

Pricing of Options and Real Options

For Arbitrary Distributions

By

Professor Wayne L. Winston

Kelley School of Business

Indiana University

Bloomington IN, 47405

April 15, 1999

Introduction

Most attempts to price financial derivatives or “real options” assume that the underlying asset (usually a stock or project value) follows a lognormal random variable. This assumption often allows us to obtain a closed form solution for the value of the derivative or real option. Unfortunately recent events such as the wild swings in Internet stock prices and the values of emerging currencies have demonstrated that the lognormal assumption is often unreasonable. It is probably even more unreasonable to assume that all sources of uncertainty in a corporate investment (such as fixed and variable cost uncertainty, product lifetime uncertainty, demand uncertainty, exchange and interest rate uncertainty, and price uncertainty) will magically result in a lognormal random variable. In this paper we use the risk neutral technique discussed by Luenberger [1997] to develop a technique that can be used to price derivatives and real options involving arbitrary distributions. In Section 1 we begin with a quick summary of Luenberger’s discussion. In Section 2 we implement the technique in EXCEL (using the optimization add-in SOLVER and simulation add-in @RISK) to price a put option on Microsoft using actual return data as our inputs. In Section 3 we show how the technique can be used to price (without a lognormal assumption!) the value of a European or American abandonment option. In Section 4 we construct a simulation model of a new product and use the complex distributions of price, cost, market share, and market size to value the following two options:

- Option 1: Three years from now we can, if desired, scale down the project by 50% and save \$1 billion in operating costs.
- Option 2: If the NPV (measured at time 3 dollars and discounted at the risk-free rate) of all profits earned from Time 3 onward exceeds \$2 billion, we receive 20% of this excess.

The key to our valuation of these options is to **risk neutralize the joint distribution of all sources of uncertainty in the simulation.**

Section 1-A Summary of the Risk Neutral Technique

Let us consider an underlying asset (say a stock price or a project value) whose one period return \mathbf{R} assumes a value r_k with probability p_k ($k = 1, 2, \dots, K$). Let r denote the risk-free rate. There are an infinite number of ways to “risk neutralize” the distribution of \mathbf{R} . As Luenberger [1997] points out, if we have a utility function $U(w)$ on the one period return on our investments, then we may uniquely risk neutralize the distribution of \mathbf{R} as follows:

Step 1: Assume we begin with \$1 in initial wealth. Assume we choose an amount α to invest in the underlying and $1 - \alpha$ in the risk-free asset to maximize the expected utility of our ending wealth (since we began with \$1 this is the same as maximizing the expected utility of the return on our investments). For the optimal choice α^* let $W_k^* = \alpha^*(1 + r_k) + (1 - \alpha^*)(1 + r)$ be our ending wealth for outcome k . We will refer to this problem as Problem 1.

Step 2: The risk neutral probability q_k for return r_k is given by

$$(1) \quad q_k = \frac{p_k U'(W_k^*)}{\sum_{k=1}^{k=K} p_k U'(W_k^*)}.$$

Step 3: We can now price any derivative on the underlying as the expected (where the expectation is taken with respect to the **risk neutral probabilities**) discounted value of the cash flows from the option. The price obtained by this method is the price of the derivative that would cause a decision-maker with a utility function $U(w)$ to be indifferent about purchasing any of the derivative. In short, the price obtained by this risk-neutral procedure is the price for the derivative that would (if all decision-makers had the utility function $U(w)$) lead to zero demand for the derivative.

Of course, there are an infinite number of utility functions, and each utility function will yield a different set of risk neutral probabilities. As described in Chapter 15 of Luenberger, the logarithmic utility function $U(w) = \ln(w)$ should be adopted by decision-makers who want to maximize the expected long-term growth rate of wealth. There is also evidence (see Brown (1987)) that the logarithmic utility function is a close approximation to the utility function implied by market behavior.

