The Valuation of Options on Multiple Operating Cash Flows

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Abstract

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This paper establishes a risk neutral valuation relationship (RNVR) for the pricing of options on multiple operating cash flows assuming that there is a representative agent who has an extended power utility function. Aggregate consumption and the underlying operating cash flows are multivariate displaced lognormal distributed. This RNVR is applied to obtain closed-form expressions for the value of a new class of investment options, the event-contingent options. The formulae maintain the risk neutrality characteristic of the Black-Scholes model, and depend on the threshold parameters of the underlying cash-flows. The threshold parameter is the lower bound of the underlying stochastic variable. A negative threshold parameter assigns a positive probability to both inflow and outflow events. The paper also offers examples of event-contingent options in a global context.

JEL classification: G13; G31

Keywords: Event-contingent options; extended power utility function; operating cash flows; risk neutral valuation relationship; displaced lognormal

1. Introduction

Operating cash flows (OCF) may have negative, zero or positive values. It appears inadequate to assume that operating cash flows have a standard lognormal distribution[1]. The standard lognormal distribution is a special case of the displaced lognormal distribution when the threshold parameter or lower limit of this density is zero. If the threshold parameter is negative, then there is a positive probability for both inflows and outflows to occur. In this case, the underlying variable is a cash flow rather than an asset value[2], and the origin of the density function of the underlying is displaced from zero to its lower bound at the left of zero. The actual displaced lognormal density function of an operating cash flow is given by the following equation:

$$f(S) = \frac{1}{\sqrt{2\pi}\sigma(S-\beta)} exp\left\{-\frac{1}{2\sigma^2}\left[ln(S-\beta)-\mu\right]^2\right\},$$

where $S > \beta$[3]. That is $S \sim \Lambda^P(\beta, \mu, \sigma^2)$, where P denotes the actual probability measure. Figure 1 displays two (actual) lognormal densities with $\mu = 0$ and $\sigma = 0.5$. The densities, which have exactly the same form, have different lower limits.

Option prices obtained in this paper have implicit a multivariate displaced lognormal risk neutral density. The literature has discussed three main methods for the risk neutral pricing of derivative assets. The first method is due to Black and Scholes (1973) and Merton (1973). They assume that it is possible to construct and to maintain a riskless hedge, that implies a partial differential equation (PDE). The second method uses the martingale approach and was initiated by Cox and Ross (1976), Harrison and Kreps (1979), and Harrison and Pliska (1981). These authors show that a risk neutral measure exists in an economy where there are no arbitrage opportunities

[1] See also Dixit and Pindyck (1994, p.137), who criticize the assumption of a standard lognormal distribution for project values that can become negative.

[2] See Shimizu and Crow (1988) for a characterization of the displaced or three-parameter lognormal.

[3] In this notation, β is a threshold parameter (i.e. the lower limit), μ is a scale parameter, and σ is a shape or volatility parameter.

and investors are non satiated. The first two methods, when the state-space is continuous, assume that markets are dynamically completed by continuous trading. The third method, initiated by Rubinstein (1976), Brennan (1979), and Stapleton and Subrahmanyam (1984) uses the first order conditions of a representative agent to obtain a risk neutral formula. A discrete-time economy with a continuous state-space is assumed, which means that markets are dynamically incomplete. These authors assume either constant proportional risk aversion (CPRA) coupled with a multivariate standard lognormal distribution or constant absolute risk aversion coupled with a multivariate normal distribution. This is the approach explored by our research.

This paper derives a multivariate risk neutral valuation relation (RNVR) assuming decreasing absolute risk aversion (DARA) preferences[4], and that aggregate consumption and the underlying operating cash flows are multivariate displaced lognormal distributed. This extends previous work both for broader preferences and broader distributions. Such economic setting appears more reasonable than an economy that assumes dynamic trading, when the objective is to price contingent claims on multiple operating cash flows. The preference-based assumptions are different from the standard hedged-based assumptions, but the option pricing results might be identical. In particular, the principle of the relative valuation of contingent claims is preserved[5]. The displaced lognormal RNVR also preserves the risk neutral property of the Black-Scholes (1973) model[6], which avoids estimating the rates of discount for the claims, the expected values of the operating cash flows (OCF) under the true probability measure, and the parameters of the utility function.

[4]Elton and Gruber (1995, p.218) argue that "there is a general agreement that most investors exhibit decreasing absolute risk aversion [which characterizes the extended power utility function]".

[5]For example, the Black-Scholes valuation equation might be obtained using the original partial differential equation (PDE) approach of Black and Scholes (1973) and Merton (1973), the martingale approach of Harrison and Kreps (1979) and Harrison and Pliska (1981), and the preference-based approach of Rubinstein (1976), Brennan (1979) and Stapleton and Subrahmanyam (1984) which assumes constant proportional risk aversion (CPRA) for the preferences of the representative agent and a bivariate standard lognormal for the distribution of consumption and stock price.

[6]Brennan and Schwartz (1985) also stress the advantages of the risk neutral property in the real options area.

The paper also defines a new type of investment options, the event-contingent options. Then the RNVR is applied to derive closed-form solutions for the price of these event-contingent options. Examples to motivate the event-contingent options are offered in a global context.

The paper is organized as follows. Section 2 defines and motivates the event-contingent options. Section 3 establishes the multivariate displaced lognormal RNVR. Section 4 applies the RNVR to derive analytical expressions for the event-contingent options. Section 5 concludes.

2. Investment options

Investment options include the option to defer investment, the option to default during staged construction, the option to expand operations, the option to contract operations, and the option to abandon for salvage value. (See Trigeorgis (1996) for a comprehensive review of the literature). In general, previous work on investment options assume either a PDE approach (e.g. Dixit and Pindyck (1994)) or a no-arbitrage approach (e.g. Berk, Green, and Naik (1999)) to obtain analytical valuation equations. The exception is the paper by Childs, Ott and Triantis (1998) who apply the RNVR derived by Brennan (1979) to obtain closed-form solutions. Our paper is different from their work on two respects, however. First, our paper applies the displaced lognormal RNVR which we will establish in the next section. Second, in our paper the payoffs of the options and the closed-form solutions are both new. These options are the event-contingent options. Childs, Ott and Triantis (1998) derive analytical expressions for the value of a sequential exchange option and the value of an option on the maximum of two assets.

An event-contingent investment option is a contingent claim whose value depend on the net present value (NPV) of interacting projects of investment. We define and provide closed-form expressions for the value of four categories of event-contingent real options: (i) option to invest contingent on investment; (ii) option to invest contingent on divestment; (iii) option to divest contingent on divestment; and (iv) option to divest contingent on investment.

The reasoning underlying the event-contingent options is as follows: an option to invest in a project might be seen as a call option, whose exercise depends on several other factors besides the NPV of the project; an option to divest in a project might be seen as a put option, whose exercise depends on several other factors besides the net present value (NPV) of the project. Those factors might, in the real world, be very complex. To simplify things, we focus our attention in two sorts of events, which we define as investment and divestment in a related project. Such structure is already sufficient to show that there are very subtle interactions in strategic real options, that were not previously considered in the literature. Interactions in strategic real options were previously considered by Trigeorgis (1993) and Kulatilaka (1995).

Consider two projects, project 1 and project 2. Consider also, at the maturity date of the options, that $NPV_1 = OCF_1 - I_1$ and that $NPV_2 = OCF_2 - I_2$ are the differences between the operating cash flows and the investment values, respectively, of project 1 and project 2.

The payoff of an option to invest (in project 2) contingent on investment (in project 1) is defined by:

$$C_{II} = \begin{cases} NPV_2 & \text{if } NPV_2 > 0 \text{ and } NPV_1 > 0 \\ 0 & \text{if otherwise.} \end{cases}$$

The payoff of the option to invest contingent on investment is positive (i.e. $C_{II} = NPV_2$), reflecting the fact that the option will be exercised (i.e. a firm invests in project 2), if both $OCF_2 > I_2$ and $OCF_1 > I_1$. If at least one of these two conditions is not satisfied, then the option is not exercised (i.e. the firm does not invest in project 2), and the payoff of the option to invest contingent on investment is zero.

