STRATEGIC DELAY IN A REAL OPTIONS MODEL OF R&D COMPETITION *

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5 May 1999

Abstract

When an irreversible decision is taken under uncertainty there is an option value of delay. If a small number of agents are in competition each one's ability to delay is restricted by the fear of preemption, seeming to undermine the real options approach. We present a model in which two firms may invest in competing research projects with uncertain returns. Two distinct types of equilibria arise. In a leader-follower equilibrium option values are reduced by competition and one firm invests strictly earlier than its rival. In a symmetric equilibrium investment is more delayed compared with the single-firm counterpart, in contrast with the expected outcome.

JEL classification numbers: C61, D81, L13, O32.
Keywords: real options, preemption, R&D, investment.

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* The author thanks Chris Harris, Dan Maldoom, Robert Pindyck, John Vickers and two doctoral examiners for their comments and suggestions. All errors remain the responsibility of the author. Financial support was provided by the Americas' Scholarship at Merton College, Oxford and the Economic and Social Research Council under grant number R00429524353.

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1 Introduction

When a firm has the opportunity to invest under conditions of uncertainty and irreversibility there is an option value of delay. By analogy with a financial call option, it is optimal to delay exercising the option, or proceeding with the investment, even when it would be profitable to do so in the hope of gaining a higher payoff in the future. This insight, first applied to the analysis of natural resource extraction and real estate markets by Brennan and Schwartz (1985) and Titman (1985) respectively, improves upon the traditional NPV approach to investment by allowing the value of delay and the importance of flexibility to be incorporated explicitly into the assessment.

Real world investment opportunities, unlike financial options, are rarely held by a single firm in isolation. In many situations an agent's ability to hold the option is constrained by the possibility that another agent may exercise it instead. In a few instances a legal right such as an oil lease or patent gives a single agent an equivalent position to the holder of a financial option. Occasionally a firm has such a strong market position, such as a natural monopoly, that its investment opportunities are *de facto* proprietory. However, in most industries some degree of competition exists and the option to invest cannot be held indefinitely.

The existence of competition for the underlying asset tends to weaken option values, in some cases eliminating them altogether. When a small number of agents are in competition with an advantage to the first mover, each agent's ability to delay is undermined by the fear of preemption. Consider the case of an option where two agents have the power to exercise and the first to do so receives the underlying asset in full, leaving the second mover empty-handed. Each firm would like to exercise the option just before its rival, giving rise to discontinuous Bertrand-style reaction functions. With symmetric firms, option values are entirely eliminated and the option will be exercised as soon at the payoff from doing so becomes marginally positive. Under such circumstances real options become irrelevant and the traditional NPV rule resurfaces as the appropriate method of investment appraisal.

In this paper we develop a model of irreversible investment in which two firms have the opportunity to invest in competing research projects. Under a winner-takes-all patent system the first to invent gains monopoly profits in the relevant product market, while the runner-up receives nothing. There is both technological and economic uncertainty. Discovery occurs randomly, while the value of the patent received by the successful inventor evolves according to a stochastic process. Thus the model becomes a stochastic stopping time game, in which a firm's optimal stopping time can be formulated in terms of a critical value or 'trigger point' for the underlying stochastic process such that the firm invests at the first time that this point is reached.

Focusing on Markov perfect equilibria in pure strategies, the outcome falls into one of two regimes depending upon parameter values. In one case the outcome is an asymmetric leader-follower equilibrium in which one firm invests strictly earlier than the other. In the second there is a multiplicity of equilibria, including a range of symmetric equilibria in which both firms invest at the same trigger point. The Pareto-dominant equilibrium coincides with the cooperative outcome and entails greater delay than the single-firm counterpart. Thus, in contrast to the presumption that strategic competition undermines option values and reduces delay, a two-firm competitive equilibrium may involve more delay than the single-firm case.

The intuition behind this result can be explained as follows. When one firm invests in research its rival's option to delay is reduced in value since there is some probability that the discovery will be made before this firm invests. In a non-cooperative framework each firm ignores the destruction of its rival's option values. Taken on its own, this business-stealing effect would induce earlier investment. However, forward-looking firms take account of their rivals' reactions. Over a range of patent values investment by one firm induces its rival to invest at once, resulting in a surge of research activity and a race for the patent. Anticipating this reaction, a firm may choose to delay investment to a greater extent than in the absence of competition. In effect, an investing firm chooses the time at which the patent race will begin and it is better for both firms if this is delayed until the jointly optimal investment point is reached. A good analogy is the behaviour of contestants in a long-distance race, who typically remain in a pack proceeding at a moderate pace for most of the distance, until near the end when someone attempts to break away and the sprint for the finish begins.

The model has the following implications. The finding that rivalry between two firms may increase rather than reduce investment delays runs contrary to the usual presumption that the fear of preemption undermines option values and speeds up investment. Thus the existence of strategic competition between firms does not necessarily undermine the relevance of the real options approach. The model also provides an alternative explanation to Choi (1991) for the observation of sudden waves of research activity ending a period of stagnation: once one firm exercises its option to invest the value of delay to other firms is reduced, inducing them to follow suit. The finding of two distinct, and rather different, types of equilibria suggests that the pattern of investment in an industry may be extremely sensitive to prevailing conditions. Factors that raise the cost of preemption lead to the destruction of option values, inducing leader-follower behaviour, but greater uncertainty may raise the value of delay to such an extent that the jointly optimal outcome becomes achievable.

The paper brings together two separate strands of literature. Real options models have been used to explain delay and hysteresis arising in a wide range of contexts, but these have generally been set in a monopolistic or perfectly competitive framework.

McDonald and Siegel (1986) and Pindyck (1988) consider irreversible investment opportunities available to a single firm. Dixit (1989a, 1991) considers product market entry and exit in monopolistic and perfectly competitive settings respectively. Applying these principles in an international setting Dixit (1989b) explains hysteresis in a country's trade balance. Bentolila and Bertola (1990) apply similar ideas to labour hiring and firing, thus explaining hysteresis in unemployment.

The second strand of literature concerns timing games of entry or exit in a deterministic setting. Timing games are straightforward examples of stopping time games where the underlying process is simply time itself. Papers analysing preemption games include Fudenberg et al (1983) and Fudenberg and Tirole (1985), while wars of attrition have been modelled by Fudenberg and Tirole (1986) and Ghemawat and Nalebuff (1985).

Existing literature combining real options with strategic interactions is relatively limited. In a two-player game in which each player's exercise cost is private information, Lambrecht and Perraudin (1997) find intermediate outcomes that lie between the fully-optimising and no-uncertainty cases. Smets (1991; summarised in Dixit and Pindyck 1994, pp. 309-314), examines irreversible market entry in a duopoly facing stochastic demand. Non-cooperative behaviour results in an asymmetric leader-follower equilibrium. When the leadership role is exogenously pre-assigned so that the follower is unable to invest until after the designated leader has done so, the cooperative outcome may then be attained. Grenadier (1996) also considers the strategic exercise of options, in this case applied to the real estate market.

The paper is structured as follows. Details of the model are described in section 2. We start by considering the optimisation problem of a single firm facing no actual or potential rivalry in section 3, while section 4 describes the optimal joint-investment rule when two firms coordinate their behaviour. The non-cooperative equilibrium for the two-player case is then found by solving the model backwards, starting with the follower's optimisation problem in section 5. After solving for the value of becoming the leader in section 6, section 7 describes the equilibria of the two-player game. Numerical examples are presented in section 8. Section 9 concludes.

2 The model

Two risk-neutral firms, $i = 1, 2$, have the opportunity to invest in competing research projects. Research is directly competitive: the firms compete for the same patent and successful innovation by one entirely eliminates all potential profits for the other. The firms face both technological and economic uncertainty. Discovery by an active firm is a Poisson arrival. Meanwhile, the value of the patent received by the successful inventor evolves stochastically over time.[1] The decision to invest in a research project is assumed to be irreversible.

The value of the patent, π, evolves exogenously according to a geometric Brownian motion (GBM) with drift given by the following expression

$$d\pi_t = \mu\pi_t dt + \sigma\pi_t dW \tag{1}$$

where $\mu \in [0, r)$ is the drift parameter, measuring the expected growth rate of π,[2] $\sigma > 0$ is the instantaneous standard deviation or volatility parameter, and dW is the increment of a standard Wiener process, $dW \sim \mathrm{N}(0, dt)$.

