Modelling Real Options: A First Passage Time Approach

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Abstract

This paper introduces the first passage time approach to study optimal option exercise rule for geometric Brownian motion process to a boundary. I have derived analytical results on the first passage time probability, density and its expectation. The results on the first moment of the first passage time clarify some recent controversies on the sign of uncertainty on investment. The first passage time provides an alternative characterisation of optimal exercise rule. In addition, we establish a new framework for testing real option models. The approach is applicable to other stochastic modelling in finance and economics.

Key Words: first passage time; real options; option pricing

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1. Introduction

In an influential paper, Myers (1977) first introduced the concept of “real options” into financial economics literature. The concept did not gain currency until 1985 when Mason and Merton, Brennan and Schwartz (1985), Titman (1985), MacDonald and Siegel (1985) launched a framework for studying real options that are inherent in almost all economic and financial decisions. Reviews of this growing literature have been provided in Dixit and Pindyck (1994), Trigeorgis (1996) and Merton (1999). One of the central insight of the real options literature is that given the stochastic processes for the state variables, it is possible to employ modern option pricing technique or equivalent method to derive an optimal decision rule. Furthermore, by observing the evolution of the state variables, it is possible to implement the optimal decision rules, which typically involve a stopping boundary. Once the state variable reaches the critical boundary, action is taken and the process is either terminated or reflected depending on the nature of the problems. An important problem that has not received wide attention concerns when the state variable will reach the critical boundary. For example, in a typical real options model of investment, optimal investment rule stipulates investment when returns reaches a particular level. However no definite relation has emerged on the impact of uncertainty on investment. While real options models demonstrate that higher uncertainty increases and the optimal investment threshold, pre-real options models of Hartman (1972) and Abel

(1983) show that uncertainty increases investment. The same indeterminacy also arises from interest rate on investment in a model of Ingersoll and Ross (1992).[1]

This indeterminacy has led to some renewed controversy.[2] It has also led to some new extensions of the model. Zhao and Zilberman (1999) consider incomplete irreversibility. Rose (2000) obtains hump type relation between investment and interests rate. Using the so-called callable risk free rate for mortgage-backed securities, Berk (1999) proposes an unambiguous relation between the special rate and investment. As in the first generation models, Berk (1999) also uses a threshold to characterise optimal investment timing. There are unfortunately several drawbacks to this approach. First, as recognised by Berk, the reliance on the risk free mortgage rate assumes rational prepayment behaviour which may be unrealistic if mortgage-backed securities are not competitively traded or with substantial transactions costs. Secondly, the approach begs the whole issue of rational default and prepayment behaviour.[3] Lastly, it suffers from the same drawback from most of the real options literature, i.e., the lack of characterisation of the relationship between uncertainty and the expected time to optimal threshold. As Pattillo (1998) observes in an empirical paper, average investment in any time interval will depend on whether the optimal investment threshold is reached. Although uncertainty raises the optimal investment threshold, the time taken by the state variable to reach the optimal threshold may decrease and the net effect on investment may be indeterminate.

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[1] Abel and Eberly (1999) and Barruch *et al* (2000) discuss some of the theoretical and empirical issues in this growing literature.

[2] Caballero (1991), Pindyck (1993), Sakellaris (1994), Jaewoo and Shin (2000).

This paper presents an alternative approach to characterise the optimal option exercise boundary and clarifies some current debate on uncertainty and investment. The first passage time provides another measure of optimal option exercise timing.[4] The main contribution of this paper then consists of an explicit formulation of the first passage time for geometric Brownian motion to a horizontal boundary[5].

The paper is organised as follows. In section 2, we first present an informal discussion on the first passage time and then review the relevant literature. Section 3 derives analytic results on the first passage time for geometric Brownian motion to a horizontal boundary. Indications for potential applications to real options context of the first passage time approach are given in section 4, ranging from asset sale, entry and exit decisions, technological adoption, labour training, rational default and exchange rate realignment under a target zone regime. Section 5 concludes this paper and points out directions for future research. The Appendix contains the proof of the main result.

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[3] Kau and Keenan (1999).