The following is an example observed in practice, when the investment project of a firm is related with projects of other firms. An option to invest contingent on investment yields a given payoff for the investing firm if and only if the project is attractive for the firm itself and a related project is attractive for another company. “Nike and Reebok have prospered by concentrating on their strengths: designing and marketing high-tech, fashionable footwear for sports and fitness. Nike owns one small factory that makes some sneaker parts. Reebok owns no plants. The two rivals contract virtually all footwear production to suppliers in Taiwan, South Korea, and other Asian countries"[7]. The projects of the suppliers are highly dependent on the main projects carried out by Nike and Reebok. In this and other cases of outsourcing, the dependent or supply companies may have an option to invest (e.g. increase capacity) which they exercise if and only if the investment is profitable to them and the main investment project of the contracting company presents a reliable source of income.

The payoff of an option to invest (in project 2) contingent on divestment (project 1) is defined by:

$$C_{ID} = \begin{cases} NPV_2 & \text{if } NPV_2 > 0 \text{ and } NPV_1 < 0 \\ 0 & \text{if otherwise.} \end{cases}$$

The payoff of the option to invest contingent on divestment is positive (i.e. $C_{ID} = NPV_2$), reflecting the fact that the option will be exercised (i.e. the firm invests in project 2), if both $OCF_2 > I_2$ and $OCF_1 < I_1$. If at least one of these two conditions is not satisfied, then the option is not exercised (i.e. the firm does not invest in project 2), and the payoff of the option to invest contingent on divestment is zero.

A realistic business situation of an option to invest contingent on divestment seems to exist in the Spanish tourism industry. The largest of the Balearic Islands is Mallorca, whose best known resort is probably Magaluf. Its climate is ideal for beach tourism. The development of the tourism industry in this region has been supported by the Spanish government during the last 40 years. During the 1960's planning controls and building restrictions were relaxed and many operators began to build accommodation *en masse*.[8] This, and other factors, transformed Magaluf in a working-class

[7] See Stern, El-Ansary, and Coughlan (1996, p. 234).

[8] See Hunter-Jones, Hughes, Eastwood, and Morrison (1997).

resort. At the beginning of the 1990's as the demand for this type of tourism started to decrease, some big hotels began to register losses. Recently, the government introduced a structured and controlled land policy for the area, giving incentives to quality tourism or sustainable tourism. In particular, restrictions were introduced on the location and height of new hotel developments. From a point of view of interacting investment options, the tourism operators have an option to invest (in small hotels with enviromental improvements and accommodation upgradings) conditional on divest (in the existing large hotels of low quality). The popular press in some European countries has reported that in Magaluf, some tourism operators started to exercise this option.

The payoff of an option to divest (project 2) contingent on divestment (project 1) is defined by:

$$C_{DD} = \begin{cases} -NPV_2 & \text{if } NPV_2 < 0 \text{ and } NPV_1 < 0 \\ 0 & \text{if otherwise.} \end{cases}$$

The payoff of the option to divest contingent on divestment is positive (i.e. $C_{DD} = -NPV_2$), reflecting the fact that the option will be exercised (i.e. the firm divests project 2), if both $OCF_2 < I_2$ and $OCF_1 < I_1$. If at least one of these two conditions is not satisfied, then the option is not exercised (i.e. the firm does not divest project 2), and the payoff of the option to divest contingent on divestment is zero.

This option pricing model might be helpful to describe certain behaviour of a follower. Firms, from a strategic point of view, might be classified as leaders or followers. Hence the option to sell (e.g. a property) of a follower might be considered an option to divest (the property) contingent on divestment by the leader (of another property)[9].

The payoff of an option to divest (project 2) contingent on investment (in project 1) is defined

[9] Quigg (1995) argues that "building cycles occur because property in a given urban area have highly correlated prices processes...When it is optimal to abandon one property, others in the neighborhood often experience the same pressures, leading to urban decay".

by:

$$C_{DI} = \begin{cases} -NPV_2 & \text{if } NPV_2 < 0 \text{ and } NPV_1 > 0 \\ 0 & \text{if otherwise.} \end{cases}$$

The payoff of the option to divest contingent on investment is positive (i.e. $C_{DI} = -NPV_2$), reflecting the fact that the option will be exercised (i.e. the firm divests project 2), if both $OCF_2 < I_2$ and $OCF_1 > I_1$. If at least one of these two conditions is not satisfied, then the option is not exercised (i.e. the firm does not divest project 2), and the payoff of the option to divest contingent on investment is zero.

Realistic examples of the option to divest contingent on investment seem to exist in the US pharmaceutical industry, which invests annually in R&D worldwide more than USD 55 billion[10]. The US pharmaceutical research companies has alone more than 1000 new medicines (investment projects) in development. Since, normally, it takes between 12 and 15 years to discover and develop a new medicine (project), some non viable projects are abandoned before their initially antecipated end. Some of these research projects use living organisms, which have to be kept alive after the abandon of a project. Otherwise, such living matter would perish and it would be extremely expensive or almost impossible to replace it later. In such situation, a research project can only be abandoned if another research project is developed using such perishable resources.

We conclude this section with some parity relations, which we state in the first proposition of the paper.

Proposition 1. (Event-contingent parity relations) *Suppose that there are no arbitrage opportunities in the economy. Assume that the options mature at the same time, are written on the same underlying OCF_2, have the same strike price I_2, and the additional contingency is on NPV_1. Then the following relations hold:*

$$I. \qquad P_c = P_{II} + P_{ID}$$

[10] See PhRMA-Pharmaceutical Research and Manufacturers of America Annual Survey (1998) to obtain these and other data about the industry.

$$II. \qquad P_p = P_{DD} + P_{DI}$$

where the symbol P stands for price or current value, c and p stand for call and put, and the other symbols have the meaning given before.

Proof: The payoff of a call option on project 2 is $C_c = NPV_2$ if $NPV_2 > 0$. The payoff of a put option on project 2 is $C_p = -NPV_2$ if $NPV_2 < 0$. Constructing tables of payoffs at the maturity of the options and using arbitrage arguments leads to results I and II.□

The first parity states that a call option is a portfolio of one option to invest contingent on investment and one option to invest contingent on divestment. The second parity states that a put option is a portfolio of one option to divest contingent on divestment and one option to divest contingent on investment.

This section defined the event-contingent investment options. Latter in the paper we will derive analytical expressions for the value of these options.

3. The discrete-time option pricing model

This section establishes the multivariate displaced lognormal RNVR.

3.1. Economic assumptions

The analysis starts by assuming that there is a representative agent with a time-additive separable utility function of consumption. The representative individual optimizes her utility function of consumption both at the beginning and the end of the economy:

$$\max_{C_0, C} E^P\{U(C_0, C)\},$$

where:

$$C = (W_0 - C_0) r_f + \underline{v}^{'} [\underline{C}(\underline{S}) - \underline{P} r_f];$$

$U(C_0)$ is her utility function for initial consumption;

$U(C)$ is her utility function for end-of-period consumption;

$E^P(.)$ is the expectations operator under the true probability measure;

C is her end-of-period consumption;

W_0 is her initial wealth;

C_0 is her beginning-of-period consumption;

r_f is 1 plus the risk free rate of return.

$\underline{v}$ is the column vector of demands for units of claims, $j = 1, 2, ..., N$;

$\underline{C}(\underline{S})$ is the column vector of end-of-period payoffs associated with the contingent claims as a function of the underlying variables, $\underline{S}' = [S_1, S_2, ..., S_n]$, and

$\underline{P}$ is the column vector of the current market values of the claims $\underline{C}(\underline{S})$.

