The Option Value of Harvesting a Renewable Resource

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Abstract

We analyze multi-period harvest problems for a renewable resource under biological uncertainty when harvesting is size-dependant. First, we show that the decision to harvest can be modeled as a real option and we derive analytical expressions for the value of the resource stock and the mean time between harvests, with and without uncertainty. We then illustrate numerically how uncertainty affects the decision to harvest: when uncertainty increases from zero, the amount harvested and the stock biomass at harvest first increase, and then decrease because of the risk of extinction when uncertainty is high enough. This paper is a first step towards defining sustainable harvesting rules under uncertainty.

I. Introduction

In the analysis of harvesting decisions under uncertainty for renewable resources, it is important to distinguish between the case where a resource is harvested at fixed time intervals, and the case where it is harvested when it reaches a certain size. This distinction can be linked to the difference between age-dependent and size dependent stochastic growth. As emphasized by Clarke and Reed (1989) and Reed and Clark (1990), age-dependent growth models are more appropriate for husbanded biological assets (such as livestock or cultivated trees), because of the more limited impact of environmental factors. By contrast, for more wild resources, such as undisturbed forests, wildlife, and natural fish or shellfish populations, the size of a resource (its biomass) is likely to be better represented by density–dependent growth models.

While the analysis of density–dependent growth models in continuous time has received some attention (e.g., see Reed and Clark 1990; Reed 1993; Li 1998), the focus in the literature has mostly been on the single harvest case, probably because of the complexity of analyzing on-going harvests. An exception is Reed and Clark (1990), who derive optimal harvest rules for biological assets with stochastic growth when the price of the resource follows a geometric Brownian motion (GBM). Their work is an important step towards making stopping-rule methods more useful for practical resource issues. Their formulation, however, ignores harvest costs and assumes that the totality of the resource is harvested; it is thus inappropriate for wildlife or fisheries. In addition, Reed and Clark's assumption that the price of the resource follows a GBM independent from the stochastic evolution of the "size" of the resource is questionable; we may in fact expect resource prices to be mean-reverting when there is a possibility of entry by other producers or when substitutes are available (in the context of commodities, see for example Schwartz, 1997).

More recently, Li (1998) analyzes fishery harvesting in the context of a cooperative profit-maximizing fleet when the fish stock follows a GBM. His analysis, which relies on real options, makes a good case for the need to take into account uncertainty and irreversibility. His assumption that fish stock follows a GBM is, however, unrealistic, because it ignores environmental carrying capacity. In addition, Li does not explicitly take into account the impact of one harvest on future harvests.

In this paper, we revisit the problem of developing harvest rules for biological assets with size-dependent stochastic growth. We focus explicitly on the multi harvest problem and consider continuous time models where harvest takes place instantaneously. This is more realistic than assuming a continuous harvest when the harvesting period is small compared to the growing season. Continuous time models also permit the analysis of the impact on harvesting of biological and economic uncertainty using tools from the theory of real options (Dixit and Pindyck 1994). In addition, they typically offer more possibilities of obtaining analytical results compared with discrete time models.

We generalize Li (1998) by considering a more realistic class of diffusion processes for the resource biomass and by analyzing the impact of one harvest on all subsequent harvests. We also expand Reed and Clarke's results (1990) by allowing for partial harvest and by taking into account harvest costs (both fixed and variable costs). We do not, however, consider stochastic resource prices because of the difficulty of analyzing multi-dimensional stopping problems.

First, we draw a parallel between the management of a fishery and the Faustmann problem in forestry. The former is more complicated than the later because we need to choose both the stock level at which harvest takes place and the amount harvested. Second, we derive and interpret first order necessary conditions. Third, we show that the decision to harvest can be modeled as a real option, with and without uncertainty. We derive analytical expressions for the value of the resource stock and the mean time between harvests. We then illustrate numerically the complex impact of uncertainty on the harvesting decision: when uncertainty increases from zero, the amount harvested and the stock biomass at harvest first increase, and then decrease because of the risk of extinction when uncertainty is high enough.

This paper is organized as follows. In Section 2, we present a general model of multi-period harvesting with size-dependent stochastic growth. In Section 3, we analyze the deterministic case to obtain a benchmark for the impact of uncertainty. Section 4 deals with the case where the deterministic part of the process for the biomass stock follows Gompertz's law. Dimensionless parameters are identified and we illustrate our analytical results with a numerical application. Section 5 summarizes our main results and presents the implication of our analysis.

II. A Model of Multi-Period Harvesting Under Uncertainty

Consider a renewable natural resource whose biomass X varies stochastically due to natural factors (e.g., predators or availability of food). The manager of this resource needs to decide when harvest should take place and how much should be harvested. We assume that X varies randomly according to the diffusion process:

$$dX = Xg(X)dt + \sigma X dz \tag{1}$$

In the above, dz is an increment of a standard Wiener process (Dixit and Pindyck); and $g(z)$ is differentiable, decreasing, and $g(0)>0$. We thus have compensatory density-dependent growth. This class of processes has been widely used in population biology (for fisheries, see Clark 1990), and it includes popular models such as the logistic and Gompertz laws. As mentioned in the introduction, this formulation where age does not intervene specifically in modeling the biological growth process should be useful for developing harvesting rules for natural populations of fish or for untended forest stands (Reed and Clarke, 1990).