[4] In the finance literature, the first passage time appeared first in Louis Bachelier (1900) for standard Brownian motion process to a fixed level.

[5] After obtaining the key results reported in this paper, it came to my attention that Brown, Geozmann and Ross (1995), Granadier (1996), Yaksick (1996), Zhou (1997, 2001), Bunch and Johnson (2001) had obtained related results independently.

2. Related literature

The first passage time refers to the amount of time before a random variable reaches certain specified level and is as ancient as gambling itself. It is still best illustrated by in the gambling context by the well-known gambler's ruin where the wager is able to decide when to stop and walk away from the gambling table. The gambler's stake at any point in time is a random variable and is ruined when his stake first becomes zero or some positive amount depending on the rules of the game.

Bachelier in his 1900 thesis solved the problem of the first passage time and epoch for standard Brownian motion. The Swedish actuary Philip Lundberg independently derived and proposed it for Poisson process in 1903. These all preceded Einstein's remarkable paper of 1905 on Brownian motion. It is notable that Lundberg's work has been followed and expanded considerable by the Swedish School led by Harold Cramer. In a series of papers dating from 1930s, Cramer proposed modelling the ruin of an insurance company as a first passage time problem.[6] The first passage time also appeared in the modern theory of option pricing. Black and Cox (1976) use it to study indenture provisions and reorganisation specifications. The so-called structural approach originated from Merton (1974), further developed by Brennan and Schwartz (1995) also use first passage time to characterise optimal default and bankruptcy boundaries. Jacka (1990) formulates the valuation of American put option as an optimal stopping problem and presents some comparative results on the first passage time. One that comes closest to and independent of my paper, is that of Yaksick (1996). Yaksick (1996) is motivated by exercise behaviour for perpetual call option

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that pays continuous dividends. Unlike this paper, Yaksick (1996) employs optional sampling theorem to derive the expectation of the first passage time.

There also some very brief remarks and even some fragmentary treatments of the first passage time problem in the real options literature. Venezia and Brenner (1979) treatment it in discrete time framework in a model of investment duration. Venezia and Brenner (1979) present an example to show that optimal duration, which is equivalent to the first passage to an optimal threshold, may not increases with higher uncertainty.[7] In a specific and continuous time framework, McDonald and Siegel (1986) study optimal investment timing and like numerous subsequent papers, prove and derive an optimal investment threshold. They even refer to the first passage time for Poisson process but have not presented any results on the first passage time for geometric Brownian motion processes used in their paper. In a more general setting, Brock, Rothchild and Stiglitz (1989) formulate capital theory as optimal stopping problems. Their focus, however, is on boundary conditions. Other authors who have mentioned first passage time include Capozza and Li (1994) in a model of land development, Mauer and Ott (1996) in a model of replacement investment. The first passage time is not the central concern of these papers with the possible exceptions of two other independent contributions.[8]

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[6] These are contributions are now collected in Cramer (1994).

[7] I thank Menachem Brenner for bringing this reference and that of Brock, *et al* (1989) to my attention after completing my own study.

[8] Grenadier (1996) used first passage time to characterise option exercise strategies in the real estate market. Brown, Geozmann and Ross (1995) examine the implications of Brownian motion to a lower absorbing boundary. Unlike this paper and that of Grenadier (1996), the absorbing boundary is imposed in Brown, *et al* (1995).

3. The first passage to a time independent boundary arising from a model of asset sale

The first known result on the first passage time was obtained by Louis Bachelor for standard Brownian motion to a fixed level in 1900 and has become a standard result in stochastic processes.[9] As Paul Samuelson (1965) points out, for economic problems, geometric Brownian motion is more appropriate because absolute Brownian motion may give rise to negative values with positive probability. In the rest of this section, we shall extend the analysis of first passage time to the case of geometric Brownian motion to a horizontal boundary that makes economic sense. I shall first point out the context in which the boundary arises and then formulate the first passage to this boundary. The boundary arises from a simple real options model that may be briefly summarised as follows.

