project for the manufacturer, as given in equation (27), can be interpreted as a fraction of the value of a claim on the *square* of the random variable representing the intercept of the demand curve in period 1. The value to the retailer is exactly one half of this value. When volatility is small, $(\sigma_A^2 \leq \eta^2)$ the retailer benefits from the introduction of the reorder option. However, when there is sufficient uncertainty, $(\sigma_A^2 > \eta^2)$ then the option contracts actually reduce the benefits to the retailer. That is the retailer would prefer an equilibrium without option contracts which would force the manufacturer to charge a low price for the entire product quantity. In other words, the retailer prefers the lower price, S_0, with the possibility of retaining some product if the low state occurs, to the flexibility of costly option contracts, C_0, and a higher wholesale, S_0.

6 Pricing with Supply Chain Put Contracts

So far we have focused on the role of reorder options by a manufacturer, who acts as a Stackelberg leader, in a supply chain with a downstream retailer who has monopoly in a marketplace. We now consider what happens to this equilibrium if the manufacturer introduces put options that allow the retailer to return to the manufacturer any unsold items for a salvage price of X.

Proposition 6

The retailer's optimal ordering policy when the manufacturer offers a wholesale price of S_0, and return options with price P_0 and exercise price X is identical to the ordering policy in a problem where the manufacturer offers a wholesale price of S_0, only call options at price C_0, with strike X, provided $C_0 = S_0 + P_0 - B_0 X$.

Proof: The result is immediate from put-call parity.[9]

The manufacturer recognizes that the retailer can synthetically construct calls using put-call parity. As a result, solving the pricing problem for reordering options will then lead to identifying the price of return puts. The following Corollaries are immediate consequences of Proposition 6 and the results of Sections 3 and 4.

Corollary 1

For the problem in which the manufacturer offers only return options with salvage price X:

(i) The equilibrium wholesale price, the state dependent retail prices of the items, and the fair price of the reorder options, derived under the conditions of Proposition 2, remain unaltered.

(ii) The optimal price the manufacturer charges for the return option, P_0, is given by

$$P_0 = e_L \left(X - \frac{a_L e_L + K_0 - K_H e_H}{2e_L} \right),$$

where the strike price is curtailed as:

$$\frac{a_L e_L + K_0 - K_H e_H}{2e_L} \leq X \leq \frac{a_H + K_H}{2}.$$

(iii) A return option is less valuable than a cash settled put option on the retail price.

Corollary 2

For the problem in which the manufacturer offers both reorder options with exercise price X_1 and return options with salvage price X_2:

(i) The wholesale and retail prices are unchanged from what they would be if only reorder options were offered.

(ii) The price of the reorder (return) option does not depend on whether the manufacturer offered the return (reorder) contract provided the strike prices are curtailed as

$$\frac{a_L e_L + K_0 - K_H e_H}{2e_L} \leq X_1, X_2 \leq \frac{a_H + K_H}{2}.$$

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[9]For each put option that the retailer purchases in period 0, the immediate cost is $S_0 + P_0$. Now, at the end of period 1, the retailer can either return the unit of product and earn X or keep it and sell it to the market. Regardless of the choice, the retailer can always achieve exactly the same result by returning the product, and then deciding whether to repurchase it or not from the manufacturer at a price X. Each put option is therefore equivalent to a riskless income of X in period 1 plus a call option with strike X.

As a generalization of Corollary 2, the wholesale and retail prices will remain unchanged if the manufacturer offers an array of contracts with different strike prices in the above interval, as long as these contracts are priced accordingly.

The retailer will then construct a portfolio of reorder/return options according to his/her personal risk preferences and assessments on the probability p of the high demand curve.

7   Extension to Continuous States

So far we have limited our discussion to cases where the maximum potential size of the market, represented by α was Bernoulli. In this section, we consider the case where α has a continuous distribution and identify the optimality conditions for the retailer's and manufacturer's problems. We illustrate that the nature of our earlier results remain unchanged. To simplify the equations, in this section we assume that the manufacturer's costs are all equal to zero.

Assume that α follows a continuous distribution with density $f(a)$, $a \in [a_L, a_H]$, with $a_L \geq 0$ and $a_H \leq \infty$. Let μ_A and σ_A^2 represent the expected value and variance of the distribution.

Lemma 6

The retailer's optimal response to the manufacturer's wholesale price of S_0, and no use of options is $I^*(S_0)$, where

$$I^*(S_0) = \begin{cases} I_0(S_0) & if \quad S_0 \leq A_0 - a_L B_0 \\ \frac{A_0 - S_0}{2\delta B_0} & if \quad A_0 - a_L B_0 < S_0 \leq A_0 \\ 0 & if \quad S_0 > A_0 \end{cases}$$

where $I_0(S_0)$ is the unique solution to

$$B_0 \int_{2\delta I_0}^{a_H} (a - 2\delta I_0) f(a) da = S_0$$

and in period 1, the optimal amount to release is

$$q^* = Min[I^*(S_0), \frac{a}{2\delta}]$$

Proof: See Appendix

The following Proposition uses the above lemma to characterize the optimal policy for the manufacturer to adopt, if options are not used.

Proposition 7

(i) *The optimal pricing policy for the manufacturer to follow is*

$$S^* = \begin{cases} \frac{A_0}{2} & \text{if} \quad \frac{A_0}{2} < a_L B_0 \\ S_0^* & \text{if} \quad \frac{A_0}{2} \geq a_L B_0 \end{cases} \tag{30}$$

where S_0^* is the solution to :

$$S_0^* = 2\delta I_0(S_0^*) B_0 \int_{2\delta I_0(S_0^*)}^{a_H} f(a) da \tag{31}$$

(ii) *The optimal ordering and selling policy for the retailer when the manufacturer follows the above policy is:*

$$I^* = \begin{cases} \frac{A_0}{4\delta B_0} & \text{if} \quad \frac{A_0}{2} < a_L B_0 \\ I_0^* & \text{if} \quad \frac{A_0}{2} \geq a_L B_0 \end{cases} \tag{32}$$

where I_0^* is the solution to

$$\int_{2\delta I_0^*}^{a_H} a f(a) da = 4\delta I_0^* \int_{2\delta I_0^*}^{a_H} f(a) da. \tag{33}$$

and

$$q^* = Min[I_0^*, \frac{a}{2\delta}] \tag{34}$$

Proof: See Appendix

When the manufacturer considers the problem with reorder options, simple analytical characterizations of the optimal pricing policy are not available and numerical computations are required to establish equilibrium prices. In period 1, given that demand a is realized, the quantity released, and the amount of options exercised by the retailer remains unchanged from our earlier analysis. That is, the optimal decisions in period 1 do not depend on the distribution of demand. Specifically, we have:

$$q^*(I, U|a) = \begin{cases} \frac{a}{2\delta} & \text{if} \quad 0 < a \leq 2\delta I \\ I & \text{if} \quad a > 2\delta I \end{cases} \tag{35}$$

$$v^*(I, U|a) = \begin{cases} 0 & \text{if} \quad a \leq 2\delta I + X \\ \frac{a-X}{2\delta} - I & \text{if} \quad 2\delta I + X < a \leq 2\delta(I+U) + X \\ U & \text{if} \quad a > 2\delta(I+U) + X. \end{cases} \tag{36}$$

