Investment and Abandonment Under Economic and Implementation Uncertainty*

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1 Introduction

Whenever a firm is contemplating a new project, the investment decision hinges upon a comparison between the outlays required and the future (and unknown) benefits derived from the project once in place. Thus, the key uncertainty that managers face in their investment decisions is over the future *economic* value of a project, as this is affected by fluctuations in the underlying market demand.

The consensus on investment appraisal, as prescribed by business textbooks, dictates replacing the unknown future benefits from the project with their respective expected values, and comparison of their properly discounted sum with the investment cost. The dispersion of the actual (but unknown) benefits around their expected values used in the decision rule, should be captured by a properly defined (uncertainty–adjusted) discount rate. Depending on the result of this comparison, the project is worth undertaking

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or not, and the manager can proceed to the appraisal of the next potential investment project.

A well–established strand of literature, collectively known as Real Options[1], has stressed that the uncertainty over the economic value of a project should not only affect the expected value of the opportunity, but also its timing: a project not worth investing in today might have a positive net present value in the future, if the uncertainty over its economic value resolves favourably. Moreover, since the costs associated with investment are largely unrecoverable[2], commiting to a project today essentially relinguishes the flexibility inherent in delaying the decision until new information about the economic value of the project becomes known. *Thus, in the face of economic uncertainty and (any degree of) irreversibility, every investment project competes with the mutually exclusive alternative of itself delayed in the future.* The difference between alternatives is termed the "option value of waiting to invest". This option value is increasing in the magnitude of economic uncertainty, implying that more uncertainty discourages investment spending by firms.

However, apart from the underlying economic value which is uncertain, there are several other complexities that surround the actual implementation of a project. For example, in technology intensive industries, there is severe uncertainty concerning the timing that an innovation will be discovered. This extra source of uncertainty will have an impact on the project's economics, far and beyond that of the uncertainty in the value of the end–process, marketable product. Large–scale projects (e.g. aircraft, mining industry) have substantial time lags between the decision to invest and the realisation of cash returns, and the actual length of time lags might be an additional uncertainty influencing the desirability of the project. In product markets, the level of profitability of a new product will depend on distribution channels, accessibility to selling points and the relative time of adoption of the prod-

[1]The term is due to Myers [13]. See Dixit and Pindyck [5] and Trigeorgis [16], [15] for reviews of the literature.

[2]Most investment expenditures are considered largely as sunk costs because of the specificity of their nature towards a particular firm or industry. Even expenditures that are not firm or industry–specific (e.g. computers, automobiles, office equipment, etc.) may not recover their true economic value due to asymmetric information between sellers and buyers in second–hand markets (see Akerlof [1]).

uct by consumers. Once investment is made and funds are committed to the underlying production process, one or all of these factors could potentially affect the level and the timing of cash flows realised from product sales. *Thus in most investment decisions, there are several factors, specific to the actual implementation phase of the project, that cause additional uncertainty in the relationship between inputs and outputs of the investment process. This uncertainty drives a wedge between the firm's decision to finance a project and the level and timing of the outcome of this project.* The major aim of this paper is to consider the effects of such extra uncertainty on optimal firm investment behaviour.

We examine optimal entry and exit decisions for a firm facing uncertainty both in the economic value and in the implementation phase of a project. As in other real options models, the stochastic nature of product market returns gives rise to the "option to wait" which must be taken into consideration when making optimal investment and abandonment decisions. In addition, resolution of implementation uncertainty is a Poisson arrival. The innovation is that unlike previous research we allow implementation uncertainty to have a very general effect both on the timing and the economic value of the project. Thus, when a firm exercises its option to invest in a project it gains a second option, that of completing the implementation stage, whose exercise time is random and its payoff could be favourable or not.

The optimal investment strategy consists of a pair of time independent trigger points for investment and abandonment. As in Dixit [4], sunk entry/exit costs and economic uncertainty cause the investment trigger point to rise and the abandonment one to fall relative to their Marshallian equivalents, widening the inertia zone of hysteresis. When implementation uncertainty is also present, the firm tends to invest earlier and abandon later when the implementation stage is expected to be short. However, the effect of this uncertainty on project value has a role: If completion of the implementation stage is short and has a positive effect on the value of the completed project, the firm might actually abandon later than the certainty case. When the effect of the implementation stage is the most important uncertainty concerning project value (low economic uncertainty) the range of hysteresis is shown to decrease substantially.

In addition, our treatment of the implementation uncertainty is general enough that allows us to encompass four existing real options models, as well as generating a range of other possible outcomes. As implementation uncertainty is eliminated from the setting, the model becomes equivalent to that of Dixit [4], on which our exposition heavily relies on. If the firm cannot abandon the project and completion of the implementation stage can occur virtually as soon as the investment cost is sunk, the problem collapses to the McDonald and Siegel [11] model of a single irreversible investment opportunity. Special values for the effect of implementation uncertainty on the economic value of the project yield the results of two less–cited papers.

Real options models take into account the economic uncertainty over the return of an investment, but usually assume that the complexities over the implementation stage are either non–existant or deterministic. In a two–player game Lambrecht [9] models the creation of a patent and its commercialisation as a single deterministic step. Bar-Ilan and Strange [2] and Grenadier [8] allow an investment project to take time to build, but the length of the implementation time lag is fixed. Majd and Pindyck [10] also examine sequential investment when it takes time to implement the project. In other less related papers, Pindyck [14] and Cortazar et. al. [3] allow for a second source of uncertainty (technical and geological respectively) between the decision to invest and the returns from the project, which can be seen analogous to an implementation stage of uncertain length. However, implementation uncertainty is resolved only as the firm invests in their model, thus tending to stimulate rather than repress investment.

The structure of what follows is: Section 2 describes the setting, presents the model assumptions and solves for the optimal investment and abandonment thresholds under economic and implementation uncertainty. Section 3 shows how four existing real options models can arise as special cases of our model. Section 4 presents two numerical examples where the uncertainty over the implementation stage can resolve favourably and unfavourably for the firm, while section 5 concludes.

2 The model

A single risk–neutral firm is contemplating investment in a project, facing no actual or potential competitors in the area. There is both economic and implementation uncertainty: the lentgh of the implementation stage is random, the value of the completed project follows a stochastic process, and the level of this value that will actually be realised after implementation is a general function of the stage length.

The firm can commit to implement the project by sinking a set–up cost K. Once this decision is made, the firm incurs a flow cost of C per unit of time unless the project is abandoned. Abandonment requires a fixed cost L to be incurred, and the set–up cost K has to be sunk again if the project is to be resumed later.

Completion of the implementation stage is modelled as an *independent* Poisson arrival with intensity $\lambda > 0$. Thus, the density function for the duration of the implementation stage is $\lambda e^{-\lambda t}$. At the first occurence of the Poisson event, the implementation stage is completed and the completed project has a market value that evolves exogenously according to the following geometric Brownian motion with drift

$$dx = \mu x dt + \sigma x dz \tag{1}$$

where $\mu \in [0, r)$ is the drift rate, r is the risk free rate of return assumed constant, $\sigma > 0$ is the volatility parameter and dz is the increment of a standard Wiener process.

As a benchmark, consider the optimal strategy of the firm in the absence of any uncertainty. The *Marshallian* investment threshold is defined as the point at which the expected value of the project is equal to the flow cost, plus interest payment on the initial outlay, i.e.

$$\overline{x}_m = C + rK \tag{2}$$

Correspondingly, the Marshallian abandonment point is

$$\underline{x}_m = C - rL \tag{3}$$

In the face of uncertainty, the firm's decision problem has two state variables, the current market value of the completed project x and a discrete variable that indicates whether the firm has commenced the implementation stage (1) or not (0). Let $V_0(x)$, $V_1(x)$ denote the expected value of the firm prior to and after investment in the implementation phase respectively. The thresholds for optimal switching between the states 1 and 0 are derived first, and are then compared to their certainty equivalents in (2) and (3).