First, let an infinitely durable asset generates profits according to a geometric Brownian process:

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[9] Bachelier's contribution was mentioned by Poncare who wrote a report on it. Poncare's report, not included in John Boness's English translation, is now available in English in Courtault, *et al* (2000). Bachelier's contributions and Poncare's report have been neglected since 1960s when interest in option pricing was revived. Neither Samuelson (1965) nor McKean (1965) included any discussion on the first passage time problem. Black and Cox (1976) seem the first to have exploited the first passage time in the option pricing literature.

$$dS = \mu S dt + \sigma S dz \qquad (1)$$

where S denotes profit, μ instantaneous drift rate and σ the variance rate, dz is a standard Wiener process. Second, assume the asset could be sold in the second hand market at a fixed resale price. Then under certain conditions, song and Gao (2000) demonstrate that there exist a threshold for (1) below which the asset should be sold:

$$S^* = \frac{2r}{\sigma^2 + 2r} R \qquad (2)$$

where R denotes resale price, r the risk free rate and S* is the optimal resale threshold. As we shall demonstrate in the remainder of this section, additional information can be extracted from the first passage time that is as important as the derivation of the threshold, the latter of which constitutes the sole task of most real options models.

Define $T(S^*)$ as the first passage time of S to S* : $T(S^*) = \inf \{t: S = S^*\}$. Using a lemma from Grimmett and Stirzaker (1991: 500), it is straightforward to establish that T is stopping time for the geometric Brownian motion process defined in (1). The main result of this paper is summarised in the following theorem.

Theorem

The expectation of the first passage time depends on actual drift and variance rates of the underlying stochastic process, the risk free rate, initial profit and the resale price.

More specifically, the expected value of the first passage time increases with the resale price, risk free rate, and the actual drift rate. It decreases with initial profit. However, the influence of variance rate on the expected value of the first passage time is indeterminate. Proof see Appendix. One immediate consequence of this theorem is the following corollary.

Corollary

If profit starts from the resale price, then the expectation of the first passage to the optimal threshold will depend only on interest, actual drift rate and the variance rate of the stochastic process. This follows from a simple substitution.

$$E(T) = \frac{\ln \frac{2r}{2r+\sigma^2}}{\mu - \frac{1}{2}\sigma^2} - \frac{1}{2} \times \frac{\sigma^2}{\left(\mu - \frac{1}{2}\sigma^2\right)^2}\left[1 - \left(\frac{2r}{2r+\sigma^2}\right)^{(1-\frac{2\mu}{\sigma^2})}\right]$$

Several remarks are in order.

First, in a discrete time setup, Venezia and Brenner (1979) show through an example that the expectation of the first passage time is not a monotonic function of the variance rate. Our continuous result confirms this.

Second, from the analytic results presented in Appendix, it is clear that even if under complete rationality and no agency problems, positive NPV projects may never be undertaken if the expectation of the first passage time is infinite because this means that the investment threshold will never be never reached. This is in contrast the results of Myers (1977). Under investment or no investment occur for different reasons. Under investment is driven by the conflicts of interests among equity holders, bond holders and the management. In our model, no investment is driven by option value consideration. Any parameters that give rise to infinite expectation result in no investment, as will be the case if the drift rate and variance rate is not in certain relations, i.e. when the denominator is zero.

Third, our results throw further insights into the relationship between uncertainty and investment. This may be clearly seen from a standard model of investment as in Mcdonald and Siegel (1986) when cost of investment is fixed. The we have the following expression for the investment threshold.

$$\pi^* = \frac{\beta}{\beta - 1} I$$

where π starred denotes return threshold, I investment and β>1 is a solution to a quadratic equation and depends on drift rate, variance rate and interest rate. Because no information is revealed as when the above threshold will be reached, there is nothing to pin down investment within one particular time interval. Our result at least tells us factors that affect the average time before the threshold is reached.

Forth, our result sheds further light on interest rate on expected time to the first passage to the threshold. This is in contrast to the model of Ingersoll and Ross (1992) in which interest impacts on both investment cost and returns such that its net effects on investment is indeterminate.

