Reversible, Flow Options

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Abstract

Reversible, Flow Options.

In this paper we produce a formula for a finitely lived, perfectly reversible option on a flow. For this real option that allows frequent and costless switching between the maximum of two asset flows, we first examine the perpetual and then the finite cases in terms of switching thresholds and values. The finite option value is inferred from the perpetual using an annuity argument. Applications include energy and commodity consumption costs where switching between flows can occur frequently and costlessly.

Key words: Reversible options, real options, options on flows.

JEL: G12, G13.

1 Introduction

In this paper we examine the reversibility of the standard real option framework as a function of the fraction of the exercise proceeds (if any) that can be recovered on reverse exercise. This is important because real investment projects allow a fraction of the investment assets to be recovered in some states of the world whereas many real options models (for perpetual American options see Merton (1973) [1] and Dixit & Pindyck (1994) [2]) assume that the investment decision is completely irreversible and that the salvage value is zero. Real option exceptions to this oversight include Brennan and Schwartz (1985) [3] and Dixit (1989) [4] where investment/divestment hysteresis is generated by activation and deactivation costs.

For the partially reversible case which generates hysteresis, we reduce the four equations that determine the asset pricing system to one explicit non–linear equation whose solution can be easily determined numerically. We show that an arbitrarily small recovery fraction is shown to correspond to the Merton (1973) [1] perpetual model and exercise threshold while a new threshold corresponding to full recovery or complete reversibility is derived and shown to conform to a simple (volatility independent) yield argument for the exercise condition.

Hence we show that even when perfectly reversible, real option values, if not thresholds, still depend on the level of uncertainty. Thus perfectly reversible real option values and thresholds represent the other extreme case compared to most real options applications. Actual practical investment cases will have real option values and thresholds that are bounded by the two special cases of fully reversible investment considered here and perfectly irreversible investment most widely considered in the literature.

Having valued these so called perpetual reversible options (a problem raised by McDonald and Siegel (1985) [5]), we go on to value finite reversible options in closed form, a task which is possible only because of the known form of the reversible boundary. Comparative statics are presented.

The finite and infinite solutions we present are applicable to real option situations where a maximum or minimum flow can be continually chosen without penalty and the equivalence between these flow options and a continuum of Black Scholes (1973) [6] options is formalised.

Categorizing general reversible options by their time to maturity and recovery fraction allows Section 2 to put these new fully reversible options on a grid which also contains standard (finite) American options as well as

Dixit's perpetual hysteresis investment. Section 3 extends Dixit's (1989) [4] analysis of investment hysteresis to recover the single non–linear equation that determines the system and prices perpetual real options under perfect recovery or full reversibility. Section 4 uses the single threshold and its time invariant property established in Section 3 to value finite reversible options in closed form. Comparative statics are discussed as well as comparison of a flow option to continuum of infinitesimal Black Scholes options. Section 5 concludes.

2 Reversible options

In this section we categorize the types of reversible options discussed in this paper. We also reformulate investment, divestment hysteresis and extend the analysis to recover the single explicit equation that determines the trigger thresholds. When thresholds are close, it is this equation that Dixit (1991) [7] approximated, but did not state.

2.1 The underlying process

For an underlying project value price process V that follows a geometric Brownian motion [1]

$$\frac{dV}{V} = (\mu - \delta)\, dt + \sigma dW$$

$$\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 R}{\partial V^2} + (r - \delta)\, V \frac{\partial R}{\partial V} - rR - \frac{\partial R}{\partial T} = 0 \tag{1}$$

no arbitrage or risk–neutral valuation implies that the price of any option claim R must satisfy the Hamilton–Bellmann–Jacobi differential Equation (1). This reversible option $R\,(T, \alpha)$ is a function of the current value of V, time T and a measure of the degree of reversibility α which will defined presently by the boundary conditions. Notational dependence on the current value V will generally be suppressed, as will time subscripts.

[1] This process has expected capital gain $g = \mu - \delta$ and dividend yield δ under the real world process dW. Under the risk neutral process, its drift and yield are $r-\delta, \delta$. We assume that both r and δ are positive.

2.2 Value matching and smooth pasting

We label the option to activate or open $O(T,\alpha)$ and the option to de–activate or shut the project $S(T,\alpha)$. Both are a function of investment amount and dis–investment proceeds $\overline{X}, \underline{X}$ through their ratio $\alpha < 1$ and a function of remaining time to maturity T. Dixit considered these investment and divestment premia k and discounts l as “transaction costs” in switching to and from a base level of investment X

$$\alpha = \frac{\underline{X}}{\overline{X}} = \frac{X - l}{X + k}$$

At the opening threshold $\overline{V}(T,\alpha)$, the investment proceeds $\overline{X}$ along with the opening option O can be exchanged for the project value at that level $\overline{V}(T,\alpha)$ along with the shutting option S. At the shutting threshold $\underline{V}(T,\alpha)$, the project value and shutting option can be exchanged for the divestment proceeds $\underline{X}$ and the option to open again O.

$$\begin{array}{ccc} \overline{X} + O(T,\alpha) & \rightarrow & \overline{V}(T,\alpha) + S(T,\alpha) \\ \underline{X} + O(T,\alpha) & \leftarrow & \underline{V}(T,\alpha) + S(T,\alpha) \end{array}$$

It is therefore useful to define a local ratio γ of the investment and divestment thresholds $\underline{V}(T,\alpha), \overline{V}(T,\alpha)$. In this Section only, this ratio depends on time T but in the perpetual or fully reversible cases later, γ will is constant.

$$\gamma(T,\alpha) = \frac{\underline{V}(T,\alpha)}{\overline{V}(T,\alpha)}$$

This notation allows the general reversible option $R(T,\alpha)$ to be defined as the asset V plus shutting option S (if currently open) or investment proceeds X (either $\overline{X}$ or $\underline{X}$) plus opening option O (if currently closed)

$$R(T,\alpha) = \begin{cases} V + S(T,\alpha) & : V > \overline{V}(T,\alpha) \\ X + O(T,\alpha) & : V < \underline{V}(T,\alpha) \end{cases} \tag{2}$$

where $\overline{V}(T,\alpha)$ is the upper and $\underline{V}(T,\alpha)$ the lower threshold of conversion and reconversion (both a function of remaining time to maturity and the (dis)investment ratio). It is understood that for intermediate values of V $\underline{V}(T,\alpha) < V < \overline{V}(T,\alpha)$ the open/shut status is determined by the last boundary encountered through hysteresis. Each option value $R(T,\alpha)$ must solve the HBJ Equation (1) along with the boundary conditions.

Reversible option $R(T,\alpha)$		Recovery fraction α				
		1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	0
Time	∞	$R(\infty,1)$	$R(\infty,\frac{1}{2})$	$R(\infty,\frac{1}{4})$	$R(\infty,\frac{1}{8})$	$R(\infty,0)$
to	4	$R(4,1)$	$R(4,\frac{1}{2})$	$R(4,\frac{1}{4})$	$R(4,\frac{1}{8})$	$R(4,0)$
final	2	$R(2,1)$	$R(2,\frac{1}{2})$	$R(2,\frac{1}{4})$	$R(2,\frac{1}{8})$	$R(2,0)$
maturity	1	$R(1,1)$	$R(1,\frac{1}{2})$	$R(1,\frac{1}{4})$	$R(1,\frac{1}{8})$	$R(1,0)$
T	0	$R(0,1)$	$R(0,\frac{1}{2})$	$R(0,\frac{1}{4})$	$R(0,\frac{1}{8})$	$R(0,0)$

Table 1: General reversible options $R(T,\alpha)$ as a function of time to maturity T and recovery fraction α. Limiting cases include standard (irreversible) finite American options (right hand column), Dixit's perpetual costly hysteresis (top row) and finally the finite Reversible Options valued in this paper (left hand column). Perpetual Merton (irreversible) options reside at the top right hand corner, perpetual reversibles at the top left hand corner while the bottom row contains final option payoffs.