We assume that the harvest of X can take place instantly. We also suppose that the resource manager is risk neutral and wants to maximize the expected value of the stream of discounted rents from successive harvests:

$$V(X_0;\rho) = \underset{\{H_i, X_i^-\}_{i=1}^{+\infty}}{Max} \varepsilon\left(\sum_{n=1}^{+\infty} \pi(H_n, X_n^-)e^{-\rho T_n} \mid X_n^+ = X_n^- - H_n, n \geq 1, X(0) = X_0\right), \quad (2)$$

subject to Equation (1) for the evolution of X. In the above, ε is the expectation operator; X_n^- and X_n^+ are respectively the stocks of biomass just before and just after harvest n; for $n>1$, T_n is the random stopping time at which X reaches X_n^- for the first time since harvest $n-1$; T_1 is the random stopping time at which X reaches X_1^- for the first time given that $X(0)=X_0$; finally, H_n, $n\geq 1$, is the amount of biomass harvested during harvest n. $V(X_0;\rho)$ designates the value of the resource when its biomass is X_0 and when the resource manager's discount rate is ρ.[i]

As emphasized in Li (1998), it is fruitful to emphasize the parallel between harvesting and investment decisions. First, each is (at least partly) irreversible: the harvested stock cannot usually be returned to its habitat, nor can much of a bad investment usually be recovered. Second, each of these decisions has to be made under uncertainty; for harvesting decisions, uncertainty can be due to in or out migrations, random encroachment from outsiders, or more generally to unforeseen or unpredictable environmental changes. Finally, both the decision to invest and the decision to harvest can be delayed if the conditions are not right. The manager of a renewable natural resource can thus be seen as holding an option to harvest, which can be exercised if the conditions are right. Using concepts from the theory of real options, we thus know that, between two successive harvests, $V(X;\rho)$ verifies the Bellman equation (Dixit and Pindyck):

$$\rho V(X;\rho) = Xg(X)V_X(X;\rho) + \frac{\sigma^2 X^2}{2} V_{XX}(X;\rho). \qquad (3)$$

In financial terms, this equation equalizes the return on the resource asset (left side) with its expected capital gains (right side) during the time where it is optimal to wait for the stock of renewable resources to be replenished. Given our assumptions on the process followed by *X*, we should have $V(0)=0$ and $V'(0)>0$. $V(X_0)$ can thus also be interpreted as the net value of the option to harvest the resource when the resource stock is X_0. Moreover, when this option is exercised and only part of the biomass is harvested, the resource manager gets a similar option for the new value of the biomass.[ii]

For *H* given, we need the continuity and smooth-pasting conditions (Dixit and Pindyck) to solve for X^*, the biomass value at which the next harvest (of size *H*) should take place. These conditions can be written:[iii]

$$\begin{cases} V(X^*;\rho) = V(X^*-H;\rho) + \pi(H,X^*) \\ \dfrac{dV(X^*;\rho)}{dX} = \dfrac{dV(X^*-H;\rho)}{dX} + \dfrac{d\pi(H,X^*)}{dX} \end{cases} \qquad (4)$$

These conditions are usually interpreted separately, but when we regroup the option terms on the left hand-side and take the ratio of the second to the first equation in (4), we obtain an important necessary condition verified by X^*:

$$\frac{\dfrac{dV(X^*;\rho)}{dX} - \dfrac{dV(X^*-H;\rho)}{dX}}{V(X^*;\rho) - V(X^*-H;\rho)} = \frac{\dfrac{d\pi(H,X^*)}{dX}}{\pi(H,X^*)} \qquad (5)$$

Equation (5) states that, at the optimum biomass for harvesting *H*, the rate of change per unit of biomass of the net harvest (the right side of (5)) equals the rate of change per unit of biomass of the difference in value (before – after harvest) of the renewable resource asset (the left side of Equation (5)). It thus balances income from the current harvest with expected discounted income from future harvests.

The resource manager chooses the harvest size *H* that maximizes expected discounted profits, taking into account not only revenues from the next harvest but also the impact of the next harvest on all future harvests. Using dynamic programming, we can rewrite (2) as:

$$V(X;\rho) = \underset{H}{Max}\,\varepsilon\left(e^{-\rho T_1(X)}\left(\pi(H,X^*)+V(X^*-H;\rho)\right)\right) \qquad (6)$$

We can see clear similarities with the Faustmann problem in forestry (also see Reed and Clarke, 1990), although our problem here is more complex because not all the biomass is harvested and we allow for biological uncertainty. Harvest will thus be constant, but it will occur at random times determined by the first time that X reaches X^* after a harvest.[iv] Hence, the resource manager's objective can be simplified to:

$$V(X_0;\rho) = \underset{H}{Max}\left(\pi(H,X^*)\varepsilon\left(e^{-\rho T_1(X_0)}\right)\left[1+\sum_{n=1}^{+\infty}\varepsilon\left(e^{-n\rho T^*}\right)\right]\right) \qquad (7)$$

We use the Markov property of X (Karlin and Taylor) to separate the expected value of $e^{-\rho T_1(X_0)}$ and the expected value of $e^{-n\rho T^*}$ in (7). Figure 1 illustrates the biomass stock dynamics and the harvest decisions.

