Real Option Games with Incomplete Information and Spillovers

Spiros H. Martzoukos[1] and Eleftherios Zacharias[2]

School of Economics and Management

University of Cyprus - Nicosia

May 2001, this version December 2001

[1,2] Assistant Professor of Finance and Lecturer of Economics respectively.

JEL classification: G13, G31; C72; L00

Keywords: Real Options, Incomplete Information and Learning, R&D Coordination, 2-Stage Games

This work was partly completed while the first author visited the Centre for Financial Research (Judge Institute of Management, Cambridge U.) and is grateful for the hospitality. He wishes to acknowledge support from the Hermes Center of Excellence in Computational Finance and Economics at the U. of Cyprus. Comments by seminar participants at the *Managing Uncertainty* Program (I. Newton Institute at Cambridge U., August 2001), at the University of Cyprus, and by G. Deltas, M. Dempster, D. Goodman, B. Lambrecht, and T. Stengos, are greatly appreciated. This version replaces University of Cambridge – Judge Institute of Management WP 14/2001.

Address correspondence:

Spiros H. Martzoukos, Assistant Professor
Department of Public and Business Administration,
School of Economics and Management – University of Cyprus
P.O.Box 20537, CY 1678 Nicosia, CYPRUS. Tel.: 357-2-892474, Fax: 357-2-892460.
Email: baspiros@ucy.ac.cy

Real Option Games with Incomplete Information and Spillovers

Abstract

We model in a game theoretic context managerial intervention directed towards value enhancement in the presence of uncertainty and spillover effects. Two firms face real investment opportunities, and before making the irreversible investment decisions, they have options to enhance value by doing more R&D and/or acquiring more information. Due to spillovers, firms act strategically by optimizing their behavior, conditional on the actions of their counterpart. They face two decisions that are solved for interdependently in a two-stage game. The first-stage decision is: what is the optimal level of coordination between them? The second-stage decision is: what is the optimal effort for a given level of the spillover effects and the cost of information acquisition? For the solution we adopt an option pricing framework that allows analytic tractability.

JEL classification: G13, G31; C72; L00

Keywords: Real Options, Incomplete Information and Learning, R&D Coordination, 2-Stage Games

Introduction

We discuss in a game theoretic context managerial intervention directed towards value enhancement in the presence of uncertainty and spillover effects. Two firms face real investment options. Embedded in these are (optional) actions that allow firms to enhance value of their prospects directly (through R&D that improves product attributes or reduces costs, advertisement, etc.) or indirectly through information acquisition (exploratory drilling, marketing research, etc.) Due to spillovers, each firm´s action affects the other firm. In this framework both firms can act strategically and take advantage of the positive spillovers (or take pre-emptive action against the negative spillovers). In equilibrium, the degree of coordination can be higher or lower. The implementation of strategy by each firm can either be implicit, or explicit, i.e., by forming a research joint venture. Each firm must thus decide: (a) How much from this effort to share with its counterpart if this level can vary (the first-stage decision) and (b) How much to spend for such (R&D, etc.) actions, given the spillover effects (the second-stage decision). For the optimal effort (a tactical) decision we allow for a continuous set of alternatives, whereas for the optimal coordination (a strategic) one, we only allow a discrete set of choices. Let for example a pharmaceutical and a chemical firm trying to enhance the value of their investment prospect, by taking R&D actions to develop new technologies. Each one can learn at least part of the results of the effort of the other for free even if both act on their own. However the two firms will optimize the value of their investment options if they strategically determine the amount of R&D they share (in forming for example a research joint venture and deciding the degree of coordination that will take place and the type of research they will engage in). Another example would be two oil firms that own rights in adjacent oil fields. Knowledge resulting from exploratory drilling could be shared benefiting both.

In the paper we adopt the real options framework from the financial economics literature and connect it with the research joint ventures literature of industrial organization. The literature on real options or otherwise irreversible investments under uncertainty (see

Dixit and Pindyck, 1994, and Trigeorgis, 1996) examines the value of flexibility in investment and operating decisions under uncertainty where the traditional net present value (NPV) approach fails. Although the importance of learning actions like exploration, experimentation, and R&D was recognized early on (e.g., Roberts and Weitzman, 1981), the real options literature has paid little attention to management´s ability to intervene in order to acquire more information and/or enhance value. Majd and Pindyck (1989), and Pennings and Lint (1997) examine real options with passive learning, while Childs, et al. (1999), and Epstein, et al. (1999) use a filtering approach towards learning. Recently, Sundaresan (2000) emphasizes the need for adding an incomplete information framework to real options valuation problems. Although many state-variables are usually treated as observable, it is often more realistic to assume that they are only estimates of quantities that will be actually observed or realized later. Martzoukos (2000) considers true value as a random variable with a known probability distribution. He examines real options with controlled jumps of random size (random controls) to model intervention of management as intentional actions with uncertain outcome (learning). He assumes that sequential actions are independent of each other. Martzoukos and Trigeorgis (2000) extend this framework to one with path-dependent actions in order to explain the learning behavior of the (single) firm. They demonstrate that activating learning actions before an investment decision is made, is the solution to an optimization problem that actually captures the trade-off between learning early and paying a cost for it, or leaving information to reveal itself ex-post at a potential cost (of exercising seemingly profitable investment options that actually have a negative NPV, or leaving seemingly unprofitable investment options to expire unexercised when they actually have a positive NPV). We adopt this random controls methodology to examine in a game theoretic framework the behavior of two firms in the existence of spillovers.

Real options papers in a game-theoretic context include Grenadier (1996) with strategic option exercise in real asset markets, where development might be sequential; Williams (1993) with symmetric and simultaneous exercise strategies for real estate developers; Smit and Ankum (1993) with an exogenously determined set of alternative corporate strategies; and Smit and Trigeorgis (2001) that include uncertainty under various types of competition in the product markets. In this paper the interaction that results in a game theoretic framework comes from the existence of spillovers that affect the decision of the other player. The importance of intra-industry spillovers has been documented in the literature. Foster and Rosenzweig (1995) for example emphasize the importance of learning spillovers in agriculture, and Carey and Bolton (1996) argue that collusion in advertisement can be successful due to significant spillovers from generic advertising. Spillover effects are significant even among different sectors, as discussed in Bernstein and Nadiri (1988) and Jaffe and Trajtenberg (1998), and among different countries, as postulated in Thompson (2000). Branstetter (1996) investigates the US and Japan to see whether technological externalities are more international or intranational in scope and finds more support for the latter, while Johnson and Evenson (1999) investigating 14 less developed or developing countries find empirical evidence that both international and interindustrial R&D spillovers can be significant.

Firms may coordinate their R&D efforts without necessarily colluding in the product markets (in a case like that we have a semi-collusion to use the term in Fershtman and Gandal, 1994, Brod and Shivakumar, 1999, etc.) Coordination can take also either the form of research joint ventures (RJV) or non-equity co-development (COD) according to the access the firms have to the innovation (see for example Tao and Wu, 1997). The seminal theoretic model of R&D choice in the presence of spillovers under various structures in the product market is d´Aspremont and Jacquemin (1988), where the degree of spillover is the same. In Kamien et.al. (1992) the degree of spillovers varies. Poyago-Theotoky (1999) recommends that firms endogenously determine the optimal degree of spillovers. In all papers firms operate in the same product market, and authors search for symmetric equilibria.

In our paper, both the optimal effort to exert and the choice of the level of spillovers are endogenously determined. Since we adopt an option pricing framework, we could easily allow at the maturity of this investment option interactions in the product market (joint determination of equilibrium quantity and price). For purposes though of analytic tractability, we focus on the case where the firms can not affect each other in the product market (i.e., they either have monopoly power over their investment option, or prices are exogenously determined). This assumption will allow us the use of models isomorphic to the familiar Black and Scholes (1973). Since firms can operate in different product markets, we allow for asymmetric equilibria in the degree of spillovers between them. We first demonstrate the general model for costly controls at a pre-investment stage that allows for different types of actions for each agent, and we introduce the framework for the two-stage game. We then present in detail the real option game with pure *learning* controls (where value enhancing comes through information acquisition, i.e., exploration, pharmaceutical experimentation, but also marketing research, etc.), and the game with *impact* controls (that allow each player direct value enhancing outcomes, i.e., R&D for attribute improvements or cost reduction, but also advertisement, etc.). We show that unlike in the impact control case where in equilibrium both players exert a positive effort, in the pure learning case there exist equilibria where one player *delegates* all effort to the other (*free lunch*). In both cases we give applications with numerical results and discussion. The final section concludes. The second order and stability conditions for the equilibria are provided in the Appendix.

The real option game with incomplete information, multiplicative controls, and spillovers.

We consider costly R&D (control) actions that managers use to affect the value of an investment opportunity at the pre-investment stage. The outcome of these actions will be observed instantaneously. Information is incomplete in the sense that the controls´ outcome is random. We classify them into two types: pure *learning* control actions with the sole purpose of information acquisition that reduce uncertainty about the estimated investment value; and *impact* control actions with a direct value enhancement (or similarly a cost reduction) purpose. Ex ante we simply know the probability distribution of the outcome, thus we call them random controls. The impact control is the most obvious one, since it is an impulse type with random outcome. Advertisement, process improvement, product attribute enhancement, etc., are actions that result directly in adding value, increasing sales volume and/or price per item, enhancing market share, or reduce production costs. In contrast, pure learning actions are intended to improve the information about (but not to directly affect) a quantity, potential sales price, etc. Exploratory drilling for example will improve information about the value of an oil field, and marketing research will help to better assess market share, etc. Learning actions are thus activated when a parameter significant for the decision making process is estimated with error. Management intends to eliminate or at least reduce this error in order to make optimal investment decisions. If uncertainty is fully resolved, exercise of an investment option on stochastic asset S^* with exercise cost X yields $S^* - X$. Has a learning action not been taken before the investment decision is made, resolution of uncertainty (learning) would occur ex post. Ex ante, the investment decision must be made based solely on expected (instead of actual) outcomes. In this case exercise of the real option is expected to provide $E[S^*] - X$. The real investment prospect is a claim contingent on $S = E[S^*]$, and we assume that $E[S^*]$ follows a geometric Brownian motion, just like S^*. Thus, S will follow the same process before and after learning. Consider for example the case where the underlying asset is a product of two variables, a stochastic but observable one, and a constant but unobservable one. We seek to learn about the unobservable entity, and in doing so we will not affect the stochastic process of the product of the two. At learning, we will simply revise our estimate of the product (which will occur as a jump). *Fully revealing* learning actions are designed to resolve uncertainty completely (assuming this is economically or technically feasible), but in the most general case *partly revealing* actions will be employed either because complete resolution of uncertainty is infeasible, or it is too costly. Each firm faces an investment decision, and either $S = E[S^*]$ is common for both (or simply differs by a constant), or each firm´s claim is contingent on a different asset, simply necessitating separate notation for S_1 and S_2 which again follow geometric Brownian motions. In both types of action (pure learning or impact), the control is modeled as an impulse (jump) with random outcome, activated at a cost.