Over the range of market values x where it is optimal for the firm not to invest, $V_0(x)$ only comprises of the firm's option to enter the implementation stage when it decides to do so. Thus, its return comes entirely in the form of capital gains or losses as the value of x changes. Under risk–neutrality, the firm's value equilibrium return condition will be

$$rV_0(x)\,dt = E[dV_0(x)]$$

Expanding $dV_0(x)$ using Itô's lemma and (1), and taking the expectation yields

$$\frac{1}{2}\sigma^2 x^2 V_0''(x) + \mu x V_0'(x) - rV_0(x) + \lambda E[V_0(x|\lambda) - V_0(x)] = 0 \tag{4}$$

This is the second–order ordinary differential equation that the value of the firm before investment must satisfy. The last term in (4) captures the effect (if any) of a Poisson occurance on the value of the firm before the implementation stage is commenced. We make the assumption that in such an event, our firm loses the opportunity to invest in the project, i.e. $\lambda E[V_0(x|\lambda) - V_0(x)] = -\lambda V_0(x)$ and (4) becomes

$$\frac{1}{2}\sigma^2 x^2 V_0''(x) + \mu x V_0'(x) - (r+\lambda)V_0(x) = 0 \tag{5}$$

We claim that this assumption is crucial in order to avoid introducing an "asymmetry" that will critically influence the position of the optimal thresholds relative to their Marshallian counterparts. Namely, if the Poisson arrival has an effect only on the value of a firm in the implementation stage $V_1(x)$, and not on $V_0(x)$, (i.e. $\lambda E[V_0(x|\lambda) - V_0(x)] = 0$ but $\lambda E[V_1(x|\lambda) - V_1(x)] \neq 0$) the position of the optimal investment and abandonment thresholds relative to $\overline{x}_m$, $\underline{x}_m$ will be determined by this asymmetry. The role of this assumption will become more apparent in section 3, more specifically in the discussion preceding Proposition 5 on page 13.

If x ever tends to zero, the possibility of rising to a level high enough to induce investment is very remote, and therefore the option to commence implementation is almost worthless. We can thus impose the following end–point condition

$$\lim_{x \to 0_+} V_0(x) = 0 \tag{6}$$

Solving (4) subject to (6) gives

$$V_0(x) = Ax^{\alpha} \tag{7}$$

where $A > 0$ is a constant whose value is determined as part of the solution, and α is given by

$$\alpha = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} > 1 \tag{8}$$

Correspondingly, over the range of values of x where our firm has invested K to enter the implementation stage of the project, the differential equation that $V_1(x)$ must satisfy would be

$$\frac{1}{2}\sigma^2 x^2 V_1''(x) + \mu x V_1'(x) - rV_1(x) + \lambda E\left[V_1(x|\lambda) - V_1(x)\right] = 0 \tag{9}$$

We assume that the successful completion of the implementation stage (the Poisson event) will affect *not only the timing but also the economic value* of the completed project. Specifically, let $\lambda E\left[V_1(x|\lambda) - V_1(x)\right] = \left[\zeta(\lambda)x - C\right] - \lambda V_1(x)$. This says that at the (random) time that the option to complete implementation is exercised, our firm earns in exchange $\zeta(\lambda)x - C$. The function $\zeta(\lambda)$ captures the expected effect of implementation uncertainty on the economic value of the completed project. The only structure we impose for the time being is that it is a positive real function of the Poisson intensity, i.e. $\zeta(\lambda) > 0 \ \forall \lambda \in \mathbb{R}_+^*$.

Under this, equation (9) becomes

$$\frac{1}{2}\sigma^2 x^2 V_1''(x) + \mu x V_1'(x) - (r+\lambda)V_1(x) + \zeta(\lambda)x - C = 0 \tag{10}$$

The above has to be solved subject to the following end–point condition

$$\lim_{x \to +\infty} V_1(x) = E\left[\int_0^{+\infty} e^{-(r+\lambda)t}\left[\zeta(\lambda)x_t - C\right]dt\right] \tag{11}$$

which merely reflects the fact that if x is very large, the value of the option to abandon tends to zero and the value of the firm is simply the expected NPV of the project. This yields

$$V_1(x) = Bx^{\beta} + \frac{\zeta(\lambda)\,x}{r+\lambda-\mu} - \frac{C}{r+\lambda} \tag{12}$$

where $B > 0$ is an unknown constant and

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} < 0 \tag{13}$$

Note that in (12), the first term on the right–hand side represents the option of a firm that has entered the implementation stage to abandon the project.

The optimal investment strategy is defined as two, time independent, trigger points: an upper $\overline{x}$ at which the implementation stage is optimally commenced, and a lower $\underline{x}$ at which the project (completed or not) is optimally abandoned. At these trigger values, the following "boundary" conditions are satisfied

$$\begin{aligned} V_0(\overline{x}) + K &= V_1(\overline{x}) \\ V_0(\underline{x}) &= V_1(\underline{x}) + L \end{aligned} \tag{14}$$

$$\begin{aligned} V_0'(\overline{x}) &= V_1'(\overline{x}) \\ V_0'(\underline{x}) &= V_1'(\underline{x}) \end{aligned} \tag{15}$$

Equation (14), also known as the *value–matching* condition, states that at the boundaries, our firm optimally incurs the set–up and abandonment costs to switch between states. Equation (15), referred to as the *high–contact* or *smooth pasting* condition, states that for the trigger values to be optimal, not only the value functions in (14), but also their first derivatives must meet at $\overline{x}$, $\underline{x}$.[3] A graphical demonstration of the value functions $V_0(x)$, $V_1(x)$ and boundary conditions (14), (15) is provided in figure 1.

Substituting for the value functions (7) and (12) in (14)–(15) yields a system of four equations which uniquely determines the four unknowns: the

[3] See Dumas [6] for a rigorous treatment of these conditions and Merton [12, p.171, n.60] for a discussion in the option pricing problem. Dumas and Luciano [7] also discuss these in another two sided transaction cost control problem.

two option constants A, B and the optimal trigger values $\overline{x}$, $\underline{x}$.

$$
\begin{aligned}
A\overline{x}^{\alpha} + K &= B\overline{x}^{\beta} + \frac{\zeta(\lambda)\,\overline{x}}{r+\lambda-\mu} - \frac{C}{r+\lambda} \\
A\underline{x}^{\alpha} &= B\underline{x}^{\beta} + \frac{\zeta(\lambda)\,\underline{x}}{r+\lambda-\mu} - \frac{C}{r+\lambda} + L \\
\alpha A\overline{x}^{\alpha-1} &= \beta B\overline{x}^{\beta-1} + \frac{\zeta(\lambda)}{r+\lambda-\mu} \\
\alpha A\underline{x}^{\alpha-1} &= \beta B\underline{x}^{\beta-1} + \frac{\zeta(\lambda)}{r+\lambda-\mu}
\end{aligned}
\tag{16}
$$

Unfortunately, the system is highly non–linear in trigger values and analytic solutions can not be found. For any set of parameter values though, a numerical solution can readily be determined.