Let us now show that $\varepsilon\left(e^{-n\rho T^*}\right)=\dfrac{V(X^*-H;n\rho)}{V(X^*;n\rho)}$. Let $G_n(X_s)=\varepsilon\left(e^{-n\rho T_{ab}}\right)$, where T_{ab} is the minimum time at which the biomass stock reaches either $a>0$ or $b>a$, starting from X_s with

$a < X_s < b$. From Karlin and Taylor (pages 202-203), G_n verifies the following equation and boundary conditions:

$$\begin{cases} \frac{\sigma^2 X^2}{2} G''(X) + Xh(X)G'(X) - n\rho G(X) = 0 \\ G(a) = 1,\ G(b) = 1 \end{cases} \tag{8}$$

This is just our Bellman Equation (3) with $n\rho$ instead of ρ. Let $F_1(X)$ be a solution of (8) such that $F_1(0)=0$ and $F_1^{'}(0)>0$. We know (see Appendix A) that there exists another solution of (8), denoted $F_2(X)$, such that $\lim_{X \to 0+} F_2(X) = \infty$. $F_1(X)$ is thus proportional to $V(X,\rho)$. Since $F_2(X)$ is clearly independent from $F_1(X)$, we have:

$$\begin{cases} G(X) = A_1 F_1(X) + A_2 F_2(X) \\ A_1 = \frac{F_2(a) - F_2(X^*)}{F_1(X^*)F_2(a) - F_1(a)F_2(X^*)}, A_2 = \frac{F_1(X^*) - F_1(a)}{F_1(X^*)F_2(a) - F_1(a)F_2(X^*)}, \end{cases} \tag{9}$$

where we have substituted X^* for b. Let us now decrease a towards 0 so that T^* becomes the first time that X reaches X^*. Indeed, it can be shown that X cannot reach 0 starting from $X(0)>0$ if σ is small enough (if $\sigma^2 \le 2\kappa \ln(K)$; see Cox, Ingersoll, and Ross 1985). Since $F_1(a)=0$ and $\lim_{X \to 0+} F_2(X) = \infty$, we get that: $\lim_{a \to 0+} A_1 = \frac{1}{F_1(X^*)}$, $\lim_{a \to 0+} A_2 = 0$. Hence, $G_n(X_s) = \frac{V(X_s; n\rho)}{V(X^*; n\rho)}$ and:

$$\varepsilon\left(e^{-\rho T_1(X_0)}\right) = \frac{V(X_0;\rho)}{V(X^*;\rho)},\ \varepsilon\left(e^{-n\rho T^*}\right) = \frac{V(X^* - H; n\rho)}{V(X^*; n\rho)}, \tag{10}$$

since the biomass value just after harvest is X^*-H. From (10) we see that, when harvest depends on biological size, the discount factor for future harvest "n" can be interpreted as the ratio of the value of the discounted net present value of the biomass just after harvest divided by the discounted net present value of the biomass just before harvest, both with an interest rate equal to *n* times the resource manager 's interest rate.

The resource manager's objective can thus be rewritten:

$$V(X_0;\rho)=\underset{H}{Max}\left(\pi(H,X^*)\frac{V(X_0;\rho)}{V(X^*;\rho)}\left[1+\sum_{n=1}^{+\infty}\frac{V(X^*-H;n\rho)}{V(X^*;n\rho)}\right]\right) \qquad (11)$$

The corresponding necessary first conditions for *H* such that *0<H<X** are:

$$\frac{d\pi}{dH}\frac{1}{V(X^*;\rho)}\left[1+\sum_{n=1}^{+\infty}\frac{V(X^*-H;n\rho)}{V(X^*;n\rho)}\right]+\pi\frac{d}{dH}\left(\frac{1}{V(X^*;\rho)}\left[1+\sum_{n=1}^{+\infty}\frac{V(X^*-H;n\rho)}{V(X^*;n\rho)}\right]\right)=0. \qquad (12)$$

Equation (12) balances the present value of a marginal change in harvest size (1st term on the left hand-side), and the present value of a change in the interval between consecutive harvests (2nd term on the left hand-side), for all future harvests.

III. The Deterministic Case

It is useful to analyze the deterministic case if only to get a benchmark for the impact of uncertainty. In this case, in addition to finding the biomass level (*X**) at which harvest should take place, the resource manager must also decide how much should be harvested (*H* in our notation). Here, we allow for a more general growth function than in the stochastic case. We assume that the biomass changes according to:

$$dX = f(X)dt, \qquad (13)$$

We require *f* to be strictly concave, with $f(0)=0$, $f'(0)>0$ and $f(K)=0$, where *K* is the environmental carrying capacity.