Assuming that the representative agent is nonsatiated and risk averse, the vector of the current market values of the contingent claims can be written as follows:

$$\underline{P} = r_f^{-1} E^P[\underline{C}(\underline{S}) Z(\underline{S})],$$

where:

$$Z(\underline{S}) = \frac{E^P[U'(C) \mid \underline{S}]}{E^P[U'(C)]} \tag{1}$$

defines the pricing kernel. The current value of an individual contingent claim can be written in terms of its end-of-period payoff $C(\underline{S})$ as follows:

$$P = r_f^{-1} E^P[C(\underline{S}) Z(\underline{S})]. \tag{2}$$

The underlying operating cash flows can themselves be valued, by applying the general formula, yielding:

$$\underline{V} = r_f^{-1} E^P[\underline{S} Z(\underline{S})], \tag{3}$$

where $\underline{V}$ is the vector of values of the underlying operating cash flows.

The general valuation formulae (1), (2) and (3) will be used to establish a RNVR in an economy where the representative agent has an extended power utility function of the HARA family of utility functions. Aggregate consumption and the underlying operating cash flows are multivariate displaced lognormal distributed.

Definition 1. (The marginal extended power utility function) *The marginal extended power utility function of end-of-period consumption is:*

$$U'(C) = (C - \beta_c)^{\varphi}, \tag{4}$$

where both $C > \beta_c$ and $\varphi < 0$ denote decreasing absolute risk aversion (DARA). The utility function displays increasing proportional risk aversion (IPRA) if $\beta_c < 0$, constant proportional risk aversion (CPRA) if $\beta_c = 0$, and decreasing proportional risk aversion (DPRA) if $\beta_c > 0$.

The joint distribution of consumption and underlying operating cash flows is defined next:

Definition 2. (The multivariate displaced lognormal) *Let the $1+N$-dimensional random variable $(C, \underline{S})$ have the joint probability density function (PDF):*

$$\begin{bmatrix} C \\ \underline{S} \end{bmatrix} \sim \Lambda^P \left(\begin{bmatrix} \beta_c \\ \underline{\beta} \end{bmatrix}, \begin{bmatrix} \mu_c \\ \underline{\mu} \end{bmatrix}, \begin{bmatrix} \sigma_c^2 & \sum_{cs} \\ \sum_{sc} & \sum_s \end{bmatrix} \right),$$

where $\beta_c < C < \infty$, $\underline{\beta} < \underline{S} < \infty$, $\beta_c > 0$, $-\infty < \underline{\beta} < \infty$, $-\infty < \mu_c < \infty$, $-\infty < \underline{\mu} < \infty$, $\sigma_c > 0$, $\sum_s > 0$, and $-\infty < \sum_{cs} < \infty$. Then the random variable $(C, \underline{S})$ is defined to have a multivariate displaced lognormal distribution.

The matrix $\sum_{cs}$ represents the covariances between the normal variates underlying consumption and the operating cash flows, while $\sum_s$ is the matrix variances-covariances of the normal variates underlying the operating cash flows.

Basic assumption. *The representative agent has an extended power utility function of consumption and, in particular, a marginal utility function of consumption given by equation (4), and aggregate consumption and the underlying operating cash-flows are multivariate displaced lognormal distributed and, in particular, have a joint density given by definition 2.*

3.2. Pricing results

The next lemma links preferences and distributions to evaluate the pricing kernel. The lemma will play a fundamental role in the proof of the RNVR.

Lemma 1. (The pricing kernel) *Suppose that the basic assumption holds. Then the pricing kernel is given by the following equation:*

$$Z(\underline{S}) = exp\left\{\varphi \sum\nolimits_{cs} \sum\nolimits_{s}^{-1} [ln(\underline{S} - \underline{\beta}) - \underline{\mu}] - \frac{1}{2}\varphi^2 \sum\nolimits_{cs} \sum\nolimits_{s}^{-1} \sum\nolimits_{sc}\right\}. \tag{5}$$

Proof: See appendix.

As one can see the pricing kernel $Z(\underline{S})$ is a scalar that depends on the preference parameter φ and on the random payoff of multiple underlying cash flows $\underline{S}$. The pricing kernel is a positive stochastic variable and, hence, rules out arbitrage opportunities from the economy.

The next proposition evaluates the basic valuation formulae of the economy considering the extended power utility function and the multivariate displaced lognormal distribution, and derives the RNVR. This RNVR will be later applied to derive closed-form valuation equations for the event-contingent options.

Proposition 2. (The RNVR for the multivariate displaced lognormal) *Suppose that the basic assumption holds. Then a risk neutral valuation relationship (RNVR) exists for operating cash flows displaced lognormally distributed. That is:*

$$E^P[C(\underline{S})Z(\underline{S})] = E^Q[C(\underline{S})],$$

where $E^Q(.)$ is the expectations operator under the risk neutral probability measure.

Proof: All the marginal distributions of the underlying cash flows are displaced lognormal. In particular, the marginal distribution of the vector $\underline{S}$ is a multivariate displaced lognormal as given by:

$$f(\underline{S}) = \frac{1}{(2\pi)^{n/2} \mid \sum_s \mid^{1/2} \prod_{j=1}^{n}(S_j - \beta_j)}$$
$$exp\left\{-\frac{1}{2}[ln(\underline{S}-\underline{\beta})-\underline{\mu}]' \sum\nolimits_s^{-1}[ln(\underline{S}-\underline{\beta})-\underline{\mu}]\right\}, \tag{6}$$

where $\underline{S} > \underline{\beta}$ and $\mid \sum_s \mid$ is the determinant of the matrix $\sum_s$.

The equilibrium value of the underlying stochastic cash flows (3) can be rewritten as:

$$\underline{V}r_f = E^P[\underline{S}Z(\underline{S})] \tag{7}$$

$$= \int_{\beta_n}^{\infty}\int_{\beta_{n-1}}^{\infty}...\int_{\beta_1}^{\infty} \underline{S}Z(S_1, S_2, ..., S_n)f(S_1, S_2, ..., S_n)dS_1dS_2...dS_n,$$

where $f(\underline{S})$ is the multivariate density of the underlying cash flows as given by equation (6).

Substituting both the pricing kernel $Z(\underline{S})$ from equation (5) and the multivariate density of the underlying cash flows from equation (6) into equation (7) yields, after simplification, the following relation:

$$\underline{V}r_f = \int_{\beta_n}^{\infty}\int_{\beta_{n-1}}^{\infty}...\int_{\beta_1}^{\infty} \frac{\underline{S}}{(2\pi)^{n/2} \mid \sum_s \mid^{1/2} \prod_{j=1}^{n}(S_j - \beta_j)}$$
$$exp\left\{-\frac{1}{2}\left[ln(\underline{S}-\underline{\beta})-(\underline{\mu}+\varphi\sum\nolimits_{sc})\right]' \sum\nolimits_s^{-1}\left[ln(\underline{S}\right.\right.$$
$$\left.\left. - \underline{\beta})-(\underline{\mu}+\varphi\sum\nolimits_{sc})\right]\right\} dS_1dS_2...dS_n. \tag{8}$$

The market equilibrium relation is obtained when this expression is evaluated. Since the underlying cash flows are multivariate displaced lognormal distributed, the underlying equilibrium is given by:

$$ln(\underline{V}r_f - \underline{\beta}) - \frac{1}{2}\underline{X} = \underline{\mu} + \varphi\sum\nolimits_{sc}, \tag{9}$$

where $\underline{X}' = [\sigma_1^2, \sigma_2^2, ..., \sigma_n^2]$.