Most generally the activation costs and deactivation proceeds (forward and reverse exercise) are not equal $\left(\underline{X} < \overline{X}\right)$ [2] so forward and reverse exercise may be separated in underlying project value V and therefore in time. This investment hysteresis is similar to Brennan and Schwartz (1985) [3] and Dixit (1991) [7] when the continuous operating costs are also rolled up with the investment cost.

Table 1 shows the reversible option $R(T,\alpha)$ as a function of T,α including certain special cases. The right hand column contains irreversible (zero recovery) American options $R(T,0)$ of finite maturity with the perpetual American solution $R(\infty,0)$ (Merton (1973) [1]) that is used in many real options models at the top.

The top row contains the perpetual investment/dis–investment hysteresis $R(\infty,\alpha)$ described by Dixit (1989) [4] as a function of the recovery fraction α. If the recovery fraction is zero, the problem reverts to the perpetual Merton solution while as the recovery fractions tends to one, the option tends to the perpetual fully reversible case described in Section 3 this paper.

[2] if costs $k,l>0$. Note that we require $l<X$ i.e. $\underline{X}>0$

Moreover the left hand column contains fully reversible options of finite maturity $R(T,\alpha)$ that are treated in this paper. These can be valued in closed form because the boundary of action is time and volatility independent. The bottom row contains option payoffs while the interior contains the general cases.

The next Section examines the perpetual investment hysteresis (top row) and shows how the four boundary conditions that determine the problem can be reduced to one non–linear equation. It also examines the perpetual reversible case (top left) and present solution forms. Finally Section 4 solves for the finite reversible options (left column) in closed form, as task which is only possible because the boundary is exogenous.

3 Infinite reversible options

We proceed to examine option values under the perfectly reversible case when the two threshold collapse to one common switching level. These perfectly reversible options have similar value form to Merton's (1973) perpetual American options although they have a very different threshold. Their applications include situations where reversion can occur costlessly and arbitrarily often.

Treating the perpetual hysteresis case, the time partial in Equation (1) can be set to zero and a general solution of the form $R(\infty,\alpha) = AV^a + BV^b$ can be used (constants A, B are determined using boundary conditions, a, b $(a > 1, b < 0)$ solve the fundamental equation[3]).

Two boundary conditions are immediately determined because the option to open O goes to zero as V tends to zero and the option to shut S goes to zero as V becomes large. Therefore these open and shut options conform to the two general solutions AV^a, BV^b respectively. The remaining boundary conditions must be determined endogenously through optimality conditions.

The optimal policy is determined by two value matching and two smooth

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[3]

$$0 = \frac{1}{2}\sigma^2\beta(\beta-1) + (r-\delta)\beta - r$$

$$a,b = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} \pm \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

pasting conditions (see Dumas (1991) [8] for a treatment of these conditions and Dumas and Luciano (1991) [9] for another two sided transaction cost control problem) at two time independent thresholds, one an upper $\overline{V} = \overline{V}(\infty, \alpha)$ and one a lower threshold $\underline{V} = \underline{V}(\infty, \alpha)$. From this point on the thresholds will always be time invariant and so will simply be labelled $\overline{V}, \underline{V}$, they still depend on α.

3.1 Hysteresis solution in four equations

This yields a system of four equations in four variables $\left(A, B, \overline{V}, \underline{V}\right)$ representing the remaining boundary conditions. The four equations co–determine the intervention points $\overline{V}, \underline{V}$ and the option constants A, B. All four are a function of the inputs to the system, the (de)activation amounts $\left(\overline{X}, \underline{X}\right)$, although we will see that it is actually easier to evaluate $\left(A, B, \overline{X}, \underline{X}\right)$ as a function of $\left(\overline{V}, \underline{V}\right)$. Writing the two value matching conditions and the smooth pasting conditions (made homogenous by a $\overline{V}, \underline{V}$ multiplication) out in matrix form allows inversion for $\left(A, B, \overline{X}, \underline{X}\right)$

$$\begin{array}{rcl} A\overline{V}^a + \overline{X} & = & B\overline{V}^b + \overline{V} \\ A\underline{V}^a + \underline{X} & = & B\underline{V}^b + \underline{V} \\ Aa\overline{V}^a & = & Bb\overline{V}^b + \overline{V} \\ Aa\underline{V}^a & = & Bb\underline{V}^b + \underline{V} \end{array} \iff \begin{bmatrix} 1 & 0 & \overline{V}^a & -\overline{V}^b \\ 0 & 1 & \underline{V}^a & -\underline{V}^b \\ 0 & 0 & a\overline{V}^a & -b\overline{V}^b \\ 0 & 0 & a\underline{V}^a & -b\underline{V}^b \end{bmatrix} \begin{bmatrix} \overline{X} \\ \underline{X} \\ A \\ B \end{bmatrix} = \begin{bmatrix} \overline{V} \\ \underline{V} \\ \overline{V} \\ \underline{V} \end{bmatrix} \tag{3}$$

Inverting the matrix (see the Appendix) to recover $\left(A, B, \overline{X}, \underline{X}\right)$ as a function of $\left(\overline{V}, \underline{V}\right)$, the matrix product is most easily evaluated as a function of the fraction $\gamma(\alpha) \equiv \underline{V}/\overline{V}$ (a time independent ratio of the lower to the upper intervention thresholds)

$$\begin{bmatrix} \overline{X} \\ \underline{X} \\ A \\ B \end{bmatrix} = \frac{\overline{V}}{ab\left(\gamma^b - \gamma^a\right)} \begin{bmatrix} ab\left(\gamma^b - \gamma^a\right) - b\gamma^b + a\gamma^a - (a-b)\gamma \\ (ab-a)\gamma^{b+1} + (b-ab)\gamma^{a+1} + (a-b)\gamma^{a+b} \\ b\left(\gamma^b - \gamma\right)\overline{V}^{-a} \\ a\left(\gamma^a - \gamma\right)\overline{V}^{-b} \end{bmatrix}$$

3.2 Hysteresis solution in one equation

Thus if $\underline{V}, \overline{V}$ were known a priori instead of $\overline{X}, \underline{X}$ it would be a simple matter to determine $\left(A, B, \overline{X}, \underline{X}\right)$. Since $\overline{V} \neq 0$ and $\gamma^b \neq \gamma^a$, the variable $\alpha = \underline{X}/\overline{X}$

(the ratio of the deactivation and activation amounts) can be expressed as a function of γ by dividing the first two lines.

$$\alpha(\gamma) = \frac{(ab-a)\gamma^{b+1} + (b-ab)\gamma^{a+1} + (a-b)\gamma^{a+b}}{(ab-b)\gamma^{b} + (a-ab)\gamma^{a} - (a-b)\gamma} \tag{4}$$

This is the one equation that determines the entire system.