To gauge the size of the optimal investment and abandonment thresholds $\overline{x}$, $\underline{x}$ relative to their Marshallian counterparts $\overline{x}_m$, $\underline{x}_m$, define $G(x)$ as the difference of the two value functions

$$
\begin{aligned}
G(x) &= V_1(x) - V_0(x) \\
&= Bx^{\beta} - Ax^{\alpha} + \frac{\zeta(\lambda)\,x}{r+\lambda-\mu} - \frac{C}{r+\lambda}
\end{aligned}
$$

From (5) and (10) the following expression for $G(x)$ can be written

$$
\frac{1}{2}\sigma^2 x^2 G''(x) + \mu x G'(x) - (r+\lambda)\,G(x) + \zeta(\lambda)\,x - C = 0 \tag{17}
$$

and conditions (14)–(15) become

$$
\begin{aligned}
G(\overline{x}) &= K \\
G(\underline{x}) &= -L
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
G'(\overline{x}) &= 0 \\
G'(\underline{x}) &= 0
\end{aligned}
\tag{19}
$$

From the above, the following proposition concerning the optimal thresholds under uncertainty can be deduced.

Proposition 1 *The optimal investment and abandonment trigger points under economic and implementation uncertainty can be expressed as*

$$\begin{aligned} \overline{x} &= \tfrac{1}{\zeta(\lambda)} \left[\overline{x}_m + k(\lambda) \right] \\ \underline{x} &= \tfrac{1}{\zeta(\lambda)} \left[\underline{x}_m - l(\lambda) \right] \end{aligned} \tag{20}$$

where $k(\lambda)$, $l(\lambda)$ are positive and increasing functions of the Poisson intensity.

Proof. Consider the investment threshold first and evaluate equation (17) at $\overline{x}$. From (18)–(19) we know that $G(\overline{x}) = K$, $G'(\overline{x}) = 0$ and $G''(\overline{x}) < 0$, thus

$$\zeta(\lambda)\,\overline{x} = -\frac{1}{2}\sigma^2 \overline{x}^2 G''(\overline{x}) + (r+\lambda) K + C$$

Substitute the Marshallian trigger from (2) to get the result of the proposition with $k(\lambda) = \lambda K - \frac{1}{2}\sigma^2 \overline{x}^2 G''(\overline{x})$.

Correspondingly for the abandonment threshold, evaluate (17) at $\underline{x}$. From (18)–(19) $G(\underline{x}) = -L$, $G'(\underline{x}) = 0$ and $G''(\underline{x}) > 0$, thus

$$\zeta(\lambda)\,\underline{x} = -\frac{1}{2}\sigma^2 \underline{x}^2 G''(\underline{x}) - (r+\lambda) L + C$$

Substitute the value of $\underline{x}_m$ from (3) to get the result, with $l(\lambda) = \lambda L + \frac{1}{2}\sigma^2 \underline{x}^2 G''(\underline{x})$. ■

It is obvious from proposition 1 that the location of the optimal thresholds under uncertainty compared to their Marshallian counterparts is ambiguous. The functions $k(\lambda)$, $l(\lambda)$ tend to increase the range of hysteresis, but the effect of implementation uncertainty on the value of the completed project, $\zeta(\lambda)$, plays a crucial role. The net effect will highly depend on the functional form of $\zeta(\lambda)$. In the next section we turn to examine some existing real options models that arise as nested cases from our model as λ assumes some extreme values, or some more structure is assumed for the effect of implementation uncertainty.

3 Some cases

The structure of the model, especially the way implementation uncertainty affects investment timing and value, is general enough that four existing models in the literature can be nested as special cases.

First, consider the case where the only uncertainty concerning the value of the project comes from equation (1). As implementation uncertainty is eliminated from the model, two changes essentially take place: (a) our firm receives the value of a completed project as soon as the investment of the set–up cost K is made and not after a random–length period of time and (b) there is no implementation stage and thus the expected project value.should be independent of the Poisson intensity. In the absence of implementation uncertainty our model collapses to the Dixit [4] model of a firm's entry and exit decision under (only economic) uncertainty.

Proposition 2 *The limiting case where there is no uncertainty over the implementation of the project is the Dixit [4] model of a firm's optimal product market entry and exit decisions.*

Proof. Set $\zeta(\lambda) = 1$ so that the expected project value is independent of the implementation hazard rate λ, i.e. $\lim_{\lambda \to 0} E\left[\int_0^{+\infty} e^{-(r+\lambda)t} \zeta(\lambda)\, x_t dt\right] = \frac{x}{r-\mu}$. The result is immediate by taking the limit as $\lambda \to 0$ of equation (17), which yields the differential equation (20) in Dixit [4]. Alternatively, as $\lambda \to 0$, the roots α and β in (8) and (13) become

$$a, b = \frac{1}{2} - \frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{21}$$

as those in Dixit [4, p. 626]. Verify that the option constants can be written as $A = \frac{\zeta(\lambda)}{r+\lambda-\mu}\mathcal{A}$ and $B = \frac{\zeta(\lambda)}{r+\lambda-\mu}\mathcal{B}$ and in the limit $\lim_{\lambda \to 0} A = \mathcal{A}$ and $\lim_{\lambda \to 0} B = \mathcal{B}$. Thus, as $\lambda \to 0$, $\alpha \to a$ and $\beta \to b$ and the system of equations (16) becomes equivalent to that in Dixit [4, eq. (12)-(15)]. ∎

Abstracting from implementation complexities restores the well documented result that hysteresis unambiguously increases with economic uncertainty (σ), as the following demonstrates.

Corollary 3 *In the limiting case where there is no uncertainty over the implementation of the project, the optimal investment and abandonment trigger points from Proposition 1 become*

$$\overline{x} = \overline{x}_m - \tfrac{1}{2}\sigma^2 \overline{x}^2 G''(\overline{x}) > \overline{x}_m$$

$$\underline{x} = \underline{x}_m - \tfrac{1}{2}\sigma^2 \underline{x}^2 G''(\underline{x}) > \underline{x}_m$$

Secondly, as implementation uncertainty is again eliminated from the model, consider the case that $C = 0$. In the absence of flow costs, our model collapses to the McDonald and Siegel [11] model of a single, irreversible investment opportunity.

Proposition 4 *If the firm does not incur operating flow costs, the limiting case where the implementation stage can be completed instantaneously is the McDonald and Siegel [11] model of an irreversible investment opportunity with a constant investment cost K.*

Proof. Let $C = 0$ so that the project is never optimally abandoned for $L > 0$ (i.e. $B = 0$).[4] As before let $\zeta(\lambda) = 1$ so that the expected project value is independent of the implementation hazard rate λ and take the limit as $\lambda \to 0$. Then, the system of equations (16) becomes

$$A\overline{x}^a + K = \frac{\overline{x}}{r-\mu}$$

$$aA\overline{x}^{a-1} = \frac{1}{r-\mu}$$

where a is as in (21). The above conditions uniquely determine the optimal investment trigger

$$\overline{x} = \frac{a}{a-1}(r-\mu)K$$

and the constant option coefficient

$$A = \overline{x}^{1-a}\left[\frac{\overline{x}}{r-\mu} - K\right]$$

[4] More correctly, even if $C > 0$, the project will never be abandoned if $L > \frac{C}{r}$.

which are identical to the solution in McDonald and Siegel [11] for a constant investment cost K. ■

Next turn to general case where implementation uncertainty is present, and let us assume that $\zeta(\lambda) = \lambda$, i.e. the value of the completed project is linearly increasing, the shorter the expected implementation lag. Weeds [18], in her Ph.D. thesis, treats a similar case in an R&D context, where the random implementation stage is interpreted as the random time it takes for a patentable discovery to be made. Her major conclusion is that the extra source of uncertainty over the completion of the project unambiguously increases the investment trigger compared to the Marshallian one, while the relative position of the abandonment trigger compared to the Marshallian can not be firmly established. She refers to this possibility, that $\underline{x} > \underline{x}_m$ while $\overline{x} > \overline{x}_m$ as "reverse hysteresis". In what follows, we first establish the conditions under which our setting yields the model of Weeds [18] and then comment on the "reverse hysteresis" phenomenon.