As in the Faustmann problem, the resource manager faces a similar problem harvest after harvest. Hence, if X_0 designates the initial biomass, the objective of the resource manager is to find *H* and *X** that maximize $V(X_0;\rho)$:

$$V(X_0;\rho) = \underset{H,X^*}{Max}\left(e^{-\rho T_1}\pi(H,X^*)\sum_{n=0}^{+\infty} e^{-\rho nT} \right), \tag{14}$$

subject to (13), with $0<H<X^*$. In the above, T_1 is the time needed for the biomass to increase from X_0 to X (T_1=0 if X_0>X*) and T is the time between successive harvests. After simplification, the summation in (14) can be replaced with $\frac{1}{1-e^{-\rho T}}$. Moreover, integrating (13), we find:

$$T_1 = \int_{X_0}^{X} \frac{dy}{f(y)} \text{ and } T = \int_{X^*-H}^{X^*} \frac{dy}{f(y)}. \tag{15}$$

Since the approach described in the previous section is valid for all levels of uncertainty, it is also valid at the limit when uncertainty goes to zero. Let us prove, however, that the option approach is similar to the conventional approach.

In the conventional approach, the first order necessary condition with respect to X is given by:

$$\frac{\frac{\partial \pi}{\partial X}}{\pi} = \frac{\rho}{e^{\rho T}-1}\left[\frac{e^{\rho T}}{f(X)} - \frac{1}{f(X-H)} \right]. \tag{16}$$

We also know that $V(.)$ verifies the Bellman equation:

$$\rho V(X;\rho) = f(X)V'(X;\rho), \tag{17}$$

so that:

$$V(X;\rho) = \exp\left(\rho \int^{X} \frac{dy}{f(y)} \right) \tag{18}$$

A multiplicative constant is needed to fully determine (18).[v] As in the stochastic case, the discount term $e^{-\rho nT}$, n>0 integer, can be expressed as:

$$e^{-n\rho T} = \exp\left(-n\rho\int_{X-H}^{X}\frac{dy}{f(y)}\right) = \frac{\exp\left(-n\rho\int^{X}\frac{dy}{f(y)}\right)}{\exp\left(-n\rho\int^{X-H}\frac{dy}{f(y)}\right)} = \frac{V(X;n\rho)}{V(X-H;n\rho)}. \qquad (19)$$

From (18) and (15), we derive:

$$V(X;\rho) - V(X-H;\rho) = \exp\left(-\rho\int^{X-H}\frac{dy}{f(y)}\right)\left[e^{\rho T}-1\right], \qquad (20)$$

and after some algebra:

$$\frac{\partial V(X;\rho)}{\partial X} - \frac{\partial V(X-H;\rho)}{\partial X} = \rho\exp\left(\rho\int^{X-H}\frac{dy}{f(y)}\right)\left[\frac{e^{\rho T}}{f(X)} - \frac{1}{f(X-H)}\right]. \qquad (21)$$

Taking the ratio of (21) and (20), we see that the right hand-side of (16) equals the right hand-side of (5), so the first order condition with respect to X is equivalent to (16). We could then view X* as a function of H as in the stochastic case. Writing the first order condition with respect to H would give (12) again.

This completes the parallel between the deterministic and the stochastic cases. It shows that we can define an option term under certainty. The deterministic option term can be seen as a degenerate case of the stochastic option term. Its value is equal to the net present revenues obtained from harvesting the resource.

IV. A Special Case

Additional assumptions

To obtain further results, we need to be more specific about the evolution of stock of biomass and the profit function. In this section, we assume that the deterministic part of the biomass process follows Gompertz's law, i.e. that:

$$dX = rXLn\left(\frac{K}{X}\right)dt + \sigma Xdz, \qquad (22)$$

where $\sigma=0$ in the deterministic case. This is a convenient formulation because r, K, and σ can be estimated by maximum likelihood. In addition, it can be shown that $X(t)$ given $X(0)=X_0$ is lognormally distributed with parameters $\left[Ln\left(\frac{X_0}{K}\right)+\frac{\sigma^2}{2r}\right]e^{-rt}+LnK-\frac{\sigma^2}{2r}$ and $\frac{1-e^{-2rt}}{2r}\sigma^2$, which are respectively the mean and the variance of the normal distribution followed by $LnX(t)$ conditional on LnX_0.

To keep out formulation as simple as possible, we also suppose that:

- As in the Schaefer model, harvest H is proportional to both the stock of biomass X and harvest effort E:

$$E = \frac{H}{qX}. \qquad (23)$$

In the above, q is a positive constant.

- All of the harvested biomass can be sold at a fixed price p.

- Variable harvest costs are proportional to harvest effort, E, and there are non-zero fixed harvest costs, c_F. Net profits $\pi = p\,H - c_V E - c_F$ from harvesting a quantity H when the resource biomass has size X can thus be written:

$$\pi(H, X) = pH - \frac{c_V}{q} HX^{-1} - c_F . \qquad (24)$$

In (24), c_V is the per unit effort cost.