It would be preferable to determine the optimal intervention threshold ratio that solves Equation (4) $\gamma(\alpha)$ as a function of the ratio α of the amounts $\underline{X}, \overline{X}$ as oppose to determining the ratio of the amounts as a function of the thresholds $\alpha(\gamma)$ but a numerical solution for the inverse is always easy to obtain for any particular values. This polynomial that represents α as a function of γ is monotonic and increasing in γ and therefore there is a unique α for every γ and vice versa. For each α, the optimal γ can be retrieved numerically from Equation (4) and then the thresholds and option constants recovered.

$$\begin{aligned}
\begin{bmatrix} \overline{V} \\ \underline{V} \end{bmatrix} &= ab\left(\gamma^b - \gamma^a\right) \begin{bmatrix} \frac{\overline{X}}{ab\left(\gamma^b-\gamma^a\right) - b\gamma^b + a\gamma^a - (a-b)\gamma} \\ \frac{\underline{X}}{ab\left(\gamma^b-\gamma^a\right) + b\gamma^a - a\gamma^b + (a-b)\gamma^{a+b-1}} \end{bmatrix} \\
\begin{bmatrix} A \\ B \end{bmatrix} &= \frac{\overline{V}}{ab\left(\gamma^b - \gamma^a\right)} \begin{bmatrix} b\left(\gamma^b - \gamma\right)\overline{V}^{-a} \\ a\left(\gamma^a - \gamma\right)\overline{V}^{-b} \end{bmatrix}
\end{aligned}$$

3.3 The reversible limit

We now examine the properties of Equation (4) for the special cases at the corners of Table (1). The function that determines the (dis)investment costs as a function of thresholds $\alpha(\gamma)$ has asymptotes which correspond to degenerate cases when the matrix determinant is zero. For values of γ that are small or close to one, the limiting behaviour of α/γ is given respectively by

$$\begin{aligned}
\lim_{\gamma \to 0} \frac{\alpha}{\gamma} &= \frac{a(b-1)}{b(a-1)} \\
\lim_{\gamma \to 1} \frac{\alpha}{\gamma} &= 1
\end{aligned}$$

The first corresponds to the Merton perpetual American irreversible calls and puts that are widely used in the real options literature. This can be seen by evaluating the intervention thresholds for small α and γ corresponding to small $\underline{X}$[4]. For small γ, taking the lowest powers of γ in denominator and numerator yields a form for the other threshold $\underline{V}$ as a function of $\underline{X}$ (or $\overline{V}$ as a function of $\overline{X}$)

$$\text{Critical thresholds} \begin{bmatrix} \overline{V} \\ \underline{V} \end{bmatrix} = \begin{bmatrix} \frac{a}{a-1}X \\ \frac{b}{b-1}X \end{bmatrix} \begin{matrix} \text{call} \\ \text{put} \end{matrix}$$

For extremely low α and γ the two variable reversible problem is reduced to a one way problem since one threshold becomes unattainable as it goes to zero. This conforms to the Merton perpetual American options that are not reversible.

However the other asymptote $\gamma = 1$ preserves the ability to reverse and is indeed perfectly reversible in the sense that forward and reverse exercise consumes no value. As $\gamma, \alpha \rightarrow 1$ both the critical thresholds collapse to a common value as the exercise proceeds become equal. In this case there is no longer any hysteresis and switching occurs infinitely often at the common activation and deactivation threshold. Using l'Hopital's rule to evaluate this limiting case and labelling this common threshold $K\left(= \overline{V} = \underline{V}\right)$ and the common (de)activation amount $X\left(= \underline{X} = \overline{X}\right)$ this implies

$$\overline{V}, \underline{V} \rightarrow K = \frac{ab}{ab-(a+b)+1}X = \frac{rX}{\delta}$$

This means that the optimal exercise strategy is now to activate the project when the opportunity cost on the project δV exceeds the opportunity cost on the required investment rX and to deactivate it when $rX > \delta V$. The change happens at $V = K$ when the so called Jorgenson (1963) [10] user costs of capital are equal. Thus a current yield criteria $\delta V \lesseqgtr rX$ is employed, not a naive or myopic present value condition $V \lesseqgtr X$. Because there is no penalty for early exercise, waiting does not have to be deferred to the Merton threshold. Although not obvious at the outset, this yield argument is not surprising.

3.4 Costless reversion

For this fully reversible case we can specialise the opening and shutting options with $\alpha = 1$ to produce a fully reversible opening option $O(T, 1)$ which

[4] i.e. for a fixed exercise amount, the reverse exercise amount could be arbitrarily small corresponding to the irreversible Merton case.

on exercise yields a fully reversible shutting option $S(T,1)$. In this Section we treat the infinite cases $O(T,\infty), S(T,\infty)$ and in the next Section we will treat the finite cases. The equation for the option thresholds and constants shows how to evaluate the perpetual reversibles where opening and shutting occurs at the new joint level of $K = rX/\delta$ as the limit as $\alpha, \gamma \to 1$ and hysteresis collapses[5].

$$\begin{aligned} O(\infty,1) &= K\frac{b-1}{a(b-a)}\left(\frac{V}{K}\right)^a \\ S(\infty,1) &= K\frac{a-1}{b(b-a)}\left(\frac{V}{K}\right)^b \end{aligned} \tag{5}$$

These are proportional to the irreversible Merton option call $O(\infty,0)$ (open with no reverse option) and put (shut with no reverse option) values $S(\infty,0)$ (both functions of $\overline{V}, \underline{V}$)

$$\begin{aligned} O(\infty,0) &= \frac{X}{a-1}\left(\frac{V(a-1)}{Xa}\right)^a = \left(\overline{V}-X\right)\left(\frac{V}{\overline{V}}\right)^a \\ S(\infty,0) &= \frac{X}{1-b}\left(\frac{V(1-b)}{Xb}\right)^b = \left(X-\underline{V}\right)\left(\frac{V}{\underline{V}}\right)^b \end{aligned}$$

The perpetual reversible options are always worth a constant multiple more than their Merton counterparts and they include the set of investment strategies of the irreversibles

$$\begin{aligned} \frac{O(\infty,1)}{O(\infty,0)} &= \frac{r}{\delta}\frac{(a-1)(b-1)}{a(b-a)}\left(\frac{a}{a-1}\right)^a > 1 \\ \frac{S(\infty,1)}{S(\infty,0)} &= \frac{r}{\delta}\frac{(a-1)(b-1)}{b(b-a)}\left(\frac{b}{1-b}\right)^b > 1 \end{aligned}$$

and their thresholds are always encountered sooner, i.e. they are lower for the opening option and higher for the shutting option. Since the opening

[5] The option constants are given by

$$\begin{bmatrix} A \\ B \end{bmatrix} = \lim_{\gamma\to 1}\frac{1}{ab\left(\gamma^b-\gamma^a\right)}\begin{bmatrix} b\left(\gamma^b-\gamma\right)\overline{V}^{1-a} \\ a\left(\gamma^{a+b-1}-\gamma^b\right)\underline{V}^{1-b} \end{bmatrix} = \frac{1}{ab(b-a)}\begin{bmatrix} b(b-1)\left(\frac{rX}{\delta}\right)^{1-a} \\ a(a-1)\left(\frac{rX}{\delta}\right)^{1-b} \end{bmatrix}$$

(closing) boundary is only ever activated from below (above) it can be said that the reversible opening and shutting options are always activated earlier in time than the irreversible.

$$\begin{aligned}\frac{\overline{V}(\infty,1)}{\overline{V}(\infty,0)} &= \frac{K(a-1)}{Xa} = \frac{r(a-1)}{\delta a} < 1 \\ \frac{\underline{V}(\infty,1)}{\underline{V}(\infty,0)} &= \frac{K(b-1)}{Xb} = \frac{r(b-1)}{\delta b} > 1\end{aligned}$$

4 Finite reversible options

In this Section we value finite reversible options, a task which is only possible due to the known and constant exercise boundary K.