The major difference from our setting is that in her R&D context, the Poisson arrival, which marks the completion of the discovery process, is assumed not to have any effect whatsoever if it occurs prior to investment in the research project. In our notation, this means that $\lambda E[V_0(x|\lambda) - V_0(x)] = 0$, i.e. the Poison event is only relevant in the post–investment stage of the project. Which assumption is more realistic is an open issue and will probably depend on the nature of the project under review. However we claim that it is this assumption that causes an "asymmetric" effect on the optimal thresholds, giving rise to reverse hysteresis.

Proposition 5 *If the Poisson arrival is only relevant for firm value post investment, the limiting case where the value of the completed project is linear in the hazard rate is the Weeds [18] model of R&D investment.*

Proof. With $\lambda E[V_0(x|\lambda) - V_0(x)] = 0$ and $\zeta(\lambda) = \lambda$, the firm's option to enter the implementation stage in (7) becomes

$$V_0(x) = \text{Å} x^a$$

with Å> 0 a different constant to be determined and a as in (21). The value of a firm active in the implementation stage will be identical to (12) with

$\zeta(\lambda)$ replaced by λ. Substituting these two value functions in the boundary conditions (14)–(15) yields the system of simultaneous equations in Weeds [18, eq. (9)-(12)]. ■

Figure 2 demonstrates graphically the effect of the different assumptions in our model and in Weeds [18]. The optimal investment threshold in Weeds is unambiguously above the Marshallian threshold for any set of parameter values. However, in our model, as Proposition 1 predicts, depending on parameter values the investment threshold can be above ($\sigma = 0.05$, $\lambda = 0.7$ for any K) or below ($\sigma = 0.1$, $\lambda = 0.9$ for low K) the Marshallian one.[5] We can thus conclude that the assumption concerning the effect of implementation uncertainty on firm value before investment is crucial on the relative positions of the trigger values vis-à-vis the Marshallian ones.

Finally, concentrating on entry ($C = 0$), it has be shown in Tsekrekos [17] that for several alternatives for the function $\zeta(\lambda)$, the optimal entry threshold can be above or below the certainty case. There, several effects of implementation uncertainty on project value are treated and the conditions that determine the direction of hysteresis are demonstrated.

4 Numerical examples

In this section we demonstrate the generality of the model with some numerical examples. We examine two distinct cases for the effect of implementation uncertainty on project value. These two cases are depicted in figure 3. In the first case (panel (a)), we assume that upon entry (time T) and during the implementation stage ($\tau - T$) the firm earns a constant fraction $\gamma < 1$ of the value of the completed project x. When the implementation stage is completed at the random time determined by the Poisson arrival (time τ), the firm receives the whole value of the project. We term this case, where our firm experiences a gain from resolution of implementation uncertainty as *"favourable"* (the "good–news" scenario). In contrast, in the *"unfavourable"* case of panel (b), our firm loses a fraction $\gamma < 1$ of the project

[5] Note that in figure 2, $\zeta(\lambda)$ is set equal to λ as in Weeds [18] so that the graph only demonstrates the effect of the assumption concerning $\lambda E[V_0(x|\lambda) - V_0(x)]$.

value once implementation uncertainty is resolved. It is shown in the Appendix that the two scenarios correspond to setting $\zeta(\lambda)$ in equation (12) equal to $\gamma + \lambda(1-\gamma)$ and $1 - \lambda\gamma$ in the favourable and unfavourable cases respectively.

In both cases, we use the parameter values of Dixit [4] as a base case, namely $\mu = 0$, $r = 0.05$, $K = 0.5$, $L = 0$, $C = 1$ and $\gamma = 0.5$. In an effort to disentangle the effects of economic (σ) and implementation (λ) uncertainty, figures 4–7 plot the ratios of optimal investment and abandonment thresholds to their Marshallian counterparts $(\frac{\overline{x}}{\overline{x}_m}, \frac{\underline{x}}{\underline{x}_m})$ as a function of the set–up cost K, for combinations of high ($\sigma = 0.4$, $\lambda = 0.9$) and low ($\sigma = 0.1$, $\lambda = 0.4$) uncertainty parameter values.

First concentrate in the case where implementation uncertainty is resolved favourably (figures 4 and 5). The effect of economic uncertainty is depicted in figure 4. As predicted by real options models, more uncertainty in the economic value of the completed project widens the range of hysteresis, making the firm proner to delaying investment/abandonment decisions. This value of waiting seems more important when the implementation uncertainty parameter λ is low (panel (a)).

The effect of implementation uncertainty however is different as figure 5 demonstrates. A shorter implementation stage (higher λ) makes the firm more eager to commit sooner and abandon later. As λ increases there are essentially two effects in action: Firstly, the implementation stage is completed sooner and our firm receives the value of the completed project quicker. Secondly, our firm is willing to enter the implementation stage earlier because a higher λ means a higher probability of losing the opportunity to invest in the project if the implementation stage has not commenced.[6] When supplemented with high economic uncertainty (σ), the project is actually abandoned at a level lower than the Marshallian (panel (b)).

Turning to the unfavourable scenario, the effect of economic uncertainty in figure 6 is qualitatively identical. A higher σ increases the range of inaction. However, unfavourable resolution of implementation uncertainty does not necessarily imply earlier or later optimal entry/exit. For low values of λ (longer expected implementation stage), an unfavourable development dic-

[6] This essentially highlights the effect of our assumption that $\lambda E\left[V_0(x|\lambda) - V_0(x)\right] = -\lambda V_0(x)$.

tates delayed investment and sooner abandonment (compare figures 5 and 7 with $\lambda = 0.4$). However, if the implementation stage is expected to be short (high λ), a firm can commit to investment early even if completion of this stage has an adverse effect on project value (compare figures 5 and 7 with $\lambda = 0.9$). When set–up costs are sufficiently low, the firm will optimally invest before the Marshallian level, even under the "bad news" scenario. This demonstrates the ambiguity of the position of the optimal investment–abandonment trigger values relative to their Marshallian counterparts, as Proposition 1 implies.

To summarise, the numerical examples demonstrate that economic and implementation uncertainty have distinctly different effects on optimal investment/abandonment decisions. The former increases the range of inaction as the consensus of the "investment under uncertainty" literature predicts. However, when implementation uncertainty affects both the timing and the desirability of the project, the firm might optimally invest and abandon earlier or later compared to the certainty case depending on parameter values. In all cases, the shorter the implementation stage the sooner our firm wants to commit to the project. Moreover, when the major project uncertainty comes from the implementation stage (high λ and low σ) the range of hysteresis is seen to decrease considerably. Lastly, the direction of uncertainty resolution plays an important role, however favourable and unfavourable news do not necessarily imply unambiguous positions for the optimal thresholds relative to their Marshallian counterparts.

5 Conclusions

The future economic benefits that may arise from new investment (or disinvestment) are the main uncertainty that managers face in their everyday decisions. The higher this economic uncertainty, the more reluctant are firms to commit resources to new projects or terminate operating ones as the real options literature has stressed. However, business reality is far more complex than that: there is a wide range of economy–wide (e.g. interest rates) or firm–specific (e.g. investment/abandonment costs) factors which might not be known with certainty, and which necessarily influence the economic value and desirability of any decision.