Section 2-Pricing A Put on Microsoft

We now apply Luenberger's methodology to price a put on Microsoft. We will use the **bootstrapping** approach. To bootstrap we will model future returns of Microsoft by assuming that each daily return for the last four months has the same chance of being Microsoft's return for a given day in the future. Then we use the EXCEL SOLVER (the new EVOLUTIONARY version) to solve for α^* . Then we use (1) to generate a risk-neutralized distribution for Microsoft's monthly returns. Then @RISK is used to estimate the price of a European put as the expected discounted value of the cash flows from the put. Our work is in the file msft.xls.

We obtained from Yahoo daily returns on Microsoft for January-April 1999. See Figure 1

Figure 1

	A	B	C	D	E	F	G	H	I
1	rfree rate	0.076961				rf	0.000317		
2		stdev(daily	0.0258			alpha	6.765668	total	
3		annual	0.4092			mean utility	0.01522	0.9997	
4		Date	Close	Return	port return	ln(port return)	ln(1+return)	risk neutral index	risk neutral probabilities
5		9-Apr-99	94.25	-0.0033	0.975811	-0.02448618	-0.00331	0.0155	0.015532027
6		8-Apr-99	94.563	0.0134	1.088801	0.08507754	0.013307	0.0139	0.013920191
7		7-Apr-99	93.313	-0.008	0.944224	-0.05739175	-0.00801	0.016	0.016051619
8		6-Apr-99	94.063	-0.0092	0.935813	-0.06633935	-0.00926	0.0162	0.016195887
9		5-Apr-99	94.938	0.02428	1.162407	0.15049288	0.023985	0.013	0.013038742
10		1-Apr-99	92.688	0.03417	1.229354	0.20648847	0.033599	0.0123	0.012328695
11		31-Mar-99	89.625	-0.0363	0.752641	-0.28416648	-0.03697	0.0201	0.020137514
12		30-Mar-99	93	0.00677	1.043945	0.04300725	0.006743	0.0145	0.014518311
13		29-Mar-99	92.375	0.03719	1.249805	0.22298755	0.036518	0.0121	0.012126952
14		26-Mar-99	89.063	-0.0101	0.930016	-0.07255379	-0.01012	0.0163	0.016296848
15		25-Mar-99	89.969	0.05073	1.341395	0.29371045	0.049486	0.0113	0.011298924
16		24-Mar-99	85.625	0.02814	1.188577	0.17275699	0.027754	0.0127	0.012751654
17		23-Mar-99	83.281	-0.0362	0.75348	-0.283053	-0.03684	0.0201	0.020115103
18		22-Mar-99	86.406	0.00949	1.062385	0.06051629	0.009447	0.0143	0.014266322
19		19-Mar-99	85.594	-0.0072	0.949125	-0.05221445	-0.00728	0.016	0.015968729
20		18-Mar-99	86.219	0.03179	1.213238	0.19329276	0.031293	0.0125	0.012492459
21		17-Mar-99	83.563	-0.0115	0.920637	-0.08268916	-0.01153	0.0165	0.016462863
22		16-Mar-99	84.531	0.01922	1.128177	0.12060262	0.019033	0.0134	0.013434356
61		19-Jan-99	77.813	0.03923	1.263601	0.23396534	0.038482	0.012	0.011994553
62		15-Jan-99	74.875	0.05644	1.380006	0.32208806	0.054902	0.011	0.010982794
63		14-Jan-99	70.875	-0.0143	0.901144	-0.10409041	-0.01444	0.0168	0.016818986
64		13-Jan-99	71.906	0.01143	1.075482	0.07276892	0.011362	0.0141	0.014092589
65		12-Jan-99	71.094	-0.036	0.754495	-0.28170599	-0.03668	0.0201	0.020088026
66		11-Jan-99	73.75	-0.0158	0.890957	-0.11545889	-0.01597	0.017	0.017011283
67		8-Jan-99	74.938	-0.0042	0.970073	-0.03038395	-0.00416	0.0156	0.015623902
68		7-Jan-99	75.25	-0.005	0.964621	-0.03602015	-0.00497	0.0157	0.01571221
69		6-Jan-99	75.625	0.03242	1.217534	0.19682775	0.031909	0.0124	0.012448376
70		5-Jan-99	73.25	0.03901	1.262079	0.23276011	0.038266	0.012	0.012009018
71		4-Jan-99	70.5						

We will use our methodology to price a three-month European put on Microsoft We assume an annual risk-free rate of 8%, current price of \$90, and exercise price of \$80.