The total quantity of items released into the market is

$$q^* + v^* = \begin{cases} \frac{a}{2\delta} & \text{if} \quad 0 < a \leq 2\delta I \\ I & \text{if} \quad 2\delta I < a \leq 2\delta I + X \\ \frac{a-X}{2\delta} & \text{if} \quad 2\delta I + X < a \leq 2\delta(I+U) + X \\ I + U & \text{if} \quad a > 2\delta(I+U) + X \end{cases} \tag{37}$$

and the retail price is $S_R(I, U|a)$ where

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$$S_R(I,U|a) = \begin{cases} \frac{a}{2} & \text{if} \quad 0 < a \le 2\delta I \\ a - \delta I & \text{if} \quad 2\delta I < a \le 2\delta I + X \\ \frac{a+X}{2\delta} & \text{if} \quad 2\delta I + X < a \le 2\delta(I+U) + X \\ a - \delta(I+U) & \text{if} \quad a > 2\delta(I+U) + X \end{cases} \tag{38}$$

Now consider the retailer's problem in period 0. We have:

$$\begin{aligned} R_0(I,U) &= -IS_0 - UC_0 + B_0 \int_0^{2\delta I} \frac{a^2}{4\delta} f(a)da \\ &+ B_0 \int_{2\delta I}^{2\delta I + X} I(a - \delta I)f(a)da + B_0 \int_{2\delta I + X}^{2\delta(I+U)+X} [\frac{a^2 - X^2}{4\delta} - \frac{X(a-X)}{2\delta + XI}]f(a)da \\ &+ B_0 \int_{2\delta(I+U)}^{\infty} [(I+U)(a - \delta(I+U) - XU]f(a)da \end{aligned}$$

Using Leibnitz rule, and simplifying, the first order conditions are:

$$\begin{aligned} \frac{\partial R_0}{\partial I} &= -S_0 + B_0 \int_{2\delta I}^{2\delta I + X} (a - 2\delta I)f(a)da + B_0 X \int_{2\delta I + X}^{2\delta(I+U)+X} f(a)da \\ &+ B_0 \int_{2\delta(I+U)+X}^{\infty} [a - 2\delta(I+U)]f(a)da = 0 \end{aligned} \tag{39}$$

$$\frac{\partial R_0}{\partial U} = -C_0 + B_0 \int_{2\delta(I+U)+X}^{\infty} [a - 2\delta(I+U)]f(a)da = 0 \tag{40}$$

Let $\pi \equiv (S_0, C_0)$, and let $I^*(\pi)$ and $U^*(\pi)$ be the optimal ordering response by the retailer. Then, the manufacturer's profit at date 0, is

$$\begin{aligned} M(\pi) &= I^*(\pi)S_0 + U^*(\pi)C_0 + B_0 X \int_0^{\infty} v^*(I^*, U^*|a)f(a)da \\ &= I^*(\pi)S_0 + U^*(\pi)C_0 + B_0 \int_{2\delta I^*(\pi)+X}^{2\delta(I^*(\pi)+U^*(\pi))+X} [\frac{a-X}{2\delta} - I^*(\pi)]f(a)da \\ &+ B_0 \int_{2\delta(I^*(\pi)+U^*(\pi))+X}^{\infty} [a - \delta(I^*(\pi) + U^*(\pi))]f(a)da \end{aligned} \tag{41}$$

For general distributions on α it is not possible to obtain simple solutions for the wholesale price and the cost of the reorder options. However, for specific distributions, the first order conditions may simplify.

To illustrate the results for continuous states, consider the case where α has a uniform distribution. For this special case, the optimal policies in Proposition 4 simplify. The equilibrium wholesale price has the following form:

$$S^* = \begin{cases} \frac{A_0}{2} & \text{if} \quad \frac{A_0}{2} < a_L B_0 \\ \frac{2B_0 a_H^2}{9(a_H - a_L)} & \text{if} \quad \frac{A_0}{2} \ge a_L B_0 \end{cases} \tag{42}$$

Now substituting $a_L = \mu_A - \sqrt{3}\sigma_A$, and $a_H = \mu_A + \sqrt{3}\sigma_A$, and letting $c = \frac{\sigma_A}{\mu_A}$ be the coefficient of variation, the above policy can be reexpressed as

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$$S^* = \begin{cases} \frac{A_0}{2} & \text{if} \quad c < \frac{1}{2\sqrt{3}} \\ \frac{A_0(1+\sqrt{3}c)^2}{9\sqrt{3}c} & \text{if} \quad c \geq \frac{1}{2\sqrt{3}} \end{cases} \tag{43}$$

The above equation shows that as the uncertainty in the demand curve increases, the whole-sale price either stays the same or it decreases.

For this pricing policy, the retailer's optimal response is

$$I^* = \begin{cases} \frac{A_0}{4\delta B_0} & \text{if} \quad c < \frac{1}{2\sqrt{3}} \\ \frac{A_0(1+\sqrt{3}c)}{6\delta B_0} & \text{if} \quad c \geq \frac{1}{2\sqrt{3}} \end{cases} \tag{44}$$

The retailer's optimal response to an increase in uncertainty in the demand curve is either unchanged, or to order more units.

The equilibrium retail price, $S_R(a)$ say, is given by

$$S_R(a) = \begin{cases} a - \gamma_c & \text{if} \quad a > 2\gamma_c \\ \frac{a}{2} & \text{if} \quad a \leq 2\gamma_c \end{cases} \tag{45}$$

where

$$\gamma_c = \begin{cases} \frac{A_0}{4B_0} & \text{if} \quad c < \frac{1}{2\sqrt{3}} \\ \frac{A_0}{6B_0}(1+\sqrt{3}c) & \text{if} \quad c \geq \frac{1}{2\sqrt{3}} \end{cases}$$

For any given σ_A, the retail price in period 1, increases as a increases, at a linear rate of one half, up to a point, γ_c, and then the sensitivity of prices to demand increases to one.

To illustrate the advantages of introducing reordering options, reconsider the case where α has a uniform distribution. Figure 3 compares the equilibrium wholesale prices as uncertainty in the demand curve increases, for the case where reorder options are used and not used. When no reordering options are used, the price stays unchanged, up to a critical point, and then declines. In contrast, when reordering options are used, the equilibrium wholesale price is insensitive to the uncertainty in the demand curve.

Figure 3 Here

Figure 4 compares the volatility of retail prices as uncertainty in the demand curve increases. The figure confirms the fact that volatility of retail prices are lower when the manufacturer introduces reordering options.

Figure 4 Here

Figure 5a compares the value of the project with and without options for the manufacturer, and Figure 5b repeats the analysis for the retailer. The option program is always attractive for the manufacturer, but not so for the retailer.

Figure 5a and 5b Here

The example with a continuous distribution reconfirms the results that we obtained when uncertainty in the demand curve was represented by a simple Bernoulli random variable.

8   Conclusion

This article has considered the problem of option pricing when the demand curve is downward sloping. Our particular application arises in a supply chain setting, where a manufacturer produces an item that is sold through a retailer. In this setting the manufacturer charges a fixed wholesale price in period 0. The retailer responds to this price by ordering a quantity in period 0. The retailer bears quantity risk, and in period 1, based on the demand curve, determines the optimal amount of inventory to release. We have shown that if the manufacturer introduces option contracts, that shift some of the quantity risk away from the retailer, then the equilibrium prices adjust, in a way that benefits the manufacturer, and may benefit or harm the retailer. When volatility of the demand curve is low, then the retailer benefits from supply chain options. On the other hand, when volatility is high, the retailer is worse off. The manufacturer is always keen to issue supply chain option contracts as long as the strike prices are curtailed, and their prices dictated by the pricing equation we derived.

We have derived the equilibrium prices for the supply chain options. These contracts are less valuable than otherwise identical cash settled financial contracts on the retail price. With the real option, the only reason to exercise the contract, is to obtain an additional unit of inventory to release into the market. But because the demand curve is downward sloping, releasing an extra unit would result in lowering the retail price. Hence, exercise decisions alter retail prices, and this has a feedback effect into the value of the option.