In this paper we consider optimal investment and abandonment decisions for a firm that faces both an economy–wide and a project–specific uncertainty. The former arises from the unknown future value of the project once in place. The latter concerns all factors that might interfere between the decision to initiate investment and the actual project implementation. Namely some projects take time to construct, and the actual time lag between investment and completion might be a key unknown factor. Other projects depend highly on critical, one–time events like discovery or customer acceptance, events that will ultimately determine both the timing and the economic value of the project.

We allow this implementation uncertainty to be resolved randomly and to affect both the timing and the value of the completed project. The way implementation uncertainty affects project value is general enough that allows us to nest four existing real options models as special cases of our setting.

In the general case, it has been shown that the position of the investment and abandonment thresholds cannot be unambiguously determined relative to the thresholds of the certainty case. The firm might optimally commit to investment earlier or abandon later under implementation uncertainty depending on parameter values. Our main result in Proposition 1 shows analytically how each source of uncertainty influences the position of the optimal thresholds.

Numerical simulations of two cases explicitly addressed show that a firm will abandon a project later than Marshallian theory would predict if the probability of favourable uncertainty resolution is relatively high. On the other hand, a firm might commit to implement a project sooner even if bad news are expected during the implementation stage of the project. In both cases, when implementation is the major source of project uncertainty, the range of hysteresis is seen to decrease considerably, in sheer contrast to the well–documented positive relation of economic uncertainty and inaction.

A Appendix

In the favourable scenario, over the range of values that it is optimal for the firm to commit to the project implementation equation (9) becomes

$$\frac{1}{2}\sigma^2 x^2 V_1''(x) + \mu x V_1'(x) - rV_1(x) + \lambda E\left[V_1(x|\lambda) - V_1(x)\right] + \gamma x = 0$$

where the last term represents the constant fraction γ of the completed project that the firm earns over the implementation stage. At the arrival of the Poisson event the firm experiences a gain of $(1-\gamma)x$ with probability λdt, i.e. $\lambda E\left[V_1(x|\lambda) - V_1(x)\right] = \lambda(1-\gamma)x - \lambda V_1(x)$. Substitute this and collect terms to get equation (10) with $\zeta(\lambda) = \gamma + \lambda(1-\gamma)$.

Correspondingly for the unfavourable scenario

$$\frac{1}{2}\sigma^2 x^2 V_1''(x) + \mu x V_1'(x) - rV_1(x) + \lambda E\left[V_1(x|\lambda) - V_1(x)\right] + x = 0$$

and at the random implementation uncertainty resolution the firm loses γx with probability λdt, i.e. $\lambda E\left[V_1(x|\lambda) - V_1(x)\right] = -\lambda\gamma x - \lambda V_1(x)$. Thus in the "bad news" case $\zeta(\lambda) = 1 - \lambda\gamma$ in (10).

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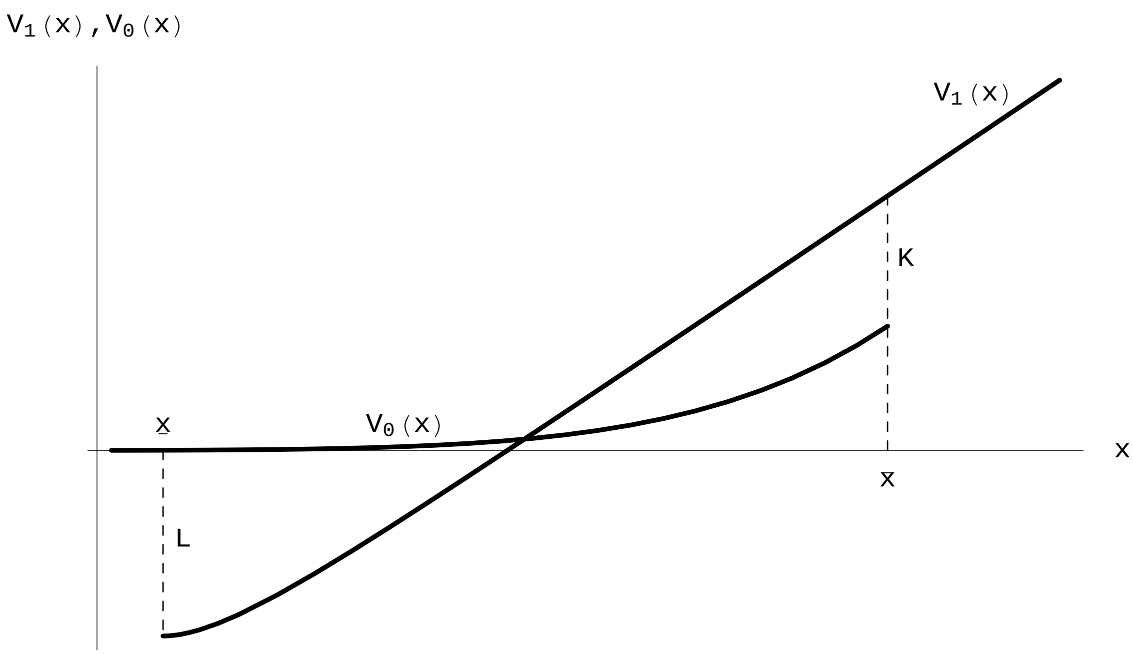

Figure 1: The value of idle and active firm, $V_0(x)$, $V_1(x)$, as a function of the state variable x. At the investment and abandonment thresholds $\overline{x}$, $\underline{x}$, the firm optimally switches between the idle and active states by sinking the set–up and exit costs, K and L.

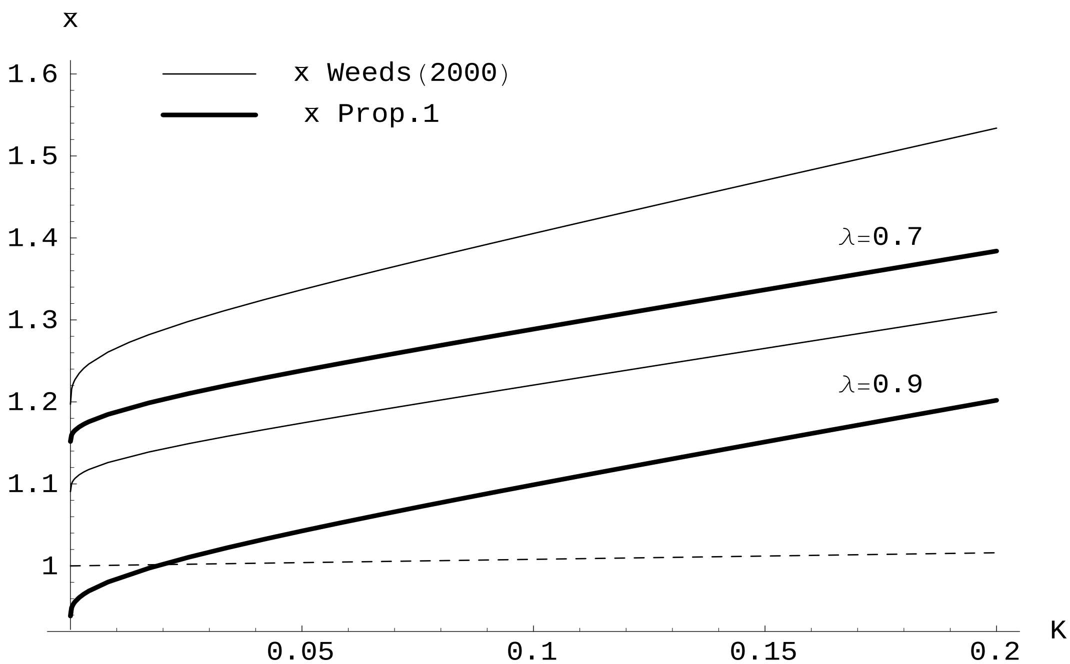

Figure 2: The optimal investment threshold $\overline{x}$ as a function of the set–up cost K as predicted by Proposition 1 (thick line)and Weeds [18]. The optimal trigger can be above or below its certainty equivalent (dashed line) depending to parameter values, in contrast to Weeds [18] where the threshold is unambiguously above. The parameters used are $\mu = 0$, $r = 0.08$, $\sigma = 0.1$, $\zeta(\lambda) = \lambda$, $L = 0$ and $C = 1$.