Formally, we assume that each underlying asset (project) value, S, subject to i optional (and often costly) controls, follows a stochastic process of the form:

$$\frac{dS}{S} = \mu dt + \sigma dZ^R + \sum_{i=1}^{N} k_i dq_i \ , \tag{1}$$

where μ is the instantaneous expected return (drift) and σ the instantaneous standard deviation, dZ^R is an increment of a standard Wiener process in the real probability measure, and dq_i is a jump counter for managerial activation of action i -- a control (not a random) variable. Under risk-neutral valuation, the asset value S follows the process

$$\frac{dS}{S} = \left(r - \delta\right) dt + \sigma dZ + \sum_{i=1}^{N} k_i dq_i \tag{1a}$$

where r is the riskless rate of interest, while the parameter δ represents any form of a "dividend yield" (e.g., in McDonald and Siegel, 1984, δ is a deviation from the equilibrium required rate of return, while in Brennan, 1991, δ is a convenience yield). As in Constantinides (1978) we need to assume that an intertemporal capital asset pricing model as in Merton (1973) holds. As in Merton (1976), we assume the jump (control) risk to be diversifiable (and hence not priced).

For each control i, we assume that the distribution of its size, $1 + k_i$, is log-normal, i.e., $\ln(1 + k_i) \sim \boldsymbol{N}(\gamma_i - .5\sigma_{C_i}^2,\ \sigma_{C_i}^2)$, with $\boldsymbol{N}(.,.)$ denoting the normal density function with mean $\gamma_i - .5\sigma_{C_i}^2$ and variance $\sigma_{C_i}^2$, and $E[k_i] \equiv \bar{k}_i = \exp(\gamma_i) - 1$. The control outcome is assumed independent of the Brownian motion -- although in a more general setting it can be dependent on time and/or the value of S. In general we can assume any plausible form, but the log-normality assumption will allow analytic tractability. Stochastic differential equation (1a) can alternatively be expressed in integral form as:

$$\ln\left[S(T)\right]-\ln\left[S(0)\right]=\int_0^T\left(r-\delta-0.5\sigma^2\right)dt+\int_0^T\sigma dZ(t)+\sum_{i=1}^N dq_i\ln\left(1+k_i\right). \quad (2)$$

Conditional on control activation

$$E[S^* \mid \text{activation of control } i] = E[S^*](1+\bar{k}_i) = S(1+\bar{k}_i)$$

and if the control is a pure learning (information acquisition) action ($\bar{k}_i = 0 = \gamma_i$)

$$E[S^* \mid \text{activation of control } i] = S\ .$$

Each firm's management seeks to optimally activate controls that belong to an admissible set *C*, so that each firm´s claim *F* on the underlying asset must satisfy (subject to the actions of the other firm) the following optimization problem:

$$Maximize\left[F\left(t,S,C\right)\right] \quad (3)$$

subject to:

$$\frac{dS}{S}=\left(r-\delta\right)dt+\sigma dZ+\sum_{i=1}^N k_i dq_i$$

and

ln(1 + k_i) is normally distributed with mean: $\gamma_i - 0.5\,\sigma_{C_i}^2$, and variance: $\sigma_{C_i}^2$.

The next follows directly from the log-normality assumption of the multiplicative controls.

Lemma 1: *Assuming independence between the controls′ outcome and the increment dZ of the standard Wiener process, the conditional solution to the European call option (excluding the controls′ cost) is given by:*

$$F(S, X, T, \sigma, \delta, r; \gamma_i, \sigma^2_{C_i}) = e^{-rT} E[(S^*_T - X)^+ \mid \text{activation of } N \text{ controls}]. \quad (4)$$

The present value of the risk-neutral expectation E[.] conditional on activation of the controls at t = 0, is isomorphic to the Black-Scholes (1973) model:

$$E[(S^*_T - X)^+ \mid \text{activation of } N \text{ controls}] =$$

$$S\, e^{\left(rT - \delta T + \sum_{i=1}^{N} g_i(f_i\gamma_i)\right)} N(d_1) - X\, N(d_2) \quad (5)$$

where

$$d_1 \equiv \frac{\ln(S/X) + (r-\delta)T + \sum_{i=1}^{N} g_i(f_i\gamma_i) + .5\sigma^2 T + .5\sum_{i=1}^{N} g_i\left(f_i\sigma^2_{C_i}\right)}{\left(\sigma^2 T + \sum_{i=1}^{N} g_i\left(f_i\sigma^2_{C_i}\right)\right)^{1/2}}$$

and

$$d_2 \equiv d_1 - \left(\sigma^2 T + \sum_{i=1}^{N} g_i\left(f_i\sigma^2_{C_i}\right)\right)^{1/2}.$$

The ***N***(*d*) denotes the cumulative standard normal density evaluated at *d*. The degree of spillovers (parameter) *f* may differ between firms. The functions *g*(.) with arguments parameters of the controls′ distribution and the degree of spillovers will be shown more clearly in the next sections. The exact form depends on the type of control that is activated (impact or pure learning control). Controls′ outcome can be generated intentionally due to a firm′s own action, or unintentionally due to a spillover effect from the actions of the other firm. Finally the controls′ cost $\beta = \theta g(.)$ where θ is a cost parameter, must be subtracted from the firm′s *conditional* real option value. Given activation of learning controls, the (risk-neutral) probability $P(S_T > X)$ that the call option will be exercised at maturity T equals $\boldsymbol{N}(d_2)$. At the moment of activation of the controls and due to the controls only, the probability that the new value of S will exceed some value X equals $\boldsymbol{N}(d_2)$ with $T \to 0$, since the outcome of the controls is observed instantaneously (impulse type control). In pure learning actions intended just to resolve uncertainty about the true value of the unobservable variable S^* the impact parameters $\gamma_i = 0$. In the most general R&D case the impact parameters would differ from zero, and they can be positive if they affect revenues or negative if they affect fixed or variable costs. The spillover impact from the other firm can have either sign.

The Tactical Resource Allocation Decision.

The two firms must solve their optimization problem simultaneously, seeking thus an equilibrium in this (tactical) decision (see Figure 1). Let us denote with β_1 and β_2 the cost (effort) of the first and the second firm′s actions. The impact on option value is a function of that cost. Given the actions β_2 of the second firm, the impact on the net option value of firm one equals $F_1(\beta_1 \mid \beta_2) - \beta_1$, and given β_2 the first firm must maximize $F_1 - \beta_1$ through the first order condition

$$\frac{\partial\left(F_1\left(\beta_1 \mid \beta_2\right) - \beta_1\right)}{\partial \beta_1} = 0. \qquad (6)$$

Similarly, firm two conditional on the first firm′s action β_1 must maximize $F_2(\beta_2 \mid \beta_1) - \beta_2$ through the first order condition

$$\frac{\partial\left(F_2\left(\beta_2 \mid \beta_1\right)-\beta_2\right)}{\partial\beta_2}=0\,. \tag{6a}$$

The first order conditions are necessary for the existence of a maximum. Furthermore, if the second order conditions (that the second derivative is negative) are satisfied everywhere (or at least in some admissible range), this maximum, is unique (in the admissible range).

As shown in Figures 1(b) and 1(a), the optimal cost effort functions $\beta_1^*(\beta_2)$ and $\beta_2^*(\beta_1)$ for each firm are depended on the other firm's actual effort. In this duopolistic game, both firms optimize their actions simultaneously and the equilibrium solution pair β_1^{**} and β_2^{**} is shown in Figure 1(c) at the intersection of $\beta_1^*(\beta_2)$ and $\beta_2^*(\beta_1)$. Since the cost efforts β_1 and β_2 affect F_1 and F_2 through the *impact* (γ_i) and *learning* (σ_{C_i}) parameters and some cost parameter θ, we must define the mappings $\beta_1(\theta_1, \gamma_1, \sigma_{C_1}^2)$ and $\beta_2(\theta_2, \gamma_2, \sigma_{C_2}^2)$. As will be seen in the examples presented later, it is more intuitive to optimize directly with respect to the learning (or the impact) parameter. In order to solve numerically these two equations in the most general case, the 2x2 Jacobian matrix of the 2nd order analytic derivatives is needed and an iterative two-dimensional Newton-Raphson scheme is implemented. The following proposition holds for cost functions that are homothetic of degree one to the cost parameter θ.

Proposition 1: *The equilibrium efforts of the tactical decision are invariant to identically proportional changes in the price of the underlying asset S, the exercise price X, and the cost β (or equivalently a cost parameter θ) of the control. The constant of proportionality may differ between the two firms.*

Proof: We can verify through equations (6) – (6a) that the property of option prices to be homogenous of degree one in the underlying asset and the exercise price, can also be preserved in this game theoretic context, due to the multiplicative nature of the random control. With the proper choice of the cost function, the conditional option prices (for each firm) can be homogeneous of degree one in the underlying asset, the exercise price, and the control's cost θ, as clearly seen in equations (10a, b) and (12a, b).

The Strategic Coordination Decision.

We now must consider the strategic coordination choice. Firms must decide on the optimal degree of coordination of their R&D efforts. For example, in a 2x2 game, each firm can decide to exert high (H) or low (L) coordination effort (see Figure 2). The degree of coordination determines the extent of spillover effects (through the parameter f), and potentially the cost of R&D (through the cost parameter θ). We assume for ease of exposition two strategies available for each firm, but more than two (or even a continuous set of alternatives) could exist. The choice sets (H, H), (H, L), etc. uniquely determine the degree of spillover effects. The optimal choice for the two firms is provided by the pure *Nash* equilibrium(a), or alternatively the mixed strategies equilibrium. Equilibria off the diagonal can occur because of the asymmetry in the direct spillover effects and the cost reduction results of coordination. This is justifiable when the two firms operate in different product markets, and can be for example technology dependent. The solution to the tactical decision in Figure 1 is nested to the solution of the strategic one in Figure 2. The next follows directly from Proposition 1.

Corollary 1: *If the constant of proportionality (as discussed in proposition* 1*) is the same for both firms, then the Nash equilibrium strategy is invariant to the choice of this constant.*

The real options game with costly information acquisition.

Let us consider the case of pure learning actions. Two companies face an investment opportunity each. Before they decide to invest they also have the option to invest in order to acquire information about the true value or at least a better estimate of the investment. Thus, both impact parameters equal zero since they do not pursue to directly enhance value but they do so indirectly by reducing uncertainty. We consider i (= 1, 2) pure learning actions ($\gamma_i = 0$), one from each firm j (= 1, 2), with $i = j$ implying the firm's own action. To find equilibrium, given the action of the other firm, each one must maximize the conditional option value given below as an application of Lemma 1:

$$F_j - \beta_j = S_j e^{-\delta_j T} N(d_1) - X_j e^{-rT} N(d_2) - \beta_j \tag{7}$$

where

$$d_1 \equiv \frac{\ln(S_j / X_j) + (r - \delta_j)T + 0.5\sigma_j^2 T + 0.5\sum_{i=1}^{2} f_{i\to j}\sigma_{C_i}^2}{\left(\sigma_j^2 T + \sum_{i=1}^{2} f_{i\to j}\sigma_{C_i}^2\right)^{1/2}}$$

and

$$d_2 \equiv d_1 - \left(\sigma_j^2 T + \sum_{i=1}^{2} f_{i\to j}\sigma_{C_i}^2\right)^{1/2}.$$

The information revelation potential is bounded from above (as well as positive)

$$\sum_{i=1}^{2} f_{i\to j}\sigma_{C_i}^2 \le \sigma_{Cmax_j}^2 ,$$

and

$$\sigma_{Cmax_j}^2 - \sum_{i=1}^{2} f_{i\to j}\sigma_{C_i}^2$$

defines exactly the unresolved uncertainty (implying that the *experiment* produces only information and no noise). The parameters $f_{i\to j}$ define the degree of spillovers. For the influence of own actions, most often $f_{1\to 1} = f_{2\to 2} = 1$. Uncertainty is resolved at a cost, and the cost function is defined for simplicity quadratic in the learning effort

$$\beta_j = \theta_j f_{j\to j} \sigma_{C_j}^2 . \tag{8}$$

Note that the cost parameter θ and the spillover parameters f are conditional on the strategic decision. In this application we assume that the optimal learning efforts are below the upper bound for the maximum feasible learning (non-binding constraint) without loss of generality. It is natural to assume that complete elimination of uncertainty would almost be infeasible, or, after some point, exceedingly costly. Else, if the constraint were binding, we would simply incorporate it explicitly in the numerical solution.