Step 1: In column G we compute the logarithm of (1 + each daily return). In C2 and C3 we find this yields an annualized historical volatility estimate of 40.9%. From BS this volatility level implies a price of \$2.58.

Step 2: In Column E we compute the return on a portfolio investing alpha in Microsoft and (1 - alpha) in the risk-free investment by copying from E5 to E6:E70 the formula

$$= \text{alpha}*(1+D5)+(1-\text{alpha})*(1+rf)$$

Step 3: In F5 through F70 we compute the Logarithm of our portfolio return. We need to ensure that if the portfolio return is negative, a very low utility (say -100,000) will be computed. Otherwise our SOLVER model will clash. Therefore we copy from F5 to F6:F70 the formula

$$= \text{IF}(E5<0,-100000,\text{LN}(E5)).$$

Step 6: If we assign an equal probability to each month's return our expected utility is computed in cell G3 with the formula

$$= \text{AVERAGE}(F5:F70).$$

Step 7: We now use the EXCEL EVOLUTIONARY SOLVER to choose alpha to maximize cell G3 (expected utility). We find alpha = 6.77 does the trick. Note the =IF statement causes us to need a genetic based SOLVER.

Step 8: In H5:H70 we compute the numerator of (1) by copying from H5 to H6:H70 the formula

$$=(1/66)*(1/E5).$$

Step 9: In H3 we compute the sum of H5:H70. This is the denominator of (1). In I5:I70 we normalize Column G to obtain the risk-neutralized probabilities. Copy from I5 to I6:I70 the formula

$$=H5/\$H\$3.$$

Note that the risk-neutral probabilities of the worst returns are larger than 1/66 while the risk-neutral probabilities of the best returns are less than 1/66.

Step 10: We now use our risk neutral probabilities to simulate three months of Microsoft returns and obtain the expected discounted cash flows from the put. See Figure 2.

Figure 2

	J	K
8	Ours	
9	0	90
10	1	90.03491
11	2	90.06984
12	3	90.10478
13	4	90.13973
14	5	90.1747
15	6	90.20968
16	7	90.24467
17	8	90.27968
18	9	90.3147
19	10	90.34973
20	11	90.38478
21	12	90.41984
22	13	90.45492
61	52	91.83352
72	63	92.22615
73	dcf	0

After entering the current price of \$90 in cell K9 we generate Microsoft prices for the next three months (63 days) by copying from K10 to K11:K72 the formula

$$=K9*(1+RiskDiscrete(\$D\$5:\$D\$70,\$I\$5:\$I\$70)).$$

Step 11: In cell K73 we compute the discounted cash flow from the put with the formula

$$=(1/((1.08)^{0.25}))*MAX(80-K72,0).$$

Step 12: After running 16,000 iterations of @RISK we found our estimate for the put value to be \$2.48. Recall that the BS price was \$2.58.

How dependent was our price on our choice of utility function?

For a square root utility function, however, I obtained a price of \$2.50 and for $U(w) = w^{-9}$ I obtained a price of \$2.52. Therefore the level of risk aversion of our utility function appears to play little role in determining the option price. The advantage of our approach is, of course, **we make no assumptions about the distribution governing the return on the underlying asset**. This means we do not have to worry about whether returns follow a lognormal or jump diffusion or whatever. We just assume that future returns will look something like the recent past.