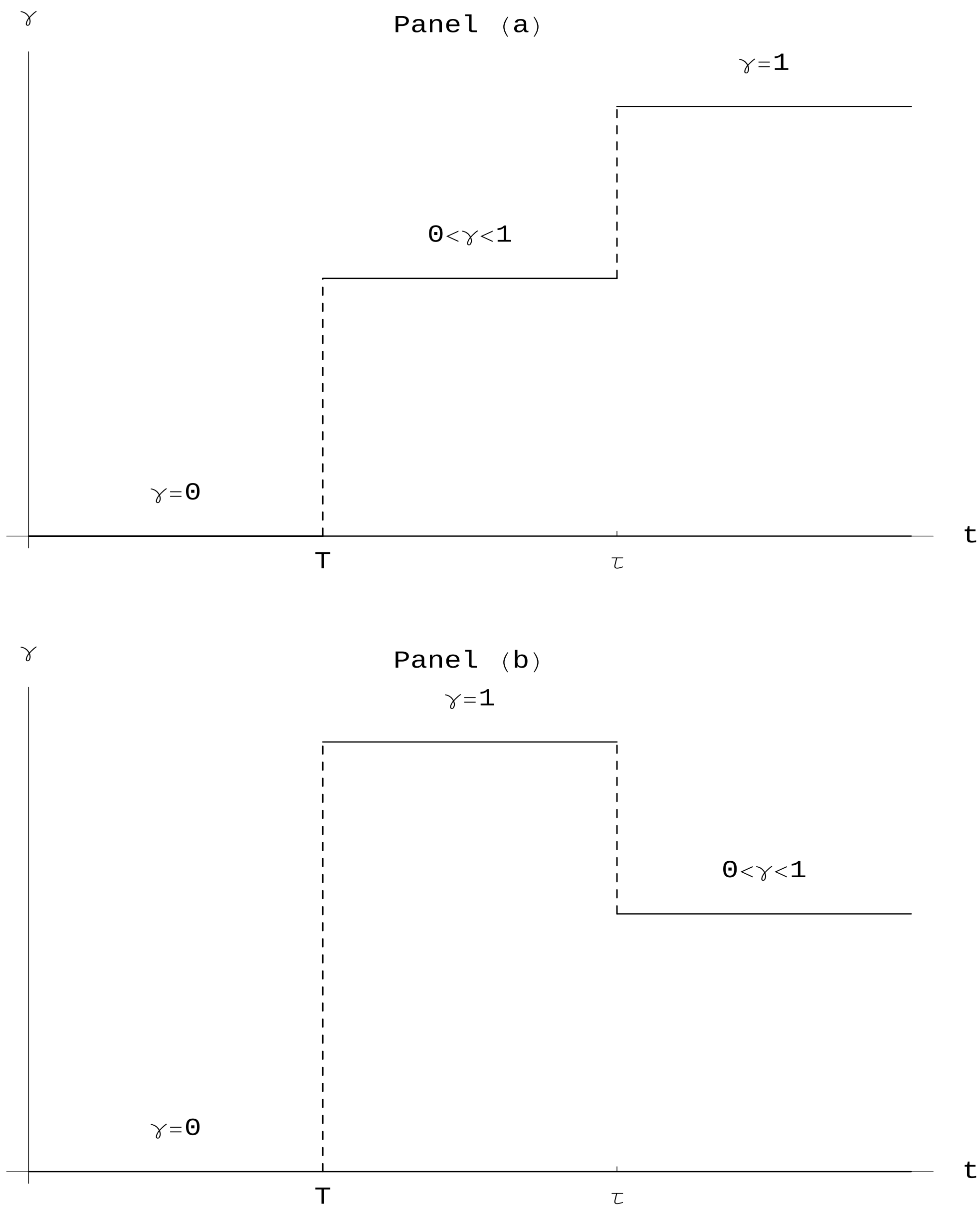

Figure 3: The two cases of implementation uncertainty resolution treated in the numerical examples. In the *"favourable"* scenario (panel a) the firm earns a fraction $\gamma \in (0, 1)$ of the completed project value until the implementation stage is completed. In the *"unfavourable"* scenario (panel b) the firm loses a fraction $\gamma \in (0, 1)$ when implementation uncertainty is resolved.