It is convenient to introduce the dimensionless variables:

$$z = \frac{X^*}{K} \text{ and } h = \frac{H}{K}, \qquad (25)$$

and the dimensionless parameters:

$$\nu = \frac{\rho}{r},\ \eta = \frac{\sigma^2}{r},\ d = \frac{c_V}{qpK},\ \text{and } f = \frac{c_F}{pK}. \qquad (26)$$

In the above, ν is the ratio of the discount rate to a measure of how fast X reverts to K; η characterizes the magnitude of uncertainty; d represents dimensionless variable costs; and f is dimensionless fixed costs.

The profit function then becomes:

$$\pi(h, z) = h - dhz^{-1} - f, \qquad (27)$$

if we assume, without loss of generality, that $pK=1$.

Deterministic case

With these assumptions, the deterministic option term (Equation (18)) is:

$$V(X;\rho) = c_0 \left(Ln\left(\frac{K}{X}\right)\right)^{-\nu}, \qquad (28)$$

where $c_0>0$ is a constant.[vi] From this expression, we see that the option term is well defined only for $X \geq K$. When $X \geq K$, the biomass is decreasing so it is best to never harvest, if costs are too high, or to harvest immediately. In the later case:

$$V(X;\rho) = V(X^* - H;\rho) + \pi(X - X^* + H, X). \qquad (29)$$

The time interval between consecutive harvests is:

$$T^* = \frac{1}{r} Ln\left(\frac{Ln(z-h)}{Ln(z)} \right). \qquad (30)$$

Moreover, the two necessary first order conditions ((5) and (12)) become:

$$\begin{cases} \dfrac{\nu}{z-h} \dfrac{(z-h)(-Ln(z))^{-\nu-1} - z(-Ln(z-h))^{-\nu-1}}{(-Ln(z))^{-\nu} - (-Ln(z-h))^{-\nu}} = \dfrac{hd}{h(z-d) - fz} \\ \dfrac{\nu h}{z-h} \dfrac{(-Ln(z-h))^{-\nu-1}}{(-Ln(z))^{-\nu} - (-Ln(z-h))^{-\nu}} = \dfrac{h(z-d)}{h(z-d) - fz} \end{cases} \qquad (31)$$

The dimensionless deterministic problem is thus completely determined by three dimensionless parameters: ν, *d*, and *f*.

Stochastic case

The stochastic case is slightly more complex. To solve (3), we perform the change of variables $Y = \frac{r}{\sigma^2}\left(\frac{\sigma^2}{2r} - Ln\left(\frac{K}{X} \right) \right)^2$ and obtain Kummer's equation (see Slater 1960). Since we want $V(0;\rho)=0$ and $V'(0;\rho)>0$, the solution is:

$$V(X;\rho) = V_0 \psi\left(\frac{\rho}{2r}, \frac{1}{2}, \frac{r}{\sigma^2}\left[Ln\left(\frac{K}{X} \right) - \frac{\sigma^2}{2r} \right]^2 \right), \qquad (32)$$

where ψ is the confluent hypergeometric function of the second kind and V_0 is a positive constant.[vii] It disappears in (5) so it is not needed to calculate X^* and *H*, which have to be obtained numerically. Equation (5) becomes:

$$\frac{U'(\nu,\eta,z) - U'(\nu,\eta,z-h)}{U(\nu,\eta,z) - U(\nu,\eta,z-h)} = \frac{1}{z} \frac{hd}{h(z-d) - fz}, \qquad (33)$$

where $U(\nu,\eta,z)=\psi\left(\frac{\nu}{2},\frac{1}{2},\frac{1}{\eta}\left(Ln(z)+\frac{\eta}{2}\right)^2\right)$, and $U'(\nu,\eta,z)=\frac{dU(\nu,\eta,z)}{dz}$. Using results in Slater (1960), it is easy to show that: $U'=\frac{dU}{dz}=-\frac{\nu}{\eta}\frac{Ln(z)+\frac{\eta}{2}}{z}\psi\left(\frac{\nu}{2}+1,\frac{3}{2},\frac{1}{\eta}\left(Ln(z)+\frac{\eta}{2}\right)^2\right)$.

With this notation, the resource manager's objective becomes:

$$\underset{h}{Max}\left\{\left(h-h\frac{d}{z}-f\right)\frac{U(\nu,\eta,z_0)}{U(\nu,\eta,z)}\left[1+\sum_{n=1}^{+\infty}\frac{U(n\nu,\eta,z-h)}{U(n\nu,\eta,z)}\right]\right\}, \qquad (34)$$

where z is an implicit function of h based on (33).