Note that geometric Brownian motion is a Markov process with continuous sample paths. The probability distribution for the value of the process at any future date depends only on its own current value, being unaffected either by past values of the process or by any other current information. Thus all that is needed to make a best estimate of the future value of the process is its own current value, along with the parameter values μ and σ.

Each firm has the opportunity to invest in its own research project. When a firm invests it pays a set-up cost $K_i > 0$ at the start of the project and an on-going flow cost of $C_i \geq 0$ per unit time thereafter. The flow cost is incurred for as long as research activity continues, until a breakthrough is achieved by either firm. Following investment, discovery takes place randomly according to a Poisson distribution with parameter (or hazard rate) $h_i > 0$. Thus when firm i acts alone its conditional probability of making the breakthrough in a short time interval of length dt, given that it has not done so before this time, is $h_i dt$ and the density function for the duration of research is $h_i e^{-h_i t}$. The probabilities of discovery by each of the firms are stochastically independent. Thus when both firms engage in research the density function for discovery by firm i is given by $h_i e^{-(h_i + h_j)t}$.

With uncertainty in the research technology the danger of preemption by the rival is less severe than in the case of deterministic discovery. In effect, the hazard rate drives a wedge between the decision to invest in research and the outcome of this project, allowing option values to be preserved to some extent. It is implicitly assumed that a firm can react 'instantaneously' to its rival's action, thus a situation in which one firm copies its rival at once entails a negligible expected loss to the follower. However, this concept is not unproblematic in a continuous time setting, since 'the time immediately following *t*' is then undefined. Simon and Stinchcombe (1989) provide a framework for specifying pure strategies in continuous time that conform as closely as possible to the discrete-time analogue and resolve difficulties such as this.

The possible states of each firm are denoted $\theta_i \in \{0,1\}$ for the idle and active states respectively. There is a common risk-free interest rate *r*. All parameter values are common knowledge. We shall focus on the symmetric case in which $h_i = h$, $C_i = C$ and $K_i = K$ for *i* = 1, 2; however, in most derivations we continue to use the more general notation to avoid confusion between a firm's own and its rival's parameters.

The following assumptions are made:

Assumption 1. If $\theta_i(\tau) = 1$ then $\theta_i(t) = 1 \ \forall t \geq \tau$.

Assumption 2. $E\left(\int_0^\infty e^{-(r+h)t} \left(h\pi_t - C \right) dt \right) - K < 0$.

Assumption 1 formalises the irreversibility of investment: if firm *i* has invested by date τ, it then remains active at all dates subsequent to τ. Assumption 2 states that the initial value of the patent is sufficiently low that the expected return from immediate investment is negative, thus ensuring that immediate investment is not worthwhile.

We now consider the actions available to firm *i* at time *t*. If *i* has not commenced research at any time $\tau < t$, its action set is $A_i(t)$ = {invest, don't invest}. If, on the other hand, *i* has invested at some $\tau < t$, then $A_i(t)$ is the null action 'don't move.' Thus, the firm faces a control problem in which its only choice is when to choose the action 'stop' – or rather in this case to commence research. After taking this action the firm can make no further moves to influence the outcome of the game.

In a duopoly setting the optimal control problem becomes a stopping time game. These games have been analysed by Dutta and Rustichini (1991). In a stopping time game each player has an irreversible action such that, following this action by one or more players, expected payoffs in the subsequent subgame are fixed. Dutta and Rustichini's formulation allows for the possibility that the stochastic process continues to evolve after the leader's action and that the follower still has a move to make, as is the case in this game. The stopping time game is described by the stochastic process (1) and the payoff functions of the leader and follower, which will be derived in sections 5 and 6 respectively.

At time $t \geq 0$, the history of the game has two components: the sample path of the state variable π and the actions of the two firms. With irreversible investment the history of play in the game at *t* is summarised by the fact that the game is still continuing at *t* (i.e. $\theta_i = 0$ for *i* = 1, 2). However, the history of the state variable is more complex since its current value could have been preceded by any one of a huge number of possible paths.

Firms are assumed to use stationary Markovian strategies. A stationary Markovian strategy consists of actions that depend on only the current state, and where the strategy formulation itself does not vary with time. Since π follows a Markov process, Markovian strategies incorporate all payoff-relevant factors in this game. Furthermore, if one player uses a Markovian strategy then its rival has a best response that is Markovian as well. Hence, a Markovian equilibrium remains an equilibrium when history-dependent strategies are also permitted, although other non-Markovian equilibria may then also exist. For further explanation see Maskin and Tirole (1988) and Fudenberg and Tirole (1991, chapter 13). With the Markovian restriction a player's strategy is a stopping rule specifying a critical value or 'trigger point' for the exogenous variable π at which the firm invests, depending on the state of the rival firm.[3]

As usual for dynamic games, the game is solved backwards using subgame perfection. Combining this with the assumption that players use stationary Markovian strategies, we seek Markov perfect equilibria (MPE) in pure strategies.[4] After one firm has invested the game is effectively over: the subgame from this point onwards is a single-agent optimisation problem for the remaining idle firm. This maximisation problem can be solved straightforwardly and the solution then used to solve subgames in which neither firm has yet invested in research.

3 Optimal investment timing for a single firm

As a comparative case, we start by deriving the optimal stopping time for a single firm which makes its investment decision unilaterally in the absence of competition. This can be found by solving the following stochastic optimal stopping problem

$$V = \max_{T} E\left\{ e^{-rT} \left(\int_{T}^{\infty} e^{-(r+h)t} \left(h\pi_t - C \right) dt - K \right) \right\} \tag{2}$$

where E denotes the expectation, T is the unknown future stopping time at which the investment is made, π_t is the value of the prize π at time t and other parameters are as given above. Note that the exponential term within the integral takes account of the active firm's survival probability e^{-ht} in addition to the discount factor e^{-rt}.

Prior to investment the firm holds the opportunity to invest. It has no cashflows but may experience a capital gain or loss on the value of its option. Hence, in the continuation region (values of π for which it is not yet optimal to invest) the Bellman equation for the value of the investment opportunity $V_0(\pi)$ is given by

$$rV_0 dt = E\left(dV_0\right). \tag{3}$$

Expanding dV_0 using Itô's lemma we can write

$$dV_0 = V_0'(\pi)\,d\pi + \frac{1}{2}V_0''(\pi)(d\pi)^2 .$$

Substituting from (1) and noting that $E(dW) = 0$, we can write

$$E(dV_0) = \mu\pi V_0'(\pi)\,dt + \frac{1}{2}\sigma^2\pi^2 V_0''(\pi)\,dt .$$

Thus the Bellman equation (3) gives rise to the following second-order differential equation

$$\frac{1}{2}\sigma^2\pi^2 V_0''(\pi) + \mu\pi V_0'(\pi) - rV_0 = 0 . \qquad (4)$$

From (1) it can be seen that if π ever goes to zero it then stays there forever. Therefore the option to invest has no value when $\pi = 0$ and $V_0(\pi)$ must satisfy the following boundary condition

$$V_0(0) = 0 . \qquad (5)$$

Solving the differential equation (4) subject to the boundary condition (5) the following solution for the value of the option to invest in research is obtained

$$V_0(\pi) = B_0\pi^{\beta_0} \qquad (6)$$

where $B_0 \geq 0$ is a constant whose value is yet to be determined,

and β_0 is the positive root of the characteristic equation $\varepsilon^2 - \left(1 - \frac{2\mu}{\sigma^2}\right)\varepsilon - \frac{2r}{\sigma^2} = 0$,

$$\beta_0 = \frac{1}{2}\left\{1 - \frac{2\mu}{\sigma^2} + \sqrt{\left(1 - \frac{2\mu}{\sigma^2}\right)^2 + \frac{8r}{\sigma^2}}\right\} > 1.$$

Next we consider the value of the firm in the stopping region (values of π for which is it optimal to undertake the investment at once). Since investment is irreversible the value of the firm in the stopping region $V_1(\pi)$ is given by the expected value alone with no option value terms. Recalling that discovery is a Poisson arrival the survival probability at t for a firm that invests at date 0 (i.e. the probability that discovery has *not* been achieved by date t) is given by the negative exponential distribution e^{-ht}. Thus the expected value of the active project when the current value of the stochastic process is π_t is given by

$$V_1(\pi_t) = E\left(\int_t^\infty e^{-(r+h)\tau}(h\pi_\tau - C)\,d\tau\right). \tag{7}$$

Since π is expected to grow at rate μ we can write

$$V_1(\pi) = \int_t^\infty e^{-(r+h)\tau}\left(h\pi e^{\mu\tau} - C\right)d\tau$$

which yields the following expression for the expected value of the project

$$V_1(\pi) = \frac{h\pi}{r+h-\mu} - \frac{C}{r+h}. \tag{8}$$

Note that the hazard rate h enters the denominators of the terms in this expression in the form of an 'augmented discount rate,' $r+h$. This result is found in many models involving Poisson arrival functions; for other examples of this characteristic in the context of R&D see, *inter alia*, Loury (1979), Dasgupta and Stiglitz (1980), Lee and Wilde (1980) and Dixit (1988).