The two first order conditions

$$\frac{\partial\left(F_1 - \theta_1 f_{1\to 1} \sigma_{C_1}^2\right)}{\partial \sigma_{C_1}} = 0 , \tag{9a}$$

and

$$\frac{\partial\left(F_2 - \theta_2 f_{2\to 2} \sigma_{C_2}^2\right)}{\partial \sigma_{C_2}} = 0 \tag{9b}$$

are conditional on the other firm's move, and must be solved simultaneously. Specifically we get,

$$\frac{\partial\left(F_1-\theta_1 f_{1\to 1}\sigma_{C_j}^2\right)}{\partial\sigma_{C_1}}=$$

$$X_1 e^{-rT}\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\frac{f_{1\to 1}\sigma_{C_1}}{\left(\sigma_1^2 T+f_{2\to 1}\sigma_{C_2}^2+f_{1\to 1}\sigma_{C_1}^2\right)^{0.5}}-2\theta_1 f_{1\to 1}\sigma_{C_1}=0 \tag{10a}$$

and

$$\frac{\partial\left(F_2-\theta_2 f_{2\to 2}\sigma_{C_2}^2\right)}{\partial\sigma_{C_2}}=$$

$$X_2 e^{-rT}\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\frac{f_{2\to 2}\sigma_{C_2}}{\left(\sigma_2^2 T+f_{2\to 2}\sigma_{C_2}^2+f_{1\to 2}\sigma_{C_1}^2\right)^{0.5}}-2\theta_2 f_{2\to 2}\sigma_{C_2}=0 \tag{10b}$$

After solving for the optimal learning efforts $\sigma_{C_1}^{2**}$ and $\sigma_{C_2}^{2**}$, we find the optimal cost efforts β_1^{**} and β_2^{**}. The second order conditions so that the solution is a maximum are

$$\frac{\partial^2\left(F_j-\theta_j f_{j\to j}\sigma_{C_j}^2\right)}{\partial\left(\sigma_{C_j}\right)^2}<0$$

and are given in the Appendix. The general form of the response functions is as in Figure 3a with the unique solution at *E*. Figure 3b presents the case with the unique equilibrium *E* on the horizontal axis when firm 2 gets a free lunch by exerting zero effort and benefiting from the spillovers from the actions of firm 1 (symmetrically when firm 1 gets the free lunch). We observe that the first order conditions are always satisfied at $\sigma_{C_j}^2=0$, so if also the second order conditions are satisfied, this is a solution. Notice that by the model construction this actually is an interior and not a corner solution.

These figures indicate the existence of a solution. The uniqueness of this solution depends on the slope and the curvature of the response functions. Figure 3c for example, presents a case with three equilibria (E_1, E_2 and E_3), in which case equilibrium E_2 is not a stable one (in the numerical examples presented in this paper, such a case has not been observed). Still, we cannot exclude the possibility of the existence of an infinite number of equilibria when the response functions coincide (such cases are observed and identified with asterisk in Tables 2 and 3). In the stable equilibrium point E in Figure 3a the slope of the second player´s response function is smaller than that of the first player, unlike E_2 in Figure 3c where the opposite holds. We have the following stability condition for each equilibrium point

$$\left|\frac{\partial \sigma^*_{C_2}}{\partial \sigma_{C_1}}\right| < \frac{1}{\left|\frac{\partial \sigma^*_{C_1}}{\partial \sigma_{C_2}}\right|}$$

which is given in the Appendix. In the discussion that follows, all solutions satisfy the SOC and the stability conditions.

In the numerical example we assume that the spillovers are 50% (for $1 \rightarrow 2$) and 25% (for $2 \rightarrow 1$). Tables 1A and 1B show the learning effort and the cost involved for each firm. As we see in Figure 3 equilibrium is when the first firm spends 1.2047 and the second spends 3.0802. Their spending, results in information acquisition equivalent to σ_{C_1} = 0.10976 and σ_{C_2} = 0.20266, providing (through the use of equation 4) a net investment option value equal to 5.2756 and 5.5246 to firm 1 and 2 respectively.

[Enter Tables 1A and 1B, and Figures 3, 3a, 3b, 3c about here]

In Table 2 we see the optimal decisions for a wide variety of asymmetric spillover effects and asymmetric costs. The degree of influence of these parameters on optimal effort is profound. When the underlying investment options and the R&D expenses are symmetric but the spillover effects are not, the firm that receives less spillover benefits must spend more on R&D. When the spillover effects are symmetric but the R&D costs are not, the firm that faces a steeper cost curve would rather reduce R&D spending, and oftentimes, we encounter *free-lunch* (zero effort, *delegation* to the other player), as the preferred choice.

[Enter Tables 2, 3, and 4 about here]

While Table 2 investigates the case where the two firms face symmetric investment decisions, Table 3 investigates the case where the investment alternative of the first one is expected to be of larger scale than that of the second one, and Table 4 investigates the opposite case. As a result of Proposition 1, in the example discussed we can multiply the price of the underlying asset S_1, the exercise price X_1, and the cost θ_1 with a positive constant, then multiply S_2, X_2, and θ_2 with another positive constant, and the results regarding the equilibrium effort will not change (compare for example the lower third of Table 2 with the middle third of Table 3). Optimal option values will of course change by the relevant constant.

Figure 4 presents the results for the strategic decision where the two had to choose the optimal degree of R&D coordination. We see that both decide on the maximum degree of coordination. The first one however, decides not to spend on learning at all (see the results in parenthesis), whereas the second exhibits a very high effort. Figure 5 presents a case where the second one concedes to a high degree and the first to a low degree of coordination. Finally, figure 6 shows a case where the Nash equilibrium is a Low/Low strategy and figure 7 shows a case where the Nash equilibrium is a High/High strategy. Under no Nash equilibrium, mixed strategies could be considered as an alternative approach, providing the probabilities that each firm would play a High or a Low strategy.

[Enter Figures 4, 5, 6 and 7 about here]

As a result of Corollary 1, if for the two firms the constants of multiplication are the same, then the equilibrium strategy for the games in figures 5-7 will be unaffected, since all payoffs will be multiplied by the same positive constant.

The real options game with costly impact controls.

In the previous section we focused on the pure learning (information acquisition) case. If the impact parameters are not zero (a direct effort to enhance value), this case of random control would be similarly solved through the use of the first order conditions (6) and (6a). Again we define the conditional option value by applying Lemma 1

$$F_j - \beta_j = S_j e^{\left(-\delta_j T + \sum_{i=1}^{2} f_{i\to j}\gamma_i\right)} N\left(d_1\right) - X_j e^{-rT} N\left(d_2\right) - \beta_j \qquad (11)$$

where

$$d_1 \equiv \frac{\ln\left(S_j / X_j\right) + \left(r - \delta_j\right)T + \sum_{i=1}^{2} f_{i\to j}\gamma_i + 0.5\sigma_j^2 T + 0.5\sum_{i=1}^{2} | f_{i\to j}\gamma_i | \sigma_{C_i}^2}{\left(\sigma_j^2 T + \sum_{i=1}^{2} | f_{i\to j}\gamma_i | \sigma_{C_i}^2\right)^{1/2}},$$

$$d_2 \equiv d_1 - \left(\sigma_j^2 T + \sum_{i=1}^{2} | f_{i\to j}\gamma_i | \sigma_{C_i}^2\right)^{1/2},$$

and a cost function quadratic in the impact effort $\beta_j = \theta_j \left(f_{j\to j}\gamma_j\right)^2$.

For the tactical decision we consider the two first order conditions

$$\frac{\partial\left(F_1-\theta_1\left(f_{1\to 1}\gamma_1\right)^2\right)}{\partial\gamma_1}=S_1 e^{\left(-\delta_1 T+\sum_{i=1}^{2} f_{i\to 1}\gamma_i\right)} f_{1\to 1} N\left(d_1\right)$$

$$+X_1 e^{-rT}\frac{e^{-d_2^2/2}}{2\sqrt{2\pi}}\frac{s_{1\to 1} f_{1\to 1}\sigma_{C_1}^2}{\left(\sigma_1^2 T+s_{1\to 1} f_{1\to 1}\gamma_1\sigma_{C_1}^2+s_{2\to 1} f_{2\to 1}\gamma_2\sigma_{C_2}^2\right)^{0.5}}-2\theta_1 f_{1\to 1}^2\gamma_1=0 \tag{12a}$$

and

$$\frac{\partial\left(F_2-\theta_2\left(f_{2\to 2}\gamma_2\right)^2\right)}{\partial\gamma_2}=S_2 e^{\left(-\delta_2 T+\sum_{i=1}^{2} f_{i\to 2}\gamma_i\right)} f_{2\to 2} N\left(d_1\right)$$

$$+X_2 e^{-rT}\frac{e^{-d_2^2/2}}{2\sqrt{2\pi}}\frac{s_{2\to 2} f_{2\to 2}\sigma_{C_2}^2}{\left(\sigma_2^2 T+s_{1\to 2} f_{1\to 2}\gamma_1\sigma_{C_1}^2+s_{2\to 2} f_{2\to 2}\gamma_2\sigma_{C_2}^2\right)^{0.5}}-2\theta_2 f_{2\to 2}^2\gamma_2=0 \tag{12b}$$

where the constants $s_{i\to j}$ simply guarantee the positivity of the variance components. Subsequently and similarly with the pure learning case we solve the two equations simultaneously and we get the optimal impact efforts γ_1^{**} and γ_2^{**} and through them the optimal cost efforts β_1^{**} and β_2^{**}. Since each player´s intension is to enhance value, the admissible impact effort is non-negative (for a put option it would be non-positive). It is also bounded from above by the minimum of γ_{max_j} (which defines an economically or technically feasible range) and the point where the second derivative becomes positive (which guarantees uniqueness).

Proposition 2: *Let $f_{j\to j}>0$ and the cost function $\beta_j=\theta_j\left(f_{j\to j}\gamma_j\right)^2$. Then in any equilibrium in the impact control case, both players exert a positive effort (thus excluding delegation or free lunch).*

Proof: Notice that equations 12a and 12b get positive values at $\gamma_j=0$ given that $f_{j\to j}>0$. Therefore, conditional solutions ($\gamma_1^*|\ \gamma_2$) and ($\gamma_2^*|\ \gamma_1$) are positive, and (unlike the case of pure learning) in any equilibrium, both players must exert a *positive* effort.