The equilibrium option price that we establish does not depend on the slope of the linear demand curve, as long as the curve is downward. We also showed that once reordering call options were introduced, the equilibrium wholesale price would not be influenced by the addition of return put options, as long as the strike prices are appropriately curtailed.

An important feature of our modeling process is that our results are independent of risk attributes of the retailer and manufacturer. If either the manufacturer or the retailer is risk averse and wanted to hedge the uncertainty of cash flows in this project, then, with a linear demand curve, precise hedging could be accomplished with financial instruments that have payoffs linked to the square of the price of a traded instrument.

In our analysis, we assumed away the manufacturer's costs of production. If a fixed setup cost and a per unit variable cost is included, the analysis goes through with only minor changes and little additional insights. If, on the other hand, the costs of production depend on whether the items were ordered in a regular manner, in period 0, or through exercise of an option, in period 1, then the analysis becomes more complex. Such cost differentials might arise if the manufacturer has to expedite orders that arise through the late exercising of reordering contracts. It remains for future research to extend the analysis to cases where the manufacturer offers the retailer American reorder options and return options that extend over multiple time periods. Other extensions include examining the consequence of allowing for more complex supply chains, where a manufacturer distributes a product through a network of retailers who compete in an oligopolistic market.

Appendix

Proof of Lemma 1

In period 1 the retailer solves the problem:

$$R_1(q^*|I, a) = Max_{0 \leq q \leq I} q(a - \delta q)$$

The optimal solution is:

$$q^* = \begin{cases} \frac{a}{2\delta} & \text{if} & \frac{a}{2\delta} < I \\ I & \text{if} & \frac{a}{2\delta} \geq I \end{cases}$$

Now consider the retailer's problem in period 0. We have:

$$R_0(I) = -IS_0 + e_L R_1(q_L^*|I) + e_H R_1(q_H^*|I).$$

There are two cases that need to be considered.

1. $0 \leq I \leq \frac{a_L}{2\delta}$

2. $\frac{a_L}{2\delta} \leq I \leq \frac{a_H}{2\delta}$

For case 1 we have

$$R_0(I_1^*) = Max_{0 \leq I \leq \frac{a_L}{2\delta}} \{-IS_0 + e_L I(a_L - \delta I) + e_H I(a_H - \delta I)\}.$$

The optimal solution is

$$I_1^* = \begin{cases} \frac{A_0 - S_0}{2\delta B_0} & \text{if} & (a_H - a_L)e_H \leq S_0 \leq A_0 \\ \frac{a_L}{2\delta} & \text{if} & S_0 < (a_H - a_L)e_H \end{cases}$$

For the second case, we have:

$$R_0(I_2^*) = Max_{\frac{a_L}{2\delta} \leq I \leq \frac{a_H}{2\delta}} \{-IS_0 + e_L \frac{a_L}{2\delta}(a_L - \delta \frac{a_L}{2\delta} + e_H I(a_H - \delta I)\}.$$

The optimal solution is

$$I_2^* = \begin{cases} \frac{a_H e_H - S_0}{2\delta e_H} & \text{if} & 0 < S_0 < (a_H - a_L)e_H \\ \frac{a_L}{2\delta} & \text{if} & S_0 \geq (a_H - a_L)e_H \end{cases}$$

The result then follows.

Proof of Proposition 1

Given the results of Lemma 1 the manufacturer's profit as a function of S_0 is

$$M_0(S_0) = \begin{cases} M_1(S_0) & \text{if } 0 < S_0 \le (a_H - a_L)e_H \\ M_2(S_0) & \text{if } (a_H - a_L)e_H < S_0 \le A_0 \\ 0 & \text{if } S_0 > A_0 \end{cases},$$

where

$$M_1(S_0) = \frac{e_H a_H - S_0}{2\delta e_H}(S_0 - K_0)$$

and

$$M_2(S_0) = \frac{A_0 - S_0}{2\delta B_0}(S_0 - K_0).$$

Therefore, the maximum value of $M_0(S_0)$ is given by

$$M_0^* = M_0(S_0^*) = \max\{M_1^*, M_2^*\},$$

where

$$M_1^* = M_1(S_1^*) = Max_{0 \le S_0 \le (a_H - a_L)e_H} M_1(S_0) \tag{A.1}$$

and

$$M_2^* = M_2(S_2^*) = Max_{(a_H - a_L)e_H \le S_0 \le A_0} M_2(S_0). \tag{A.2}$$

The maximizing S_0 value for the problem in (A.1) is

$$S_1^* = \begin{cases} \frac{e_H a_H + K_0}{2} & \text{if } K_0 \le \overline{K} \\ (a_H - a_L)e_H & \text{if } K_0 > \overline{K} \end{cases},$$

where the value of $\overline{K}$ is determined by solving the inequality $\frac{e_H a_H + K_0}{2} \le (a_H - a_L)e_H$ in K_0, i.e.,

$$\overline{K} = a_H e_H - 2a_L e_H = (a_H - a_L)e_H - a_L e_H.$$

In this case the manufacturer's profits are:

$$M_1^* = \begin{cases} \frac{(a_H e_H - K_0)^2}{8\delta e_H} & \text{if } K_0 \le \overline{K} \\ \frac{[(a_H - a_L)e_H - K_0]a_L}{2\delta} & \text{if } K_0 > \overline{K}. \end{cases}$$

Similarly, the maximizing S_0 value for the problem in (A.2) is

$$S_2^* = \begin{cases} \frac{A_0 + K_0}{2} & \text{if } K_0 \ge \underline{K} \\ (a_H - a_L)e_H & \text{if } K_0 < \underline{K}, \end{cases},$$

where the value of $\underline{K}$ is determined by solving the inequality $\frac{A_0 + K_0}{2} \ge (a_H - a_L)e_H$ in K_0, i.e.,

$$\underline{K} = a_H e_H - 2a_L e_H - a_L e_L = (a_H - a_L)e_H - a_L e_H(1 + \rho),$$

where $\rho = e_L/e_H$.

In this case the manufacturer's profits are:

$$M_2^* = \begin{cases} \frac{(A_0-K_0)^2}{8\delta B_0} & \text{if} \quad K_0 \geq \underline{K} \\ \frac{[(a_H-a_L)e_H - K_0]a_L}{2\delta} & \text{if} \quad K_0 < \underline{K} \end{cases}$$

Since $\underline{K} < \overline{K}$, the solution can be summarized as follows

$$S_0^* = \begin{cases} \frac{a_H e_H + K_0}{2} & \text{if} \quad K_0 \leq \underline{K} \\ \frac{A_0 + K_0}{2} & \text{if} \quad K_0 \geq \overline{K} \\ \frac{e_H a_H + K_0}{2} & \text{if} \quad \underline{K} < K_0 < \overline{K} \text{ and } M_1^* > M_2^* \\ \frac{A_0 + K_0}{2} & \text{if} \quad \underline{K} < K_0 < \overline{K} \text{ and } M_1^* \leq M_2^* \end{cases} \tag{A.3}$$

The solution can be simplified. Substituting the values of M_1^* and M_2^* for the case where $\underline{K} < K_0 < \overline{K}$, the relationship $M_1^* - M_2^* \leq 0$ can be written as the following quadratic inequality in K_0

$$K_0^2 + 2(a_H - a_L)e_H K_0 - 2a_H a_L e_H^2 - a_L^2 e_H e_L + a_H^2 e_H^2 \leq 0.$$