In the continuation region we have the Bellman equation

$$\frac{1}{2}\sigma^2\pi^2 V_0''(\pi) + \mu\pi V_0'(\pi) - rV_0 = 0;$$

and an inequality reflecting the fact that the value of waiting exceeds the value obtained by investing immediately

$$V_0(\pi) \geq V_1(\pi) - K.$$

In the stopping region we have the inequality

$$\frac{1}{2}\sigma^2\pi^2 V_0'' + \mu\pi V_0' - rV_0 \leq 0$$

and an equality known as the 'value-matching condition'

$$V_0(\pi) = V_1(\pi) - K. \tag{9}$$

The boundary between the continuation region and the stopping region is given by a critical value of the stochastic process, or trigger point, π^* such that continued delay is optimal for $\pi < \pi^*$ and immediate investment is optimal for $\pi \geq \pi^*$. The optimal stopping time T^* is then defined as the first time that the stochastic process π hits the interval $[\pi^*, \infty)$. By arbitrage, the critical value π^* must satisfy the value-matching condition (9). Optimality requires a second condition, known as 'smooth-pasting,' to be satisfied. This condition requires the value functions $V_0(\pi)$ and $V_1(\pi)$ to meet smoothly at π^* with equal first derivatives

$$V_0'(\pi^*) = V_1'(\pi^*). \tag{10}$$

If smooth-pasting were violated and instead a kink arose at π^*, a deviation from the supposedly optimal policy would raise the firm's expected payoff. By delaying for a small interval of time after the stochastic process first reached π^*, the next step $d\pi$ could be observed. If the kink were convex, the firm would obtain a higher expected payoff by entering if and only if π has moved (strictly) above π^*, since an average of points on either side of the kink give it a higher expected value than the kink itself. If the kink were concave, on the other hand, second order conditions would be violated. Continuation along the initial value function would yield a higher payoff than switching to the alternative function and switching at π^* could not be optimal. More detailed explanation of this condition can be found in appendix C of chapter four in Dixit and Pindyck (1994).

Substituting expressions for the value functions from (6) and (8), the value-matching and smooth-pasting conditions for the single-firm optimisation problem can be written as follows

$$B_0(\pi^*)^{\beta_0} = \frac{h\pi^*}{r+h-\mu} - \frac{C}{r+h} - K \,;$$

$$\beta_0 B_0(\pi^*)^{\beta_0 - 1} = \frac{h}{r+h-\mu}.$$

The critical value π^* and the unknown coefficient B_0 are determined uniquely by these conditions. Denoting the unilateral trigger point for a single firm by π_U the following expressions are obtained

$$\pi_U = \frac{\beta_0}{(\beta_0 - 1)}\left(\frac{C}{r+h} + K\right)\frac{(r+h-\mu)}{h}; \tag{11}$$

and

$$B_0 = \frac{h\pi_U^{1-\beta_0}}{(r+h-\mu)\beta_0}. \tag{12}$$

Thus the optimal stopping time at which the single firm invests, T_U, can be defined as follows

$$T_U = \inf\{t \geq 0 : \pi \geq \pi_U\}. \tag{13}$$

Considering briefly the properties of this trigger value, we see that as economic uncertainty is eliminated (i.e. as $\sigma \to 0$), β_0 approaches r/μ and the optimal stopping point becomes the NPV breakeven level given by

$$\pi_B = \frac{r}{(r-\mu)}\left(\frac{C}{r+h} + K\right)\frac{(r+h-\mu)}{h}. \tag{14}$$

As economic uncertainty σ is increased, β_0 falls (in the limit as $\sigma \to \infty$, $\beta_0 \to 1$) raising the critical value π_U above π_B and increasing the optimal stopping time T_U. Thus greater uncertainty over the value of the patent delays investment. This finding is similar to results found in the papers by McDonald and Siegel (1986), Pindyck (1988), Dixit (1989) and others.

Note that

$$\lim_{h\to\infty} \pi_U = \frac{\beta_0}{(\beta_0 - 1)} K.$$

This limiting value is the trigger point for an irreversible investment opportunity with constant investment cost *K* derived by McDonald and Siegel (1986), as presented in Dixit and Pindyck (1994, p142). This result is unsurprising: as *h* becomes very large, discovery occurs (almost) immediately following investment and the model collapses to the case with no technological uncertainty in which the return to investment is obtained as soon as investment takes place.

4 The optimal joint investment rule

We now consider the value of each firm when both continue to delay and then invest at the same time. Then, on the assumption that the firms can commit to a joint investment time, the optimal stopping rule for the coordinated case is derived.

We start by considering the case where both firms adopt the same stopping rule, each investing at some arbitrary trigger point π_J. Prior to investment the value of each firm satisfies the Bellman equation (3), as in the case of a single firm. Using the notation $V_{\theta_i,\theta_j}(\pi)$ to denote the value of firm *i* when its own and its rival's states are θ_i and θ_j respectively and the current value of the patent is π, the value of each firm in the continuation region can be derived as before to yield

$$V_{0,0}(\pi) = B_J \pi^{\beta_0} \tag{15}$$

where $B_J \geq 0$ is a constant whose value is yet to be determined and β_0 is as defined above.

Following investment by both firms, the value of firm *i* is the expected value of its investment project, taking into account the possibility of prior discovery by its rival

$$V_{1,1}(\pi) = E\left(\int_t^\infty e^{-(r+h_i+h_j)\tau}(h_i \pi_\tau - C_i)\, d\tau\right).$$

Solving as before we obtain

$$V_{1,1}(\pi) = \frac{h_i \pi}{r + h_i + h_j - \mu} - \frac{C_i}{r + h_i + h_j}\,. \tag{16}$$

At the (joint) trigger point π_J the following value-matching condition must be satisfied

$$V_{0,0}(\pi_J) = V_{1,1}(\pi_J) - K_i\,.$$

Note that in the absence of coordination to achieve the jointly optimal outcome there is no corresponding smooth-pasting condition. Solving for the unknown constant B_J and imposing symmetry yields

$$B_J = \pi_J^{-\beta_0}\left(\frac{h\pi}{r+2h-\mu} - \frac{C}{r+2h} - K\right). \tag{17}$$

Thus the value of the firm prior to investment at any arbitrary joint investment point π_J can be specified.

$$V_J\left(\pi;\pi_J\right)=B_J\pi^{\beta_0}. \tag{18}$$

We now derive the optimal investment rule for the coordinated case in which the two firms commit to a joint investment time. The optimal stopping problem in this case is to find a joint investment time *T* satisfying

$$V=\max_T E\left\{e^{-rT}\left(\int_T^\infty e^{-(r+h_i+h_j)t}\left\{(h_i+h_j)\pi_t-(C_i+C_j)\right\}dt-(K_i+K_j)\right)\right\}. \tag{19}$$

The optimal joint investment trigger is denoted π_C. In this case, unlike that of an arbitrary trigger π_J described above, there is a smooth-pasting condition to ensure optimality of the derived trigger point π_C. The problem can be solved using the method set out in section 3 to yield (again imposing symmetry)

$$\pi_C=\frac{\beta_0}{(\beta_0-1)}\left(\frac{C}{r+2h}+K\right)\frac{(r+2h-\mu)}{h}. \tag{20}$$

The value of each firm in the continuation region is given by

$$V_C=B_C\pi^{\beta_0} \tag{21}$$

where $B_C=\dfrac{h\pi_C^{1-\beta_0}}{(r+2h-\mu)\beta_0}$.