Using the cost function $\beta_j = \theta_j (f_{j\to j}\gamma_j)^2$, the second order conditions are

$$\frac{\partial^2 \left(F_j - \theta_j \left(f_{j\to j}\gamma_j \right)^2 \right)}{\partial \left(\gamma_j \right)^2} < 0$$

and the stability condition is

$$\left| \frac{\partial \gamma_2^*}{\partial \gamma_1} \right| < \frac{1}{\left| \frac{\partial \gamma_1^*}{\partial \gamma_2} \right|} .$$

In the Appendix again we provide the second order and the stability conditions in detail. In the discussion that follows, all solutions again satisfy the SOC and the stability conditions.

An example for the cost function $\beta_j = \theta_j (f_{j\to j}\gamma_j)^2$, with the range of admissible impact parameters positive and bounded below 100% is given in Figure 8 and Tables 4A and 4B (for the solution to the tactical decision). Figures 9 and 10 present two examples where the Nash equilibrium for the strategic decision is the highest (H/H) and the lowest (L/L) degree of coordination respectively.

[Enter figures 8, 9, 10 and Tables 4A and 4B about here]

In Figure 9 the impact spillover is positive (like generic advertisement) whereas in Figure 10 the impact spillover can be negative when advertisement is more competitive and less costly.

Conclusions

This paper presents and solves a real options game that jointly addresses at the pre-investment stage the strategic decision about the extent of coordination between two firms, and the decision about the optimal effort invested in R&D in the presence of uncertainty and spillover effects. We assume that the two firms can influence each other´s decision at the pre-investment stage, whereas at the investment decision each firm has monopoly power over its investment and there is no further interaction between the two. Firms want to enhance value and to resolve (or reduce) uncertainty of real (investment) opportunities, before they make a commitment. Managerial actions are treated as controlled jumps of random size whose realization is a random variable with a known probability distribution. We used a contingent claims framework with incomplete information and costly control actions, and without loss of generality or any sacrifice in insights gained we made the assumption that the two firms face investment opportunities of the European type, allowing thus the use of analytic models isomorphic to Black and Scholes (1973). Alternatively, fully numerical methods like lattice or numerical solutions to partial differential equations could have been used, but the iterative solution to this continuous game would have been much more intensive computationally and less accurate. Within such a numerical solution framework it would be easy to incorporate further firm interactions in the product markets. Note that the model discussed above pertains to call options where players try to enhance the value of the underlying asset *S*. Symmetrically we could have worked for a put option where the players could pursue a cost-reduction strategy.

Finally, the solution to the firms' optimal strategic and tactical R&D decision-making is found as the solution of a two-stage game. This decision, as expected, is heavily dependent on the effectiveness of R&D investments, their cost, and the degree of coordination that is optimal for the two firms. Some times *high* coordination and other times *low* coordination will be optimal. The degree of coordination affects both the degree of spillovers, and the parameters of the cost function. In the cases of pure learning actions, there are instances where a firm will *delegate* research by agreeing on a high degree of coordination (lowering thus the R&D cost of the other firm and increasing the degree of spillovers) and reap afterwards the rewards. In contrast, in the cases of impact control, players always exert a positive effort with the trivial exception when control is prohibitively expensive. In general we have shown that optimal coordination and optimal R&D effort are essential for value enhancement and optimal investment decision-making.

Appendix: Second Order (SOC) and Stability Conditions.

We demonstrate the general SOCs and the stability conditions for the continuous game of the tactical decision. The conditions are shown for player one, and those for player two can be derived symmetrically. In all we assume that the two players own investment options of the European call type, and the cases of put options can be analyzed similarly. For the conditional optimization of player one, the second order condition for a maximum in the case of pure learning actions, is

$$\frac{\partial^2\left(F_1-\theta_1 f_{1\to 1}\sigma_{C_1}^2\right)}{\partial\left(\sigma_{C_1}\right)^2}<0 \qquad \text{A.1}$$

where

$$\frac{\partial^2\left(F_1-\theta_1 f_{1\to 1}\sigma_{C_1}^2\right)}{\partial\left(\sigma_{C_1}\right)^2}=\frac{A}{\sqrt{2\pi}}X_1 e^{-rT}\frac{f_{1\to 1}\sigma_{C_1}}{\left(\hat{\sigma}_1^2 T\right)^{0.5}}$$
$$-X_1 e^{-rT}\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\frac{\left(f_{1\to 1}\sigma_{C_1}\right)^2}{\left(\hat{\sigma}_1^2 T\right)^{3/2}}+X_1 e^{-rT}\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\frac{f_{1\to 1}}{\left(\hat{\sigma}_1^2 T\right)^{0.5}}-2\theta_1 f_{1\to 1} \qquad \text{A.1a}$$

with

$$A=\frac{e^{-d_2^2/2}d_2 f_{1\to 1}\sigma_{C_1}}{\left(\hat{\sigma}_1^2 T\right)^{0.5}}\left(1+\frac{d_2}{\left(\hat{\sigma}_1^2 T\right)^{1/2}}\right)$$

$$\hat{\sigma}_1^2 T\equiv\sigma_1^2 T+\sum_{i=1}^{2} f_{i\to 1}\sigma_{C_i}^2$$

and

$$d_2\equiv\frac{\ln\left(S_1/X_1\right)+\left(r-\delta_1\right)T-0.5\hat{\sigma}_1^2 T}{\left(\hat{\sigma}_1^2 T\right)^{1/2}}.$$

Then we derive the SOC for the case of impact control for player one

$$\frac{\partial^2\left(F_1-\theta_1\left(f_{1\to1}\gamma_1\right)^2\right)}{\partial\left(\gamma_1\right)^2}<0 \qquad \text{A.2}$$

where

$$\begin{aligned}
\frac{\partial^2\left(F_1-\theta_1\left(f_{1\to1}\gamma_1\right)^2\right)}{\partial\left(\gamma_1\right)^2} &= S_1 e^{\left(-\delta_1 T+\sum_{i=1}^{2} f_{i\to1}\gamma_i\right)} f_{1\to1}^2 N\left(d_1\right) \\
&+BS_1 e^{\left(-\delta_1 T+\sum_{i=1}^{2} f_{i\to1}\gamma_i\right)} \frac{e^{-d_1^2/2}}{\sqrt{2\pi}} f_{1\to1}\left(1-\frac{s_{1\to1}\sigma_{C_1}^2 d_2}{2\left(\hat{\sigma}_1^2 T\right)^{0.5}}\right) \\
&-X_1 e^{-rT}\frac{e^{-d_2^2/2}}{4\sqrt{2\pi}}\left(s_{1\to1}f_{1\to1}\sigma_{C_1}^2\right)^2\frac{1}{\left(\hat{\sigma}_1^2 T\right)^{1.5}} \\
&+X_1 e^{-rT}\frac{e^{-d_2^2/2}}{4\sqrt{2\pi}}\left(s_{1\to1}f_{1\to1}\sigma_{C_1}^2\right)^2\frac{d_2}{\left(\hat{\sigma}_1^2 T\right)}-2\theta_1 f_{1\to1}^2
\end{aligned} \qquad \text{A.2a}$$

with

$$B=\frac{f_{1\to1}}{\left(\hat{\sigma}_1^2 T\right)^{0.5}}+\frac{0.5 s_{1\to1}f_{1\to1}\sigma_{C_1}^2}{\left(\hat{\sigma}_1^2 T\right)^{0.5}}\left(1-\frac{d_1}{\left(\hat{\sigma}_1^2 T\right)^{0.5}}\right)$$

$$\hat{\sigma}_1^2 T\equiv\sigma_1^2 T+\sum_{i=1}^{2} s_{i\to1}f_{i\to1}\sigma_{C_i}^2$$

$$d_1\equiv\frac{\ln\left(S_1/X_1\right)+(r-\delta_1)T+\sum_{i=1}^{2} f_{i\to1}\gamma_i+0.5\hat{\sigma}_1^2 T}{\left(\hat{\sigma}_1^2 T\right)^{0.5}}$$

and

$$d_2 \equiv d_1 - \left(\hat{\sigma}_1^2 T\right)^{0.5}$$

Next we provide the stability conditions for equilibrium. Note that the stability conditions differ from the ones for the d´Aspremont and Jacquemin (1988) (conditions which were discussed in Henriques, 1990), due to the asymmetry and non-linearity of our case (see also Seade, 1980). For the local stability of the equilibrium for the case of pure learning we require

$$\left|\frac{\partial \sigma_{C_2}^*}{\partial \sigma_{C_1}}\right| < \frac{1}{\left|\frac{\partial \sigma_{C_1}^*}{\partial \sigma_{C_2}}\right|} \qquad \text{A.3}$$

where $\sigma_{C_2}^* \equiv \sigma_{C_2}^*(\sigma_{C_1})$ and $\sigma_{C_1}^* \equiv \sigma_{C_1}^*(\sigma_{C_2})$. The terms are derived from

$$\frac{\partial \sigma_{C_1}^*}{\partial \sigma_{C_2}} = \frac{\dfrac{\partial^2\left(F_1 - \theta_1 f_{1\to 1}\sigma_{C_1}^2\right)}{\partial\left(\sigma_{C_1}\right)\partial\left(\sigma_{C_2}\right)}}{\dfrac{\partial^2\left(F_1 - \theta_1 f_{1\to 1}\sigma_{C_1}^2\right)}{\partial\left(\sigma_{C_1}\right)^2}} \text{ and } \frac{\partial \sigma_{C_2}^*}{\partial \sigma_{C_1}} = \frac{\dfrac{\partial^2\left(F_2 - \theta_2 f_{2\to 2}\sigma_{C_2}^2\right)}{\partial\left(\sigma_{C_1}\right)\partial\left(\sigma_{C_2}\right)}}{\dfrac{\partial^2\left(F_2 - \theta_2 f_{2\to 2}\sigma_{C_2}^2\right)}{\partial\left(\sigma_{C_2}\right)^2}}. \qquad \text{A.3a}$$

To complete calculations we need the cross-partial derivative

$$\frac{\partial^2\left(F_1 - \theta_1 f_{1\to 1}\sigma_{C_1}^2\right)}{\partial\left(\sigma_{C_1}\right)\partial\left(\sigma_{C_2}\right)} =$$

$$\frac{A}{\sqrt{2\pi}} X_1 e^{-rT} \frac{f_{1\to 1}\sigma_{C_1}}{\left(\hat{\sigma}_1^2 T\right)^{0.5}} - X_1 e^{-rT} \frac{e^{-d_2^2/2}}{\sqrt{2\pi}} \frac{\left(f_{1\to 1}\sigma_{C_1} f_{2\to 1}\sigma_{C_2}\right)}{\left(\hat{\sigma}_1^2 T\right)^{3/2}} \qquad \text{A.3b}$$

with

$$A = \frac{e^{-d_2^2/2} d_2 f_{2\to 1}\sigma_{C_2}}{\left(\widehat{\sigma}_1^2 T\right)^{0.5}} \left(1 + \frac{d_2}{\left(\widehat{\sigma}_1^2 T\right)^{0.5}}\right)$$

$$\widehat{\sigma}_1^2 T \equiv \sigma_1^2 T + \sum_{i=1}^{2} f_{i\to 1}\sigma_{C_i}^2$$

and

$$d_2 \equiv \frac{\ln\left(S_1 / X_1\right) + \left(r - \delta_1\right)T - 0.5\widehat{\sigma}_1^2 T}{\left(\widehat{\sigma}_1^2 T\right)^{1/2}} .$$