Fifth, the fact that the expectation of the first passage time depends on the actual drift rate rather than the risk-neutral drift come as a surprise given the well established result that option value does not depends on the drift rate. The intuition for this contrast is that the valuation problem and the first passage time to a specific boundary are entirely separate issues. Furthermore, the first passage time is more general and always refers to actual processes rather then risk neutral process. This dependence on actual drift has also been pointed out by Kau and Keenan (1999) in an important paper on default probability and default severity. Thus this appealing property enables us to implement option exercise strategy more easily, as for monitoring purposes considered in Clark (2000).

Lastly, our results are not subject to some recent criticisms of the real options literature caused by game-theoretic considerations. This is clearly illustrated by a recent paper by Grenadier (1996) in context of real estate market. Grenadier (1996) uses the first passage time to characterize the equilibria strategies under duopoly.

To end this section, we show that it is equally important to have some information on the probability of state variable never reaching the threshold because in this case no decision is ever to be made and the existence of optimal threshold means nothing. This probability maybe obtained by applying a result in Collin-Dufresne and Goldstein (2000). Plugging (2) and after substitution, we have the following proposition.

Proposition

The probability of option never being exercised depends on actual drift rate, the variance rate, exercise price and the risk free rate. It does not depend on the initial value of the state variable, however.

$$P(T > \infty) = 1 - e^{-\frac{2\mu S^*}{\sigma^2}} = 1 - e^{-\frac{4r\mu}{\sigma^2(\sigma^2+2r)}R} \quad (5)$$

Furthermore, this probability exhibits the following comparative statistics:

$$\frac{\partial P}{\partial S^*} > 0$$

$$\frac{\partial P}{\partial \mu} > 0$$

$$\frac{\partial P}{\partial \sigma^2} < 0$$

Thus higher exercise boundary increases the likelihood of the option not being exercised. The intuition is that it takes longer for stock price to fall below this critical level at which the option is exercised. A higher drift rate makes it more likely that stock price will not fall to the critical level either. However the effect of the variance rate works in the opposite direction: more volatile stock price increases the likelihood of reaching the optimal boundary, decreasing its not reaching the boundary.

4. Some applications

In this section, I briefly indicate some of the applications of the first passage time approach to a variety of problems in finance, economics, industrial organisation and international finance.

4.1.Entry and exit decisions under uncertainty

If we introduce stochastic resale price, the model could easily be converted into Dixit's entry and exit model under uncertainty (1989). Then operating profit will be constrained by two-sided reflecting barriers: the upper barrier being higher than operating cost and the lower barrier being lower than scrap value. It is then easy to calculate when entry will occur and when exit will take place on average in the sense of probability. Then it is straightforward to calculate how long the firm will stay in this optimal zone once it has entered it.

4.2.Asset sale and asset life

As remarked at the outset of this paper, I was initially motivated to investigate the problem of first passage time in the context of asset sale for transitional economies. Having derived the first moment of the first passage time, it is easy to derive the average life for any specific activity. As external circumstances change, the same asset may change hands and maybe employed for other purposes. The life of the asset continues. Using our approach we may calculate one circle of the asset life. It is possible to calculate successive lives of the asset once the underlying processes and its resale prices are known. Song and Gao (2000) have employed this approach to the case of asset sale with random resale price and obtained results similar to that of MacDonald and Siegel (1986). Song and Gao (2000) use the first passage time to calculate when the critical ratio will be reached.

4.3.When to cancel an operating lease?

Many operating lease contracts contain a cancellation clause at leasee's discretion. The valuation of this type of contract involves developing equilibrium rate and the optimal cancellation strategy. Gao and Song (2000) provide such a framework. In particular, they derive a threshold profit level for leasee and calculate when cancellation is likely to take place.

4.4. Technology adoption under uncertainty and exchange rate realignment

Bessen (2000) considers when to adopt a new technology when returns are uncertain. Collins (1992) studies exchange rate realignment under the Exchange Rate Mechanism for the period of 1989-1993. Both employed the notion of first passage time.