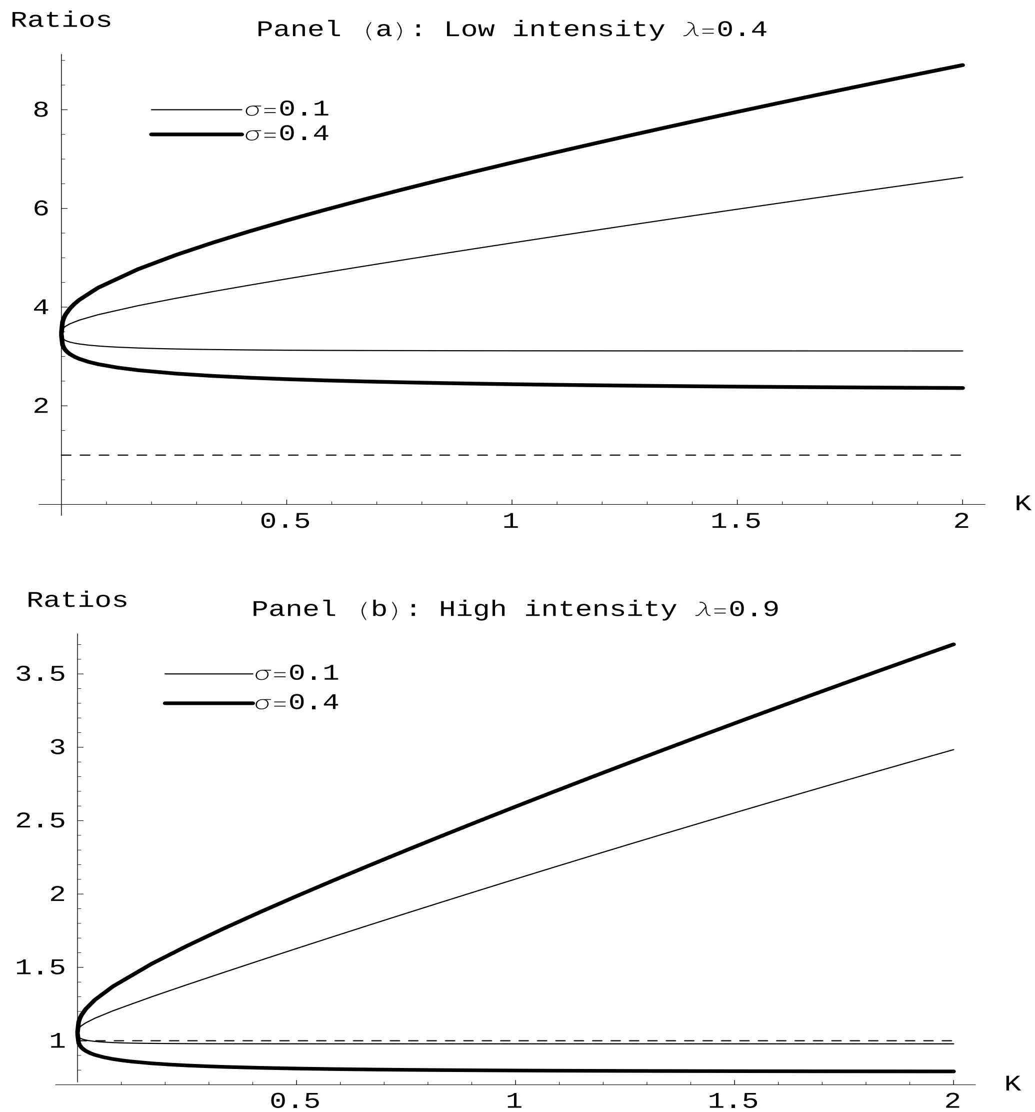

Figure 4: The ratios of the optimal investment and abandonment thresholds to their Marshallian counterparts, $\frac{\overline{x}}{\overline{x}_m}$, $\frac{\underline{x}}{\underline{x}_m}$ as a function of the set–up cost K under the *"favourable"* scenario for two values of the economic uncertainty parameter σ. In panels (a) and (b) the implementation stage length is expected to be long and short respectively. The higher the economic uncertainty (thick line), the further apart the threshold ratios are drawn. The firm might optimally abandon the project later than in the certainty case (dashed line). The parameters used are $\mu = 0$, $r = 0.05$, $\zeta(\lambda) = \gamma + \lambda(1-\gamma)$, $L = 0$ and $C = 1$.

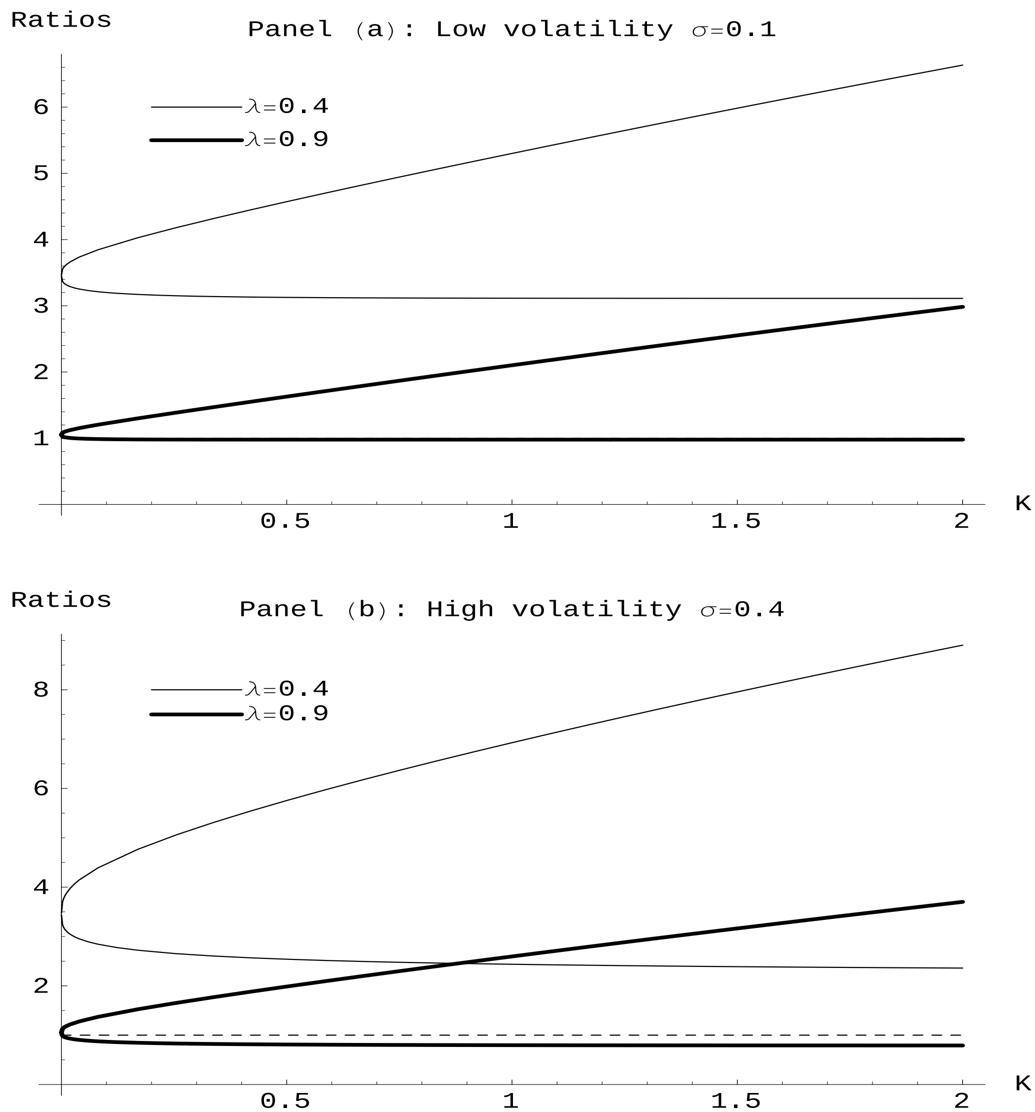

Figure 5: The ratios of the optimal investment and abandonment thresholds to their Marshallian counterparts, $\frac{\overline{x}}{\overline{x}_m}$, $\frac{\underline{x}}{\underline{x}_m}$ as a function of the set–up cost K under the *"favourable"* scenario for two values of the implementation uncertainty parameter λ. In panels (a) and (b) the economic uncertainty is low and high respectively. The shorter the expected implementation stage (thick line), the closer the threshold ratios are drawn. The firm might optimally abandon the project later than in the certainty case (dashed line). The parameters used are $\mu = 0$, $r = 0.05$, $\zeta(\lambda) = \gamma + \lambda(1 - \gamma)$, $\gamma = 0.5$, $L = 0$ and $C = 1$.