Since the time between successive harvests if stochastic, it is useful to obtain an expression for its expected time. From Karlin and Taylor (1981), we know that if *Y* follows a diffusion process with infinitesimal mean $\mu(y)$ and infinitesimal variance $\sigma^2(y)$, then for $a<y_0<b$, the expected time (denoted by $v(y_0)$) for *Y* to reach either b_L or $b_U > b_L$ starting from y_0 ($b_L < y_0 < b_U$) verifies:

$$\begin{cases}\frac{\sigma^2(y)}{2}v''(y)+\mu(y)v'(y)+1=0\\ v(b_L)=v(b_U)=0\end{cases} \qquad (35)$$

We solve this equation for $\sigma(y)=\sigma y$ and $\mu(y)=ry(Ln(K)-Ln(y))$, as in (22), with $0<b_L<b_U$. We then take the limit of $v(y)$, *y* given, as b_L goes to zero, and replace b_U with X^*. We find that the expected time for *X* to reach X^* starting from X^*-H is:

$$E(T^*)=\frac{1}{r}\int_{z_1}^{z_2}\left[2z\phi(1,\frac{3}{2},z^2)+\sqrt{\pi}e^{z^2}\right]dz, \qquad (36)$$

where $z_1 = \sqrt{\frac{1}{\eta}}\left(Ln(z-h)+\frac{\eta}{2}\right)$ and $z_2 = \sqrt{\frac{1}{\eta}}\left(Ln(z)+\frac{\eta}{2}\right)$.[viii] Details of the calculations are provided in Appendix B.

Numerical illustration.

For the numerical application, we consider the following values: d=0.2, f=0.2, ν=0.1, and η=0.1. Using Mathcad on a personal computer, we solve the system of equations (31) for the deterministic problem. For the stochastic problem, we solve equation (33) to get *z* as a value of *h*, and then maximize (34) using 20 terms in the summation. Results obtained to-date are preliminary and incomplete.[ix] They are shown in Table 1.

We see that as uncertainty increases (i.e., as η increases), the optimal harvest increases and then decreases. The same holds for the optimal biomass at which harvest should take place. In addition, as uncertainty increases, the time between consecutive harvests decreases, so harvest takes place more frequently. The explanation is that when uncertainty increases from zero, it allows the stock of biomass to move back more quickly to *K*, the biomass' environmental carrying capacity, initially without a risk of extinction.[x] When uncertainty is large enough, however, the stock of biomass can fall to zero thus depriving the resource manager from the revenues of all future harvests. The resource manager thus reduces harvest size *H*, but to compensate, (s)he also reduces X*, the stock of biomass at which harvest takes place. Since X* is reduced more than H, T* decreases.

These result differs from Li (1998) who found that an increase in uncertainty lead to a decrease in harvest because he assumed that the stock of biomass follows a GBM and ignored the impact of one harvest on all future harvests.

V. Conclusions

In this paper, we have focused on multi-period harvest problems for a renewable resource under biological uncertainty when harvesting is size-dependant and thus does not take place at fixed dates. Although this class of problem has attracted the attention of researchers because of its very important policy implications, much remains to be done in order to incorporate more real world features in harvesting decisions. This paper is a first step towards designing more realistic, sustainable harvest rules.

This paper makes several contributions and generalizes the existing literature. First, it shows how to incorporate biological uncertainty in a manageable harvesting rule. We simplify the objective function of the resource manager and derive first order necessary conditions. Second, we make a link with real options theory and show how the decision to harvest can be seen as a real option. For Gompertz's law, we obtain analytical expressions for the value of the stock of resource and the mean time between harvests. Third, our numerical application (incomplete) illustrates the complex impact of uncertainty on the decision to harvest.

Planned additions to this paper include analyzing how low levels of uncertainty affect the harvest decision, using perturbation techniques (e.g., see Nayfeh 1981) as recommended by Ludwig (1979), and performing a sensitivity analysis of the results with respect to the resource growth rate and the level of biological uncertainty.

Future research will consider uncertainty in the parameters describing the evolution of the stock of biomass as well as rules where the decision to harvest has to be taken well before harvest (as in fisheries with quota systems for example).

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Equation Section 1**Appendix A**

In this appendix, we prove the following result:

Lemma. If Equation (3) has a solution $F_1(X)$ such that $F_1(X)=0$ and $F\,'(0)>0$, then Equation (3) has another solution $F_2(X)$ such that $\lim_{X\to 0+} F\,(X)=\infty$.

To find another solution of (3), we consider $F_2(x)=f(x).F_1(x)+c.F_1(x)$, where c is a convenient constant introduced to simplify the expression of $F_2(x)$. Since $F_1\,'(0)>0$ and $F_1\,'$ is continuous, and since h is strictly decreasing with $h(0)>0$, we can find $x_0>0$ such that $\forall x\in(0,x_0),\ F_1(x)>0$ and $h(x_0)>0$. We solve for $f(x)$ using the Bellman equation (3). After simplifications, we obtain:

$$F_2(x)=c_2F_1(x)\int_x^{x_0}\frac{\exp\left(\frac{2}{\sigma^2}\int_z^{x_0}\frac{h(y)}{y}dy\right)}{F_1^2(z)}dz, \tag{A1}$$

where c_2 is a positive constant. We know that, since h is decreasing and $h(x_0)>0$, $\forall y\in(0,x_0),\ h(y)\geq h(x_0)$, so $\forall z\in(0,x_0)$:

$$\exp\left(\frac{2}{\sigma^2}\int_z^{x_0}\frac{h(y)}{y}dy\right)\geq\exp\left(\frac{2h(x_0)}{\sigma^2}\int_z^{x_0}\frac{dy}{y}\right)=x_0^{\frac{2h(x_0)}{\sigma^2}}z^{-\frac{2h(x_0)}{\sigma^2}} \tag{A2}$$