The two roots of this quadratic function in K_0 are

$$\begin{aligned} k_1 &= (a_H - a_L)e_H - a_L e_H \sqrt{1+\rho} \\ k_2 &= (a_H - a_L)e_H + a_L e_H \sqrt{1+\rho} \end{aligned}$$

and the inequality is valid for $k_1 \leq K_0 \leq k_2$. In addition, in this case it must be true that $\underline{K} < K_0 < \overline{K}$. It is easy to see from the definitions of $\underline{K}$ and $\overline{K}$ that $k_2 > \overline{K}$ and $\underline{K} < k_1 < \overline{K}$. Summarizing the above relationships, it follows that, in the case where $\underline{K} < K_0 < \overline{K}$, $M_1^* - M_2^* \leq 0$ is true if and only if $k_1 < K_0 < \overline{K}$. Based on this, equation (A.3) can simplified as follows.

$$S_0^* = \begin{cases} \frac{a_H e_H - K_0}{2} & \text{if} \quad K_0 \leq k_1 \\ \frac{A_0 - K_0}{2} & \text{if} \quad K_0 > k_1 \end{cases} \tag{A.4}$$

Further, the condition $K_0 \leq k_1$ can be reexpressed as an equivalent condition involving the volatility of the demand curve. In particular, substituting $a_H = \frac{A_0}{B_0} + \sqrt{\rho}\sigma_A$ and $a_L = \frac{A_0}{B_0} - \frac{\sigma_A}{\sqrt{\rho}}$ into the expression for k_1, we obtain:

$$K_0 \leq k_1$$

if and only if

$$\sigma_A \geq \eta(K_0) \equiv \frac{\sqrt{\rho}}{1 + \rho + \sqrt{1+\rho}}\left[(1+\rho)\frac{K_0}{B_0} + \sqrt{1+\rho}\frac{A_0}{B_0}\right].$$

Therefore equation (9) for the optimal wholesale price is established, with the corresponding optimal value M_0^* given by (26).

To complete the proof, it remains to determine the retailer's policy in each of the two cases.

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In the case when $K_0 \leq k_1$, we have that $S_0^* = \frac{a_H e_H + K_0}{2} < (a_H - a_L)e_H$, and the results of Lemma 1 imply that

$$I^* = \frac{a_H e_H - S_0^*}{2\delta e_H} = \frac{a_H e_H - K_0}{4\delta e_H}.$$

Furthermore, it is easy to see that $I^* < \frac{a_H}{2\delta}$, therefore, $q_H^* = I^*$. On the other hand,

$$I^* - \frac{a_L}{2\delta} = \frac{a_H e_H - 2a_L e_H - K_0}{4\delta e_H} = \frac{\overline{K} - K_0}{4\delta e_H} > 0,$$

since $K_0 \leq k_1 < \overline{K}$. Thus, $I^* > \frac{a_L}{2\delta}$ and $q_L^* = \frac{a_L}{2\delta}$.

Finally, in the case when $K_0 > k_1$, it follows that

$$(a_H - a_L)e_H < S_0^* = \frac{A_0 + K_0}{2} < A_0,$$

and the results of Lemma 1 imply that

$$I^* = \frac{A_0 - S_0^*}{2\delta B_0} = \frac{A_0 - K_0}{4\delta B_0}.$$

Furthermore,

$$I^* - \frac{a_L}{2\delta} = \frac{A_0 - 2a_L B_0 - K_0}{4\delta B_0} = = \frac{a_H e_H - 2a_L e_H - a_L e_L}{4\delta B_0} = \frac{\underline{K} - K_0}{4\delta B_0} < 0,$$

since $K_0 > k_1 > \underline{K}$. Hence $I^* < \frac{a_L}{2\delta} < \frac{a_H}{2\delta}$, and $q_L^* = q_H^* = I^*$. This completes the proof.

Proof of Lemma 2

Take (q, v), with $q < I$, and $v > 0$. Let (q', v') be set such that $q' = q + \epsilon$, $v' = v - \epsilon$, with $0 < \epsilon < Min[I - q, v]$. Then, $q' + v' = q + v$, and

$$R_1(q', v'|a) = ((q'+v')(a-\delta(q'+v')) - Xv' = (q+v)(a-\delta(q+v)) - Xv + X\epsilon = R(q, v) + X\epsilon > R(q, v).$$

That is (q', v') improves over (q, v).

Proof of Lemma 3

Let

$$
\begin{aligned}
R_1^{(1)}(q) &= R_1(q,0) = q(q - \delta q) \\
R_1^{(2)}(v) &= R_1(I,v) = (I+v)(a - \delta(I+v)) - Xv.
\end{aligned}
$$

Then, Lemma 2 implies that

$$
\max_{0 \le q \le I, 0 \le v \le U} R_1(q,v) = \max\{\max_{0 \le q \le I} R_1^{(1)}(q), \max_{0 \le v \le U} R_1^{(2)}(v)\}.
$$

We also have

$$
\begin{aligned}
\frac{dR_1^{(1)}}{dq}(I) &= a - 2\delta I \\
\frac{dR_1^{(2)}}{dv}(0) &= a - 2\delta I - X
\end{aligned}
$$

First, assume $\frac{dR_1^{(1)}}{dq}(I) < 0$. This implies that $R_1^{(1)}(q)$ is maximized for $q = q^* = \frac{a}{2\delta} < I$. Further, for this case, $\frac{dR_1^{(2)}}{dv}(0) < 0$. This and the concavity of $R_1^{(2)}(v)$ imply that $\frac{dR_1^{(2)}}{dv}(v) < 0$ for all $v > 0$. Hence $Max_{0 \le v \le U}[R_1^{(2)}(v)] = R_1^{(2)}(0)$.

This implies that:

$$
R_1^{(2)}(v) \le R_1^{(2)}(0) = R_1(I,0) = R^{(1)}(I) \le R_1^{(1)}(q^*)
$$

which means that $q = q^*, v = 0$ is optimal in this case.

Second, consider the case where $I < \frac{a}{2\delta}$. In this case, $\frac{dR_1^{(1)}}{dq}(I) > 0$. Hence:

$$R_1^{(1)}(q) \leq R_1^{(1)}(I) = R_1^{(2)}(0) \leq R_1^{(2)}(v^*)$$

where v^* is the value maximizing $R_1^{(2)}(v)$. Specifically, using the first order conditions for $R_1^{(2)}(v)$ we obtain:

$$v^* = \begin{cases} 0 & \text{if} \quad I + U > \frac{a-X}{2\delta} \\ \frac{a-X}{2\delta} - I & \text{if} \quad \frac{a-X}{2\delta} > I \end{cases}$$

Proof of Lemma 4

First, consider the regions

1. $I < \frac{a_L - X}{2\delta}$ and $U > \frac{a_H - X}{2\delta} - I$.

2. $\frac{a_L - X}{2\delta} < I < \frac{a_H - X}{2\delta}$ and $U > \frac{a_H - X}{2\delta} - I$.

3. $I > \frac{a_H - X}{2\delta}$ and $U > 0$.

In each of t hese regions, regardless of which state occurs in the future, the maximum number of options that can be exercised, U, is never attained. Hence, if $C_0 > 0$, then clearly U can be reduced and the retailer can obtain savings. Hence the optimal solution for the retailer will never lie in these regions.

Second, we show that if $U > 0$ and $I < \frac{(a_L - X)}{2\delta}$, then (I, U) cannot be optimal.