Although the infinite asymptotes for the thresholds are available as the solution to a non–linear equation, for any partially reversible, finite horizon problem T (non time homogeneous) the true boundaries can only be found using numerical techniques.

However, for costless reversion, the boundaries are flat at $K = \frac{rX}{\delta}$ for all time because there is no penalty for early exercise and the problem is again time homogeneous even for a finite maturity T. This means that the boundaries are known a priori for the reversible case and that the problem solution can be formulated in closed form.

4.1 Black Scholes flow representation

For a finite horizon T, the finite reversible real option $R(T,1)$ has a time homogeneous optimal policy that is to open the project V if it is priced more than rX/δ or shut it if not. This can be written as a risk–neutral expectation of the integral of the discounted maximum flow $\max(rX, \delta V_t)$ over time until T which (by carrying expectations under the integral) can be written as an integral of a continuum of Black Scholes (1973) [6] European option formulae[6]

$$
\begin{aligned}
R(T,1) &= E_0^Q\left[\int_0^T e^{-rt}\max(rX,\delta V_t)\,dt\right] \qquad (6)\\
&= \begin{cases} X\left(1-e^{-rT}\right)+ & \delta\int_0^T\left[Ve^{-\delta t}N(+d_{K1})-Xe^{-rt}N(+d_{K0})\right]dt \quad : V<K \\ V\left(1-e^{-\delta T}\right)+ & r\int_0^T\left[Xe^{-rt}N(-d_{K0})-Ve^{-\delta t}N(-d_{K1})\right]dt \quad : V>K \end{cases}
\end{aligned}
$$

d_{K1}, d_{K0} are like Black Scholes parameters (see later definition and correspondence to d_1, d_2) for options but with a strike K. If $V<K$ the perfectly reversible option is interpreted as the investment amount X plus a dividend yield δ on a continuum of intermediate life $(0,T)$ Black Scholes call options struck at K. If however $V>K$ the perfectly reversible real option is interpreted as the project value itself V plus a risk free yield r on a continuum of Black Scholes puts struck at K.

Thus a reversible option gives the excess maximum yield or flow δV_t or rX i.e. optimal investment in V or X, while allowing for expected capital gains associated with all future optionality associated with the continuum of Black Scholes puts or calls. The integral of a continuum of Black Scholes options (Equation (6)) is difficult to evaluate directly, moreover the integral limit as $T\to 0$ is different for differing moneyness $V \gtrless K$. However once the result is derived by another means, it is relatively easy to show by differentiation that the reversible flow option does indeed conform to a continuum of Black Scholes options.

4.2 Risk–neutral valuation

Thinking of the finite flow option as an annuity, it can be valued by subtracting (the risk–neutral expectation of) a forward start perpetuity from a current start perpetuity. As $T\to\infty$ this expression will converge toward the known infinite horizon solution. The current start perpetual flow option is given as a function of one of the two perpetual opening and shutting values

[6]

$$\max(rX,\delta S) = rX + \delta\max\left(V-\frac{r}{\delta}X,0\right) = \delta V + r\max\left(X-\frac{\delta}{r}V,0\right)$$

in Equation (5) conditional on $V \gtrless K$

$$R(\infty,1) = \begin{cases} V + S(\infty,1) & : V > K \\ X + O(\infty,1) & : V < K \end{cases} \tag{7}$$

The **finite** reversible real option is equal to the infinite reversible less the (discounted) risk–neutral expectation of the forward start perpetual $E_0^Q[R_T(\infty,1)]$ (R_T, V_T etc are unknown time T values)

$$R(T,1) = R(\infty,1) - e^{-rT}E_0^Q \begin{cases} V_T + S_T(\infty,1) & : V_T > K \\ X + O_T(\infty,1) & : V_T < K \end{cases}$$

The risk–neutral expectation elements can be calculated because each of the value elements (X, O_T, V_T, S_T) has a special elasticity $(0, a, 1, b)$ with respect to V. Moreover the discount rates associated with those instruments with no cashflows (S, O) cancel with the risk free and those that have flows (V, X) (δ, r) respectively reduce to $e^{-\delta T}, e^{-rT}$. Thus the finite case can be shown to be

$$R(T,1) = \begin{bmatrix} X \\ O(\infty,1) \\ V \\ S(\infty,1) \end{bmatrix} \cdot \begin{bmatrix} 1_{V<K} & - & e^{-rT}N(-d_{K0}) \\ 1_{V<K} & - & N(-d_{Ka}) \\ 1_{V>K} & - & e^{-\delta T}N(d_{K1}) \\ 1_{V>K} & - & N(d_{Kb}) \end{bmatrix} \tag{8}$$

where the Black Scholes like Normal parameters $d_{K\beta}$ are specific to each of the (elasticities of the) four assets given by

$$\beta \quad : \quad d_{K\beta} = \frac{\ln V - \ln K + \left(r - \delta + \sigma^2\left(\beta - \frac{1}{2}\right)\right)T}{\sigma\sqrt{T}}$$

$$\begin{matrix} \beta = 0\,(X) \\ \beta = a\,(O) \\ \beta = 1\,(V) \\ \beta = b\,(S) \end{matrix} \quad : \quad \begin{bmatrix} d_{K0} \\ d_{Ka} \\ d_{K1} \\ d_{Kb} \end{bmatrix} = \frac{\ln V - \ln K + \left(r - \delta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} + \sigma\sqrt{T}\begin{bmatrix} 0 \\ a \\ 1 \\ b \end{bmatrix}$$

$$\begin{matrix} R(\infty,1) \\ - \\ e^{-rT}E_0^Q[R_T(\infty,1)] \end{matrix} = \begin{bmatrix} AN(X, V<K) & - & AN(X, V_T<K) \\ AN(O, V<K) & - & AN(O_T, V_T<K) \\ AN(V, V>K) & - & AN(V_T, V_T>K) \\ AN(S, V>K) & - & AN(S_T, V_T>K) \end{bmatrix} \tag{9}$$

Each of the four elements has a representation as an **asset or nothing option** which pay if an upper or lower condition is satisfied, $AN\left(Y_T, V_T \gtreqless K\right)$. These are instruments which at time T pay the underlying asset $Y_T \in (X, O_T, V_T, S_T)$ only when the condition $V_T > K$, or $V_T < K$ is met, otherwise they pay nothing. Since one or the other will be exercised for sure, they satisfy a summation parity condition

$$AN\left(Y_T, V_T > K\right) + AN\left(Y_T, V_T < K\right) = e^{-rT} E_0^Q \left[Y_T\right] = Y$$

The zero maturity asset or nothing option represents the current reversible option position

$$\begin{aligned} AN\left(Y, V > K\right) &= Y : V > K \\ AN\left(Y, V < K\right) &= Y : V < K \end{aligned}$$

Thus the asset or nothing option prices play the role of annuity factors that convert the current perpetuity into an annuity. The finite maturity reversible option is represented by its current perpetual value less the annuity discount factors associated with its termination at time T.

Differentiation of Equation (8) w.r.t. to time yields the integrand of Equation (6) although care needs to be taken to treat the two moneyness cases differently. Equation (8) also satisfies Equation (1).