Moreover, a Taylor expansion with Lagrange remainder (e.g., see Sprecher 1970) of F_1 in the neighborhood of 0+ gives (we use that $F_1(0)=0$):

$$\forall z\in(0,x_0),\exists\xi\in(0,z),F_1(z)=zF_1^{'}(0)+z^2\frac{F_1^{"}(\zeta)}{2} \tag{A3}$$

Since $F_1^{"}$ is continuous on $[0,x_0]$, it is bounded: $\exists B>0,\forall\xi\in(0,x_0),\left|F_1^{"}(\zeta)\right|<B$. Then:

$$\forall z\in(0,x_0),F_1^2(z)<z^2\left(F_1^{'2}(0)+F_1^{'}(0)Bx_0+\frac{B^2}{4}x_0^2\right)\equiv z^2c_3. \tag{A4}$$

Using (A2) and (A4) into (A1), we find:

$$F_2(x) \geq \frac{c_2}{c_3} x_0^{\frac{2h(x_0)}{\sigma^2}} F_1(x) \int_x^{x_0} z^{-\frac{2h(x_0)}{\sigma^2}-2} dz = c_4 \left(x^{-\frac{2h(x_0)}{\sigma^2}-1} - x_0^{-\frac{2h(x_0)}{\sigma^2}-1} \right) F_1(x) \qquad \text{(A5)}$$

In the above, $c_4>0$ is just a constant. Since $F_1(x)=O(x)$ in the neighborhood of 0:

$$\lim_{x \to 0+} c_4 \left(x^{-\frac{2h(x_0)}{\sigma^2}-1} - x_0^{-\frac{2h(x_0)}{\sigma^2}-1} \right) F_1(x) = +\infty \qquad \text{(A6)}$$

and thus $\lim_{x \to 0+} F_2(x) = +\infty$, which proves our lemma.

Equation Section 2**Appendix B**

In this appendix, we derive the expression of the expected time between consecutive harvests. Combining Equations (22) and (35), we have to solve:

$$\begin{cases} \frac{\sigma^2 x^2}{2} v''(x) + rxLn\left(\frac{K}{x}\right) v'(x) + 1 = 0 \\ v(a) = v(b) = 0 \end{cases} \qquad \text{(B1)}$$

First, we change variables. Let $y = Lnx - LnK + \frac{\sigma^2}{2r}$, $f(y) = v(x)$. Then the equation in (B1) becomes:

$$\frac{\sigma^2}{2} f''(y) - ryf'(x) + 1 = 0 \qquad \text{(B2)}$$

To find a particular solution, we try a series expansion $f_P(y) = \sum_0^{+\infty} a_n y^n$ and get:

$$f_P(y) = \frac{-1}{r}\sum_{p=1}^{+\infty}\frac{1}{\left(\frac{3}{2}\right)_{p-1}(p-1)}\left(\frac{r}{\sigma^2}y^2\right)^p = \frac{-1}{r}\int_0^{\sqrt{\frac{r}{\sigma^2}}y} 2z\phi(1,\frac{3}{2},z^2)dz, \qquad \text{(B3)}$$

where ϕ(a,c,z) is the confluent hypergeometric function of the first kind (Slater 1960). A general solution of the homogeneous equation associated with (B2) is:

$$f_H(y) = C_0 \int_{Ln\left(\frac{a}{K}\right)+\frac{\sigma^2}{2r}}^{y} e^{\frac{r}{\sigma^2}z^2}dz + C_1, \qquad \text{(B4)}$$

where C_0 and C_1 are two constants. The sum of f_H and f_P gives a general solution of (B2) and thus also of (B1) by reverting to *v(x)*.

$$C_1 = \frac{-1}{r}\int_A^0 2z\phi(1,\frac{3}{2},z^2)dz,\ C_0 = \frac{1}{r}\sqrt{\frac{r}{\sigma^2}}\frac{\int_A^B 2z\phi(1,\frac{3}{2},z^2)dz}{\int_A^B e^{\frac{r}{\sigma^2}z^2}dz}, \qquad \text{(B5)}$$

where $A = \sqrt{\frac{r}{\sigma^2}}\left(Ln\left(\frac{a}{K}\right)+\frac{\sigma^2}{2r}\right)$, $B = \sqrt{\frac{r}{\sigma^2}}\left(Ln\left(\frac{b}{K}\right)+\frac{\sigma^2}{2r}\right)$. Taking the limit when a goes to 0+ and using that $\phi(a,c,z) = \frac{\Gamma(c)}{\Gamma(a)}e^z z^{a-c}\left\{1+O(\frac{1}{z})\right\}$, as $z \to +\infty$, we obtain Equation (36).

Table 1: Preliminary numerical results

η	H	z	rT*
0.00	0.760	0.820	2.66
0.12	0.786	0.935	*2.66*
0.14	0.802	0.932	2.65
0.16	0.807	0.923	2.64
0.18	0.800	0.912	2.58
0.20	0.779	0.904	2.48

Note: These results were generated with MathCad on a PC. Parameter values are d=0.2, f=0.2, ν=0.1, and η=0.1.