If $X > a_L$, then $a_L - X < 0$ and $I > (a_L - X)/2\delta$. Now consider $X \leq a_L$. Take (I, U) such that $U > 0$ and $I \leq \frac{a_L - X}{2\delta}$. Then $v_L^*, v_H^* > 0$, for all $U > 0$. Now

$$R_0(I, U) = -S_0 I - C_0 U + e_L(I + v_L^*)(a_L - \delta(I + v_L^*) - X e_L v_L^* + e_H(I + v_H^*)(a_H - \delta(I + v_H^*) - X e_H v_H^*$$

Let $I' = I + \epsilon$, $U' = U - \epsilon$, where $0 < \epsilon < Min[v_L^*, v_H^*]$. Then $v_L' = v_L^* - \epsilon < U - \epsilon = U'$ and $v_H' = v_H^* - \epsilon < U - \epsilon = U'$ are feasible, but perhaps not optimal, exercise policies. Also, $I' + v_L' = I + v_L^*$, and $I' + v_H' = I + v_H^*$. Then:

$$R_0(I', U') \geq -S_0 I' - C_0 U' + e_L(I + v_L^*)(a_L - \delta(I + v_L^*)) - X e_L(v_L^* - \epsilon) + e_H(I + v_H^*)(a_H - \delta(I + v_H^*)) - X e_H(v_H^* - \epsilon)$$

Hence, $R_0(I', U') - R_0(I, U) \geq (C_0 - S_0 + X B_0)\epsilon > 0$. Therefore, (I, U) is not optimal.

The only region that remains when $U > 0$ is the region where $I > \frac{a_L - X}{2\delta}$ and $U \leq \frac{a_H - X}{2\delta} - I$. This completes the proof.

Proof of Lemma 5

First, consider the optimal solution for the retailer in the region $\frac{a_L - X}{2\delta} \leq I \leq \frac{a_H - X}{2\delta}$, and $0 < U \leq \frac{a_H - X}{2\delta}$ with $a_L > a_H - X$. In this region, corresponding to region R3 in Figure 1, we have

$$R_0(I_1^*, U_1^*) = Max[-IS_0 - UC_0 + e_L(I)(a_L - \delta I) + e_H(I + U)(a_H - \delta(I + U)) - e_H UX] \quad (A.5)$$

The optimal solution for this problem is given by equations (10) and (11). Further, for the solution to be interior, the conditions in equations (12), (13) and (14) must apply. Note that equation (12) is required in order for $I^* > 0$.

Second, consider the optimal solution for the retailer in the region $\frac{a_L - X}{2\delta} \leq I \leq \frac{a_L}{2\delta}$, and $0 < U \leq \frac{a_H - X}{2\delta}$ with $a_L < a_H - X$. In this region, corresponding to region R3 in Figure 2, the objective function is the same as in equation (A.5). However, for the solution to be interior, the condition in equations (12), (15) and (16) must now apply.

Finally, we consider the solution for the retailer in the region $\frac{a_L}{2\delta} \leq I \leq \frac{a_H - X}{2\delta}$, and $0 < U \leq \frac{a_H - X}{2\delta}$ with $a_L < a_H - X$. Over this region we have:

$$R_0(I_3^*, U_3^*) = Max[-IS_0 - UC_0 + e_L \frac{a_L}{2\delta}(a_L - \delta \frac{a_L}{2\delta}) + e_H(I + U)(a_H - \delta(I + U)) - e_H UX]$$

Let (I, U) be a point that is interior in this region. Now consider the point $(I - \epsilon, U + \epsilon)$ where $0 < \epsilon < min[I - \frac{a_L}{2\delta}, \frac{a_H - X}{2\delta} - I - U]$. Then

$$R_0(I - \epsilon, U + \epsilon) - R_0(I, U) = (S_0 - C_0 - Xe_H)\epsilon$$

If $0 < C_0 < S_0 - Xe_H$, then substituting options for inventory is beneficial, and the optimal solution will be at $I^* = \frac{a_L}{2\delta}$. If $C_0 \geq S_0 - Xe_H$, then there always will be a solution with no options, that at worst is equivalent. Hence, there is no solution that is strictly interior in this region, and the only candidate with a positive number of options is $I^* = \frac{a_L}{2\delta}$, when $0 < C_0 < S_0 - Xe_H$. However, under this condition, it follows that the solution defined in (10), (11) satisfies $I_2^* < \frac{a_L}{2\delta}$ and the optimal solution is then I_2^*, U_2^*. This completes the proof.

Proof of Proposition 2

Consider the manufacturer's profit function with options offered:

$$M_0 = I^*(S_0 - K_0) + U^*C_0 + e_H v_H^*(X - K_H) + e_L v_L^*(X - K_L).$$

From Lemma 5 it follows that, when S_0, C_0, X satisfy equation (12) and either (13) and (14), or (15) and (16), the retailer's optimal response is given by equations (10) and (11). Furthermore, for these values of I^* and U^*, the optimal number of options exercised is equal to $v_L^* = 0$ and $v_H^* = U^*$. Therefore,

$$M_0 = I^*(S_0 - K_0) + U^*(C_0 + e_H X - e_H K_H).$$

Substituting for I^* and U^* from (10) and (11) and differentiating with respect to S_0 and C_0 lead to first order conditions, with the following solution:

$$S_0^* = \frac{A_0 + K_0}{2} \tag{A.6}$$

$$C_0^* = \frac{a_H e_H + K_H e_H - 2X e_H}{2}. \tag{A.7}$$

For U^* and I^* to be nonnegative, the constraint from equation (12), in Lemma 5, reduces to

$$K_0 - K_H e_H \leq a_L e_L. \tag{A.8}$$

Further, to ensure that under this pricing policy it is optimal for the retailer to purchase options, we must check that the above solution satisfies equations (12), (13) and (14), or (12), (15) and (16). We first identify the conditions that ensure that the above solution does satisfy these equations. We then consider the case where the solution does not satisfy the conditions. In this second case, at the optimum, at least one of the above constraints must be binding.

Case 1:

From (A.7) it follows that $C_0^* > 0$ if $X < \frac{a_H + K_H}{2}$. In addition, $C_0^* > S_0^* - B_0 X$ if $X > \frac{a_L}{2} + \frac{K_0}{2e_L} - \frac{K_H e_H}{2e_L}$. Therefore, the lower bound on C_0 in (14) and (16) is valid if $\frac{a_L}{2} + \frac{K_0}{2e_L} - \frac{K_H e_H}{2e_L} \leq X \leq \frac{a_H + K_H}{2}$.

Regarding the upper bound on C_0, substituting (A.6) and (A.7) into the inequality

$$C_0^* - [S_0^* - XB_0 + e_L(a_H - a_L)]\frac{e_H}{B_0} \leq 0$$

reduces after some algebra to

$$B_0 K_H - K_0 \leq (a_H - a_L)e_L. \tag{A.9}$$

Finally substituting (A.6) and (A.7) into the inequality

$$C_0^* - (S_0^* - Xe_H) \leq 0$$

reduces to

$$K_0 - K_H e_H \geq -a_L e_L$$

which is always true, since we have assumed that $K_H e_H < K_0$. Summarizing the above discussion, the constraints on the costs, K_0 and K_H, ensure that bounds on C_0 are always valid, so that the retailer does use options. Regarding the retailer's response to this pricing policy, substituting (A.6) and (A.7) into (10) and (11) it follows that

$$I^* = \frac{a_L}{4\delta} + \frac{K_H e_H - K_0}{4\delta e_L} \quad \text{and} \quad U^* = \frac{a_H - a_L}{4\delta} - \frac{B_0 K_H - K_0}{4\delta e_L}.$$

Since $I^* < a_L/2\delta$, the quantity of product sold from inventory in period 1 is equal to $q_L = q_H = I^*$. The number of options exercised $v_L = 0, v_H = U^*$.