The optimal investment time in the coordinated case, T_C, can thus be defined as

$$T_C=\inf\left\{t\geq 0:\pi\geq\pi_C\right\}. \tag{22}$$

Comparing the coordinated trigger point (20) with (11) for the single firm, it can be seen that $\pi_C>\pi_U$. Given that the initial value π_0 is sufficiently low that neither firm wishes to invest at once, the ranking of trigger points entails that $T_C>T_U$. Thus when two identical firms can commit to a joint investment time, investment takes place strictly later than the case in which a single firm acts alone.

5 The follower's optimisation problem

We now move on to solve the non-cooperative two-player game. As usual in dynamic games the stopping time game is solved backwards. Thus we start by considering the optimisation problem of the follower (here denoted *i*), who invests strictly later than its rival (*j*). Given that the rival has invested irreversibly the follower faces the conditional probability $h_j dt$ that the leader will make a breakthrough in any time interval *dt*. The probability that the rival has not yet made the discovery by date *t* is $e^{-h_j t}$; this probability is independent of whether the follower itself has or has not invested. Thus the follower's optimal stopping problem is described by

$$V = \max_T E\left\{ e^{-(r+h_j)T} \left(\int_T^\infty e^{-(r+h_i+h_j)t} \{h_i \pi_t - C_i\} dt - K_i \right) \right\}. \tag{23}$$

Note that, in effect, the follower acts as a single firm but facing the augmented discount rate $r + h_j$.

In the continuation region the follower holds the option to invest but also faces the possibility of innovation by its rival. Thus in any short time interval *dt* the follower experiences a capital gain or loss dV with probability $1 - h_j dt$, while with probability $h_j dt$ the rival innovates and its option expires with no value. The Bellman equation for the value of the investment opportunity in the continuation region is therefore given by

$$(r + h_j) V_{0,1} dt = E(dV_{0,1}).$$

Expanding using Itô's lemma and substituting from (1) as before, the Bellman equation entails the following ODE

$$\frac{1}{2}\sigma^2 \pi^2 V''_{0,1} + \mu \pi V'_{0,1} - (r + h_j) V_{0,1} = 0 .$$

As before, the differential equation is solved subject to the boundary condition $V_{0,1}(0) = 0$ to derive

$$V_{0,1}(\pi) = B_F \pi^{\beta_1} \tag{24}$$

where $B_F \geq 0$ is a constant whose value is yet to be determined, and β_1 is the positive root of the characteristic $\varepsilon^2 - \left(1 - \frac{2\mu}{\sigma^2}\right)\varepsilon - \frac{2(r + h_j)}{\sigma^2} = 0$.

Imposing symmetry, β_1 can be written as $\beta_1 = \frac{1}{2}\left\{1 - \frac{2\mu}{\sigma^2} + \sqrt{\left(1 - \frac{2\mu}{\sigma^2}\right)^2 + \frac{8(r+h)}{\sigma^2}}\right\}$.

In the stopping region the value of the project undertaken by the follower is given by its expected value, taking account of the probability of discovery by the rival firm. This value was derived in section 4 and is given by expression (16). As in previous cases value-matching and smooth-pasting conditions are used to determine the critical value of π describing the boundary between the continuation and stopping regions, along with the unknown coefficient B_F. Denoting the follower's optimal investment trigger by π_F these conditions can be stated as

$$V_{0,1}(\pi_F) = V_{1,1}(\pi_F) - K_i \tag{25}$$

and

$$V'_{0,1}(\pi_F) = V'_{1,1}(\pi_F). \tag{26}$$

Solving the simultaneous system and imposing symmetry yields

$$\pi_F = \frac{\beta_1}{\beta_1 - 1}\left(\frac{C}{r + 2h} + K\right)\frac{(r + 2h - \mu)}{h} \tag{27}$$

and

$$B_F = \frac{h\pi_F^{1-\beta_1}}{(r + 2h - \mu)\beta_1}. \tag{28}$$

Note that the follower's trigger point is similar to the single-firm trigger π_U given by (11) but with the interest rate *r* replaced by the augmented discount rate $r + h$ and the root β_1 replacing β_0. Note also that π_F is independent of the point at which the leader invests: given that the firm invests second the precise location of the leader's trigger point is irrelevant.

Thus, writing the leader's investment time as T_L (this being the first time that the leader's trigger point π_L is reached, to be derived later in section 7), the follower's optimal investment time can be written as

$$T_F = \inf\{t \geq T_L : \pi \geq \pi_F\}. \tag{29}$$

6 The leader's payoff

We now consider the payoff to a firm that becomes the leader, given that neither firm has invested so far and that the follower will act optimally in the future in accordance with the stopping rule derived in section 5. Once the payoffs to becoming the leader, acting as the follower, or continuing to delay while one's rival does so too, have been derived the outcome of the game can then be determined.

After the leader has sunk the investment cost K_i it has no further decision to take and its payoff is described by the expected value of its research project. However, this payoff is affected by the action of the rival firm investing later at T_F. Taking account of subsequent investment by the follower the leader's post-investment payoff is given by

$$V(\pi_t) = E\left(\int_t^{T_F} e^{-(r+h_i)\tau}\left(h_i\pi_\tau - C_i\right)d\tau + \int_{T_F}^{\infty} e^{-\left(r+h_i+h_j\right)\tau}\left(h_i\pi_\tau - C_i\right)d\tau\right). \tag{30}$$

Two separate value functions must therefore be considered depending upon whether the follower has already invested or is yet to do so. Subsequent to investment by the follower, for $t \in [T_F, \infty)$, the leader's (and also the follower's) value function is simply the expected value of research taking account of the probability of discovery by the rival. This value is $V_{1,1}(\pi)$ given by equation (16) above.

Prior to investment by the follower, for $t \in [T_L, T_F)$, the leader's value function consists of two components: the expected flow payoff from research and an option-like term that anticipates investment by the follower at T_F. The Bellman equation for the leader is given by

$$V_{1,0} = \left(h_i\pi - C_i\right)dt + e^{-(r+h_i)\,dt}E\left(V_{1,0} + dV_{1,0}\right).$$

Expanding using Itô's lemma, substituting from (1) and simplifying we obtain

$$\left(r + h_i\right)V_{1,0} = \frac{1}{2}\sigma^2\pi^2V''_{1,0} + \mu\pi V'_{1,0} + h_i\pi - C_i\,.$$

Solving the differential equation subject to the boundary condition $V_{1,0}(0) = 0$ (as the prize value π tends to zero the project becomes worthless and the follower will never invest) yields

$$V_{1,0}(\pi)=\frac{h_i\pi}{r+h_i-\mu}-\frac{C_i}{r+h_i}-B_L\pi^{\beta_1} \tag{31}$$

where $B_L > 0$ is a constant whose value is yet to be determined and $\beta_1 > 1$ is as previously defined.

The value of the unknown constant B_L is found by considering the impact of the follower's investment on the payoff to the leader. When π_F is first reached the follower invests and the leader's expected flow payoff is reduced, since there is now a positive probability that its rival will make the discovery instead. Since value functions are forward-looking, $V_{1,0}$ anticipates the effect of the follower's action and must therefore meet $V_{1,1}$ at π_F. Hence, a value-matching condition holds at this point (for further explanation see Harrison (1985)). However, there is no optimality on the part of the leader; thus there is no corresponding smooth-pasting condition.

$$V_{1,0}(\pi_F)=V_{1,1}(\pi_F). \tag{32}$$

Solving for B_L and imposing symmetry yields

$$B_L=\pi_F^{-\beta_1}\left(h\pi_F z_2-Cz_1\right)>0 \tag{33}$$

where $z_1=\frac{1}{r+h}-\frac{1}{r+2h}>0$

and $z_2=\frac{1}{r+h-\mu}-\frac{1}{r+2h-\mu}\geq z_1$.