Remarks

1. Our method allows us to give more weight to more recent observations. Simply weight Microsoft's return from t days ago by β^t . If we postulate an exponential utility function of the form $U(x) = -e^{-ax}$ we may use current prices of options on Microsoft to simultaneously estimate β and a (a measures the decision-maker's risk aversion). In effect, we will have inferred a utility function that investors are applying to Microsoft and Microsoft related derivatives. This utility function could later be applied to price real options involving Microsoft's projects.
2. Often daily returns are autocorrelated. We can easily build into our simulation the correlation between successive day's returns. This could have a significant impact on an

option's value. For our Microsoft data the correlation between yesterday's return and today's was only .02, so this was unnecessary.

- We can also use this method to price foreign exchange options. For example, suppose we are pricing an option keying off the \$/yen exchange rate. Then α = amount put in US dollars and $1 - \alpha$ = amount put in yen. If we let r = US interest rate and r_{yen} = Japanese interest rate it turns out that our risk-neutralized distribution will yield an average growth rate of $r - r_{\text{yen}}$ for the \$/yen exchange rate. This agrees, of course, with the usual risk neutralization of an exchange rate that follows a lognormal random variable.

Section 3-Pricing An Abandonment Option

In this section we use our methodology to price a simple one- year European abandonment option and also a three-year American abandonment option. Let's suppose (see Figure 3 and workbook abd1.xls)

Figure 3

	C	D	E	F	G	H	I	J
1	Abandonment value	\$ 460,000.00						
2	Current Value	\$ 500,000.00						
3	rf	0.08	alpha	-0.63751				
4	One year from Now							
5	Prob	Value	Return	Port return	Ln(Port return)	Rn index	Rn Prob	Abandonment value
6	0.1	\$ 400,000.00	-0.2	1.258504	0.229923344	0.079459	0.0858162	60000
7	0.2	\$ 450,000.00	-0.1	1.194752	0.17793886	0.167399	0.1807907	10000
8	0.3	\$ 500,000.00	0	1.131001	0.12310309	0.265252	0.286472	0
9	0.2	\$ 550,000.00	0.1	1.06725	0.06508501	0.187398	0.2023894	0
10	0.15	\$ 650,000.00	0.3	0.939747	-0.062144351	0.159617	0.1723868	0
11	0.05	\$ 800,000.00	0.6	0.748493	-0.289692843	0.066801	0.0721449	0
12								
13					0.08472174			
14							DCF from abandoning	6441.556

that we own a company currently worth \$500,000. We have the right to sell this company for \$460,000 one year from now. The current risk-free rate is 8%. Our view is that one year from now there is a 10% chance the company will be worth \$400,000, a 20% chance that the company will be worth \$450,000, a 30% chance the company will be worth \$500,000, a 20% chance the company will be worth \$550,000, a 15% chance the company will be worth \$650,000 and a 5% chance that the company will be worth \$800,000. What is the value of our option to sell the company?

As in our Microsoft example we begin by solving problem (1). In Column F we compute the return for each possible project return on a portfolio where α is invested in the project and $(1 - \alpha)$ is invested in the risk-free asset. Using SOLVER we find $\alpha = -.64$ maximizes expected utility. In Column I we compute the risk-neutral probabilities for this project. In J6:J11 we compute the value of our option for each possible project value by copying from cell J6 to J7:J11 the formula

$$=IF(D6 < \$D\$1, \$D\$1 - D6, 0).$$

In cell J14 we compute the expected discounted value of our cash flows from the option with the formula

$$=(1/(1+rf))*SUMPRODUCT(I6:I11,J6:J11).$$

Note, of course, that our expectation is with respect to the risk-neutral probabilities. We find that our option to sell is worth \$6442.

In the sheet Wealth⁹ we priced the option for the less risk-averse utility function $U(w) = w^9$. We obtained (see Figure 4) a virtually identical price of \$6882.