4.6. Land development and real estate market

Capozza and Li (1994), Grenadier (1996) presents real option models for the real estate market.   Capozza and Li (1994) allow land rent to evolve randomly and derive an optimal reservation rent above which development takes place. Capozza and Li (1994) use the moment generating function to obtain the expectation of the first passage time to the threshold. Grenadier (1996) presents a real options model with a game-theoretic aspect when investment opportunity is shared among economic agents using the first passage time to characterise development lag between the leader and its follower.   The paper by Grenadier (1996) shows that our approach is equally applicable to the situations in which strategic behaviours are involved despite the fact that some game-theoretic factors might change the basic insights of the real option models.

4.7. Rational default

In an important paper, Kau and Keenan (1999) moved closer to the interests of empirical research by providing the entire distribution of default's severity. In their setting default and prepayment occur at lower and higher housing prices. Since housing price follows a log normal process. Then it is straightforward to calculate the expected time to default and prepayment using our result. The same technique may be used to calculate other default boundaries such as that of Black and Cox (1976), Longstaff and Schwartz (1995), Briys and de Varenne (1997).

4.8. Training in labour market

Using the real options framework, Booth and Zoega (1999) study if quits by employees cause under-training by employers. Assuming aggregate productivity evolves as geometric Brownian motion, Booth and Zoega (1999) obtain a critical training threshold expressed in terms of other parameters of their model. It is easy to derive the average time in the sense of probability before an employer undertakes training.

5. Conclusion

This paper advances a new approach to characterise option exercise strategies. The new approach is based on the notion of the first passage time. I have presented analytic results on the first passage time of geometric Brownian motion to a time-independent endogenous boundary arising from a model of asset sale. This new approach does not suffer from the drawbacks of most real option models. The approach has wide applicability and this is indicated in this paper for some examples in economics. The framework is also applicable to financial option pricing and exercise as in Bunch and Johnson (2001). The analysis has been used to study first passage time for Ornstein-Uhlenbeck process by Leblanc *et al* (2000).

The threshold considered in this paper is independent of time and the underlying state variables. Although this is sufficient for many real option situations, it has limitations. Many option valuation and exercise problems involve more complicated boundaries. For instance, the pricing of finite American put option requires the evaluation of first passage tome to a boundary depending both on time and stock price as demonstrated recently by Bunch and Johnson (2000) with some success. For this and other applications, other thresholds may be appropriate.   Moerbeke (1976), Sato, *et al* (1987) and Durbin (1992) consider other boundaries.

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Appendix

The following lemmas are used for the proof.

Lemma 1: Harrison (1985: 13-4)

The probability distribution of the first passage time T is

$$P(T > t) = \phi(d_1) - e^{\{2\mu S^* / \sigma^2\}} \phi(d_2)$$
$$d_1 = \frac{S^* - \mu\ t}{\sigma\ \sqrt{t}}$$
$$d_2 = \frac{-S^* - \mu\ t}{\sigma\ \sqrt{t}}$$

$\phi(d)$ is the cumulative standard normal density.

Lemma 2: Ingersoll (1987)

The first passage time for geometric Brownian motion to a linear boundary has the following density:

$$p(T, S^*) = \frac{\left|\log^2 (S^* / S)\right|}{2\sqrt{\pi t^3}} \exp\{-\frac{\log^2 (S^* / S)}{4t}\}$$

Lemma 3: Expectation of the first passage time: Kannan (1979)

The proof proceeds in three steps. First, use lemma 1 and lemma 2 to obtain the distribution functions and the density function for the first passage time. Next, apply the definition for the expectation, with lemma 3 and equation two to obtain an analytic expression for the expected value of the first passage time as follows.

$$E(T)=\frac{\ln[\frac{2r}{2r+\sigma^2}\frac{R}{S}]}{\mu-\frac{1}{2}\sigma^2}-\frac{1}{2}\times\frac{\sigma^2}{\left(\mu-\frac{1}{2}\sigma^2\right)^2}\left[1-\left(\frac{2r}{2r+\sigma^2}\frac{R}{S}\right)^{(1-\frac{2\mu}{\sigma^2})}\right]$$

Finally partial differentiation complete the proof of the theorem.