4.3 Example

For $(r, \delta, \sigma) = (10\%, 5\%, 20\%)$ and for V values from 100 to 300 and times T from 0 to 10 years (and beyond to the perpetual solution via 20 and 50 years) the finite reversible option is plotted by maturity and moneyness. Naturally the reversible flow call increases with V & T.

4.4 Comparative statics

Comparative statics w.r.t. $T, V,$ (and X) are relatively easy to evaluate and correspond to the elements of the HBJ partial differential Equation (1). Sensitivities w.r.t. to σ, r, δ are more difficult because of the complex dependency of a, b on these variables.

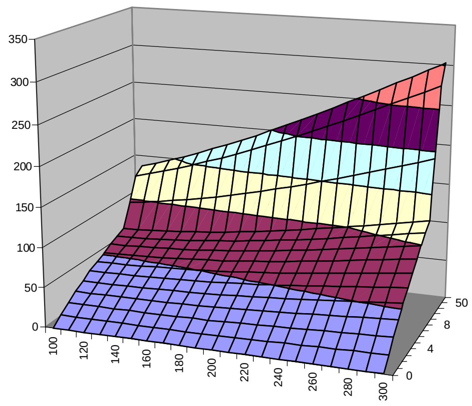

Figure 1: Reversible option values for V from 100 to 300 and T from 0 to 10 years (and 20, 50 and ∞), all for (r, δ, σ) equal to $(10\%, 5\%, 20\%)$.

5 Conclusions

This paper produces a formula for the present value of the continuous maximum of two flows over a finite horizon. If the stochastic dividend flow from a project $f_t = \delta V_t$ or the constant flow from some investment proceeds $k = rX = \delta K$ can be costlessly and continuously selected, a reversible flow option can be used as a valuation basis. This is because the exercise and re–exercise threshold is constant in time. Furthermore a finite reversible option can be valued using the difference between a current start and the expectation of a forward start perpetual flow option.

This finite call option (which satisfies the HBJ asset pricing equation and all relevant boundary conditions) also has a representation as an integral of a continuum of infinitesimal Black Scholes call options. Taking the time derivative of the formula presented in this paper yields the Black Scholes integrand.

As expected the flow call option increases in value with the underlying stochastic flow and also with the time to expiry of the option.

Applications include commodity costs where the flow consumption decision can be costlessly and arbitrarily frequently chosen from the max (or min) of two input costs $\max [f_t, k]$. This allows pricing of the flexibility generated by industrial plant which could run either on a variable price commodity input such as oil or a fixed energy price input.

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6 Appendix

.1 Inverting the system matrix

$$
\begin{bmatrix} \overline{X} \\ \underline{X} \\ A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & \overline{V}^a & -\overline{V}^b \\ 0 & 1 & \underline{V}^a & -\underline{V}^b \\ 0 & 0 & a\overline{V}^a & -b\overline{V}^b \\ 0 & 0 & a\underline{V}^a & -b\underline{V}^b \end{bmatrix}^{-1} \begin{bmatrix} \overline{V} \\ \underline{V} \\ \overline{V} \\ \underline{V} \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & \frac{-\overline{V}^a b\underline{V}^b + \overline{V}^b a\underline{V}^a}{ab\left(\overline{V}^a\underline{V}^b - \overline{V}^b\underline{V}^a\right)} & -\overline{V}^a\overline{V}^b \frac{-b+a}{ab\left(\overline{V}^a\underline{V}^b - \overline{V}^b\underline{V}^a\right)} \\ 0 & 1 & \underline{V}^a\underline{V}^b \frac{-b+a}{ab\left(\overline{V}^a\underline{V}^b - \overline{V}^b\underline{V}^a\right)} & -\frac{-\underline{V}^a b\overline{V}^b + \underline{V}^b a\overline{V}^a}{ab\left(\overline{V}^a\underline{V}^b - \overline{V}^b\underline{V}^a\right)} \\ 0 & 0 & \frac{\underline{V}^b}{a\left(\overline{V}^a\underline{V}^b - \overline{V}^b\underline{V}^a\right)} & -\frac{\overline{V}^b}{a\left(\overline{V}^a\underline{V}^b - \overline{V}^b\underline{V}^a\right)} \\ 0 & 0 & \frac{\underline{V}^a}{b\left(\overline{V}^a\underline{V}^b - \overline{V}^b\underline{V}^a\right)} & -\frac{\overline{V}^a}{b\left(\overline{V}^a\underline{V}^b - \overline{V}^b\underline{V}^a\right)} \end{bmatrix} \begin{bmatrix} \overline{V} \\ \underline{V} \\ \overline{V} \\ \underline{V} \end{bmatrix}
$$

.2 Evaluating the risk–neutral expected value of the forward perpetuity

We need to calculate the risk neutral expectation of a claim V_T^β which has general elasticity β. For a normally distributed variable $v = \ln V_T$ with mean μ and sd Σ

$$
\begin{aligned}
v &= \ln V_T \\
v &\sim n(\mu, \Sigma)
\end{aligned}
$$

the expectation is evaluated by integrating over states

$$
\begin{aligned}
E_0^Q\left[V_T^\beta\right] &= \int_A^B V^\beta n(\ln V)\, d\ln V \\
&= \int_A^B e^{\beta v} n(v)\, dv \\
&= \int_A^B \exp\left(-\frac{(v-\mu)^2}{2\Sigma^2} + \beta v\right) dv
\end{aligned}
$$

Firstly complete the square within the normal density[7]

$$\frac{(v-\mu)^2-2\beta v\Sigma^2}{2\Sigma^2}=\frac{\left(v-\mu-\beta\Sigma^2\right)^2}{2\Sigma^2}-\frac{2\mu\beta\Sigma^2+\beta^2\Sigma^4}{2\Sigma^2}$$

so that the expectation becomes

$$\begin{aligned} E_0^Q\left[V_T^\beta\right] &= \int_A^B \exp\left(\mu\beta+\frac{1}{2}\beta^2\Sigma^2\right)\exp\left(-\frac{\left(v-\mu-\beta\Sigma^2\right)^2}{2\Sigma^2}\right)dv \\ &= \exp\beta\left(\mu+\frac{1}{2}\beta\Sigma^2\right)\left(N\left(\frac{B-(\mu+\beta\Sigma^2)}{\Sigma}\right)-N\left(\frac{A-(\mu+\beta\Sigma^2)}{\Sigma}\right)\right) \end{aligned}$$

Now for a lognormal stock price process under the risk neutral density, $\mu = \ln V + \left(r-\delta-\frac{1}{2}\sigma^2\right)T$ and $\Sigma = \sigma\sqrt{T}$ so that

$$\begin{aligned} \mu+\frac{1}{2}\beta\Sigma^2 &= \ln V+\left(r-\delta-\frac{1}{2}\sigma^2\right)T+\frac{1}{2}\beta\sigma^2 T \\ \exp\beta\left(\mu+\frac{1}{2}\beta\Sigma^2\right) &= V^\beta\exp\left(\left(r-\delta+\frac{1}{2}\sigma^2(\beta-1)\right)T\right) \end{aligned}$$

$$\frac{E_0^Q\left[V_T^\beta\right]}{V^\beta e^{\left(r-\delta+\frac{1}{2}\sigma^2(\beta-1)\right)\beta T}} = \begin{array}{c} N\left(\frac{B-\left(\ln V^\beta+\left(r-\delta+\sigma^2\left(\beta-\frac{1}{2}\right)\right)T\right)}{\sigma\sqrt{T}}\right) \\ -N\left(\frac{A-\left(\ln V^\beta+\left(r-\delta+\sigma^2\left(\beta-\frac{1}{2}\right)\right)T\right)}{\sigma\sqrt{T}}\right)\end{array} \quad (10)$$