Similarly we get the stability condition for the case of impact control

$$\left|\frac{\partial \gamma_2^*}{\partial \gamma_1}\right| < \frac{1}{\left|\frac{\partial \gamma_1^*}{\partial \gamma_2}\right|} \qquad \text{A.4}$$

where $\gamma_1^* \equiv \gamma_1^*(\gamma_2)$ and $\gamma_2^* \equiv \gamma_2^*(\gamma_1)$, and the terms are again derived from

$$\frac{\partial \gamma_1^*}{\partial \gamma_2} = \frac{\dfrac{\partial^2\left(F_1 - \theta_1\left(f_{1\to 1}\gamma_1\right)^2\right)}{\partial\left(\gamma_1\right)\partial\left(\gamma_2\right)}}{\dfrac{\partial^2\left(F_1 - \theta_1\left(f_{1\to 1}\gamma_1\right)^2\right)}{\partial\left(\gamma_1\right)^2}} \text{ and } \frac{\partial \gamma_2^*}{\partial \gamma_1} = \frac{\dfrac{\partial^2\left(F_2 - \theta_2\left(f_{2\to 2}\gamma_2\right)^2\right)}{\partial\left(\gamma_1\right)\partial\left(\gamma_2\right)}}{\dfrac{\partial^2\left(F_2 - \theta_2\left(f_{2\to 2}\gamma_2\right)^2\right)}{\partial\left(\gamma_2\right)^2}} . \qquad \text{A.4a}$$

To complete, we provide the cross-partial derivative

$$
\begin{aligned}
\frac{\partial^2\left(F_1-\theta_1\left(f_{1\to1}\gamma_1\right)^2\right)}{\partial(\gamma_1)\partial(\gamma_2)} &= S_1 e^{\left(-\delta_1 T+\sum_{i=1}^{2} f_{i\to1}\gamma_i\right)} f_{1\to1} f_{2\to1} N(d_1) \\
&+B S_1 e^{\left(-\delta_1 T+\sum_{i=1}^{2} f_{i\to1}\gamma_i\right)} \frac{e^{-d_1^2/2}}{\sqrt{2\pi}} f_{1\to1}\left(1-\frac{s_{1\to1}\sigma_{C_1}^2 d_2}{2\left(\hat{\sigma}_1^2 T\right)^{0.5}}\right) \\
&-X_1 e^{-rT}\frac{e^{-d_2^2/2}}{4\sqrt{2\pi}}\frac{\left(s_{1\to1}f_{1\to1}\sigma_{C_1}^2\right)\left(s_{2\to1}f_{2\to1}\sigma_{C_2}^2\right)}{\left(\hat{\sigma}_1^2 T\right)^{1.5}} \\
&+X_1 e^{-rT}\frac{e^{-d_2^2/2}}{4\sqrt{2\pi}}\frac{\left(s_{1\to1}f_{1\to1}\sigma_{C_1}^2\right)\left(s_{2\to1}f_{2\to1}\sigma_{C_2}^2\right)d_2}{\left(\hat{\sigma}_1^2 T\right)}
\end{aligned} \qquad \text{A.4b}
$$

with

$$
B=\frac{f_{2\to1}}{\left(\hat{\sigma}_1^2 T\right)^{0.5}}+\frac{0.5 s_{2\to1}f_{2\to1}\sigma_{C_2}^2}{\left(\hat{\sigma}_1^2 T\right)^{0.5}}\left(1-\frac{d_1}{\left(\hat{\sigma}_1^2 T\right)^{0.5}}\right)
$$

$$
\hat{\sigma}_1^2 T \equiv \sigma_1^2 T+\sum_{i=1}^{2} s_{i\to1} f_{i\to1}\sigma_{C_i}^2
$$

$$
d_1 \equiv \frac{\ln\left(S_1/X_1\right)+(r-\delta_1)T+\sum_{i=1}^{2} f_{i\to1}\gamma_i+0.5\hat{\sigma}_1^2 T}{\left(\hat{\sigma}_1^2 T\right)^{0.5}}
$$

and

$$
d_2 \equiv d_1-\left(\hat{\sigma}_1^2 T\right)^{0.5}
$$

References

Bernstein, J. I., and M. I. Nadiri, 1988. Interindustry R&D Spillovers, Rates of Return, and production in High-Tech Industries. *American Economic Review* 78, 429-434.

Black, F., and M. Scholes, 1973. The pricing of options and corporate liabilities. *Journal of Political Economy* 81, 637-659.

Branstetter, L., 1996. Are knowledge spillovers international or intranational in scope? Microeconomic evidence from the US and Japan. National Bureau of Economic Research Working Paper: 5800.

Brennan, M.J., 1991. The price of convenience and the valuation of commodity contingent claims. In Lund D., and Øksendal, B. (Eds.), *Stochastic Models and Option Values*. North-Holland, 33-72.

Brod, A., and R. Shivakumar, 1999. Advantageous semi-collusion. *Journal of Industrial Economics* XLVII, 221-230.

Carey, C., and D. L. Bolton, 1996. Brand versus generic advertising and the decision to advertise collectively. *Review of Industrial Organization* 11, 93-105.

Childs, P. D., S. H. Ott, and T. J. Riddiough, 1999. Valuation and information acquisition policy for claims written on noisy real assets. Working Paper presented at the 3rd Annual International Conference on Real Options, NIAS – Leiden, The Netherlands.

Constantinides, G., 1978. Market risk adjustment in project valuation. *Journal of Finance* 33, 603-616.

d´Aspremont, C., and A. Jacquemin, 1988. Cooperative and noncooperative R&D in duopoly with spillovers. *American Economic Review* 78, 1133-1137.

Dixit, A.K., and R.S. Pindyck, 1994. *Investment Under Uncertainty*. Princeton University Press, Princeton: New Jersey.

Epstein, D., N. Mayor, P. Schonbucher, E. Whaley, and P. Wilmott, 1999. The value of market research when a firm is learning: Real option pricing and optimal filtering. In *Real options and business strategy. Applications to decision making*, L. Trigeorgis (ed.). Risk Books, London, UK.

Fershtman, C., and N. Gandal, 1994. Disadvantageous semicollusion. *International Journal of Industrial Organization* 12, 141-154.

Foster, A. D., and M. R. Rosenzweig, 1995. Learning by doing and learning from others: Human capital and technical change in agriculture. *Journal of Political Economy* 103, 1176-1209.

Grenadier, S. R., 1996. The strategic exercise of options: Development cascades and overbuilding in real estate markets. *Journal of Finance* 51, 1653-1679.

Henriques, I., 1990. Cooperative and noncooperative R&D in duopoly with spillovers: Comment. *American Economic Review* 80, 638-640.

Jaffe, A. B., and M. Trajtenberg, 1998. International knowledge flows: evidence from patent citations. National Bureau of Economic Research Working Paper: 6507.

Johnson, D. K. N., and R. E. Evenson, 1999. R&D spillovers to agriculture: Measurement and application. *Contemporary Economic Policy* 17, 432-456.

Kamien, M. I., E. Muller, and I. Zang, 1992. Research joint ventures and R&D cartels. *American Economic Review* 82, 1293-1306.

Majd, S., and R. Pindyck, 1989. The learning curve and optimal production under uncertainty. *RAND Journal of Economics* 20, 331-343.

Martzoukos, S.H., 2000, Real options and the value of learning. *Annals of Operations Research* 99, 305-323.

Martzoukos, S.H., and L. Trigeorgis, 2000. Resolving a real options paradox with incomplete information: After all, *why learn*? Working paper, Department of Public and Business Administration, University of Cyprus. Presented at the 5th Annual International Conference on Real Options, UCLA (July 2001).

McDonald, R., and D. Siegel, 1984. Option pricing when the underlying asset earns a below-equilibrium rate of return: A note. *Journal of Finance* 39, 261-265.

Merton, R. C., 1973. An intertemporal capital asset pricing model. *Econometrica* 41, 867-887.

Merton, R. C., 1976. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* 3, 125-144.

Paddock, J. L., D. Siegel, and J.L. Smith, 1988. Option valuation of claims on real assets: The case of offshore petroleum leases. *Quarterly Journal of Economics* 103, 479-508.

Pennings, E., and Lint, O., 1997. The option value of advanced R&D. *European Journal of Operational Research* 103, 83-94.

Poyago-Theotoky, J., 1999. A Note on Endogenous Spillovers in a Non-Tournament R&D Duopoly. *Review of Industrial Organization* 15, 253-262.

Roberts, K., and M. Weitzman, 1981. Funding criteria for research, development and exploration projects. *Econometrica* 49, 1261-1288.

Seade, J. K., 1980. The stability of Cournot revisited. *Journal of Economic Theory* 23, 15-27.

Smit, H. T. J., and L. A. Ankum, 1993. A real options and game-theoretic approach to corporate investment strategy under competition. *Financial Management* 22, 241-250.

Smit, H. T. J., and L. Trigeorgis, 2001. R&D option strategies. Working Paper presented at the 5th Annual Conference on Real Options at UCLA.

Sundaresan, S. M., 2000. Continuous-time methods in finance: A review and an assessment. *The Journal of Finance* 55, 1569-1622.

Tao, Z., and C. Wu, 1997. On the organization of cooperative research and development: theory and evidence. *International Journal of Industrial Organization* 15, 573-596.

Thompson, P., 2000. Learning from the experience of others: Parameter uncertainty and economic growth in a model of creative destruction. *Journal of Economic Dynamics and Control* 24, 1285-1314.

Trigeorgis, L., 1996. *Real Options: Managerial Flexibility and Strategy in Resource Allocation*. The MIT Press.

Williams, J., 1993. Equilibrium and options on real assets. *Review of Financial Studies* 6, 825-850.