Equation (9) is the equilibrium relation of this economy, and is defined as the multivariate displaced lognormal extension of the capital asset pricing model (CAPM). The equilibrium value of a contingent claim is given from equation (2) by:

$$Pr_f = E^P[C(\underline{S})Z(\underline{S})] \tag{10}$$
$$= \int_{\beta_n}^{\infty}\int_{\beta_{n-1}}^{\infty} ... \int_{\beta_1}^{\infty} C(S_1, S_2, ..., S_n)Z(S_1, S_2, ..., S_n)f(S_1, S_2, ..., S_n)dS_1dS_2...dS_n.$$

To evaluate this expression, the pricing kernel (5) and the joint density of the underlying cash flows (6) are both substituted into equation (10), yielding after simplification the following result:

$$\begin{aligned} Pr_f &= \int_{\beta_n}^{\infty}\int_{\beta_{n-1}}^{\infty} ... \int_{\beta_1}^{\infty} \frac{C(\underline{S})}{(2\pi)^{n/2} \mid \sum_s \mid^{1/2} \prod_{j=1}^{n}(S_j - \beta_j)} \\ & exp\left\{-\frac{1}{2}\left[ln(\underline{S} - \underline{\beta}) - (\underline{\mu} + \varphi \sum\nolimits_{sc})\right]' \sum\nolimits_s^{-1} [ln(\underline{S} \right. \\ - & \left. \underline{\beta}) - (\underline{\mu} + \varphi \sum\nolimits_{sc})]\right\} dS_1dS_2...dS_n. \end{aligned} \tag{11}$$

Substituting the multivariate displaced lognormal extension of the CAPM (9) into the equilibrium value of the claim (11) yields the following relation:

$$\begin{aligned} Pr_f &= \int_{\beta_n}^{\infty}\int_{\beta_{n-1}}^{\infty} ... \int_{\beta_1}^{\infty} \frac{C(\underline{S})}{(2\pi)^{n/2} \mid \sum_s \mid^{1/2} \prod_{j=1}^{n}(S_j - \beta_j)} \\ & exp\left\{-\frac{1}{2}\left[ln(\underline{S} - \underline{\beta}) - (ln(\underline{V}r_f - \underline{\beta}) - \frac{1}{2}\underline{X})\right]' \sum\nolimits_s^{-1} [ln(\underline{S} \right. \\ - & \left. \underline{\beta}) - (ln(\underline{V}r_f - \underline{\beta}) - \frac{1}{2}\underline{X})]\right\} dS_1dS_2...dS_n. \end{aligned} \tag{12}$$

Equation (12) is the multivariate RNVR. □

The RNVR shows that the value of options on operating cash flows, in our economy, does not depend on preference parameters. This extends the Black-Scholes valuation equation.

4. Event-contingent options

This section applies the multivariate RNVR (12) to obtain analytic expressions for a new class of real options: the event-contingent (interacting) investment options. It will be assumed that the basic assumption holds.

Corollary 1. (Option to invest contingent on investment) *The payoff function of an option to invest contingent on investment is, by definition, equal to either* $S_2 - K_2$ *if* $S_2 > K_2$ *and* $S_1 > K_1$ *or zero if otherwise. The value of the option to invest contingent on investment is given by the following equation:*

$$P_{I,I} = (V_2 - \beta_2 r_f^{-1})N[d_1 + \sigma_2, d_2 + \rho\sigma_2, \rho] - (K_2 - \beta_2)r_f^{-1}N[d_1, d_2, \rho], \tag{13}$$

where $N[.,.,.]$ *is the cumulative bivariate standard normal random variable, and*

$$d_1 = \frac{ln(\frac{V_2 r_f - \beta_2}{K_2 - \beta_2}) - \frac{1}{2}\sigma_2^2}{\sigma_2},$$
$$d_2 = \frac{ln(\frac{V_1 r_f - \beta_1}{K_1 - \beta_1}) - \frac{1}{2}\sigma_1^2}{\sigma_1}.$$

Proof: See appendix.

Some details of equation (13) are better analyzed by its simulation. The simulations of this section assume that $E^Q(S_1) = E^Q(S_1) = 110$, $Var^Q(S_1) = Var^Q(S_1) = 493.81$, $V_1 = V_2 = 100$, $r_f = 1.1$, $\beta_1 = \beta_2 = \beta$, $\sigma_1 = \sigma_2 = \sigma$, $K_1 = K_2 = K$, where $E^Q(S) = Vr_f$ and $Var^Q(S) = (Vr_f - \beta)^2(exp(\sigma^2) - 1)$. We say that an option to invest (divest) is in-the-money if $V_2 r_f > K_2$ ($V_2 r_f < K_2$). Equation (13) is simulated for several combinations of β and σ. There are three main conclusions from the analysis of the results of the simulations presented in table 1. First, the value of an option to invest contingent on investment increases with the coefficient of correlation. Second, when the correlation is positive, the value of an in-the-money option to invest contingent on investment decreases when the shape and threshold parameters increase. Third, if $\rho = -1$ then the value of an out-of-the-money option to invest contingent on investment is zero. This point will be later analyzed in detail.

Corollary 2. (Option to invest contingent on divestment) *The payoff function of an option to invest contingent on divestment is, by definition, equal to either* $S_2 - K_2$ *if* $S_2 > K_2$ *and* $S_1 < K_1$ *or zero if otherwise. The value of the option to invest contingent on divestment is given* *by the following equation:*

$$P_{I,D} = (V_2 - \beta_2 r_f^{-1})N[d_1 + \sigma_2, -d_2 - \rho\sigma_2, -\rho] - (K_2 - \beta_2)r_f^{-1}N[d_1, -d_2, -\rho]. \quad (14)$$

Proof: See appendix.

Equation (14) is simulated for several combinations of β and σ. There are two main conclusions from the analysis of the results of the simulations presented in table 2. First, the value of an option to invest contingent on divestment decreases when the correlation between projects increases. Second, when the correlation is positive, the value of an out-of-the-money option to invest contingent on divestment increases when the shape and threshold parameters increase.

Corollary 3. (Call option) *The payoff function of a call option is, by definition, equal to* $S_2 - K_2$ *if* $S_2 > K_2$ *or zero if otherwise. The value of the call option*[11] *is given by the following equation:*

$$P_c = (V_2 - \beta_2 r_f^{-1})N(d_1 + \sigma_2) - (K_2 - \beta_2)r_f^{-1}N(d_1), \quad (15)$$

where $N(.)$ *is the cumulative univariate standard normal random variable.*

Proof: See appendix.

Equation (15) is simulated for several combinations of β and σ, and its results are presented in the panel A of table 3. The panel shows that the value of an in-the-money (out-of-the-money) call option decreases (increases) when the shape and threshold parameters increase. We can also see that the call option is related with the investment-contingent options in several ways:

(ia) $P_c = P_{II} + P_{ID}$, which is the parity condition I. In our numerical simulation, this relation is showed, for example, in the Panel A of table 3 and tables 1 and 2.

(iia) $P_c = P_{II}$ if $\rho = 1$ and $d_1 \leq d_2$. Our numerical simulation, when $d_1 = d_2$, is shown in the Panel A of the table 3. Hence tables 1 and 2 do not display option values when $\rho = 1$.

[11] This equation is similar to the valuation model derived by Rubinstein (1983) in a different context. Rubinstein (1983) in a no-arbitrage economy derives a closed-form expression for the value of a firm which has both risky and riskless assets.

(iiia) $P_c = P_{ID}$ if $\rho = -1$ and $d_1 + d_2 \leq 0$. Tables 1, 2, and 3 (Panel A) show that a sufficient condition for this to hold in our numerical case is that $K \geq V r_f$[12].

Corollary 4. (Option to divest contingent on divestment) *The payoff function of an option to divest contingent on divestment is, by definition, equal to either* $K_2 - S_2$ *if* $S_2 < K_2$ *and* $S_1 < K_1$ *or zero if otherwise. The value of the option to divest contingent on divestment is given by the following equation:*

$$P_{D,D} = (K_2 - \beta_2) r_f^{-1} N[-d_1, -d_2, \rho] - (V_2 - \beta_2 r_f^{-1}) N[-d_1 - \sigma_2, -d_2 - \rho\sigma_2, \rho]. \tag{16}$$

Proof: See appendix.

Equation (16) is simulated for several combinations of β and σ. There are three main conclusions from the analysis of the results of the simulations presented in table 4. First, the value of an option to divest contingent on divestment increases when the correlation between projects increases. Second, when the correlation is positive, the value of an out-of-the-money (in-the-money) option to divest contingent on divestment decreases (increases) when the shape and threshold parameters increase. Third, the value of an out-of-the-money option to divest contingent on divestment is zero when $\rho = -1$.