This verifies the Black Scholes result with $A=\ln X$ and $B=\infty$

$$\int_{\ln X}^{\infty}(V-X)\,n(\ln V)\,d\ln V=\int_{\ln X}^{\infty}Vn(\ln V)\,d\ln V-\int_{\ln X}^{\infty}Xn(\ln V)\,d\ln V$$

7

$$(v-\mu)^2-2\beta v\sigma^2 = v^2-2v\mu+\mu^2-2v\beta\sigma^2$$

$$\left(v-\mu-\beta\sigma^2\right)^2 = v^2-2v\mu-2v\beta\sigma^2+\mu^2+2\mu\beta\sigma^2+\beta^2\sigma^4$$

since the first part of the integral corresponds to $\beta = 1$

$$\int_{\ln X}^{\infty} Vn\left(\ln V\right) d\ln V = V^{\beta} e^{(r-\delta)T} N\left(\frac{\ln V^{\beta} - \ln X + \left(r - \delta + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}\right)$$

and the second part of the integral corresponds to $\beta = 0$

$$\int_{\ln X}^{\infty} Xn\left(\ln V\right) d\ln V = N\left(\frac{\ln V^{\beta} - \ln X + \left(r - \delta - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}\right)$$

Discounting by e^{-rT} yields the Black Scholes components.

Now to return to the finite reversible case. The current value of the perpetual reversible is given by, $R\left(\infty, 1\right)$

$$\begin{aligned} R\left(\infty,1\right) &= \begin{cases} V+S & V>K \\ X+O & V<K \end{cases} \\ O &= K\frac{b-1}{a\left(b-a\right)}\left(\frac{V}{K}\right)^{a} \\ S &= K\frac{a-1}{b\left(b-a\right)}\left(\frac{V}{K}\right)^{b} \end{aligned}$$

where a, b are the positive and negative solution to the fundamental

$$a, b = \frac{1}{2} - \frac{r-\delta}{\sigma^2} \pm \left(\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}\right)^{\frac{1}{2}}$$

We can value the finite reversible by taking present value expectations of a forward perpetual and subtracting it from the current perpetual

$$R\left(T,1\right) = R\left(\infty,1\right) - e^{-rT} E_0^Q\left[R_T\left(\infty,1\right)\right]$$

Taking risk neutral expectations of the forward start perpetual involves integrals to and from K the critical threshold (again $v = \ln V_T$)

$$E_0^Q\left[R_T\left(\infty,1\right)\right] = \int_{-\infty}^{\ln K} \left(X + O_T\right) n\left(v\right) dv + \int_{\ln K}^{\infty} \left(V_T + S_T\right) n\left(v\right) dv$$

When discounted at the risk free rate, this has the interpretation of four asset or nothing calls, which pay conditional on $V_T \gtrless K$. The first two pay X, O if $V_T < K$ and the second two pay V_T, S if $V_T > K$.

Using the general integral result (Equation 10) for each of the four asset or nothing parts can be evaluated

$$
\begin{aligned}
X &: \quad \beta = 0, A = -\infty, B = \ln K \\
\int_{-\infty}^{\ln K} X n(v)\, dv &= X\left(1 - N\left(\frac{\ln V - \ln K + \left(r - \delta - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}\right)\right) \\
&= X\left(1 - N\left(d_{K0}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
O &: \quad \beta = a, A = -\infty, B = \ln K \\
\int_{-\infty}^{\ln K} O n(v)\, dv &= \int_{-\infty}^{\ln K} K\frac{b-1}{a(b-a)}\left(\frac{V}{K}\right)^a n(v)\, dv \\
&= K\frac{b-1}{a(b-a)}\left(\frac{V}{K}\right)^a e^{\left(r-\delta+\frac{1}{2}\sigma^2(a-1)\right)aT}\left(1 - N\left(\frac{\ln V - \ln K + \left(r - \delta + \sigma^2\left(a\right.\right.}{\sigma\sqrt{T}}\right.\right. \\
&= O e^{\left(r-\delta+\frac{1}{2}\sigma^2(a-1)\right)aT}\left(1 - N\left(d_{Ka}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
V &: \quad \beta = 1, A = \ln K, B = \infty \\
\int_{\ln K}^{\infty} V n(v)\, dv &= V e^{(r-\delta)T} N\left(\frac{\ln V - \ln K + \left(r - \delta + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}\right) \\
&= V e^{(r-\delta)T} N\left(d_{K1}\right)
\end{aligned}
$$

$$
\begin{aligned}
S &: \quad \beta = b, A = \ln K, B = \infty \\
\int_{\ln K}^{\infty} S n(v)\, dv &= \int_{\ln K}^{\infty} K\frac{a-1}{b(b-a)}\left(\frac{V}{K}\right)^b n(v)\, dv \\
&= K\frac{a-1}{b(b-a)}\left(\frac{V}{K}\right)^b e^{\left(r-\delta+\frac{1}{2}\sigma^2(b-1)\right)bT} N\left(\frac{\ln V - \ln K + \left(r - \delta + \sigma^2\left(b - \frac{1}{2}\right)\right) T}{\sigma\sqrt{T}}\right. \\
&= S e^{\left(r-\delta+\frac{1}{2}\sigma^2(b-1)\right)bT} N\left(d_{Kb}\right)
\end{aligned}
$$

This representation uses Black Scholes like parameters all evaluated at a strike price of K but with different elasticities $(a > 1 > 0 > b)$ that correspond to the four cases of β

$$
\begin{aligned}
\beta &: \; d_{K\beta} = \frac{\ln V - \ln K + \left(r - \delta + \sigma^2 \left(\beta - \frac{1}{2}\right)\right) T}{\sigma\sqrt{T}} \\
O\left(\beta = a\right) &: \; d_{Ka} = \frac{\ln V - \ln K + \left(r - \delta + \sigma^2 \left(a - \frac{1}{2}\right)\right) T}{\sigma\sqrt{T}} \\
V\left(\beta = 1\right) &: \; d_{K1} = \frac{\ln V - \ln K + \left(r - \delta + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \\
X\left(\beta = 0\right) &: \; d_{K0} = \frac{\ln V - \ln K + \left(r - \delta - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \\
S\left(\beta = b\right) &: \; d_{Kb} = \frac{\ln V - \ln K + \left(r - \delta + \sigma^2 \left(b - \frac{1}{2}\right)\right) T}{\sigma\sqrt{T}}
\end{aligned}
$$

Finally the parameters a, b are special in that they cancel with the risk free rate when discounting is applied, i.e.