Table 1A
Pure learning response functions
Firm 1 optimizing (given 2's effort)

Total learning effort		Learning cost		Net call option value	
1's effort	2's effort	1's cost	2's cost	1's option	2's option
0.14028	0.10266	1.9680	0.7904	4.5123	5.4933
0.13506	0.12766	1.8240	1.2222	4.6563	5.5611
0.12841	0.15266	1.6488	1.7478	4.8314	5.5974
0.12010	0.17766	1.4424	2.3671	5.0379	5.5888
0.10976	0.20266	1.2047	3.0802	5.2756	5.5246
0.09673	0.22766	0.9357	3.8870	5.5445	5.3964
0.07972	0.25266	0.6355	4.7876	5.8447	5.1979
0.05515	0.27766	0.3041	5.7819	6.1762	4.9241
0.00000	0.30266	0.0000	6.8700	6.5386	4.5876

Table 1B
Pure learning response functions
Firm 2 optimizing (given 1's effort)

Total learning effort		Learning cost		Net call option value	
1's effort	2's effort	1's cost	2's cost	1's option	2's option
0.00976	0.21690	0.0095	3.5284	5.3223	5.0764
0.03476	0.21561	0.1208	3.4866	5.3283	5.1181
0.05976	0.21285	0.3571	3.3980	5.3329	5.2067
0.08476	0.20857	0.7184	3.2625	5.3212	5.3422
0.10976	0.20266	1.2047	3.0802	5.2756	5.5246
0.13476	0.19497	1.8160	2.8509	5.1785	5.7538
0.15976	0.18529	2.5523	2.5748	5.0148	6.0299
0.18476	0.17328	3.4136	2.2518	4.7720	6.3529
0.20976	0.15841	4.3998	1.8820	4.4401	6.7228

Notes (for 1A and 1B). Investment options' parameters are: underlying assets $S_1 = S_2 = 100.00$, exercise prices $X_1 = X_2 = 100.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$, and costs of learning (per unit of variance) $\theta_1 = 100.00$ and $\theta_2 = 75.00$ with spillover of learning 50% (for $1 \rightarrow 2$) and 25% (for $2 \rightarrow 1$).

Table 2
Optimal R&D learning effort (firm 1 / firm 2)

Cost θ_2	Spillovers $2 \to 1$	Spillovers $1 \to 2$				
		0.00	0.25	0.50	0.75	1.00
50.00	0.00	0.14938 0.34096	0.14938 0.33268	0.14938 0.32419	0.14938 0.31547	0.14938 0.30650
	0.25	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096
	0.50	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096
	0.75	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096
	1.00	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096
75.00	0.00	0.14938 0.21701	0.14938 0.20375	0.14938 0.18957	0.14938 0.17423	0.14938 0.15741
	0.25	0.10267 0.21701	0.10604 0.21043	0.10976 0.20266	0.11390 0.19329	0.11855 0.18176
	0.50	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701
	0.75	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701
	1.00	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701
100.00	0.00	0.14938 0.14938	0.14938 0.12937	0.14938 0.10563	0.14938 0.07469	0.14938 0.00000
	0.25	0.12937 0.14938	0.13361 0.13361	0.13830 0.11292	0.14352 0.08286	0.14938 0.00000
	0.50	0.10563 0.14938	0.11292 0.13830	0.12197 0.12197	0.13361 0.09448	0.14938 0.00000
	0.75	0.07469 0.14938	0.08286 0.14352	0.09448 0.13361	0.11292 0.11292	0.14938 0.00000
	1.00	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	*0.10563 0.10563

Notes. First number for firm 1 and second for firm 2. $S_1 = X_1 = 100.00$ and $S_2 = X_2 = 100.00$. Other real investment option parameters as in Tables 1A and 1B.
* Due to the symmetry of the assumptions, the response functions coincide to provide an infinite number of equilibria, and this point is approximated at the limit from $f_{1\to2} = f_{2\to1} \to 1^-$ or $f_{1\to2} = f_{2\to1} \to 1^+$.

Table 3
Optimal R&D learning effort (firm 1 / firm 2)

Cost θ_2	Spillovers $2 \rightarrow 1$	Spillovers $1 \rightarrow 2$				
		0.00	0.25	0.50	0.75	1.00
50.00	0.00	0.14938 0.24898	0.14938 0.23751	0.14938 0.22546	0.14938 0.21273	0.14938 0.19919
	0.25	0.08256 0.24898	0.08527 0.24530	0.08826 0.24103	0.09160 0.23600	0.09534 0.23000
	0.50	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898
	0.75	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898
	1.00	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898
75.00	0.00	0.14938 0.14938	0.14938 0.12937	0.14938 0.10563	0.14938 0.07469	0.14938 0.00000
	0.25	0.12937 0.14938	0.13361 0.13361	0.13830 0.11292	0.14352 0.08286	0.14938 0.00000
	0.50	0.10563 0.14938	0.11292 0.13830	0.12197 0.12197	0.13361 0.09448	0.14938 0.00000
	0.75	0.07469 0.14938	0.08286 0.14352	0.09448 0.13361	0.11292 0.11292	0.14938 0.00000
	1.00	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	*0.10563 0.10563
100.00	0.00	0.14938 0.09078	0.14938 0.05160	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000
	0.25	0.14232 0.09078	0.14698 0.05329	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000
	0.50	0.13488 0.09078	0.14420 0.05516	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000
	0.75	0.12702 0.09078	0.14091 0.05724	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000
	1.00	0.11863 0.09078	0.13698 0.05958	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000

Notes. First number for firm 1 and second for firm 2. $S_1 = X_1 = 100.00$ and $S_2 = X_2 = 75.00$. Other real investment option parameters as in Tables 1A and 1B.

* Due to the relative symmetry of the assumptions, the response functions coincide to provide an infinite number of equilibria, and this point is approximated at the limit from $f_{1\rightarrow 2} = f_{2\rightarrow 1} \rightarrow 1^-$ or $f_{1\rightarrow 2} = f_{2\rightarrow 1} \rightarrow 1^+$.

Table 4

Optimal R&D learning effort (firm 1 / firm 2)

Cost θ_2	Spillovers $2 \rightarrow 1$	Spillovers $1 \rightarrow 2$				
		0.00	0.25	0.50	0.75	1.00
50.00	0.00	0.09078	0.09078	0.09078	0.09078	0.09078
		0.34096	0.33793	0.33487	0.33178	0.32866
	0.25	0.00000	0.00000	0.00000	0.00000	0.00000
		0.34096	0.34096	0.34096	0.34096	0.34096
	0.50	0.00000	0.00000	0.00000	0.00000	0.00000
		0.34096	0.34096	0.34096	0.34096	0.34096
	0.75	0.00000	0.00000	0.00000	0.00000	0.00000
		0.34096	0.34096	0.34096	0.34096	0.34096
	1.00	0.00000	0.00000	0.00000	0.00000	0.00000
		0.34096	0.34096	0.34096	0.34096	0.34096
75.00	0.00	0.09078	0.09078	0.09078	0.09078	0.09078
		0.21701	0.21221	0.20730	0.20227	0.19711
	0.25	0.00000	0.00000	0.00000	0.00000	0.00000
		0.21701	0.21701	0.21701	0.21701	0.21701
	0.50	0.00000	0.00000	0.00000	0.00000	0.00000
		0.21701	0.21701	0.21701	0.21701	0.21701
	0.75	0.00000	0.00000	0.00000	0.00000	0.00000
		0.21701	0.21701	0.21701	0.21701	0.21701
	1.00	0.00000	0.00000	0.00000	0.00000	0.00000
		0.21701	0.21701	0.21701	0.21701	0.21701
100.00	0.00	0.09078	0.09078	0.09078	0.09078	0.09078
		0.14938	0.14232	0.13488	0.12702	0.11863
	0.25	0.05160	0.05329	0.05516	0.05724	0.05958
		0.14938	0.14698	0.14420	0.14091	0.13698
	0.50	0.00000	0.00000	0.00000	0.00000	0.00000
		0.14938	0.14938	0.14938	0.14938	0.14938
	0.75	0.00000	0.00000	0.00000	0.00000	0.00000
		0.14938	0.14938	0.14938	0.14938	0.14938
	1.00	0.00000	0.00000	0.00000	0.00000	0.00000
		0.14938	0.14938	0.14938	0.14938	0.14938

Notes. First number for firmr 1 and second for firm 2. $S_1 = X_1 = 75.00$ and $S_2 = X_2 = 100.00$. Other real investment option parameters as in Tables 1A and 1B.