Figure 4

	C	D	E	F	G	H	I	J
1	Abandonment value	\$ 460,000.00						
2	Current Value	\$ 500,000.00						
3	rf	0.08	alpha	-2.07079				
4	One year from Now							
5	Prob	Value	Return	Port return	U(Port return)	Rn index	Rn Prob	Abandon V
6	0.1	\$ 400,000.00	-0.2	1.659822	1.577812753	0.085553	0.0926022	60000
7	0.2	\$ 450,000.00	-0.1	1.452743	1.399490346	0.173402	0.187689	10000
8	0.3	\$ 500,000.00	0	1.245664	1.218598614	0.264134	0.2858965	0
9	0.2	\$ 550,000.00	0.1	1.038584	1.034659638	0.17932	0.1940946	0
10	0.15	\$ 650,000.00	0.3	0.624425	0.654534298	0.14151	0.153169	0
11	0.05	\$ 800,000.00	0.6	0.003187	0.005662881	0.07996	0.0865486	0
12								
13					1.108654145			
14							DCF from abandoning	6882.43

Suppose we have an option to sell the company for \$450,000 any time in the next three years. What is a fair price for this option? We assume that the percentage change in company value each year will be governed by the previously given probabilities. To value our option we assume the company value will grow each year according to the risk-neutral probabilities. We use RISKOPTIMIZER to solve for a "cutoff" point each year that will maximize (cell C16) the expected discounted cash flow from the option. See sheet American and Figure 5. Our Adjustable cells for RISKOPTIMIZER are D11:E11 which we constrain to be integers between \$0 and \$450,000. D11 is a year 1 company value below which we should abandon. E11 is a year 2 company value below which we should abandon. F11 is clearly equal to \$450,000. In Row 10 we generate company values for years 1-3 using the **risk neutral probability distribution of returns**. For example, in D11 we generate the Year 2 company value with the formula

$$=C10*(1+RiskDiscrete(A2:A7,B2:B7)).$$

In row 12 we use =IF statements to determine if we have already exercised the option. In Row 13 we key off row 12 and whether or not the current company value is below the cutoff to determine if we exercise the option. In row 14 we record the profit obtained if the option is exercised. Finally, in C16 we compute the discounted (at risk-free rate) value of our profits. A fair price for the option is the mean discounted cash flow associated with the optimal set of cutoff points. RISKOPTIMZER tells us a fair price is \$10,340.

Figure 5

	A	B	C	D	E	F
1	Return	Prob				
2	-0.2	0.085816226				
3	-0.1	0.180790655				
4	0	0.286471954				
5	0.1	0.202389409				
6	0.3	0.172386819				
7	0.6	0.072144937				
8						
9	Time		0	1	2	3
10	Value		500000	550000	605000	665500
11	Cutoff point		does not apply	446491	425000	450000
12	Already Ex?		no	no	no	no
13	Should we Ex?		no	no	no	no
14	Profit		0	0	0	0
15					0	
16	DCF			\$0.00		
17						
18						
19						
20						Mean = 10340.7922

Remark

If we felt the company's value would grow during year t according a random variable R_t , we could risk neutralize the joint distribution of (R_1, R_2, R_3) and then price the option. This could be done even if the R_t are not independent random variables. To risk neutralize the situation we would solve Problem (1) for a three year period. Our possible values for portfolio returns would be generated from each possible triple of values for R_1, R_2 , and R_3 .

Section 4-How to Risk Neutralize a Monte Carlo Simulation

Most investments in the corporate world involve multiple sources of uncertainty such as fixed and variable cost uncertainty, product lifetime uncertainty, demand uncertainty, exchange and interest rate uncertainty, and price uncertainty. It would appear difficult indeed to simultaneously risk neutralize all sources of uncertainty. The following approach appears to surmount these difficulties.

Step 1: Build a Monte Carlo simulation of the uncertainty involved in the investment. This is standard procedure at companies like GM, Eli Lilly and Proctor and Gamble.

Step 2: Run the simulation, say, 1000 times and obtain a distribution of IRR's (hopefully it exists) for the project. Save all 1000 iterations of IRR's **and the values of all random variables for each iteration**

Step 3: Suppose the project is, say, a ten- year project. Solve Problem (1) for a ten year problem assuming each of our 1000 IRR's has probability 1/1000.