$$
\left(r - \delta + \frac{1}{2}\sigma^2\left(a - 1\right)\right) a = r = \left(r - \delta + \frac{1}{2}\sigma^2\left(b - 1\right)\right) b
$$

$$
\begin{aligned}
e^{-rT} E_0^Q \left[R\left(V_T, \infty\right)\right] &= \begin{bmatrix} X e^{-rT}\left(1 - N\left(d_{K0}\right)\right) \\ O\left(1 - N\left(d_{Ka}\right)\right) \\ V e^{-\delta T} N\left(d_{K1}\right) \\ S N\left(d_{Kb}\right) \end{bmatrix} = \begin{bmatrix} X e^{-rT} N\left(-d_{K0}\right) \\ O N\left(-d_{Ka}\right) \\ V e^{-\delta T} N\left(d_{K1}\right) \\ S N\left(d_{Kb}\right) \end{bmatrix} \\
R\left(V, \infty\right) &= \begin{bmatrix} 1_{V<K} X \\ 1_{V<K} O \\ 1_{V>K} V \\ 1_{V>K} S \end{bmatrix} \\
R\left(V, \infty\right) - e^{-rT} E_0^Q \left[R\left(V_T, \infty\right)\right] &= \begin{bmatrix} X \\ O \\ V \\ S \end{bmatrix} \cdot \begin{bmatrix} 1_{V<K} - e^{-rT} N\left(-d_{K0}\right) \\ 1_{V<K} - N\left(-d_{Ka}\right) \\ 1_{V>K} - e^{-\delta T} N\left(d_{K1}\right) \\ 1_{V>K} - N\left(d_{Kb}\right) \end{bmatrix}
\end{aligned}
$$

.3 Comparative statics

Now redefine the specific $d_{K\beta}$ as a function of d_{K0} plus a term in β

$$\begin{aligned} d_{K\beta} &= \frac{\ln V - \ln K + \left(r - \delta + \sigma^2\left(\beta - \frac{1}{2}\right)\right)T}{\sigma\sqrt{T}} \\ &= \frac{\ln V - \ln K + \left(r - \delta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} + \beta\sigma\sqrt{T} = d_{K0} + \beta\sigma\sqrt{T} \end{aligned}$$

and similarly for the time derivative

$$\frac{\partial d_{K\beta}}{\partial T} = \frac{\partial d_{K0}}{\partial T} + \frac{1}{2\sqrt{T}}\beta\sigma$$

The definition of the normal density $n\left(d_{K\beta}\right)$ can be expressed as a function of $n\left(d_{K0}\right)$

$$\begin{aligned} n\left(d_{K\beta}\right) &= \frac{1}{\sqrt{2\pi}}\exp -\frac{1}{2}d_{K\beta}^2 \\ &= \frac{1}{\sqrt{2\pi}}\exp -\frac{1}{2}\left(d_{K0}^2 + 2\beta\sigma\sqrt{T}d_{K0} + \beta^2\sigma^2 T\right) \\ &= n\left(d_{K0}\right)\exp -\frac{1}{2}\left(2\beta\sigma\sqrt{T}d_{K0} + \beta^2\sigma^2 T\right) \\ &= n\left(d_{K0}\right)\exp -\beta\sigma\sqrt{T}\left(d_{K0} + \frac{1}{2}\beta\sigma\sqrt{T}\right) \\ &= n\left(d_{K0}\right)\exp -\beta\left(\ln V - \ln K\right) - \beta\left(r - \delta + \frac{1}{2}\left(\beta - 1\right)\sigma^2\right)T \\ &= n\left(d_{K0}\right)\left(\frac{V}{K}\right)^{-\beta}\exp -\beta\left(r - \delta + \frac{1}{2}\left(\beta - 1\right)\sigma^2\right)T \end{aligned}$$

$$\begin{array}{ll} \beta = 0 & n\left(d_{K\beta}\right) = n\left(d_{K0}\right) \\ \beta = a & n\left(d_{K\beta}\right) = n\left(d_{K0}\right)\left(\frac{V}{K}\right)^{-a}e^{-rT} \\ \beta = 1 & n\left(d_{K\beta}\right) = n\left(d_{K0}\right)\left(\frac{V}{K}\right)^{-1}e^{-(r-\delta)T} \\ \beta = b & n\left(d_{K\beta}\right) = n\left(d_{K0}\right)\left(\frac{V}{K}\right)^{-b}e^{-rT} \end{array}$$

$$r = \frac{1}{2}\sigma^2\beta\left(\beta - 1\right) + \left(r - \delta\right)\beta$$

.4 Time partial Θ

For the expected value of the forward start perpetual reversible

$$e^{-rT}E_0^Q\left[R\left(V_T,\infty\right)\right] = \begin{bmatrix} Xe^{-rT}\left(1-N\left(d_{K0}\right)\right) \\ O\left(1-N\left(d_{Ka}\right)\right) \\ Ve^{-\delta T}N\left(d_{K1}\right) \\ SN\left(d_{Kb}\right) \end{bmatrix} = \begin{bmatrix} Xe^{-rT}N\left(-d_{K0}\right) \\ ON\left(-d_{Ka}\right) \\ Ve^{-\delta T}N\left(d_{K1}\right) \\ SN\left(d_{Kb}\right) \end{bmatrix}$$

$$\frac{\partial e^{-rT}E_0^Q\left[R\left(V_T,\infty\right)\right]}{\partial T} = \begin{bmatrix} Xe^{-rT}\left(n\left(-d_{K0}\right)\frac{\partial -d_{K0}}{\partial T}-rN\left(-d_{K0}\right)\right) \\ On\left(-d_{Ka}\right)\frac{\partial -d_{Ka}}{\partial T} \\ Ve^{-\delta T}\left(n\left(d_{K1}\right)\frac{\partial d_{K1}}{\partial T}-\delta N\left(d_{K1}\right)\right) \\ Sn\left(d_{Kb}\right)\frac{\partial d_{Kb}}{\partial T} \end{bmatrix}$$

$$= Kn\left(d_{K0}\right)e^{-rT}\begin{bmatrix} -\frac{X}{K}\frac{\partial d_{K0}}{\partial T} \\ -\frac{b-1}{a(b-a)}\left(\frac{\partial d_{K0}}{\partial T}+\frac{a\sigma}{2\sqrt{T}}\right) \\ \frac{\partial d_{K0}}{\partial T}+\frac{\sigma}{2\sqrt{T}} \\ \frac{a-1}{b(b-a)}\left(\frac{\partial d_{K0}}{\partial T}+\frac{b\sigma}{2\sqrt{T}}\right) \end{bmatrix} - rN\left(-d_{K0}\right)Xe^{-rT}-\delta N\left(d_{K1}\right)Ve^{-\delta T}$$

$$= Kn\left(d_{K0}\right)e^{-rT}\left(\frac{\partial d_{K0}}{\partial T}\left(-\frac{X}{K}-\frac{b-1}{a\left(b-a\right)}+1+\frac{a-1}{b\left(b-a\right)}\right)-\frac{b-1}{a\left(b-a\right)}\frac{a\sigma}{2\sqrt{T}}+\frac{b\sigma}{2\sqrt{T}}\frac{a-1}{b\left(b-a\right)} \right.$$
$$-rN\left(-d_{K0}\right)Xe^{-rT}-\delta N\left(d_{K1}\right)Ve^{-\delta T}$$

$$\frac{\delta}{r}=\frac{1+ab-a-b}{ab}$$

since[8]