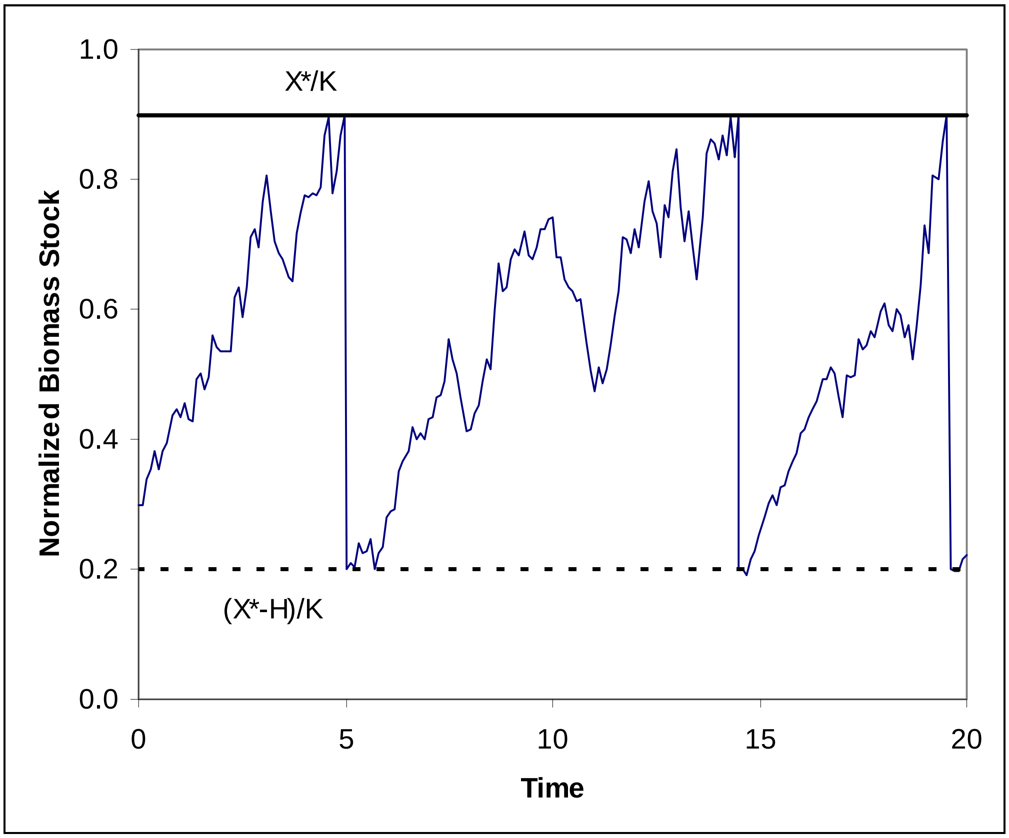

Figure 1: Illustration of the optimal stopping rule for the ongoing-rotation problem.

Note: Three harvests are shown. As soon as the stock of biomass reaches X*, harvest takes place and the stock of biomass goes down to X*-H. The stock of biomass evolves stochastically between 0 and X*. On Figure 1, the stock of biomass and the harvest are both normalized by K, which is the environmental carrying capacity of the renewable resource.

[i] The notation $V(X_0;r)$ emphasizes the value of the discount rate for a reason that will be made clear shortly.

[ii] Unless the resource cannot regenerate itself when it is harvested completely (e.g., the case of fisheries).

[iii] We omit writing *X*(H)* for simplicity but here *X** depends on *H*.

[iv] The first harvest may differ from all the others if the initial stock of biomass X_0 is greater than X^* in which case $T_1=0$; for simplicity, we assume that $X_0 \leq X^*$.

[v] We find it by jointly solving for it and X* using (4).

[vi] It can be determined jointly with *X** by solving (4).

[vii] From Slater (1960, Ψ(a,b,z)= $\Psi(a,b,z) = \dfrac{\Gamma(1-b)}{\Gamma(1+a-b)}\Phi(a,b,z) + \dfrac{\Gamma(b-1)}{\Gamma(a)} x^{1-b}\Phi(1+a-b, 2-b, z)$, where

$\Phi(a,b,z) = \sum_{n=0}^{+\infty} \dfrac{(a)_n z^n}{(b)_n n!},\ (c)_n = c(c+1)...(c+n-1)$ is the confluent hypergeometric function of the first kind and Γ is the Gamma function.

[viii] $\phi(a,c,y)$ is the confluent hypergeometric function of the first kind. See Slater, 1960.

[ix] The results shown are preliminary because ratios of discount terms are made up of two terms that increase with n. For n large enough, we should use asymptotic equivalents. However, I believe that N=20 gives a qualitatively correct indication of the impact of uncertainty on h, z, and T*.

[x] Feller, cited in Cox, Ingersoll, and Ross (1985) shows that *X* cannot reach the origin if $2\kappa\, ln(K) \geq \sigma^2$.