Case Two:

So far we have identified the conditions ensuring that the first order solution of the manufacturer's problem leads to a response by the retailer that includes options. If the first order solution to the manufacturer's optimization problem is not feasible, then, since the profit function is concave, the optimimum must occur when one or more of the boundary constraints (12), (13) and (14), or (12), (15) and (16)are tight. Since, the problem is uninteresting when $C_0 \leq Max[0, S_0 - XB_0]$, and since we know that the constraint $C_0^* \leq S_0^* - Xe_H$ will not be binding, we only need to consider the case when the constraint, $C_0 \leq [S_0 - XB_0 + e_L(a_H - a_L)]\frac{e_H}{B_0}$, is binding. In this case, it can be shown that $U_1^* = 0$, and $I_1^* = \frac{A_0 - S_0}{2\delta B_0}$. Further, from the two pricing restrictions,

$$\begin{aligned} C_0^* &< S_0^* - Xe_H \\ C_0^* &= [S_0^* - XB_0 + e_L(a_H - a_L)]\frac{e_H}{B_0}, \end{aligned}$$

we obtain $S_0^* > (a_H - a_L)e_H$. Therefore the retailer's optimal response is identical to the response with no options obtained from Lemma 1. This completes the proof.

Proof of Proposition 3

The expressions for M_0^*, M^*, R_0^*, R^* follow directly by substituting the manufacturer's and retailer's optimal policies from Proposition 1 and Lemma 6 into the corresponding profit expressions.

Proof of Proposition 4

(i) First, consider the case where $\sigma_A^2 < \eta^2(K_0)$ and $\widetilde{K}_L \leq K_H \leq \widetilde{K}_H$. Then, computing the difference between the manufacturer's profit with and without options, leads, upon simplification to:

$$M^* - M_0 = \frac{e_H}{8\delta e_L B_)}[a_L e_L - a_H e_L - K_0 + e_H K_H + K_H e_L]^2 > 0$$

which shows that, for this case, the manufacturer is better off with options.

(ii) Now consider the case where $\sigma_A^2 \geq \eta^2(K_0)$ and $\widetilde{K}_L \leq K_H \leq \widetilde{K}_H$. Then, computing the difference between the manufacturer's profit with and without options, leads, upon simplification to:

$$M^* - M_0 = \frac{N(K_H)}{8\delta e_H e_L}$$

where

$$N(K_H) = a_0 K_H^2 + b_0 K_H + c_0$$

and

$$
\begin{aligned}
a_0 &= e_H^3 + e_H^2 e_L \\
b_0 &= K_0 e_H e_L - K_0 e_H^2 - 2(a_H - a_L)e_L e_H^2 \\
c_0 &= a_L^2 e_H e_L^2 - 2K_0 a_L e_H e_L + 2K_0 a_H e_H e_L - K_0^2 e_L
\end{aligned}
$$

The smaller root of the quadratic function $N(K_H)$ is $K_H^* = \frac{-b_0}{2a_0} - \frac{\sqrt{b_0^2 - 4a_0 c_0}}{2a_0}$. Furthermore, $N(K_H)$ reaches a minimum at $K_H = \overline{K}_H$. Hence, over the interval $0 \leq K_H \leq \widetilde{\widetilde{K}}_H$, the manufacturer's profit with options is greater than the profit with no options. This completes the proof.

Proof of Proposition 5

First consider the case where $\sigma_A^2 < \eta^2(K_0)$ and $\widetilde{K}_L \leq K_H \leq \widetilde{K}_H$. In this case the manufacturer is better off using options. Moreover, the retailer's profit in this case is proportional to the manufacturer's profit, so whatever is best for the manufacturer is also best for the retailer.

Now, consider the case where $\sigma_A^2 \geq \eta^2(K_0)$ and $\widetilde{K}_L \leq K_H \leq \widetilde{\widetilde{K}}_H$.

In this case the manufacturer will introduce options. Further, the difference in the retailer's net present values is:

$$R_0^* - R^* = \frac{N(K_H)}{16\delta e_H e_L}$$

where

$$N(K_H) = a_0 K_H^2 + b_0 K_H + c_0$$

where

$$
\begin{aligned}
a_0 &= -e_H^2 B_0 \\
b_0 &= 2e_H^2(a_L e_L + a_H e_L + K_0) \\
c_0 &= K_0^2 e_L + 3a_L^2 e_L^2 e_H - 2K_0 a_H e_L e_H - 2K_0 a_L e_L e_H - k_0^2 e_H
\end{aligned}
$$

Substituting $K_H = \overline{K}_H$, we obtain:

$$N(\overline{K}_H) = \frac{e_L^2}{B_0}\left[a_1 K_0^2 + b_1 K_0 + c_1\right]$$

where

$$
\begin{aligned}
a_1 &= 1 \\
b_1 &= -2(a_H - a_L)e_H \\
c_1 &= a_H^2 e_H^2 - 2a_L a_H e_H^2 + 4a_L^2 e_H^2 + 3a_L^2 e_H e_L.
\end{aligned}
$$

Further, $b_1^2 - 4a_1 c_1 = -\frac{12a_L^2 e_H e_L^4}{B_0} < 0$. This implies that $N(\overline{K}_H) > 0$. Finally, differentiating $N(K_H)$ with respect to K_H and substituting $K_H = \overline{K}_H$, leads to $\frac{d}{dK_H}N(K_H)|_{\overline{K}_H} = 0$. This shows that the function $N(K_H)$ is positive over the interval $\widetilde{K}_L \leq K_H \leq \overline{K}_H$, and the result then follows.

Proof of Lemma 6

The retailer's objective in period 1 is to maximize:

$$R_1(q|I, a) = q(a - \delta q).$$

This leads to $q^* = Min[I, \frac{a}{2\delta}]$. Hence,

$$R_1(q^*|I, a) = \begin{cases} \frac{a^2}{2\delta} & \text{if} \quad a \leq 2\delta I \\ I(a - \delta I) & \text{if} \quad a > 2\delta I \end{cases}$$

Now consider period 0. We have:

$$R_0(I) = -IS_0 + B_0 \int_{a_L}^{2\delta I} \frac{a^2}{4\delta} f(a)da + B_0 \int_{2\delta I}^{a_H} I(a - \delta I) f(a)da$$

First, consider the case where $I \leq \frac{a_L}{2\delta}$. The above equation simplifies to

$$R_0(I) = -IS_0 + B_0 I A_0 - \delta I^2 B_0$$

from which we obtain, $I^* = \frac{A_0 - S_0}{2\delta B_0}$.

Second, consider the case where $I > \frac{a_L}{2\delta}$. For this case, using Leibnitz rule and simplifying, the first order condition is given by

$$S_0 = B_0 \int_{2\delta I}^{a_H} (a - 2\delta I) f(a) da. \tag{A.10}$$

This completes the proof.

Proof of Proposition 6

Proposition 6 follows along similar lines to Proposition 2, so only a sketch of the proof is provided. The manufacturer's profit is

$$M_0(S_0) = IS_0$$

First, consider the case where $A_0 - a_L B_0 < S_0 < A_0$. From Lemma 6,

$$M(S_0) = \frac{A_0 - S_0}{2\delta B_0} S_0.$$

The optimal solution for this problem is $S_0^* = \frac{A_0}{2}$. Now, for $S_0^* > A_0 - a_L B_0$, it must be the case that $\frac{A_0}{2} < a_L B_0$.

Second, consider the case where $S_0 < A_0 - a_L B_0$. Here,

$$M_0(S_0) = [I_0^*(S_0)] S_0$$

where $I_0^*(S_0)$ is given in lemma 6. The first order conditions lead to

$$S_0^* = B_0 \int_{2\delta I}^{a_H} f(a) da \tag{A.11}$$

This completes the characterization of the manufacturer's problem.

The retailer's problem in period 0, has been solved in lemma 6. For the case where $\frac{A_0}{2} < a_L B_0$, $I^*(S_0) = \frac{A_0 - S_0}{2\delta B_0}$. Hence, when $S_0^* = A_0/2$, $I^*(S_0) = \frac{A_0}{4\delta B_0}$.