8

$$\begin{aligned} d_{K0}+\frac{1}{2}\beta\sigma\sqrt{T} &= \frac{\ln V_0-\ln K+\left(r-\delta-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}+\frac{1}{2}\beta\sigma\sqrt{T} \\ &= \frac{\ln V_0-\ln K+\left(r-\delta+\frac{1}{2}\left(\beta-1\right)\sigma^2\right)T}{\sigma\sqrt{T}} \\ \beta\sigma\sqrt{T}\left(d_{K0}+\frac{1}{2}\beta\sigma\sqrt{T}\right) &= \beta\sigma\sqrt{T}\left(\frac{\ln V_0-\ln K+\left(r-\delta+\frac{1}{2}\left(\beta-1\right)\sigma^2\right)T}{\sigma\sqrt{T}}\right) \\ &= \beta\left(\ln V_0-\ln K\right)+\beta\left(r-\delta+\frac{1}{2}\left(\beta-1\right)\sigma^2\right)T \end{aligned}$$

$$
\begin{aligned}
d_{K\beta} &= \frac{\ln V - \ln K + \left(r - \delta + \sigma^2\left(\beta - \frac{1}{2}\right)\right)T}{\sigma\sqrt{T}} \\
\frac{\partial d_{K\beta}}{\partial V} &= \frac{1}{V\sigma\sqrt{T}} \\
\frac{\partial d_{K\beta}}{\partial K} &= \frac{1}{K\sigma\sqrt{T}} \\
\frac{\partial d_{K\beta}}{\partial T} &= -\frac{\ln V - \ln K}{2\sigma T^{\frac{3}{2}}} + \frac{\left(r - \delta + \sigma^2\left(\beta - \frac{1}{2}\right)\right)}{2\sigma\sqrt{T}} \\
&= \frac{-\ln V + \ln K + \left(r - \delta + \sigma^2\left(\beta - \frac{1}{2}\right)\right)T}{2\sigma T^{\frac{3}{2}}}
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
\frac{\partial N\left(d_{K\beta}\right)}{\partial V} &= \frac{\partial d_{K\beta}}{\partial V} n\left(d_{K\beta}\right) \\
&= \frac{n\left(d_{K0}\right)}{V\sigma\sqrt{T}}\left(\frac{V}{K}\right)^{-\beta} e^{-\beta\left(r-\delta+\frac{1}{2}(\beta-1)\sigma^2\right)T} \\
&= \frac{1}{V\sigma\sqrt{T}}\left(\frac{V}{K}\right)^{-\beta} \exp - \left(\beta\left(r - \delta + \frac{1}{2}(\beta-1)\sigma^2\right)T + \frac{1}{2}d_{K0}^2\right)
\end{aligned}
$$

Now for $V < K$ we can attack the time partial of the finite reversible, immediately subtracting the time partial of the Black Scholes

$$
\frac{\partial \text{Call}\left(V, K, T\right)}{\partial T} = \delta V e^{-\delta T} N\left(d_{K1}\right) - rXe^{-rT}N\left(d_{K0}\right)
$$

$$
\begin{aligned}
\frac{\partial\left(R\left(T\right)-X\right)}{\partial T}-\frac{\partial \mathrm{Call}\left(V,K,T\right)}{\partial T} &= 1'.\left[\begin{array}{c} -Xe^{-rT}n\left(-d_{K0}\right)\frac{\partial -d_{K0}}{\partial T} \\ -O\left(\infty\right)n\left(-d_{Ka}\right)\frac{\partial -d_{Ka}}{\partial T} \\ S\left(\infty\right)n\left(d_{Kb}\right)\frac{\partial d_{Kb}}{\partial T} \\ Ve^{-\delta T}n\left(d_{K1}\right)\frac{\partial d_{K1}}{\partial T} \end{array}\right] \\
&= n\left(d_{K0}\right)e^{-rT}\left[\begin{array}{c} -K\frac{\delta}{r} \\ -K\frac{b-1}{a(b-a)} \\ K\frac{a-1}{b(b-a)} \\ K \end{array}\right]\cdot\left[\begin{array}{c} \frac{\partial d_{K0}}{\partial T} \\ \frac{\partial d_{K0}}{\partial T}+\frac{1}{2\sqrt{T}}a\sigma \\ \frac{\partial d_{K0}}{\partial T}+\frac{1}{2\sqrt{T}}b\sigma \\ \frac{\partial d_{K0}}{\partial T}+\frac{1}{2\sqrt{T}}\sigma \end{array}\right] \\
&= n\left(d_{K0}\right)e^{-rT}\frac{\partial d_{K0}}{\partial T}K\left[-\frac{\delta}{r}-\frac{b-1}{a\left(b-a\right)}+\frac{a-1}{b\left(b-a\right)}+1=0\right] \\
&\quad +\frac{1}{2\sqrt{T}}\sigma n\left(d_{K0}\right)e^{-rT}K\left[\frac{-b+1+a-1+b-a}{b-a}=0\right] \\
&= 0!
\end{aligned}
$$

so the formula for the reversible is indeed the integral of the Black Scholes continuum of options!

$$
\begin{aligned}
1+\frac{a-1}{b\left(b-a\right)} &= \frac{\delta}{r}+\frac{b-1}{a\left(b-a\right)} \\
\frac{\delta}{r} &= 1+\frac{a-1}{b\left(b-a\right)}-\frac{b-1}{a\left(b-a\right)} \\
&= \frac{ab\left(b-a\right)+a\left(a-1\right)-b\left(b-1\right)}{ab\left(b-a\right)} \\
\frac{\delta}{r} &= \frac{1+ab-a-b}{ab}
\end{aligned}
$$

.5 Junk

suspend

.6 Costly reversion

If we wish to value an option which allows us to enjoy the higher of a variable or fixed asset return the situation can be represented either in current value or flow terms.

.6.1 Value representation

Suppose the project with stochastic value V and stochastic current cashflow δV can be activated or opened at any time and subsequently de–activated or closed at any time. Opening the project is done by investing $\overline{X}$ (i.e. V can be called for $\overline{X}$), gaining further rights to de–activate or shut the project later at any time retrieving $\underline{X}$ (i.e. V can be put back for $\underline{X} < \overline{X}$ and similarly shutting gains a right to subsequently re–open). When the former happens, the activation option together with the activation exercise amount are exchanged for the project value plus the de–activation option, when the latter happens the deactivation option and the current project value are exchanged for the deactivation proceeds plus the activation option.

.6.2 Flow representation

Alternatively the situation can be thought of in terms of flows. The reversible option gives its holder the right to enjoy the maximum of two flows; one stochastic δV and the other fixed rX subject to positive activation and de–activation costs.

Equivalently, using Dixit's notation (Dixit (1989) [4]), the opening and closing costs can be thought of as a premium k and discount l to a capitalised perpetuity cost rate rX on the base investment X where the risk free rate r is used as the opportunity cost of capital (Dixit uses $\frac{\omega}{\rho}$ for X; a perpetual flow of ω capitalised at a rate of ρ). This is equivalent to saying that the maximum of cashflows rX (on the base investment X) or δV (on the stochastic value

V) can be enjoyed subject to non–negative switching costs k, l[9].

$$\begin{array}{rl} \overline{X} = X + k & \geq \overline{X} \\ \underline{X} = X - l & \leq \underline{X} \\ \alpha \equiv \underline{X}/\overline{X} = \frac{X-l}{X+k} & \leq 1 \end{array}$$

The ratio α of the two exercise amounts will be used to determine the boundary conditions.

suspend

$$\begin{aligned} a &> 1, b < 0 \\ ab &= \frac{-r}{\frac{1}{2}\sigma^2}, \\ a + b &= -\frac{r - \delta - \frac{1}{2}\sigma^2}{\frac{1}{2}\sigma^2} \\ a + b - ab - 1 &= \frac{\delta}{\frac{1}{2}\sigma^2} \end{aligned}$$

[9] Note that we also require that $\underline{X} \geq 0$ or that $l \leq X$.