Step 4: We can now obtain risk neutral probabilities for each iteration (or scenario).

Step 5: Since we have kept track of the underlying random variables for each iteration we may now use the risk neutral approach to value any derivative as the expected discounted value of its cash flows, where the probability of each iteration is now the risk neutral probability.

We now present an example of this procedure.

Example

Suppose a new product is going to be sold for 7 years (see file nprod.xls). Year 1 is the first year of sales and development costs are incurred in Year 0. The major sources of uncertainty in this product are as follows:

- **Year 1 Market Size**-Worst case is 1,000,000; most likely case is 2,000,000; Best case is 4,000,000.
- **Year 1 Market Share**-Worst case is 5%; most likely case is 15%, Best case is 20%.
- **Year-to-Year Market Size Growth**- Year 2 market size will grow (percentage-wise) over year 1 according to a normal distribution with mean 5% and standard deviation of 2%. During later years the mean percentage growth in market size will equal last year's and have a standard deviation of 2%.
- **Year-to-Year Market Share Growth**- On average, next year's market share will equal last year's, with the change in share having a standard deviation equal to 5% of the current market share.
- **Product Variable Cost**- Year 1 cost of producing a unit has a best case of \$1700, most likely case of \$1900, and worst case of \$2500. Product cost will grow at 5% per year.
- **Product Price**- Year 1 price of product has a best case of \$3500, most likely case of \$3400, and worst case of \$3000. Product price will grow at 5% per year.
- **Development Cost**-The cost of developing the product is assumed to be incurred in Year 0. The best case is a development cost of \$1 billion, most likely case is a development cost of \$2 billion, and worst case is a development cost of \$4 billion.

We wish to value the following two options:

- Option 1: Three years from now we can, if desired, scale down the project by 50% and save \$1 billion in operating costs.
- Option 2: If the NPV (measured in time 3 dollars and discounted at the risk-free rate) of all profits earned from Time 3 onward exceeds \$2 billion, we receive 20% of this excess. This amount is received at conclusion of product life 8 years from now.

Solution

Our work is in Figure 6 and file nprod.xls. We begin by building a simulation model of the product's cash flows. The key is to use the **triangular random variable** (or RISKTRIANG function) to model the best case, worst case and most likely random variables. Major corporations use the triangular random variable far more frequently than the normal random variable. In rows 26-33 we keep track of the project's IRR. This is a bit involved, because on different iterations EXCEL might (if no guess for IRR is included) return an error message. Our technique let's EXCEL try guesses between -100% and 100% and "traps" the actual IRR (assuming one exists). See Chapter 37 of Winston (1999) for more details of this procedure. We ran 700 iterations of the simulation and saved the results (including values of IRR and all random variables for each iteration) in file nprod2.xls.

Figure 6

	A	B	C	D	E	I	K
1							
2							
3	Year		1	2	3		
4	Year 1 Market Size	2333333.333					
5	Year 1 Share	0.133333333					
6	Year to Year Growth Size		0.05	0.05	0.05		
7	Fixed Cost	2333333333					
8	Year to Year Share Growth		1	1	1		
9	Variable Cost	2033.333333					
10	Unit Price	3300					
11	Inflation	0.05					
12							
13							
14	Year	0	1	2	3	7	
15	Fixed Cost	-2333333333					
16	Price		3300	3465	3638.25	4422.316	
17	Cost		2033.333	2135	2241.75	2724.861	
18	Market Size		2333333	2E+06	2572500	3126890	
19	Market Share		0.133333	0.1333	0.13333	0.133333	
20	Unit Sales		311111.1	326667	343000	416918.6	
21	Profit	-2333333333	3.94E+08	4E+08	4.8E+08	7.08E+08	
22							
23	NPV	\$174,158,130.19					
24	irr	12%					
25							
26	guess	-0.9	-0.7	-0.5	-0.3	0.5	0.9
27		12%	12%	12%	12%	12%	12%
28		0.122085761	0.122086	0.1221	0.12209	0.122086	0.122086
29							
30		largest	0.122086				
31		smallest	0.122086				
32		diff	1.11E-16				
33		irr	0.122086				