For the case where $A_0/2 > a_L B_0$, we can combine equations (A.10) and (A.11) to establish an alternative characterization of the optimal response for the retailer, namely:

$$\int_{2\delta I_0^*}^{a_H} a f(a) da = 4\delta I_0^* \int_{2\delta I_0^*}^{a_H} f(a) da.$$
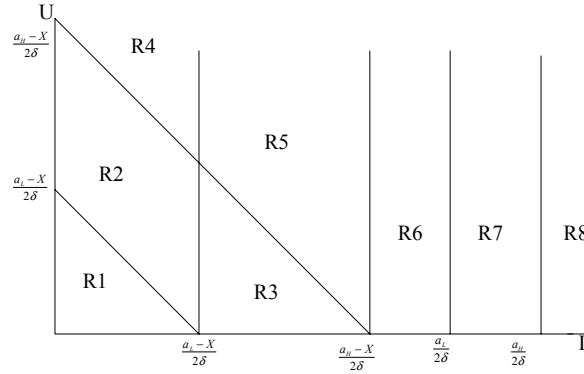
This completes the proof.

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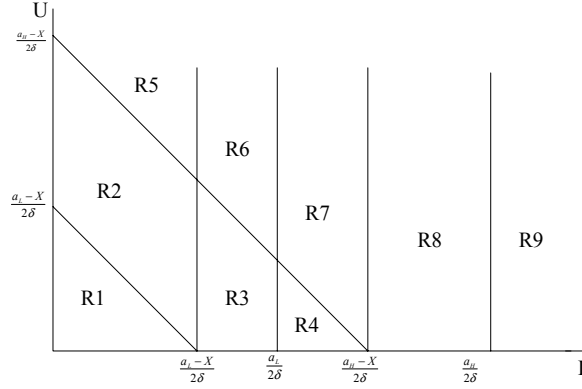**Figure 1: Retailer's Optimal Policy Regions in Period 1: Case 1***

U

$\frac{a_H - X}{2\delta}$ R4

     R5

$\frac{a_L - X}{2\delta}$ R2

         R6   R7   R8

R1

    R3

$\frac{a_L - X}{2\delta}$ $\frac{a_H - X}{2\delta}$ $\frac{a_L}{2\delta}$ $\frac{a_H}{2\delta}$ I

Region	q_L^*	v_L^*	q_H^*	v_H^*
R1	I	U	I	U
R2	I	$\frac{a_L - X}{2\delta} - I$	I	U
R3	I	0	I	U
R4	I	$\frac{a_L - X}{2\delta} - I$	I	$\frac{a_H - X}{2\delta} - I$
R5	I	0	I	$\frac{a_H - X}{2\delta} - I$
R6	I	0	I	0
R7	$\frac{a_L}{2\delta}$	0	I	0
R8	$\frac{a_L}{2\delta}$	0	$\frac{a_H}{2\delta}$	0

* Figure 1 shows the optimal policies for the retailer in period 1, conditional on inventory and option decisions taken in period 0, for each of the two realizations of the demand curve. Case 1 corresponds to the situation where $\frac{a_L}{2\delta} > \frac{a_H - X}{2\delta}$. For example, if $[I, U]$ fall in region R2, and the demand realization is high, then the optimal policy is for the retailer to release all inventory and exercise all options.

Figure 2: Retailer's Optimal Policy Regions in Period 1: Case 2[*]

U

$\frac{a_H - X}{2\delta}$

R5

R6

R2

$\frac{a_L - X}{2\delta}$

R7

R8   R9

R1   R3

R4

$\frac{a_L - X}{2\delta}$ $\frac{a_L}{2\delta}$ $\frac{a_H - X}{2\delta}$ $\frac{a_H}{2\delta}$ I

| Region	q_L^*	v_L^*	q_H^*	v_H^*
R1	I	U	I	U
R2	I	$\frac{a_L-X}{2\delta} - I$	I	U
R3	I	0	I	U
R4	$\frac{a_L}{2\delta}$	0	I	U
R5	I	$\frac{a_L-X}{2\delta} - I$	I	$\frac{a_H-X}{2\delta} - I$
R6	I	0	I	$\frac{a_H-X}{2\delta} - I$
R7	$\frac{a_L}{2\delta}$	0	I	$\frac{a_H-X}{2\delta} - I$
R8	$\frac{a_L}{2\delta}$	0	I	0
R9	$\frac{a_L}{2\delta}$	0	$\frac{a_H}{2\delta}$	0

[*] Figure 2 shows the optimal policies for the retailer in period 1, conditional on inventory and option decisions taken in period 0, for each of the two realizations of the demand curve. Case 2 corresponds to the situation where $\frac{a_L}{2\delta} < \frac{a_H-X}{2\delta}$. The regions have the same interpretations as in Figure 1.
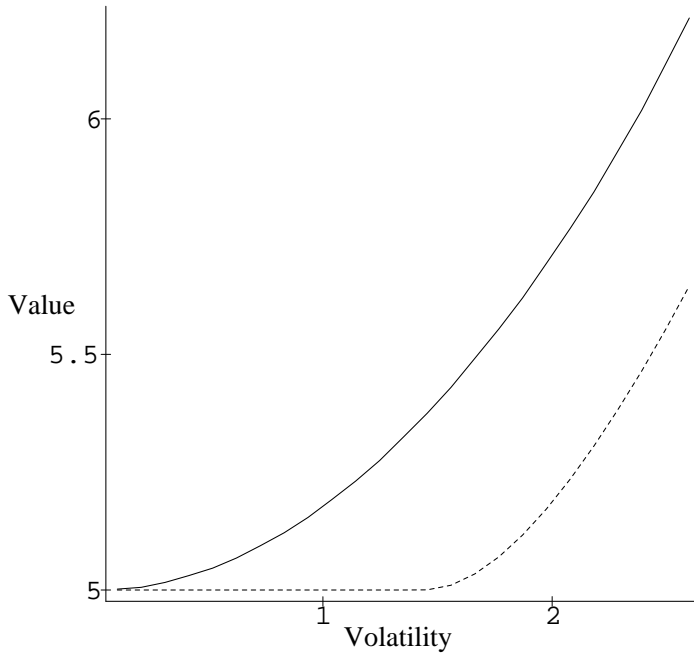
Figure 3: Equilibrium Wholesale Price*
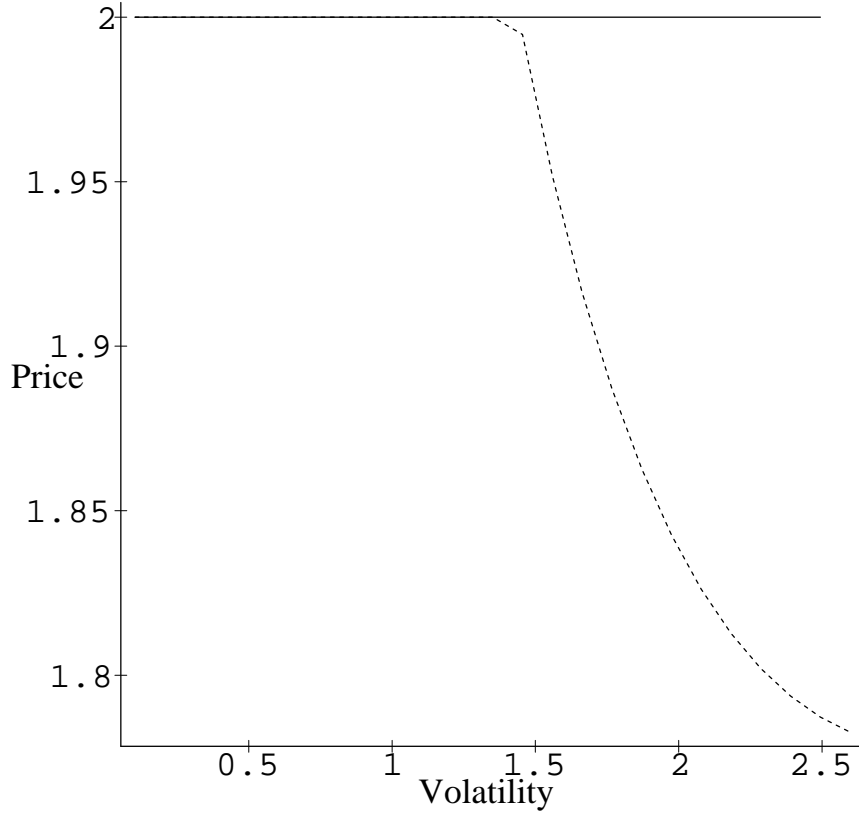
*Figure 3 shows the behavior of the wholesale price as the volatility of the demand curve increases. The solid (dashed) line corresponds to the price when options (no options) are used. The figure illustrates that with options the wholesale price is never lower than the price without options. The intercept α of the demand curve follows a uniform distribution with mean μ and variance σ_A^2. The case parameters for the problem are : $\delta = 0.5, B_0 = 0.8, \mu = 5, X = 2.5$.