We can now write down an expression for the payoff to investing as the leader when the patent value is π. This value, denoted V_L, also takes account of the initial sunk cost incurred when the investment is made.

$$V_L(\pi)=\begin{cases}\dfrac{h\pi}{r+h-\mu}-\dfrac{C}{r+h}-B_L\pi^{\beta_1}-K & \text{for } \pi<\pi_F\\[2ex] \dfrac{h\pi}{r+2h-\mu}-\dfrac{C}{r+2h}-K & \text{for } \pi\geq\pi_F.\end{cases} \tag{34}$$

7 Solving the game

In a multi-agent setting the firm's investment problem can no longer be solved using the optimisation techniques typically employed in real options analysis. Instead the problem must be solved as a game, taking account of strategic interactions between the parties. In particular, without the ability to precommit to trigger points (in contrast to the precommitment strategies used by, say, Reinganum (1981)) the leader's stopping point π_L cannot be derived as the solution to a single-agent optimisation problem. Whether a firm becomes a leader, and if so the trigger point at which it invests, is determined by the firm's incentive to pre-empt its rival and the point at which it must invest to prevent itself being pre-empted.

As in Fudenberg and Tirole (1985) the nature of equilibrium in the game depends on the relative magnitudes of the leader's value V_L and the value of delay until the optimal joint-investment point, V_C. Depending upon whether or not these functions intersect somewhere in the interval (0, π_F), two possible cases arise. If V_L *ever* exceeds V_C preemption incentives are too strong for a joint-investment equilibrium to be sustained and the outcome is a leader-follower equilibrium in which one firm invests strictly earlier than its rival, with both investing earlier than the optimal joint-investment time. If, on the other hand, V_L *never* exceeds V_C a joint-investment outcome may be sustained, although leader-follower equilibria are also possible.

Before formally describing the equilibria we must first define and prove the existence of what we shall show to be the leader's trigger point π_L. In order to overcome preemption incentives it must be the case that $V_L(\pi_L)=V_F(\pi_L)$. Thus it is necessary to prove the existence of a point, other than and strictly below π_F, at which this equality holds.

Lemma 1. There exists a unique point $\pi_L \in (0, \pi_F)$ such that

$$\begin{aligned}
V_L(\pi)&<V_F(\pi) \text{ for } \pi < \pi_L\\
V_L(\pi)&=V_F(\pi) \text{ for } \pi = \pi_L\\
V_L(\pi)&>V_F(\pi) \text{ for } \pi \in (\pi_L, \pi_F)\\
V_L(\pi)&=V_F(\pi) \text{ for } \pi \geq \pi_F .
\end{aligned}$$

Proof. See appendix.

Thus, the stopping time of the leader can be written as

$$T_L = \inf\{t \geq 0 : \pi \in [\pi_L, \pi_F]\}. \tag{35}$$

Proposition 1. (Case 1.) If $\exists\pi \in (0, \pi_F)$ such that $V_L(\pi) > V_C(\pi)$, there exist two asymmetric leader-follower equilibria, differing only in the identities of the two firms. In one equilibrium firm 1 (the leader) invests at T_L with firm 2 (the follower) investing strictly later at T_F; in the other equilibrium the firms' identities are reversed.[5]

Proof. The proof is illustrated with reference to figure 1. As π rises from its low initial value, we know from the premise that a point (labelled *A*) will eventually be reached where V_L first exceeds V_C. At this point each firm has a unilateral incentive to deviate from the continuation strategy to become the leader. However, if one firm were to succeed in pre-empting its rival at *A* the payoff to the leader would be strictly greater than that of the follower, since $V_L > V_F$ at this point. From Lemma 1 we know that the leader's payoff is strictly greater than that of the follower over the interval (π_L, π_F). Thus preemption incentives rule out any putative trigger point in this range. We know also that $V_L < V_F$ for all π below π_L; thus prior to π_L each firm prefers to let its rival take the lead. From Lemma 1 we know that π_L is unique. Once the leader has invested the follower faces a single-agent optimisation problem, the solution to which was derived in section 5. Thus, there exists a unique equilibrium configuration in which one firm (the leader) invests when π_L is first reached and the other (the follower) invests strictly later at π_F. Since the firms' identities are interchangeable there are two equilibria of this type. Q.E.D.

Note that at the leader's investment point π_L the expected payoffs of the two firms are equal. Indeed, if this were not the case one firm would have an incentive to deviate and the proposed outcome could not be an equilibrium. By investing earlier than its rival the leader has a greater likelihood of making the discovery. However the value of the prize it stands to win is likely to be lower. Hence, when viewed from the start of the game there is a trade-off between investing pre-emptively to increase one's likelihood of making the discovery first and the probable value of the prize that is gained in that case. At π_L the two effects are in balance and the expected payoffs to the firms are equal. Thus in contrast with several other games where asymmetric equilibria arise, such as Reinganum (1981), the agents in this model are indifferent between the two roles.

Before describing the set of joint-investment equilibria we first define π_S, the lowest joint-investment trigger such that there is no unilateral incentive to deviate. At π_C it is a dominant strategy to invest even though, with $\pi_C > \pi_F$, the rival will follow at once. Thus there can be no joint-investment equilibrium above π_C. Note that, depending upon the relative positions of the value functions, the critical value π_S does not necessarily exist.

$$\pi_S = \inf\left\{\pi_J \in \left(0, \pi_C\right]: V_J\left(\pi; \pi_J\right) \geq V_L\left(\pi\right)\ \forall \pi \in \left(0, \pi_J\right]\right\} \tag{36}$$

where $V_J\left(\pi; \pi_J\right)$ is as defined by (18) above.

Proposition 2. (Case 2). If $V_C(\pi) \geq V_L(\pi)\ \forall \pi \in (0,\ \pi_F)$, two types of equilibria exist. The first is the leader-follower equilibrium described in Proposition 1; two equilibria of this type exist as before. The second is a joint-investment equilibrium in which both firms invest at some trigger point $\pi \in [\pi_S, \pi_C]$; there is a continuum of such equilibrium trigger points over this interval.

Proof. The proof is illustrated with reference to figure 2. As before, fear of preemption by one's rival in the interval $(\pi_L,\ \pi_F)$ over which $V_L > V_F$ entails that the asymmetric leader-follower outcome is also an equilibrium configuration in this case. From the premise, however, there is no *unilateral* incentive to deviate from the continuation strategy anywhere in the interval $(0,\ \pi_C)$. For $\pi \geq \pi_C$ it is a dominant strategy to invest, despite the knowledge that the rival will follow at once. Thus, the joint-investment outcome in which both firms invest at π_C is also an equilibrium. From the definition of π_S any joint-investment point $\pi \in [\pi_S, \pi_C]$ has the property that no unilateral deviation is profitable, thus satisfying the criterion for being an equilibrium. Q.E.D.

We now compare payoffs to the firms under the alternative equilibria described in Proposition 2. Fudenberg and Tirole (1985) argue that if one equilibrium Pareto-dominates all others it is the most reasonable outcome to expect. Thus, using the Pareto criterion it is possible to reduce the multiplicity of equilibria described in Proposition 2 to a unique outcome.

Proposition 3. All joint-investment equilibria, if these exist, Pareto-dominate the asymmetric leader-follower equilibria.

Proof. From the definition of π_S any joint-investment point $\pi_J \in [\pi_S, \pi_C]$ has the property that no unilateral deviation is profitable; thus $V_J(\pi) \geq V_L(\pi)\ \forall \pi \in (0,\ \pi_J]$. The value of continuation is at least as great as the amount the leader could expect to gain from preemption at any π value, and the values of both firms in the pre-emptive leader-follower equilibrium are strictly lower than this amount. Q.E.D.

Essentially, the asymmetric equilibria arising in case 2 are situations in which each firm invests purely as a result of the fear that its rival will do so first. Such instances of ‘attack as a means of defence’ are somewhat irrational, as both firms achieve higher payoffs by coordinating on any one of the symmetric equilibria.

Proposition 4. The joint-investment equilibria are Pareto-ranked by their respective trigger points, with investment at π_C being Pareto optimal and trigger points closer to this level Pareto-dominating all lower ones.

Proof. For any symmetric equilibrium the payoffs of the firms are equal. Therefore maximising the payoff of an individual firm is equivalent to maximising their joint payoff, which is achieved by joint investment at π_C. The loss from joint investment at an arbitrary trigger $\pi_J < \pi_C$ compared with continuation until π_C is given by

$$Q(\pi_J) = \frac{h\pi_J}{r + 2h - \mu} - \frac{C}{r + h} - K - B_C \pi_J^{\beta_0} .$$

From the derivation of π_C we know that $Q(\pi_C) = 0$. Differentiating the function twice yields

$$Q''(\pi_J) = -\beta_0 (\beta_0 - 1) B_C \pi_J^{\beta_0 - 2} < 0.$$

The function is strictly concave, reaching its minimum at π_C. Thus it is strictly monotonic for values of π up to π_C, with trigger points further below π_C entailing greater losses relative to the optimum. The outcomes can be Pareto-ranked accordingly. Q.E.D.