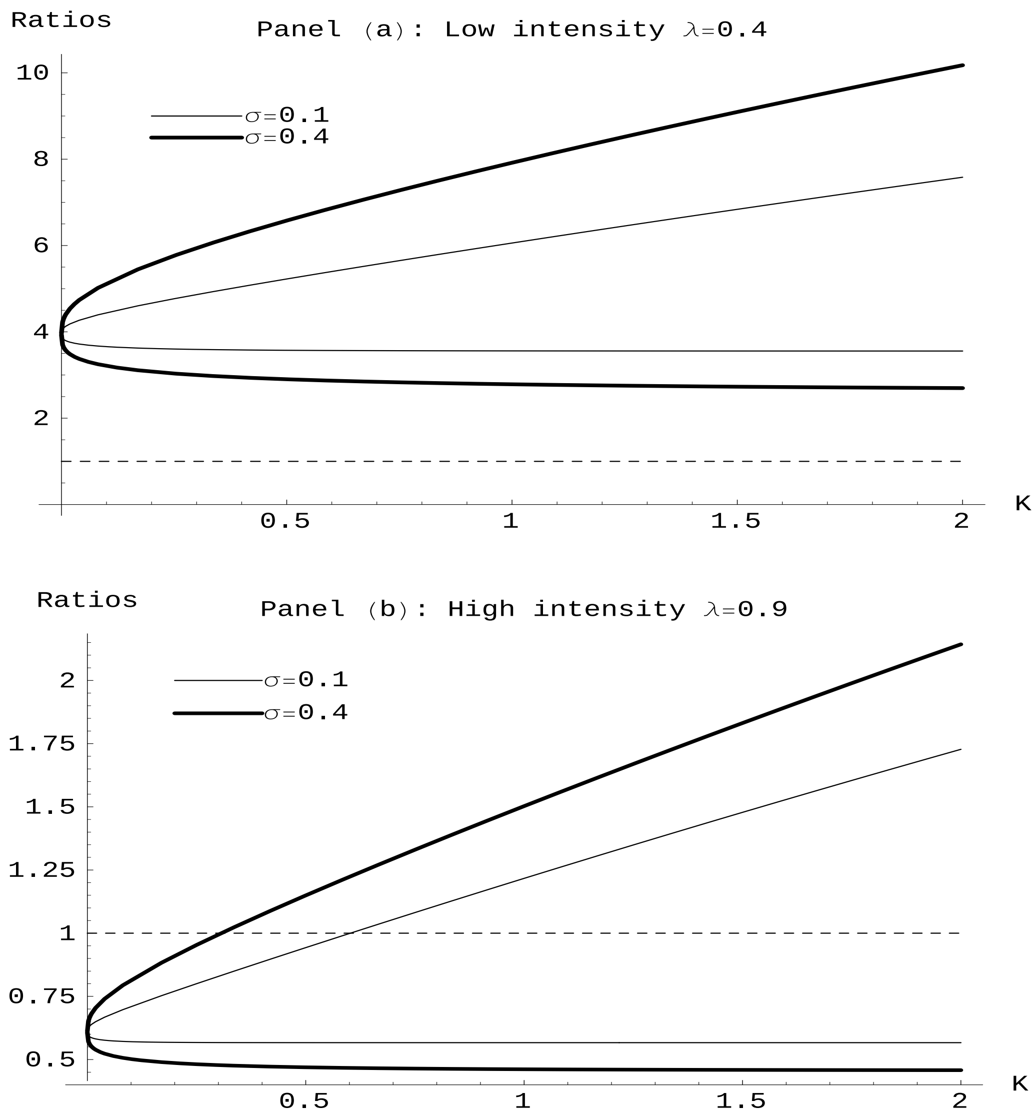

Figure 6: The ratios of the optimal investment and abandonment thresholds to their Marshallian counterparts, $\frac{\overline{x}}{\overline{x}_m}$, $\frac{\underline{x}}{\underline{x}_m}$ as a function of the set–up cost K under the *"unfavourable"* scenario for two values of the economic uncertainty parameter σ. In panels (a) and (b) the implementation stage length is expected to be long and short respectively. The higher the economic uncertainty (thick line), the further apart the threshold ratios are drawn. The firm might optimally invest sooner and abandon earlier than in the certainty case (dashed line). The parameters used are $\mu = 0$, $r = 0.05$, $\zeta(\lambda) = 1 - \lambda\gamma$, $L = 0$ and $C = 1$.

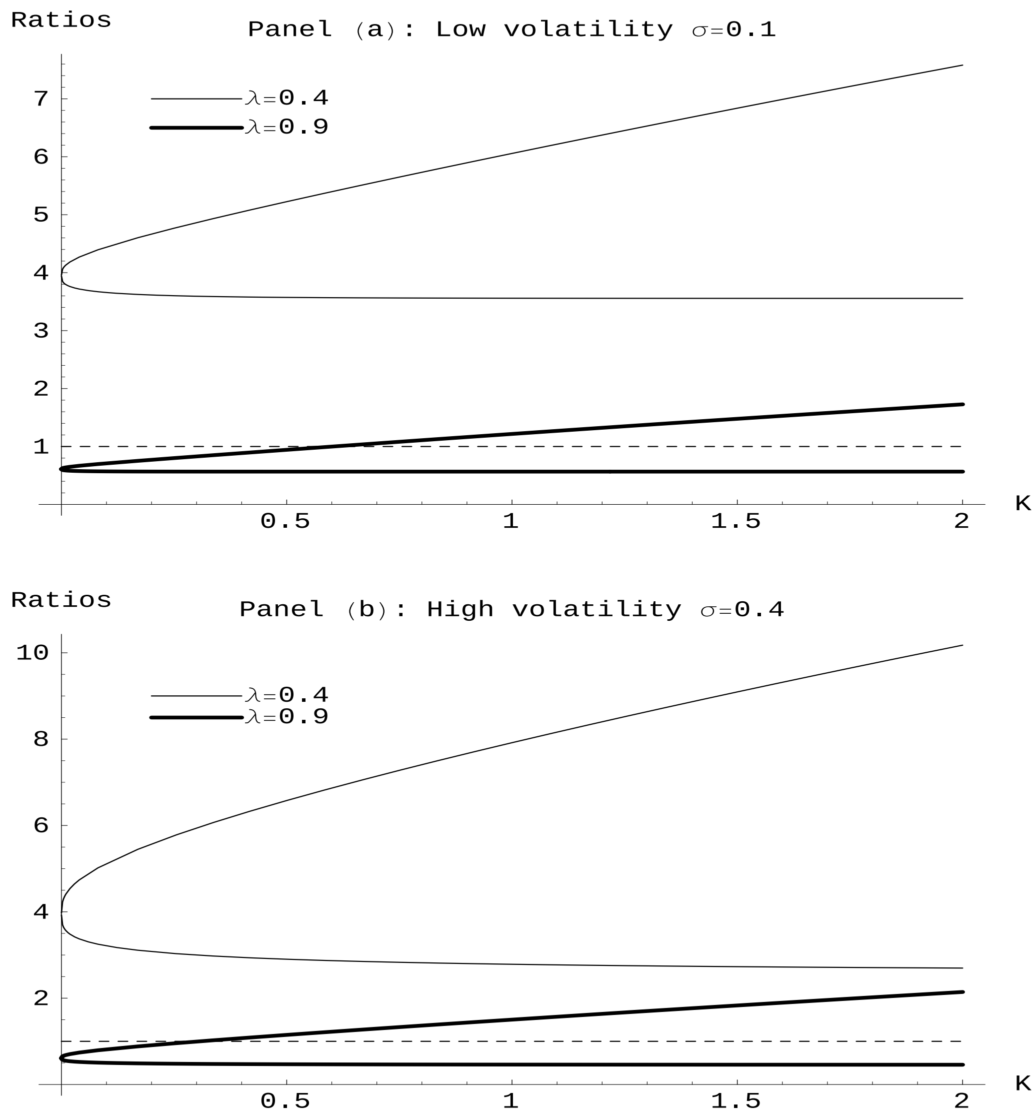

Figure 7: The ratios of the optimal investment and abandonment thresholds to their Marshallian counterparts, $\frac{\overline{x}}{\overline{x}_m}$, $\frac{\underline{x}}{\underline{x}_m}$ as a function of the set–up cost K under the *"unfavourable"* scenario for two values of the implementation uncertainty parameter λ. In panels (a) and (b) the economic uncertainty is low and high respectively. The shorter the expected implementation stage (thick line), the closer the threshold ratios are drawn. The firm might optimally invest sooner and abandon earlier than in the certainty case (dashed line). The parameters used are $\mu = 0$, $r = 0.05$, $\zeta(\lambda) = 1 - \lambda\gamma$, $\gamma = 0.5$, $L = 0$ and $C = 1$.