Corollary 5. (Option to divest contingent on investment) *The payoff function of an option to divest contingent on investment is, by definition, equal to either* $K_2 - S_2$ *if* $S_2 < K_2$ *and* $S_1 > K_1$ *or zero if otherwise. The value of the option to divest contingent on investment is given by the following equation:*

$$P_{D,I} = (K_2 - \beta_2) r_f^{-1} N[-d_1, d_2, -\rho] - (V_2 - \beta_2 r_f^{-1}) N[-d_1 - \sigma_2, d_2 + \rho\sigma_2, -\rho]. \tag{17}$$

Proof: See appendix.

Equation (17) is simulated for several combinations of β and σ. There are two main conclusions from the analysis of the results of the simulations presented in table 5. First, the value of an

[12]Technical results (iia) and (iiia) are stated formally and proved in the appendix.

option to divest contingent on investment decreases when the correlation between projects increases. Second, the value of an out-of-the-money option to divest contingent on investment decreases when the shape and threshold parameters increase.

Corollary 6. (Put option) *The payoff function of a put option is, by definition, equal to* $K_2 - S_2$ *if* $S_2 < K_2$ *or zero if otherwise. The value of the put option is given by the following equation:*

$$P_p = (K_2 - \beta_2)r_f^{-1}N(-d_1) - (V_2 - \beta_2 r_f^{-1})N(-d_1 - \sigma_2). \tag{18}$$

Proof: See appendix.

Equation (18) is simulated for several combinations of β and σ, and its results are presented in the panel B of table 3. The panel shows that the value of an in-the-money (out-of-the-money) put option increases (decreases) when the shape and threshold parameters increase. We can also see that the put option is related with the divestment-contingent options in several ways:
(ib) $P_p = P_{DD} + P_{DI}$, which is the parity condition II. In our numerical simulation, this relation is showed, for example, in the Panel B of table 3 and tables 4 and 5.
(iib) $P_p = P_{DD}$ if $\rho = 1$ and $d_1 \geq d_2$. Our numerical example, when $d_1 = d_2$, is shown in the Panel B of the table 3. Hence tables 4 and 5 do not display option values when $\rho = 1$.
(iiib) $P_p = P_{DI}$ if $\rho = -1$ and $d_1 + d_2 \geq 0$. Tables 3 (Panel B), 4, and 5 show that a sufficient condition for this to hold in our numerical case is that $K \leq V$[13].

The valuation equations derived in this section show that the interaction between projects has value. The formulae also show that an investment option will be probably misvalued when such interaction is not considered in the analysis.

[13]Technical results (iib) and (iiib) are stated formally and proved in the appendix.

5. Conclusions

This paper uses an equilibrium approach to derive a multivariate risk neutral valuation relation assuming that there is a representative individual who has an extended power utility function of the HARA family of utility functions. Aggregate consumption and underlying operating cash-flows are multivariate displaced lognormal distributed. The multivariate displaced lognormal distribution depends on a vector of threshold parameters or lower bounds of the operating cash flows. Negative threshold parameters assign positive probabilities to both inflows and outflows. The risk neutral valuation relationship is presented as a model for the valuation of strategic options on multiple operating cash flows, since these underlyings may have both negative and positive values.

The multivariate displaced lognormal risk neutral valuation relation is used to investigate a new class of investment options, the event-contingent investment options. An event-contingent option is a multivariate contingent claim which depends on a contingency that affects the exercise of the option, but which does not appear in the random payoff of the option. We define and provide closed-form expressions for the value of four categories of event-contingent investment options: (i) option to invest contingent on investment; (ii) option to invest contingent on divestment; (iii) option to divest contingent on divestment; and (iv) option to divest contingent on investment. The value of these options converges, under certain conditions, to put and call prices. The paper investigates these conditions. The simulations show that the correlation between projects has value.

Appendix

Proof of lemma 1: Suppose that the representative agent has a marginal extended power utility function of consumption given by equation (4) and that the joint distribution of aggregate consumption and underlying cash flows is multivariate displaced lognormal. That is:

$$\begin{bmatrix} C \\ \underline{S} \end{bmatrix} \sim \Lambda^P \left(\begin{bmatrix} \beta_c \\ \underline{\beta} \end{bmatrix}, \begin{bmatrix} \mu_c \\ \underline{\mu} \end{bmatrix}, \begin{bmatrix} \sigma_c^2 & \sum_{cs} \\ \sum_{sc} & \sum_s \end{bmatrix} \right),$$

for $C > \beta_c$ and $\underline{S} > \underline{\beta}$.

The marginal distribution of C is a univariate displaced lognormal. That is $C \sim \Lambda^P(\beta_c, \mu_c, \sigma_c^2)$:

$$f(C) = \frac{1}{\sqrt{2\pi}\sigma_c(C-\beta_c)} exp\left\{-\frac{1}{2\sigma_c^2}[ln(C-\beta_c)-\mu_c]^2\right\}.$$

The expected value of the marginal utility function, considering the definition of the φ moment around β_c of a univariate displaced lognormal random variable, is given by:

$$\begin{aligned} & E^P[U^{'}(C)] \\ = \;& E^P[(C-\beta_c)^{\varphi}] \\ = \;& exp[\varphi\mu_c + \frac{1}{2}\varphi^2\sigma_c^2]. \end{aligned} \tag{19}$$

The conditional distribution of consumption given the underlying cash flows, since consumption and the underlying cash flows are multivariate displaced lognormal distributed, is a univariate displaced lognormal distribution:

$$C \mid \underline{S} \sim \Lambda^P \left\{\beta_c, \mu_c + \sum\nolimits_{cs} \sum\nolimits_{s}^{-1} [ln(\underline{S}-\underline{\beta}) - \underline{\mu}], \sigma_c^2 - \sum\nolimits_{cs} \sum\nolimits_{s}^{-1} \sum\nolimits_{sc}\right\}.$$

The expected value of the conditional marginal utility function, considering the definition of the φ moment around β_c of a univariate displaced lognormal random variable, is given by:

$$
\begin{aligned}
& E^P[U^{'}(C) \mid \underline{S}] \\
= \; & E^P[(C-\beta_c)^{\varphi} \mid \underline{S}] \\
= \; & exp\left\{\varphi\mu_c + \varphi \sum\nolimits_{cs} \sum\nolimits_{s}^{-1} [ln(\underline{S}-\underline{\beta}) - \underline{\mu}] + \varphi^2 \frac{\sigma_c^2}{2} - \frac{1}{2}\varphi^2 \sum\nolimits_{cs} \sum\nolimits_{s}^{-1} \sum\nolimits_{sc} \right\}. \qquad (20)
\end{aligned}
$$

The pricing kernel (1), using equations (19) and (20), can be rewritten as:

$$
\begin{aligned}
Z(\underline{S}) &= \frac{E^P[U^{'}(C) \mid \underline{S}]}{E^P[U^{'}(C)]} \\
&= exp\{\varphi \sum\nolimits_{cs} \sum\nolimits_{s}^{-1} [ln(\underline{S}-\underline{\beta}) - \underline{\mu}] - \frac{1}{2}\varphi^2 \sum\nolimits_{cs} \sum\nolimits_{s}^{-1} \sum\nolimits_{sc}\}, \qquad (21)
\end{aligned}
$$

which concludes the proof. □

Proof of corollary 1: The value of the option, using the RNVR (12), is given by:

$$
P_{I,I} r_f = \int_{K_1}^{\infty} \int_{K_2}^{\infty} (S_2 - K_2) g(S_1, S_2) dS_1 dS_2,
$$

where $g(S_1, S_2)$ is a bivariate displaced lognormal risk neutral density with correlation coefficient ρ. Then $S_1 \sim \Lambda^Q(\beta_1, ln(V_1 r_f - \beta_1) - \frac{1}{2}\sigma_1^2, \sigma_1^2)$ and $S_2 \sim \Lambda^Q(\beta_2, ln(V_2 r_f - \beta_2) - \frac{1}{2}\sigma_2^2, \sigma_2^2)$. Let $v = ln(S_2 - \beta_2)$ and $x = ln(S_1 - \beta_1)$. Then:

$$
\begin{aligned}
P_{I,I} r_f &= \int_{ln(K_1-\beta_1)}^{\infty} \int_{ln(K_2-\beta_2)}^{\infty} e^v g(v,x) dx dv \\
&- (K_2 - \beta_2) \int_{ln(K_1-\beta_1)}^{\infty} \int_{ln(K_2-\beta_2)}^{\infty} g(v,x) dx dv.
\end{aligned}
$$