Figure 7

	A	B	C	D	E	F
1		alpha	0.751257			
2		rf	0.08			
3						
4				mean	0.874749185	
5						1
6		Project ret	Port return	ln(port)	riskneutral prob index	risk neutral prob
7	1	0.364943	9.51155	2.252507	0.000150193	0.000277997
8	2	-6.94E-03	1.170962	0.157826	0.001219998	0.002258131
9	3	0.637446	39.28651	3.670881	3.63629E-05	6.73052E-05
10	4	0.182315	3.328891	1.202639	0.000429143	0.000794314
11	5	-1.38E-02	1.132759	0.124656	0.001261143	0.002334288
12	6	-9.63E-03	1.155723	0.144726	0.001236084	0.002287906

In sheet Risk neutral (see Figure 7) we obtain risk neutral probabilities for each our 700 iterations. We treat the project as an 8-year investment yielding a return each year equal to its IRR. If we invest alpha in our portfolio and (1- alpha) in risk-free rate (assumed equal to 8%) we can compute the portfolio's return for the first iteration with the formula

$$=alpha*(1+B7)^8+(1-alpha)*(1+rf)^8.$$

We then find, as in the last two sections, the value of alpha solving (1). Then we find, as before, the risk neutral probabilities of each iteration.

We are now ready to value both Option 1 and Option 2.

Valuation of Option 1

For each iteration we use the iteration's values for the product's sources of uncertainty to compute the value of Year 3-7 profits. See Figure 8.

Figure 8

	V	W	X	Y	Z	AA
35					Value	148768771
36					No Option	With option
37					1483300005	1632068775
38	Year 4	Year 5	Year 6	Year 7	NPV	NPV with
39	7.34E+08	8.37E+08	9.26E+08	9.93E+08	\$3,360,954,526.60	3360954527
40	3.65E+08	4.12E+08	4.37E+08	4.79E+08	\$1,603,563,135.10	1801781568
41	8.23E+08	9.46E+08	1.00E+09	1.05E+09	\$3,553,166,183.11	3553166183
42	6.55E+08	7.12E+08	8.57E+08	1.01E+09	\$2,928,763,909.41	2928763909

For the first iteration we compute in cell Z39 the discounted value (in time 3 dollars) of Year 3-7 profits with the formula

=NPV(0.08,U39:Y39).

Note we are assuming cash flows are received at the end of the year.

Then in Z37 we use the risk neutral probabilities to find the expected discounted value of these profits when we have **no option to contract**. Our formula is

=SUMPRODUCT(\$A\$39:\$A\$738,Z39:Z738)/1.08^3.

Next we determine the value of the iteration to us when we have the option to contract the project. Note that we obtain at time 3 the larger of (Year 3-7 NPV) and .5*(Year 3-7 NPV) +1,000,000,000. For the first iteration the amount we obtain in time 3 dollars is computed in AA39 with the formula

=MAX(Z39,0.5*Z39+1000000000).

In cell AA37 we compute the expected discounted value of Years 3-7 to us **with the option to contract as**

=SUMPRODUCT(\$A\$39:\$A\$738,AA39:AA738)/1.08^3.

Finally, we compute the value of the option to contract as

(Mean NPV due to Year 3-7 cash flows with option to contract) - (Mean NPV due to Year 3-7 cash flows without option to contract).

We find this option is worth around \$149,000,000.

Valuing Option 2

Note that Option 2 will pay 20% of the excess over \$2 billion in any iteration where Year 3-7 profit exceeds \$2 billion. Therefore in

Figure 9

	AC
35	Pay out 20% of years 3-7 profits over \$2 billion
36	
37	26294068.35
38	Value of Option
39	272190905.3
40	0
41	310633236.6
42	185752781.9
43	0
44	0
45	0
46	436713136.4

Cell AC 39 we compute the cash flow (in time 3 dollars) from the option with the formula

=MAX(0,0.2*(Z39-200000000)).