Note that the set of symmetric equilibria can also be reduced to the unique Pareto-optimal outcome by considering the robustness of the proposed equilibria to trembling-hand perfection (though this refinement cannot on its own eliminate the pre-emptive leader-follower equilibria). Consider any equilibrium joint-investment point $\pi_J \in [\pi_S, \pi_C)$. Suppose a firm fears that its rival may ‘tremble,’ deviating from the equilibrium strategy, and compare its payoff from the equilibrium strategy with the alternative ‘wait-and-see’ strategy in which it instead delays its investment, going ahead if and only if its rival does so. Any equilibrium joint-investment point must be in stopping region of the follower, otherwise a profitable deviation would be available. In this region the expected payoff of the ‘follower’ equals that of the ‘leader,’ the two roles being somewhat notional in this case. If the rival invests just before π_J is reached, the firm's best response is to invest at once and its payoff is no lower than if both firms simply adopted the earlier joint-investment point. If the rival instead delays its investment beyond π_J the conditional strategy strictly dominates. Only at π_C itself does it become optimal to invest unconditionally, hence this equilibrium alone survives trembling-hand perfection.

Proposition 5 summarises these arguments and specifies the unique outcome of the game in case 2.

Proposition 5. Using the Pareto criterion the multiplicity of equilibria in case 2 can be reduced to a unique outcome. This is the Pareto-optimal joint-investment equilibrium where both firms invest at T_C. This outcome entails the result that investment is more delayed than in the single-firm analogue.

Whether the outcome of the game is an asymmetric leader-follower equilibrium (as in case 1) or a symmetric joint-investment equilibrium (as in case 2) depends on the parameter values. The boundary between the two regimes is found by determining whether or not the leader's value function V_L intersects V_C in the interval (0, π_F). This can be achieved numerically as follows. Defining the difference between the two value functions, $D(\pi) = V_L(\pi) - V_C(\pi)$, the maximum of this concave function is found at the point where $D'(\pi) = 0$. Denoting this point π_M, we need to determine the sign of $D(\pi_M)$. If this is positive the equilibrium of the game is the Pareto-optimal outcome involving joint investment at T_C. If it is strictly negative there is a pre-emptive leader-follower equilibrium. Due to the non-linearity of $D(\pi)$ the boundary between the two regimes cannot be expressed in an explicit analytical form. However, it can be found numerically for any given set of parameter values; conditions which tend towards each type of equilibrium are analysed numerically in section 8.

So far we have considered only initial values of the stochastic process, π_0, that are sufficiently low for immediate investment to be unprofitable. This is the most plausible starting-point since commencing the game at a higher level begs the question as to why the investment opportunity has only just become available at this point. We now consider briefly the outcome of the game with alternative starting points. One possible justification for a higher initial value could be that a prior innovation by another researcher has created a new investment opportunity in the area. In case 2, the outcome of the game (as specified in Proposition 5) is not sensitive to its starting-point. In this case the stopping region for both firms is [π_C, ∞) and their stopping time is T_C. Note that for a sufficiently high initial value, $\pi_0 \geq \pi_C$, investment will take place immediately.

In case 1, by contrast, the outcome of the game is sensitive to the initial value of π and, in some cases, its future path. For sufficiently high initial values a joint-investment equilibrium results and the Pareto-optimal outcome may even be achievable. The outcomes in the two alternative cases, in which π_0 lies in the interval (π_L, π_F) and $[\pi_F, \infty)$ respectively, are as follows.

For $\pi_0 \in (\pi_L, \pi_F)$ there are two asymmetric equilibria differing only in the identities of the leader and follower. The leader invests immediately and the follower invests strictly later at T_F. In this case, unlike that in which $\pi_0 \leq \pi_L$, the payoffs of the two firms are different. Since there is a (strict) gain to preemption over the interval (π_L, π_F), the leader's payoff strictly exceeds that of the follower and the firms are not indifferent between the roles.

For $\pi_0 \in [\pi_F, \infty)$ the unique outcome is a symmetric equilibrium in which the stopping time of both firms is given by $T = \min(T_F, T_C)$. As in other symmetric equilibria, the expected payoffs of the firms are equal. In this case the actual outcome is sensitive to the particular path taken by π as it evolves over time. If π rises significantly without falling to π_F in the meantime, the jointly-optimal investment threshold π_C may be achieved. However, if π happens to fall significantly after the start of the game, so that π_F is reached before π_C, the firms invest at a lower threshold. The fear of preemption prevents them from delaying until the Pareto-optimal trigger point is reached.

8 Numerical examples

In this section the two cases specified in section 7 are illustrated using numerical examples. For any specified set of parameter values the outcome falls into one of the two cases. We therefore consider two sets of parameter values, differing only in the degree of uncertainty, σ. In both examples the drift parameter μ of the geometric Brownian motion is zero. The risk-free interest rate r is 5%. The research technology involves a hazard rate $h = 0.2$, flow cost $C = 0.4$ and set-up cost $K = 1$ for each firm. In the first example the volatility parameter $\sigma = 0.15$; in the second it is twice this level, at $\sigma = 0.3$.

Case 1: Asymmetric equilibrium

In this example, illustrated in figure 1, V_L exceeds V_C between points A and B, and no symmetric equilibrium can be sustained. The leader's trigger point is $\pi_L = 3.48$ while that of the follower is $\pi_F = 5.25$. For comparison, the trigger point of the single firm is π_U = 5.20 and the optimal joint-investment point is π_C = 6.80. Hence in this case it is clear that the pre-emptive effect undermines option values to a significant degree, causing investment to take place sooner than in either the single-firm or the coordinated case.

Case 2: Symmetric equilibrium

In this example, illustrated in figure 2, V_L does not exceed V_C and symmetric joint-investment equilibria can be sustained. The Pareto optimal equilibrium is the one in which both firms invest at π_C = 10.6. For completeness the full set of possible equilibria are as follows: there are two leader-follower equilibria in which the leader and follower invest at π_L = 4.11 and π_F = 6.48 respectively, and a continuum of symmetric equilibria with trigger points lying in the interval [6.95, 10.6]. For comparison, the trigger point of the single firm is π_U = 8.12. Thus, given the higher trigger point, the Pareto-dominant non-cooperative equilibrium involves greater delay than the single firm optimum.

The boundary between the two regimes

The type of equilibrium that emerges in any particular case depends on the balance between two opposing forces: the option value of delay and the leader's expected gain from pre-emptive investment. Numerical analysis indicates that the joint-investment equilibrium becomes more likely as, *ceteris paribus*, the volatility parameter σ rises, the hazard rate h falls (adjusting C and K so that the project's expected value remains constant) or r rises (adjusting μ in line so that the opportunity cost of delay $\delta = r - \mu$ remains constant). Greater uncertainty raises option values while leaving preemption incentives unchanged, thus shifting the balance of incentives towards the joint-investment outcome. A pure increase in hazard rate strengthens the link between pre-emptive investment and earlier discovery, thus pushing the equilibrium towards the pre-emptive one. As with financial options, an increase in pure discounting reduces the current value of the investment cost (or strike price) paid at some date in the future, raising option values. This effect appears to outweigh the possible impact of the discount rate on each firm's incentive to pre-empt, which would be expected to go in the opposite direction.

9 Conclusions

In this paper we have shown that, contrary to initial expectations, rivalry between firms does not necessarily undermine option values. Instead the fear of sparking a patent race may internalise the competitive effect, further raising the value of delay compared with the single-firm case. Thus in situations where both option values and strategic interactions arise, it is necessary to study the circumstances carefully before forming a view on whether the incentive to pre-empt or the option value of delay will dominate.

In competitive situations investment behaviour is very sensitive to specific industry factors, such as the degree of uncertainty or likely speed of discovery, and very different outcomes may emerge. Given the stark differences in firm behaviour between the two types of equilibria the distinction is an important one. Small differences in parameter values can result in very different patterns of investment. In some cases there is also path-dependence so that the timing of investment depends also on the exact evolution of the uncertain variable. Thus simple predictions are difficult to draw.