Then using both:

$$
\int_a^{\infty} \int_b^{\infty} e^v g(v,x) dx dv = exp(\mu_v + \frac{1}{2}\sigma_v^2) N[\frac{\mu_v - a}{\sigma_v} + \sigma_v, \frac{\mu_x - b}{\sigma_x} + \rho\sigma_v, \rho]
$$

and

$$
\int_a^{\infty} \int_b^{\infty} g(v,x) dx dv = N[\frac{\mu_v - a}{\sigma_v}, \frac{\mu_x - b}{\sigma_x}, \rho]
$$

yields the desired result.□

Proof of corollary 2: The value of the option, using the RNVR (12), is given by:

$$P_{I,D}r_f = \int_{\beta_1}^{K_1}\int_{K_2}^{\infty}(S_2 - K_2)g(S_1, S_2)dS_1dS_2,$$

where $g(S_1, S_2)$ is a bivariate displaced lognormal risk neutral density with correlation coefficient ρ. Then $S_1 \sim \Lambda^Q(\beta_1, ln(V_1r_f - \beta_1) - \frac{1}{2}\sigma_1^2, \sigma_1^2)$ and $S_2 \sim \Lambda^Q(\beta_2, ln(V_2r_f - \beta_2) - \frac{1}{2}\sigma_2^2, \sigma_2^2)$. Let $v = ln(S_2 - \beta_2)$ and $x = ln(S_1 - \beta_1)$. Then:

$$\begin{aligned} P_{I,D}r_f &= \int_{-\infty}^{ln(K_1-\beta_1)}\int_{ln(K_2-\beta_2)}^{\infty} e^v g(v,x)dxdv \\ &- (K_2 - \beta_2)\int_{-\infty}^{ln(K_1-\beta_1)}\int_{ln(K_2-\beta_2)}^{\infty} g(v,x)dxdv. \end{aligned}$$

Then using both:

$$\int_a^{\infty}\int_{-\infty}^{b} e^v g(v,x)dxdv = exp(\mu_v + \frac{1}{2}\sigma_v^2)N[\frac{\mu_v - a}{\sigma_v} + \sigma_v, \frac{b-\mu_x}{\sigma_x} - \rho\sigma_v, -\rho]$$

and

$$\int_a^{\infty}\int_{-\infty}^{b} g(v,x)dxdv = N[\frac{\mu_v - a}{\sigma_v}, \frac{b-\mu_x}{\sigma_x}, -\rho]$$

yields the desired result.□

Proof of corollary 3: The value of the option, using the RNVR (12), is given by:

$$P_c r_f = \int_{K_2}^{\infty}(S_2 - K_2)g(S_2)dS_2,$$

where $g(S_2)$ is a univariate displaced lognormal risk neutral density. Then $S_2 \sim \Lambda^Q(\beta_2, ln(V_2r_f - \beta_2) - \frac{1}{2}\sigma_2^2, \sigma_2^2)$. Let $v = ln(S_2 - \beta_2)$. Then:

$$P_c r_f = \int_{ln(K_2-\beta_2)}^{\infty} e^v g(v)dv - (K_2 - \beta_2)\int_{ln(K_2-\beta_2)}^{\infty} g(v)dv.$$

Then using both:

$$\int_a^{\infty} e^v g(v)dv = exp(\mu_v + \frac{\sigma_v^2}{2})N(\frac{\mu_v - a}{\sigma_v} + \sigma_v)$$

and

$$\int_a^{\infty} g(v)dv = N(\frac{\mu_v - a}{\sigma_v})$$

yields the desired result.□

Result (iia). The event-contingent option value depends on the cumulative bivariate standard normal random variable $N(.,.,.)$, and both d_1 and d_2. If $\rho = 1$ and $d_1 \leq d_2$ then the value of an option to invest contingent on divestment is zero (i.e. $P_{ID} = 0$), and the value of an option to invest contingent on investment is equal to the value of the call option (i.e. $P_c = P_{II}$).

Proof of result (iia): In general for $N(.,.,.)$ we have $N(d_1, d_2, 1) = N[min(d_1, d_2)]$. If $d_1 \leq d_2$ then $N(d_1, d_2, 1) = N(d_1)$. By the same property $N(d_1+\sigma_2, d_2+\sigma_2, 1) = N[min(d_1+\sigma_2, d_2+\sigma_2)]$. If $d_1 + \sigma_2 \leq d_2 + \sigma_2$ i.e. $d_1 \leq d_2$ then $N(d_1 + \sigma_2, d_2 + \sigma_2, 1) = N(d_1 + \sigma_2)$.□

Result (iiia). The event-contingent option value depends on the cumulative bivariate standard normal random variable $N(.,.,.)$, and both d_1 and d_2. Let $\rho = -1$. Let also $E^Q[S_1] = E^Q[S_2]$, $Var^Q[S_1] = Var^Q[S_2]$, $K_1 = K_2 = K$, $\sigma_2 = \sigma_1 = \sigma$, and $\beta_2 = \beta_1 = \beta$. In our numerical case, if $K \geq Vr_f$ then the value of an option to invest contingent on investment is zero (i.e. $P_{II} = 0$), and the value of an option to invest contingent on divestment is equal to the value of the call option (i.e. $P_c = P_{ID}$). This result holds, in general, when $d_1 + d_2 \leq 0$.

Proof of result (iiia): By a general property of $N(.,.,.)$ we have $N(d_1, d_2, -1) = 0$ and $N(d_1 + \sigma_2, d_2 - \sigma_2, -1) = 0$ if $d_1 + d_2 \leq 0$. An application of this to our problem implies the result when $ln(Vr_f - \beta) \leq ln(K - \beta) + \frac{\sigma^2}{2}$. A sufficient condition for this to hold in our numerical case is that $Vr_f \leq K$.□

Proof of corollary 4: The value of the option, using the RNVR (12), is given by:

$$P_{D,D}r_f = \int_{\beta_1}^{K_1} \int_{\beta_2}^{K_2} (K_2 - S_2) g(S_1, S_2) dS_1 dS_2,$$

where $g(S_1, S_2)$ is a bivariate displaced lognormal risk neutral density with correlation coefficient ρ. Then $S_1 \sim \Lambda^Q(\beta_1, ln(V_1 r_f - \beta_1) - \frac{1}{2}\sigma_1^2, \sigma_1^2)$ and $S_2 \sim \Lambda^Q(\beta_2, ln(V_2 r_f - \beta_2) - \frac{1}{2}\sigma_2^2, \sigma_2^2)$. Let

$v = ln(S_2 - \beta_2)$ and $x = ln(S_1 - \beta_1)$. Then:

$$\begin{aligned} P_{D,D} r_f &= (K_2 - \beta_2) \int_{-\infty}^{ln(K_1-\beta_1)} \int_{-\infty}^{ln(K_2-\beta_2)} g(v,x) dx dv \\ &- \int_{-\infty}^{ln(K_1-\beta_1)} \int_{-\infty}^{ln(K_2-\beta_2)} e^v g(v,x) dx dv. \end{aligned}$$