Then in cell AC37 we value the option by finding the expected discounted value of the cash flows from the option with the formula

=SUMPRODUCT(\$A\$39:\$A\$738,AC39:AC738)/1.08^8.

We find a fair price for the option to be around \$26.3 million

Remarks

1. Since correlations may easily be imbedded in @RISK, our approach let's us handle complex dependencies.
2. Using RISKOPTIMIZER, we may value American options, by building a decision rule which makes a decision based on **the data from the scenario that is known at the time the decision is made**. For example, a decision to expand capacity at time 3 can be based on Year 3 profit. This allows us to more realistically value a real option because we accounting for the fact that actual exercise decisions are made in the presence of **imperfect information**. For example, consider the usual real option assumption that put and abandonment options are equivalent. They are not, because when exercising a put on a stock we **know with certainty the current stock price**, but when exercising an abandonment option **our estimate of the project's current value is surely in error**. Our method would allow us to have the exercise decision on an abandonment option depend on an observable quantity, such as current profits. The noise involved in this type of decision will, of course, reduce the value of the option.
3. Suppose each project's lifetime is uncertain, then how should we solve Problem 1? To explain this situation, suppose the project can last at most 10 years. For an iteration in which project lasts for $t < 10$ years, we assume that for t years investment in project earns project IRR, and then for $10 - t$ years the risk free rate is earned.
4. Dealing with multiple or non-existent IRR's is a problem. In these situations we probably need to assume a discount rate and estimate the project's per year return on cash invested. A possible approach might go as follows. Suppose project lasts for N years and an estimate (call it V) for the project's value is known.

Step 1: Determine the company's hurdle rate from CAPM.

Step 2: For each iteration t discount all cash outflows (at hurdle rate) back to Time 0 and future value (at hurdle rate) all cash inflows to time T . Let's suppose that for iteration t this procedure yields an NPV of c_t for cash outflows and a future value of v_t for cash inflows.

Step 3: To solve Problem 1 we define our changing cells to be α = fraction of project that we invest in and β = amount currently invested in risk-free rate. Then our portfolio return for iteration t may be written as

$$\frac{\alpha v_t + (1 + r)^N \beta}{\alpha c_t + \beta} - 1$$

We can now choose α and β and a utility function $U(W) = W^\delta$ to solve Problem 1 **with the additional constraint that under the risk neutral distribution the value of the underlying project equals V** . The solution to this problem (which includes a value of δ) yields risk neutral probabilities for our simulation that are "calibrated" by the market value of the underlying asset. Then all "derivatives" of the underlying asset may be priced from this calibrated risk neutral distribution.

Conclusion

We have introduced a new technique for valuing financial derivatives and real options. While the technique is much more computationally intensive than present methods, it is easily implemented in a spreadsheet. It also appears that the specific choice of utility function has little affect on our final valuation result. Also, our approach to valuing real options builds on simulations that are already being performed by many large companies such as GM, Eli Lilly, and Proctor and Gamble. The total time it took me to set up and solve each example in this paper was well under an hour! Of course, the rapid increase in computer speed will make our methods even easier to use in the future. Besides the ease of spreadsheet implementation, the virtue of our bootstrapping method is that we need make **no assumptions about the specific distribution of the underlying**. We can even use the RISKMETRICS approach of giving more weight to more recent observations. Much future research needs to be done, but I am convinced that many heretofore-unsolvable valuation problems can be solved by our approach.

References

- Brown, D., "Multiperiod Financial Planning", *Management Science*, Volume 33, 1987, pages 848-875.
Luenberger, D., *Investment Science*, Oxford Press, 1997.
Winston, W., *Financial Models Using Simulation and Optimization*, Palisades Corporation, 1998.