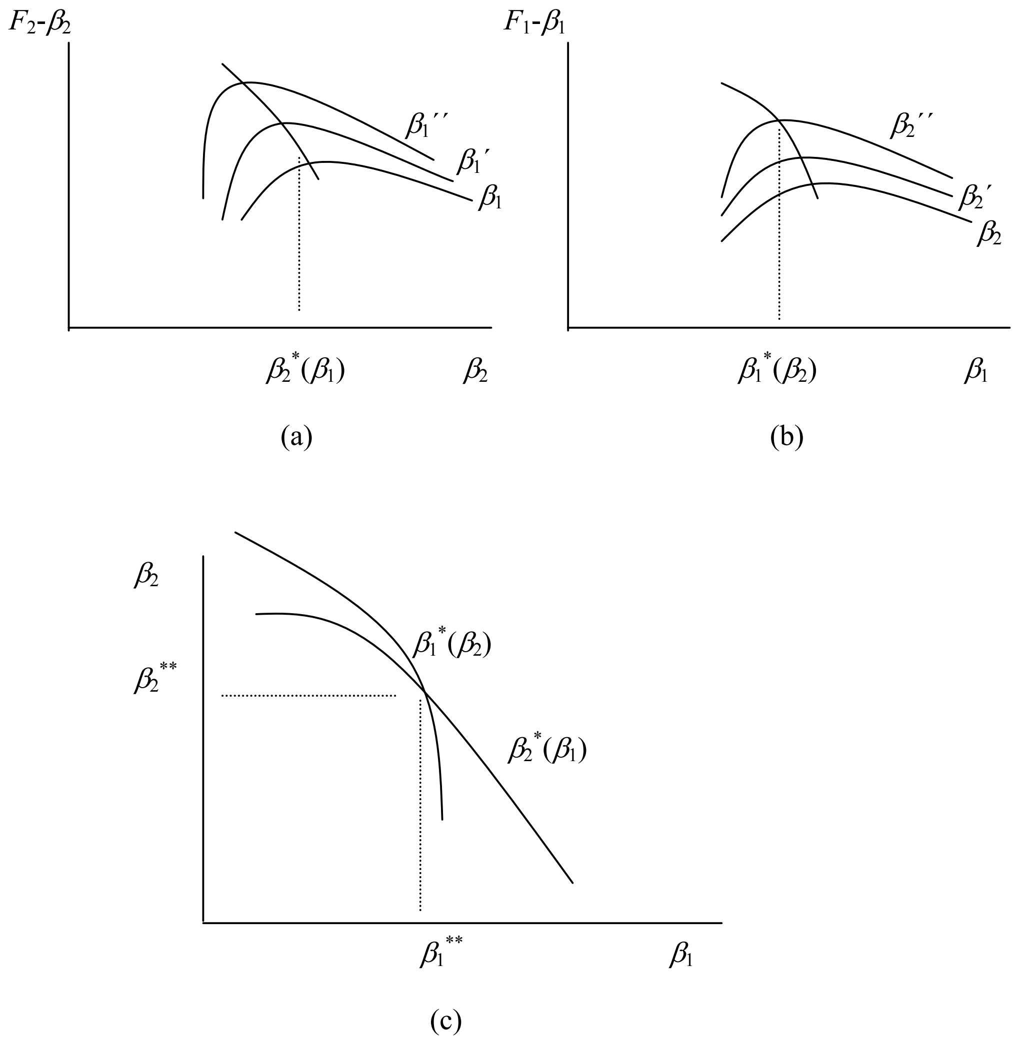


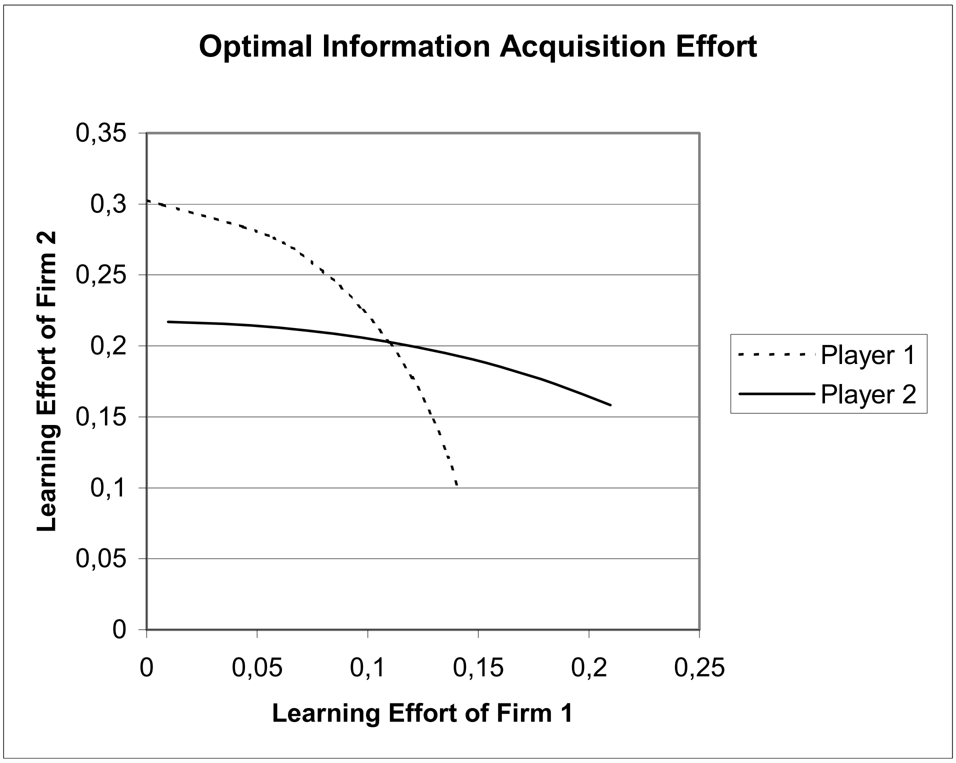
Figure 1
Game with Spillovers: Tactical Resource Allocation Decision

Notes: β_1, β_2 are the learning costs incurred by the two firms, $F_1(\beta_1 \mid \beta_2)$, $F_2(\beta_2 \mid \beta_1)$ are the investment option values before learning costs are subtracted, and $\beta_1^*(\beta_2)$, $\beta_2^*(\beta_1)$ are the optimal cost efforts (of each firm *conditional* on the effort of the other). The equilibrium solution pair (β_1^{**}, β_2^{**}) is given by the intersection of the two optimal *conditional* cost effort curves.

		FIRM 1	
		H	*L*
FIRM 2	*H*	$F_1(H, H), F_2(H, H)$	$F_1(L, H), F_2(L, H)$
	L	$F_1(H, L), F_2(H, L)$	$F_1(L, L), F_2(L, L)$

Figure 2
Game with Learning Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) or Low (*L*) degree of R&D coordination. F_1 and F_2 are the investment option values (before the cost of investments in coordinated R&D are subtracted). $F_1(H, L) = F_1[\beta_1^{**}(H, L), \beta_2^{**}(H, L)]$, $F_2(H, L) = F_2[\beta_1^{**}(H, L), \beta_2^{**}(H, L)]$.



Optimal Information Acquisition Effort
0,35
0,3
0,25
0,2
0,15
0,1
0,05
0
0
0,05
0,1
0,15
0,2
0,25
Learning Effort of Firm 2
Learning Effort of Firm 1
Player 1
Player 2

Figure 3

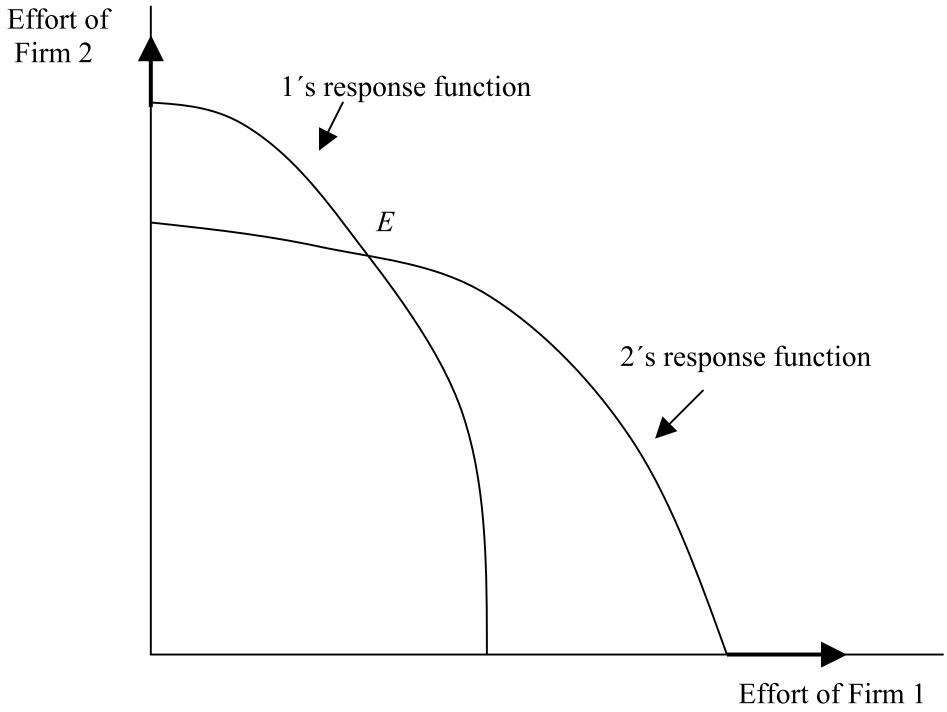


Figure 3a
Unique Optimal (Learning) R&D Solution – the general case

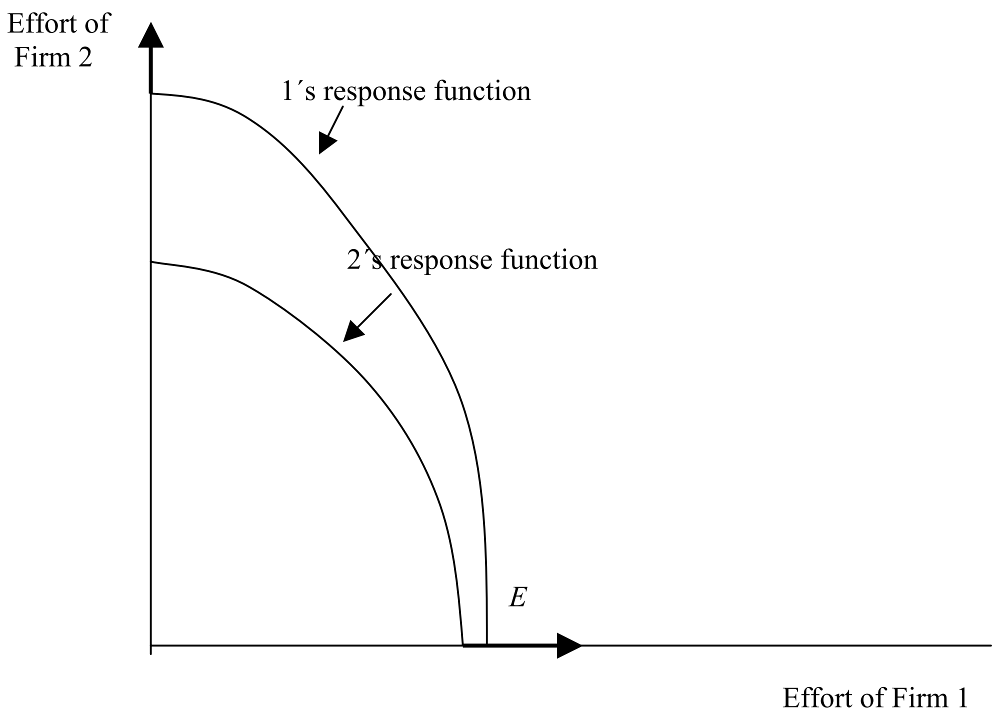


Figure 3b
Unique Optimal (Learning) R&D Solution – free lunch for firm 2

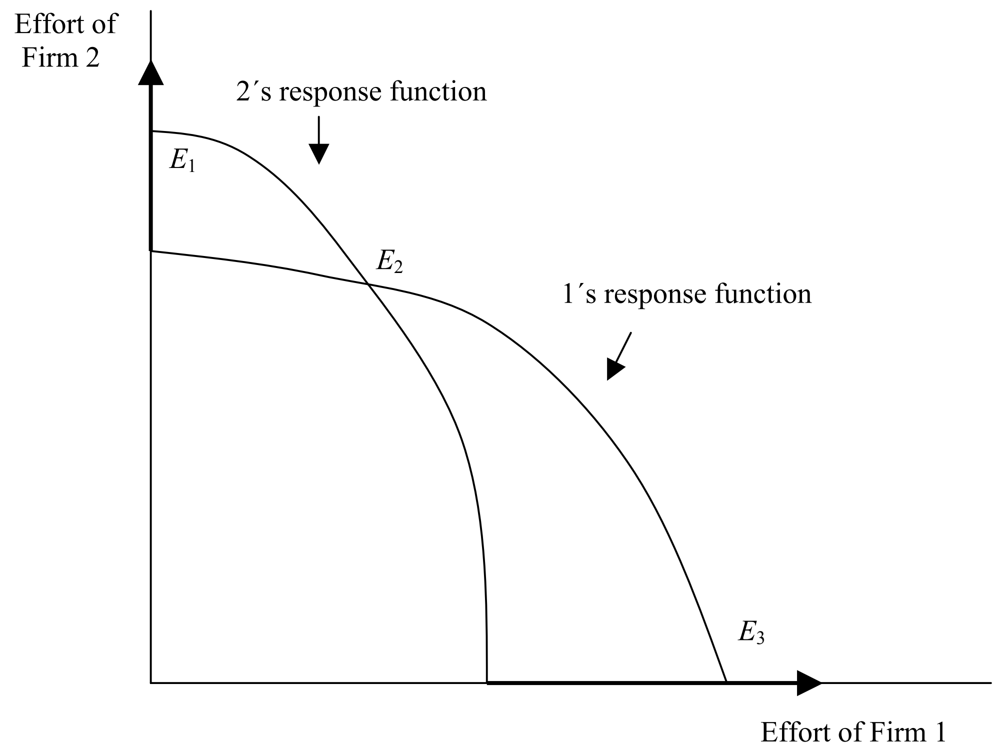


Figure 3c
Multiple Optimal (Learning) R&D Solutions

		FIRM 1	
		H	*L*
FIRM 2	*H*	**16.3155, 7.4114** (0.0000, 0.5136) 80.00, 25.00 0.75, 0.75	5.7594, 3.7296 (0.1702, 0.1228) 80.00, 75.00 0.75, 0.25
	L	9.9166, 7.4114 (0.0000, 0.5136) 100.00, 25.00 0.25, 0.75	4.5915, 3.4628 (0.1374, 0.1171) 100.00, 80.00 0.25, 0.25

Figure 4
Game with Learning Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) or Low (*L*) degree of R&D coordination. The numerical results for the investment option values under optimal effort are presented first (with the Nash equilibrium in bold). Numbers in parenthesis present the firms' effort. Below we provide the costs θ_1 and θ_2, and each firm's degree of spillover from the other firm's effort. Option parameter values are: $S_1 = X_1 = 100.00$, $S_2 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$.