Then using both:

$$\int_{-\infty}^{a} \int_{-\infty}^{b} e^v g(v,x) dx dv = exp(\mu_v + \frac{1}{2}\sigma_v^2) N[\frac{a-\mu_v}{\sigma_v} - \sigma_v, \frac{b-\mu_x}{\sigma_x} - \rho\sigma_v, +\rho]$$

and

$$\int_{-\infty}^{a} \int_{-\infty}^{b} g(v,x) dx dv = N[\frac{a-\mu_v}{\sigma_v}, \frac{b-\mu_x}{\sigma_x}, +\rho]$$

yields the desired result.□

Proof of corollary 5: The value of the option, using the RNVR (12), is given by:

$$P_{D,I} r_f = \int_{K_1}^{\infty} \int_{\beta_2}^{K_2} (K_2 - S_2) g(S_1, S_2) dS_1 dS_2,$$

where $g(S_1, S_2)$ is a bivariate displaced lognormal risk neutral density with correlation coefficient ρ. Then $S_1 \sim \Lambda^Q(\beta_1, ln(V_1 r_f - \beta_1) - \frac{1}{2}\sigma_1^2, \sigma_1^2)$ and $S_2 \sim \Lambda^Q(\beta_2, ln(V_2 r_f - \beta_2) - \frac{1}{2}\sigma_2^2, \sigma_2^2)$. Let $v = ln(S_2 - \beta_2)$ and $x = ln(S_1 - \beta_1)$. Then:

$$\begin{aligned} P_{D,I} r_f &= (K_2 - \beta_2) \int_{ln(K_1-\beta_1)}^{\infty} \int_{-\infty}^{ln(K_2-\beta_2)} g(v,x) dx dv \\ &- \int_{ln(K_1-\beta_1)}^{\infty} \int_{-\infty}^{ln(K_2-\beta_2)} e^v g(v,x) dx dv. \end{aligned}$$

Then using both:

$$\int_{-\infty}^{a} \int_{b}^{\infty} e^v g(v,x) dx dv = exp(\mu_v + \frac{1}{2}\sigma_v^2) N[\frac{a-\mu_v}{\sigma_v} - \sigma_v, \frac{\mu_x - b}{\sigma_x} + \rho\sigma_v, -\rho]$$

and

$$\int_{-\infty}^{a} \int_{b}^{\infty} g(v,x) dx dv = N[\frac{a-\mu_v}{\sigma_v}, \frac{\mu_x - b}{\sigma_x}, -\rho]$$

yields the desired result.□

Proof of corollary 6: The value of the option, using the RNVR (12), is given by:

$$P_p r_f = \int_{\beta_2}^{K_2} (K_2 - S_2) g(S_2) dS_2,$$

where $g(S_2)$ is a univariate displaced lognormal risk neutral density. Then $S_2 \sim \Lambda^Q(\beta_2, ln(V_2 r_f - \beta_2) - \frac{1}{2}\sigma_2^2, \sigma_2^2)$. Let $v = ln(S_2 - \beta_2)$. Then:

$$P_p r_f = (K_2 - \beta_2) \int_{-\infty}^{ln(K_2-\beta_2)} g(v)dv - \int_{-\infty}^{ln(K_2-\beta_2)} e^v g(v)dv.$$

Then using both:

$$\int_{-\infty}^{a} e^v g(v)dv = exp(\mu_v + \frac{\sigma_v^2}{2}) N(\frac{a - \mu_v}{\sigma_v} - \sigma_v),$$

and

$$\int_{-\infty}^{a} g(v)dv = N(\frac{a - \mu_v}{\sigma_v})$$

yields the desired result.□

Result (iib). The event-contingent option value depends on the cumulative bivariate standard normal random variable $N(.,.,.)$, and both d_1 and d_2. If $\rho = 1$ and $d_1 \geq d_2$ then the value of an option to divest contingent on investment is zero (i.e. $P_{DI} = 0$), and the value of an option to divest contingent on divestment is equal to the value of the put option (i.e. $P_p = P_{DD}$).

Proof of result (iib): In general for $N(.,.,.)$ we have $N(-d_1, -d_2, 1) = N[min(-d_1, -d_2)]$. If $d_1 \geq d_2$ then $N(-d_1, -d_2, 1) = N(-d_1)$. By the same property $N(-d_1 - \sigma_2, -d_2 - \sigma_2, 1) = N[min(-d_1 - \sigma_2, -d_2 - \sigma_2)]$. If $-d_1 - \sigma_2 \leq -d_2 - \sigma_2$ i.e. $d_1 \geq d_2$ then $N(-d_1 - \sigma_2, -d_2 - \sigma_2, 1) = N(-d_1 - \sigma_2)$.□

Result (iiib). The event-contingent option value depends on the cumulative bivariate standard normal random variable $N(.,.,.)$, and both d_1 and d_2. Let $\rho = -1$. Let also $E^Q[S_1] = E^Q[S_2]$, $Var^Q[S_1] = Var^Q[S_2]$, $K_1 = K_2 = K$, $\sigma_2 = \sigma_1 = \sigma$, and $\beta_2 = \beta_1 = \beta$. In our numerical case, if $K \leq V$ then the value of an option to divest contingent on divestment is zero (i.e. $P_{DD} = 0$), and the value of an option to divest contingent on investment is equal to the value of the put option (i.e. $P_p = P_{DI}$). This result holds, in general, when $d_1 + d_2 \geq 0$.

Proof of result (iiib): By a general property of $N(.,.,.)$ we have $N(-d_1, -d_2, -1) = 0$ and $N(-d_1 - \sigma_2, -d_2 + \sigma_2, -1) = 0$ if $-d_1 - d_2 \leq 0$ i.e $d_1 + d_2 \geq 0$. An application of this to our problem implies the result when $ln(Vr_f - \beta) \geq ln(K - \beta) + \frac{\sigma^2}{2}$. A sufficient condition for this to hold in our numerical case is that $K \leq V$.$\square$

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Table 1[a]
Options to invest contingent on investment

Panel A: Perfect negative correlation ($\rho = -1$)

β	**K** / σ	80	90	100	110	120	130	140
-60	0.130	22.543	10.830	2.446	0	0	0	0
-40	0.147	22.590	10.759	2.361	0	0	0	0
-20	0.170	22.665	10.673	2.253	0	0	0	0
0	0.200	22.794	10.569	2.114	0	0	0	0

Panel B: Partial negative correlation ($\rho = -0.5$)

β	**K** / σ	80	90	100	110	120	130	140
-60	0.130	24.201	13.708	5.844	1.743	0.349	0.047	0.004
-40	0.147	24.243	13.662	5.772	1.708	0.342	0.046	0.004
-20	0.170	24.308	13.606	5.679	1.662	0.333	0.046	0.005
0	0.200	24.414	13.539	5.554	1.600	0.321	0.046	0.005

Panel C: No correlation ($\rho = 0$)

β	**K** / σ	80	90	100	110	120	130	140
-60	0.130	25.679	16.196	8.627	3.802	1.379	0.414	0.104
-40	0.147	25.703	16.163	8.572	3.769	1.373	0.418	0.108
-20	0.170	25.739	16.122	8.500	3.726	1.364	0.423	0.113
0	0.200	25.798	16.071	8.403	3.666	1.350	0.428	0.119

Panel D: Partial positive correlation ($\rho = 0.5$)

β	**K** / σ	80	90	100	110	120	130	140
-60	0.130	26.910	18.265	11.069	5.925	2.796	1.169	0.437
-40	0.147	26.906	18.232	11.029	5.904	2.800	1.185	0.452
-20	0.170	26.903	18.189	10.976	5.877	2.804	1.205	0.472
0	0.200	26.904	18.132	10.901	5.837	2.808	1.230	0.498

[a]It is assumed that $E^Q(S_1) = E^Q(S_2) = 110$, $Var^Q(S_1) = Var^Q(S_2) = 493.81$, $V_1 = V_2 = 100$, $r_f = 1.1$, $\beta_1 = \beta_2 = \beta$, $\sigma_1 = \sigma_2 = \sigma$, and $K_1 = K_2 = K$, where $E^Q(S) = Vr_f$ and $Var^Q(S) = (Vr_f - \beta)^2(exp(\sigma^2) - 1)$.