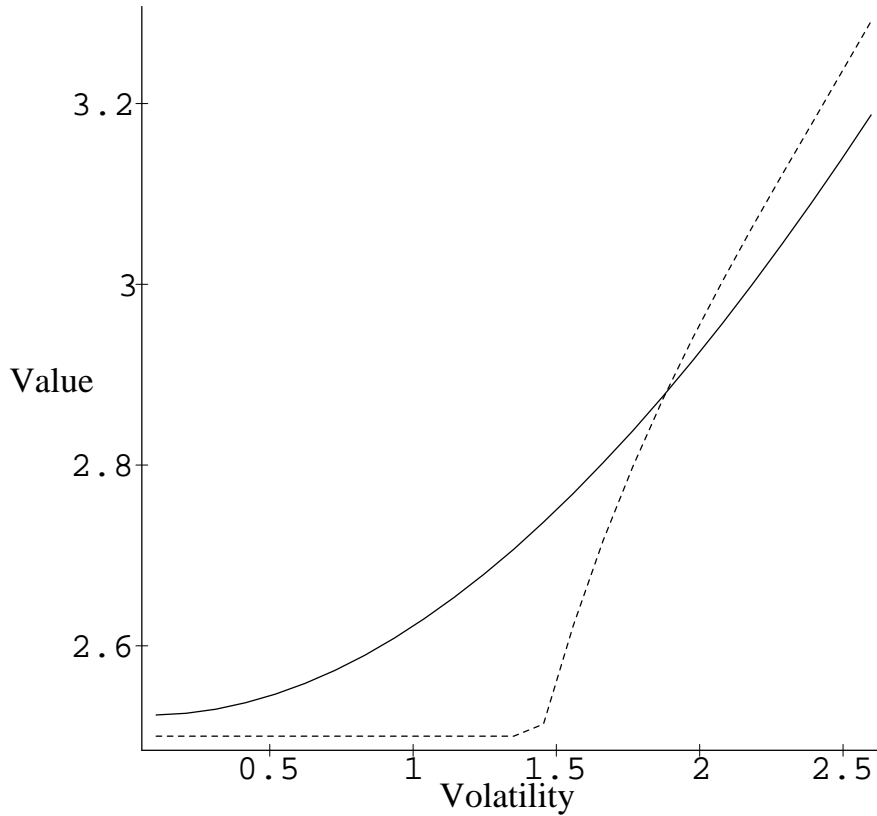**Figure 4: Variance of Equilibrium Retail Price**[*]
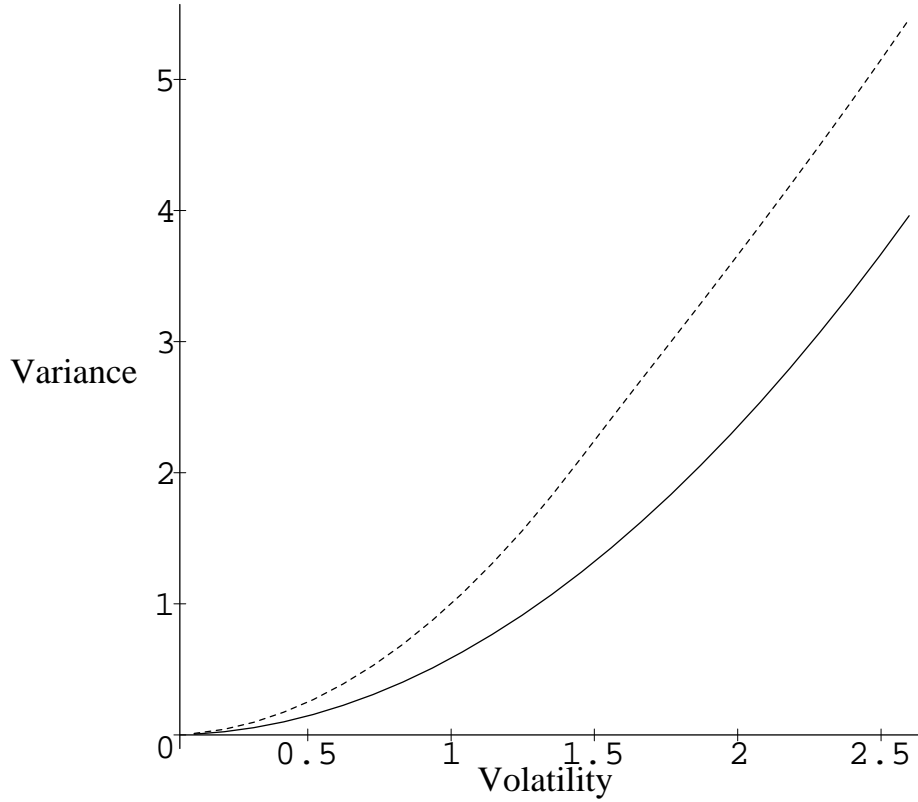
[*]Figure 4 shows the variance of the retail price as the volatility of the demand curve increases. The solid (dashed) line corresponds to the price when options (no options) are used. The figure illustrates that with options the volatility of the retail price is reduced compared to the case without options. The intercept α of the demand curve follows a uniform distribution with mean μ and variance σ_A^2. The case parameters for the problem are : $\delta = 0.5, B_0 = 0.8, \mu = 5, X = 2.5$.

Figure 5a: Project Value for Manufacturer *

*Figure 5a shows the net present value of the project for the manufacturer as the volatility of the demand curve increases. The solid (dashed) line corresponds to the price when options (no options) are used. The figure illustrates that the project value for the manufacturer is increasing in σ_A. Further, the value for the manufacturer is always higher when options are offered. The difference between the two curves, which denotes the value of the options program for the manufacturer, is increasing as the volatility of the demand curve increases. The intercept α of the demand curve follows a uniform distribution with mean μ and variance σ_A^2. The case parameters for the problem are : $\delta = 0.5, B_0 = 0.8, \mu = 5, X = 2.5$.

Figure 5b: Project Value for Retailer [*]

[*]Figure 5b shows the net present value of the project for the retailer as the volatility of the demand curve increases. The solid (dashed) line corresponds to the price when options (no options) are used. The figure illustrates that the project value for the retailer is increasing in σ_A. In addition, the option program is beneficial for the retailer only when the volatility of the demand curve is below a threshold. The intercept α of the demand curve follows a uniform distribution with mean μ and variance σ_A^2. The case parameters for the problem are : $\delta = 0.5, B_0 = 0.8, \mu = 5, X = 2.5$.