		FIRM 1	
		H	*L*
FIRM 2	*H*	4.5076, 4.8640 (0.1727, 0.0000) 90.00, 75.00 0.75, 0.75	**5.1513, 3.4829** (0.1506, 0.0977) 90.00, 85.00 0.75, 0.25
	L	4.4205, 4.3453 (0.1435, 0.0829) 100.00, 75.00 0.25, 0.75	4.5055, 3.4209 (0.1405, 0.1013) 100.00, 85.00 0.25, 0.25

Figure 5
Game with Learning Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) or Low (*L*) degree of R&D coordination. The numerical results for the investment option values under optimal effort are presented first (with the Nash equilibrium in bold). Numbers in parenthesis present the firms' effort. Below we provide the costs θ_1 and θ_2, and each firm's degree of spillover from the other firm's effort. Option parameter values are: $S_1 = X_1 = 100.00$, $S_2 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$.

FIRM 1

FIRM 2		*H*	*L*
	H	4.5186, 3.8350 (0.1401, 0.0734) 100.00, 85.00 0.50, 0.50	4.7988, 3.3586 (0.1297, 0.1049) 100.00, 85.00 0.50, 0.25
	L	4.3645, 3.9005 (0.1455, 0.0680) 100.00, 85.00 0.25, 0.50	**4.5055, 3.4209** (0.1405, 0.1013) 100.00, 85.00 0.25, 0.25

Figure 6
Game with Learning Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) or Low (*L*) degree of R&D coordination. The numerical results for the investment option values under optimal effort are presented first (with the Nash equilibrium in bold). Numbers in parenthesis present the firms' effort. Below we provide the costs θ_1 and θ_2, and each firm's degree of spillover from the other firm's effort. Option parameter values are: $S_1 = X_1 = 100.00$, $S_2 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$.

FIRM 1

		H	*L*
FIRM 2	*H*	**6.2356, 4.3422** (0.1417, 0.1886) 82.00, 57.00 0.50, 0.50	5.5622, 3.7309 (0.1552, 0.1450) 85.00, 70.00 0.50, 0.25
	L	5.1596, 4.3185 (0.1502, 0.1702) 90.00, 60.00 0.25, 0.50	4.9779, 3.7400 (0.1568, 0.1446) 90.00, 70.00 0.25, 0.25

Figure 7
Game with Learning Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) or Low (*L*) degree of R&D coordination. The numerical results for the investment option values under optimal effort are presented first (with the Nash equilibrium in bold). Numbers in parenthesis present the firms' effort. Below we provide the costs θ_1 and θ_2, and each firm's degree of spillover from the other firm's effort. Option parameter values are: $S_1 = X_1 = 100.00$, $S_2 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$.

Table 4A

Impact response functions

Firm 1 optimizing conditional on 2's effort

Total impact effort		Impact cost		Net call option value	
1's effort	2's effort	1's cost	2's cost	1's option	2's option
0.08266	0.01785	3.4168	0.1594	6.2138	6.9726
0.08719	0.03785	3.8013	0.7165	7.0636	8.0927
0.09119	0.05785	4.1579	1.6736	7.9559	8.9596
0.09472	0.07785	4.4863	3.0307	8.8858	9.5515
0.09785	0.09785	4.7878	4.7878	9.8490	9.8491
0.10065	0.11785	5.0649	6.9449	10.8418	9.8354
0.10315	0.13785	5.3204	9.5020	11.8610	9.4971
0.10543	0.15785	5.5573	12.4591	12.9041	8.8243
0.10750	0.17785	5.7787	15.8162	13.9689	7.8101

Table 4B

Impact response functions

Firm 2 optimizing conditional on 1's effort

Total impact effort		Impact cost		Net call option value	
1's effort	2's effort	1's cost	2's cost	1's option	2's option
0.017855	0.082665	0.1594	3.4168	6.9726	6.2138
0.037855	0.087192	0.7165	3.8013	8.0927	7.0636
0.057855	0.091191	1.6736	4.1579	8.9596	7.9559
0.077855	0.094724	3.0307	4.4863	9.5515	8.8858
0.097855	0.097855	4.7878	4.7878	9.8491	9.8490
0.117855	0.100647	6.9449	5.0649	9.8354	10.8418
0.137855	0.103154	9.5020	5.3204	9.4971	11.8610
0.157855	0.105426	12.4591	5.5573	8.8243	12.9041
0.177855	0.107505	15.8162	5.7787	7.8101	13.9689

Notes (for 4A and 4B). Investment options' parameters are: underlying assets $S_1 = S_2 = 100.00$, exercise prices $X_1 = X_2 = 100.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$.

Impact cost parameters $\theta_1 = 500.00$ and $\theta_2 = 500.00$ with spillover of impact 50% (for $1 \to 2$) and 50% (for $2 \to 1$). Impact induced volatility equals $\sum_{i=1}^{2} | f_{i \to j} \gamma_i | \sigma_{C_i}^2 = \sum_{i=1}^{2} | f_{i \to j} \gamma_i | 0.05^2$ for both firms. Admissible impact range is bounded below 100%.

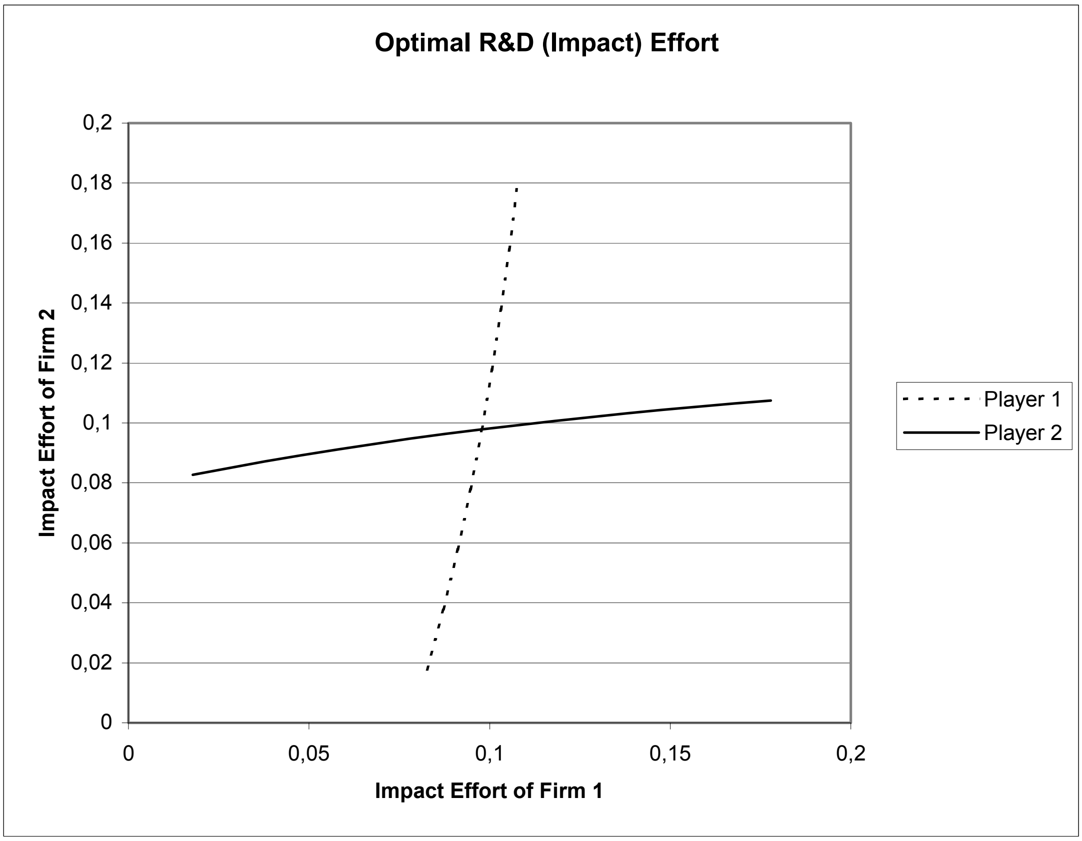

Optimal R&D (Impact) Effort
Impact Effort of Firm 2
0,2
0,18
0,16
0,14
0,12
0,1
0,08
0,06
0,04
0,02
0
0
0,05
0,1
0,15
0,2
Impact Effort of Firm 1
Player 1
Player 2

Figure 8

FIRM 1

		H	*L*
FIRM 2	*H*	**21.3807, 16.4136** (0.2566, 0.1856) 250.00, 250.00 0.50, 0.50	14.2699, 7.2619 (0.2366, 0.0702) 250.00, 500.00 0.50, 0.25
	L	9.1118, 9.3363 (0.0955, 0.1652) 500.00, 250.00 0.25, 0.50	6.7082, 4.8011 (0.0854, 0.0592) 500.00, 500.00 0.25, 0.25

Figure 9
Game with Impact Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) or Low (*L*) degree of R&D coordination. The numerical results for the investment option values under optimal effort are presented first (with the Nash equilibrium in bold). Numbers in parenthesis present the firms' effort. Below we provide the costs θ_1 and θ_2, and each firm's degree of spillover from the other firm's effort. Option parameter values are: $S_1 = X_1 = 100.00$, $S_2 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$. The impact induced volatility for both firms equals $\sum_{i=1}^{2} | f_{i \to j} \gamma_i | \sigma_{C_i}^2 = \sum_{i=1}^{2} | f_{i \to j} \gamma_i | 0.05^2$. Admissible impact range is bounded below 100%.

FIRM 1

		H	*L*
FIRM 2	*H*	7.1494, 4.5636 (0.1153, 0.0578) 400.00, 500.00 0.15, 0.15	10.6827, 1.4637 (0.2251, 0.0271) 250.00, 500.00 0.15, -0.25
	L	3.0131, 6.1968 (0.0542, 0.1508) 500.00, 250.00 -0.25, 0.15	**9.1651, 1.6091** (0.2192, 0.0384) 250.00, 400.00 -0.25, -0.25

Figure 10
Game with Impact Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) or Low (*L*) degree of R&D coordination. The numerical results for the investment option values under optimal effort are presented first (with the Nash equilibrium in bold). Numbers in parenthesis present the firms' effort. Below we provide the costs θ_1 and θ_2, and each firm's degree of spillover from the other firm's effort. Option parameter values are: $S_1 = X_1 = 100.00$, $S_2 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$. The impact induced volatility for both firms equals $\sum_{i=1}^{2} | f_{i \to j} \gamma_i | \sigma_{C_i}^2 = \sum_{i=1}^{2} | f_{i \to j} \gamma_i | 0.05^2$. Admissible impact range is bounded below 100%.