The model could be extended in a number of ways. This paper has focused on the symmetric two-firm case. If firms' research technologies are instead allowed to differ, the identities of the leader and follower may be defined uniquely and more precise predictions of firm behaviour given. If the number of potential researchers is increased the distinction between the two types of equilibria is likely to become more extreme, with little or no delay beyond the breakeven point occurring in the pre-emptive case. A cascade of activity may ensue as investment by one firm induces the next to invest in turn, resulting in a wave of research activity.

Appendix: Proof of Lemma 1

Proposition 1. There exists a unique point $\pi_L \in (0, \pi_F)$ such that

$$\begin{aligned} V_L(\pi) &< V_F(\pi) \text{ for } \pi < \pi_L \\ V_L(\pi) &= V_F(\pi) \text{ for } \pi = \pi_L \\ V_L(\pi) &> V_F(\pi) \text{ for } \pi \in (\pi_L, \pi_F) \\ V_L(\pi) &= V_F(\pi) \text{ for } \pi \geq \pi_F . \end{aligned}$$

Proof. We start by defining the function $P(\pi) = V_L(\pi) - V_F(\pi)$, describing the gain to pre-empting one's opponent as opposed to being pre-empted. From equations (24) and (34) we can write

$$P(\pi) = \frac{h\pi}{r+h-\mu} - \frac{C}{r+h} - K - \left(\frac{\pi}{\pi_F}\right)^{\beta_1}\left(\frac{h\pi_F}{r+h-\mu} - \frac{C}{r+h} - K\right) \text{ for } \pi \in (0, \pi_F).$$

The following steps are sufficient to demonstrate the existence of a root somewhere in the interval $(0, \pi_F)$.

(i) Evaluating $P(\pi)$ at $\pi = 0$ yields $P(0) = -\left(\frac{C}{r+h} + K\right) < 0$.

(ii) Evaluating $P(\pi)$ at $\pi = \pi_F$ yields $P(\pi_F) = 0$.

(iii) Evaluating the derivative $P'(\pi)$ at π_F it can be shown that

$$\text{sgn}\left\{\left.\frac{dP}{d\pi}\right|_{\pi_F}\right\} = \text{sgn}\left\{-\frac{h}{(r+h-\mu)}(\mu C + (r+h)(r+2h)K)\right\} < 0.$$

Thus, $P(\pi)$ must have at least one root in the interval $(0, \pi_F)$.

Uniqueness of the root π_L and the validity of the two inequalities can be proven by demonstrating strict concavity of $P(\pi)$ over $(0, \pi_F)$. By differentiation we can derive

$$P''(\pi) = -\beta_1(\beta_1 - 1)\pi_F^{-\beta_1}\left(\frac{h\pi_F}{r+h-\mu} - \frac{C}{r+h} - K\right)\pi^{\beta_1 - 2} < 0 \text{ for } \pi > 0.$$

Thus the root is unique, with $P(\pi) < 0$ for $\pi \in (0, \pi_L)$ and $P(\pi) > 0$ for $\pi \in (\pi_L, \pi_F)$.

The final equality is demonstrated by considering the follower's optimal behaviour over the range $[\pi_F, \infty)$. This interval is the follower's stopping region, over which its best response to investment by the leader is to invest at once. Thus, the values of the leader and follower are equal over this range. Q.E.D.

References

BENTOLILA, Samuel, and Giuseppe BERTOLA (1990), "Firing costs and labour demand: How bad is Eurosclerosis?," *Review of Economic Studies*, **57** (July), 381-402.

BRENNAN, Michael J., and Eduardo S. SCHWARTZ (1985), "Evaluating natural resource investments," *Journal of Business*, **58** (January), 135-157.

CHOI, Jay Pil (1991), "Dynamic R&D competition under 'hazard rate' uncertainty," *RAND Journal of Economics*, **22** (winter), 596-610.

DASGUPTA, Partha, and Joseph STIGLITZ (1980), "Uncertainty, industrial structure, and the speed of R&D," *Bell Journal of Economics*, **11**, 1–28.

DIXIT, Avinash K. (1988), "A general model of R&D competition and policy," *RAND Journal of Economics*, **19** (autumn), 317–326.

—— (1989a), "Entry and exit decisions under uncertainty," *Journal of Political Economy*, **97** (June), 620-638.

—— (1989b), "Hysteresis, import penetration, and exchange rate pass-through," *Quarterly Journal of Economics*, **104** (May), 205-228.

—— (1991), "Irreversible investment with price ceilings," *Journal of Political Economy*, **99** (June), 541-557.

—— and Robert S. PINDYCK (1994), *Investment Under Uncertainty*, Princeton University Press.

DUTTA, Prajit K., and Aldo RUSTICHINI (1991), "A theory of stopping time games with applications to product innovations and asset sales," Columbia University Discussion Paper Series no. 523.

FUDENBERG, Drew, Richard GILBERT, Joseph STIGLITZ and Jean TIROLE (1983), "Preemption, leapfrogging, and competition in patent races," *European Economic Review*, **22**, 3-31.

FUDENBERG, Drew, and Jean TIROLE (1985), "Preemption and rent equalisation in the adoption of new technology," *Review of Economic Studies*, **52**, 383-401.

—— and —— (1986), "A theory of exit in duopoly," *Econometrica* **54** (July), 943-960.

—— and —— (1991), *Game Theory*, MIT Press.

GHEMAWAT, Pankaj, and Barry NALEBUFF (1985), "Exit," *RAND Journal of Economics*, **16**, 184-194.

GRENADIER, Steven R. (1996), "The strategic exercise of options: Development cascades and overbuilding in real estate markets," *Journal of Finance*, **51** (December), 1653-1679.

HARRISON, J. Michael (1985), *Brownian Motion and Stochastic Flow Systems*, John Wiley & Sons, New York.

LAMBRECHT, Bart, and William PERRAUDIN (1997), "Real options and preemption," University of Cambridge, JIMS Working paper no. 15/97.

LEE, Tom, and Louis L. WILDE (1980), "Market structure and innovation: A reformulation," *Quarterly Journal of Economics*, **94** (March), 429–436.

LOURY, Glenn C. (1979), "Market structure and innovation," *Quarterly Journal of Economics*, **93** (August), 395–410.

MASKIN, Eric, and Jean TIROLE (1988), "A theory of dynamic oligopoly I: Overview and quantity competition with large fixed costs," *Econometrica*, **56**, 549-570.

McDONALD, Robert, and Daniel SIEGEL (1986), “The value of waiting to invest,” *Quarterly Journal of Economics*, **101** (November), 707-728.

PINDYCK, Robert S. (1988), “Irreversible investment, capacity choice, and the value of the firm,” *American Economic Review*, **79** (December), 969-985.

REINGANUM, Jennifer (1981), “On the diffusion of new technology: A game-theoretic approach,” *Review of Economic Studies*, **153**, 395-406.

SIMON, Leo K., and Maxwell B. STINCHCOMBE (1989), “Extensive form games in continuous time: Pure strategies,” *Econometrica*, **57** (September), 1171-1214.

SMETS, Frank (1991), “Exporting versus foreign direct investment: The effect of uncertainty, irreversibilities and strategic interactions,” Working Paper, Yale University.

TITMAN, Sheridan (1985), “Urban land prices under uncertainty,” *American Economic Review*, **75**, 505-514.