Table 2[a]
Options to invest contingent on divestment

Panel A: Perfect negative correlation ($\rho = -1$)

	K	80	90	100	110	120	130	140
β	σ							
-60	0.130	5.274	9.042	10.711	8.020	4.492	2.317	1.105
-40	0.147	5.191	9.066	10.758	8.008	4.510	2.351	1.141
-20	0.170	5.070	9.088	10.813	7.992	4.530	2.393	1.186
0	0.200	4.881	9.105	10.879	7.966	4.555	2.448	1.247

Panel B: Partial negative correlation ($\rho = -0.5$)

	K	80	90	100	110	120	130	140
β	σ							
-60	0.130	3.616	6.165	7.313	6.276	4.143	2.270	1.101
-40	0.147	3.538	6.163	7.346	6.301	4.167	2.304	1.137
-20	0.170	3.427	6.155	7.388	6.330	4.197	2.347	1.182
0	0.200	3.261	6.135	7.439	6.366	4.234	2.402	1.242

Panel C: No correlation ($\rho = 0$)

	K	80	90	100	110	120	130	140
β	σ							
-60	0.130	2.138	3.676	4.530	4.218	3.113	1.902	1.001
-40	0.147	2.079	3.662	4.547	4.239	3.137	1.932	1.033
-20	0.170	1.997	3.640	4.566	4.266	3.167	1.970	1.073
0	0.200	1.877	3.604	4.590	4.300	3.205	2.020	1.127

Panel D: Partial positive correlation ($\rho = 0.5$)

	K	80	90	100	110	120	130	140
β	σ							
-60	0.130	0.907	1.607	2.087	2.095	1.696	1.148	0.669
-40	0.147	0.875	1.593	2.089	2.104	1.709	1.166	0.689
-20	0.170	0.832	1.573	2.091	2.115	1.726	1.188	0.715
0	0.200	0.771	1.543	2.092	2.129	1.747	1.218	0.749

[a]It is assumed that $E^Q(S_1) = E^Q(S_2) = 110$, $Var^Q(S_1) = Var^Q(S_2) = 493.81$, $V_1 = V_2 = 100$, $r_f = 1.1$, $\beta_1 = \beta_2 = \beta$, $\sigma_1 = \sigma_2 = \sigma$, and $K_1 = K_2 = K$, where $E^Q(S) = Vr_f$ and $Var^Q(S) = (Vr_f - \beta)^2(exp(\sigma^2) - 1)$.

Table 3[a]

Event-contingent options with perfect positive correlation

Panel A: Call options or options to invest contingent on investment with perfect positive correlation ($\rho = 1$)

	K	80	90	100	110	120	130	140
β	σ							
-60	0.130	27.817	19.872	13.157	8.020	4.492	2.317	1.105
-40	0.147	27.781	19.825	13.118	8.008	4.510	2.351	1.141
-20	0.170	27.735	19.762	13.066	7.992	4.530	2.393	1.186
0[b]	0.200	27.675	19.675	12.993	7.966	4.555	2.448	1.247

Panel B: Put options or options to divest contingent on divestment with perfect positive correlation ($\rho = 1$)

	K	80	90	100	110	120	130	140
β	σ							
-60	0.130	0.544	1.691	4.066	8.020	13.583	20.499	28.378
-40	0.147	0.509	1.643	4.028	8.008	13.600	20.532	28.414
-20	0.170	0.463	1.580	3.976	7.992	13.621	20.575	28.459
0[b]	0.200	0.402	1.493	3.902	7.966	13.646	20.629	28.519

[a]It is assumed that $E^Q(S_2) = 110, Var^Q(S_2) = 493.81$, $V_2 = 100$, $r_f = 1.1$, $\beta_2 = \beta$, $\sigma_2 = \sigma$, and $K_2 = K$.

[b]The row presents Black-Scholes (1973) option values.

Table 4[a]

Options to divest contingent on divestment

Panel A: Perfect negative correlation ($\rho = -1$)

	K	80	90	100	110	120	130	140
β	σ							
-60	0.130	0	0	0	0.068	3.970	12.304	22.697
-40	0.147	0	0	0	0.086	4.083	12.420	22.750
-20	0.170	0	0	0	0.114	4.232	12.572	22.823
0	0.200	0	0	0	0.157	4.437	12.782	22.931

Panel B: Partial negative correlation ($\rho = -0.5$)

	K	80	90	100	110	120	130	140
β	σ							
-60	0.130	0.001	0.038	0.433	2.283	6.969	14.649	24.176
-40	0.147	0.001	0.037	0.437	2.318	7.043	14.722	24.205
-20	0.170	0.001	0.036	0.441	2.363	7.140	14.818	24.246
0	0.200	0.001	0.033	0.446	2.423	7.271	14.950	24.310

Panel C: No correlation ($\rho = 0$)

	K	80	90	100	110	120	130	140
β	σ							
-60	0.130	0.042	0.313	1.400	4.218	9.412	16.832	25.697
-40	0.147	0.038	0.303	1.396	4.239	9.460	16.878	25.717
-20	0.170	0.033	0.291	1.389	4.266	9.521	16.940	25.745
0	0.200	0.027	0.273	1.378	4.300	9.602	17.023	25.787

Panel D: Partial positive correlation ($\rho = 0.5$)

	K	80	90	100	110	120	130	140
β	σ							
-60	0.130	0.195	0.850	2.610	6.105	11.586	18.785	27.131
-40	0.147	0.180	0.825	2.590	6.108	11.613	18.818	27.153
-20	0.170	0.162	0.791	2.562	6.111	11.647	18.860	27.182
0	0.200	0.137	0.745	2.521	6.112	11.690	18.917	27.222

[a]It is assumed that $E^Q(S_1) = E^Q(S_2) = 110$, $Var^Q(S_1) = Var^Q(S_2) = 493.81$, $V_1 = V_2 = 100$, $r_f = 1.1$, $\beta_1 = \beta_2 = \beta$, $\sigma_1 = \sigma_2 = \sigma$, and $K_1 = K_2 = K$, where $E^Q(S) = Vr_f$ and $Var^Q(S) = (Vr_f - \beta)^2(exp(\sigma^2) - 1)$.

Table 5[a]

Options to divest contingent on investment

Panel A: Perfect negative correlation ($\rho = -1$)

	K	80	90	100	110	120	130	140
β	σ							
-60	0.130	0.544	1.691	4.066	7.952	9.613	8.195	5.681
-40	0.147	0.509	1.643	4.028	7.922	9.517	8.113	5.664
-20	0.170	0.463	1.580	3.976	7.878	9.389	8.003	5.636
0	0.200	0.402	1.493	3.902	7.808	9.209	7.848	5.588

Panel B: Partial negative correlation ($\rho = -0.5$)

	K	80	90	100	110	120	130	140
β	σ							
-60	0.130	0.543	1.652	3.633	5.736	6.614	5.849	4.203
-40	0.147	0.507	1.606	3.591	5.690	6.557	5.810	4.209
-20	0.170	0.462	1.544	3.534	5.629	6.481	5.757	4.213
0	0.200	0.402	1.460	3.456	5.543	6.375	5.680	4.210

Panel C: No correlation ($\rho = 0$)

	K	80	90	100	110	120	130	140
β	σ							
-60	0.130	0.503	1.378	2.666	3.802	4.171	3.667	2.681
-40	0.147	0.471	1.339	2.632	3.769	4.140	3.654	2.697
-20	0.170	0.429	1.289	2.586	3.726	4.100	3.635	2.714
0	0.200	0.375	1.219	2.524	3.666	4.043	3.607	2.732

Panel D: Partial positive correlation ($\rho = 0.5$)

	K	80	90	100	110	120	130	140
β	σ							
-60	0.130	0.349	0.841	1.456	1.915	1.997	1.714	1.247
-40	0.147	0.328	0.818	1.438	1.900	1.988	1.715	1.261
-20	0.170	0.301	0.789	1.414	1.881	1.975	1.715	1.277
0	0.200	0.265	0.748	1.380	1.854	1.956	1.713	1.297

[a]It is assumed that $E^Q(S_1) = E^Q(S_2) = 110$, $Var^Q(S_1) = Var^Q(S_2) = 493.81$, $V_1 = V_2 = 100$, $r_f = 1.1$, $\beta_1 = \beta_2 = \beta$, $\sigma_1 = \sigma_2 = \sigma$, and $K_1 = K_2 = K$, where $E^Q(S) = Vr_f$ and $Var^Q(S) = (Vr_f - \beta)^2(exp(\sigma^2) - 1)$.