Figure 1: Asymmetric equilibrium

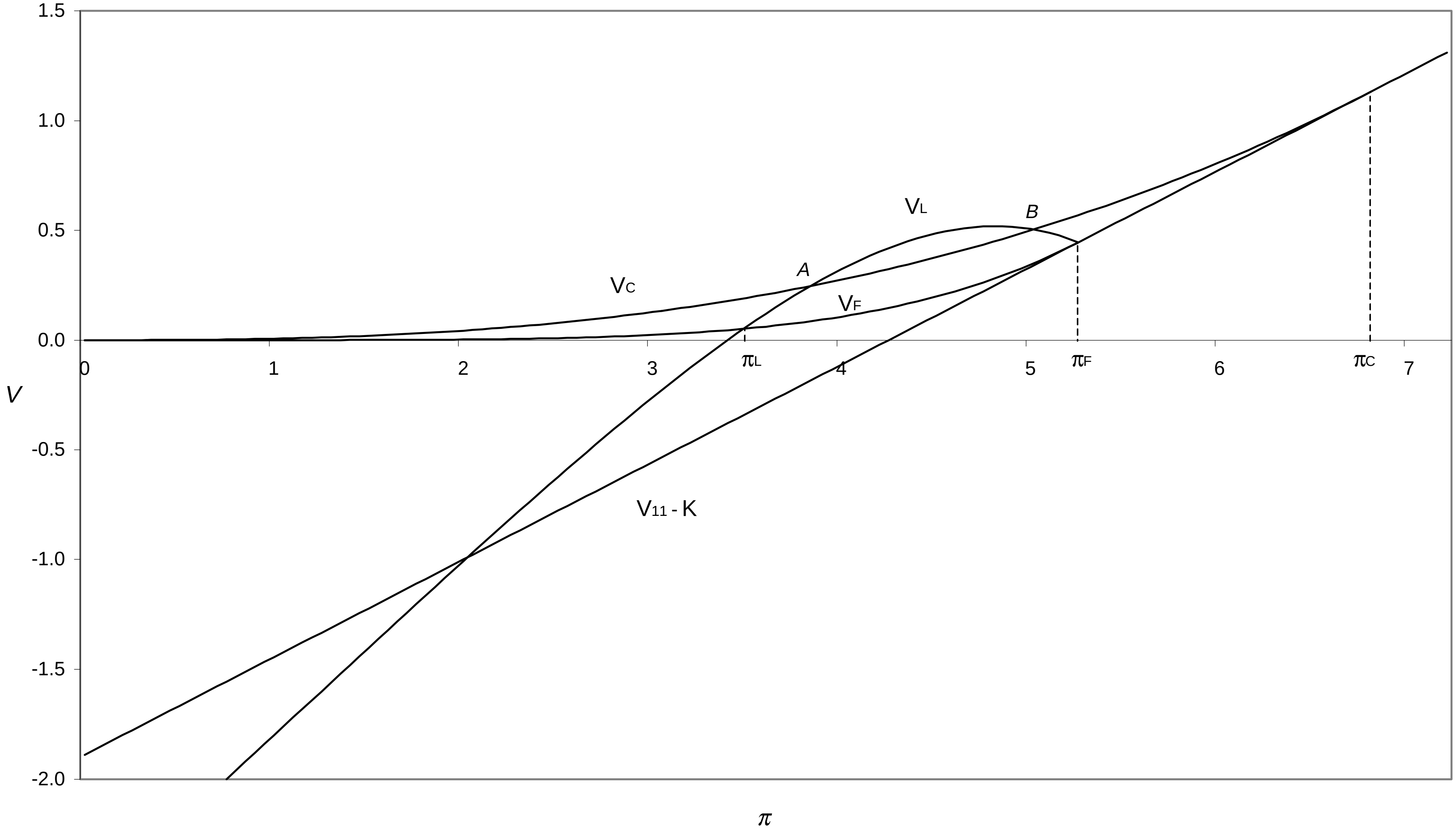
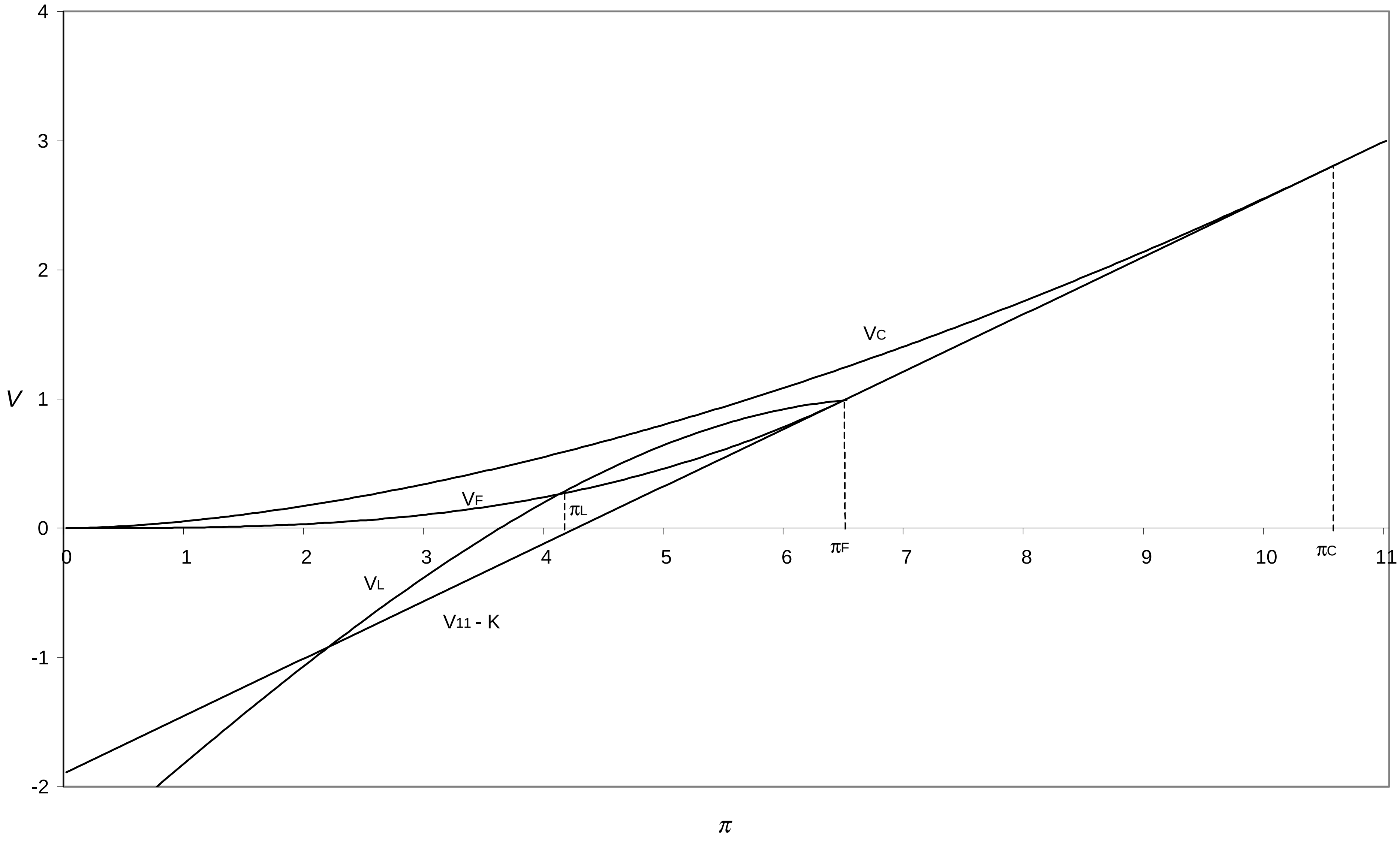


Figure 2: Symmetric equilibrium



[1] This value could be interpreted as the expected NPV of profits in the relevant product market or, if further sunk costs are required, might itself be the value of the option to invest in this market, making investment in the research stage a compound option.

[2] The restriction that $\mu < r$, commonly found in real options models, is necessary to ensure that there is a positive opportunity cost to holding the option, so that it will not be held indefinitely. A large negative drift term would, *ceteris paribus*, encourage earlier investment to raise the probability of winning the prize before its value declines significantly, counteracting the option effects in the model. To avoid such an outcome we make the assumption that μ is non-negative. Since the model is concerned with the effects of uncertainty, not expected trends, the conclusions from the analysis are unaffected by this assumption.

[3] To be precise, the statement that a firm invests at a trigger point π^* means that the firm invests at the time when the stochastic process π first hits the value π^*, approaching this point from below.

[4] Although mixed strategy equilibria may also exist, these are ignored. In a general timing game in continuous time, a mixed strategy is a cumulative probability distribution $G_i(t)$ on $[0, \infty)$, specifying the probability that player i stops at or before time t. For more details see Fudenberg and Tirole (1991, chapter 4.5).

[5] Note that if a symmetric equilibrium is desired, the roles of leader and follower could be allocated between the two firms at T_L according to a 50:50 probability, as is the case in the diffusion equilibrium specified by Fudenberg and Tirole (1985).