Preemption with an endogenous sunk cost

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Abstract

Greater divisibility allows a firm to maintain a rival on the brink of entry by accumulating capital gradually so as to enter the product market at a time which is both a myopic optimum and a preemptive equilibrium. This conduct is both the optimal policy of a firm that regulates its rival's entry threat through incremental capital accumulation and the limit of equilibrium outcomes of asymmetric preemption as investment steps become arbitrarily small. If the price of capital follows a stochastic process greater volatility delays the firm's jump to completion so option value overrides the strategic investment motive. An emblematic historical example of strategic commitment is discussed in light of this analysis.

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1 Introduction

Consider for a moment the following scenario. Two entrepreneurs plan to build competing rail lines between a pair of distant cities. These projects are both economically sound as the demand for travel between both cities is sufficient to sustain a duopoly. Currently the cost of the necessary capital goods –the cost of building either of the proposed rail links– is prohibitively high so it is not viable to build even one. But this fixed cost is expected to decrease steadily over time so that building a first line and then the second will eventually become profitable. The entrepreneurs are identical for all intents and purposes, their routes are of equal length and run through comparable terrain, but their projects differ in one critical respect. Of the two ways to connect the cities, the Northern route that one entrepreneur plans to follow runs through an expanse of uninhabited land, while the Southern route selected by the other runs through a more populated region. The farmers whose fields lie along this other route are too few to have an impact on the railroad either as workers or as passengers, but keep a keen eye on their surroundings and gossip enough among themselves that should a section of track be built in their backyards, word of this will quickly travel back to both cities.

Under such conditions one expects economics to inform us as to which entrepreneur should succeed in completing a railroad first, and to a large extent it does. The asymmetry in information sets –the only public information available up until the cities have been connected by one railroad or another is the advancement of the line that runs through the populated region– favors the Southern firm by allowing it to make a strategic commitment that the Northern firm cannot. What is less clear, taking a dynamic perspective, is the pattern that such a commitment should follow over time as industry conditions evolve. In the situation described above, the Southern entrepreneur's objective is not to shift product market reaction functions as described in textbook treatments of strategic commitment, but to use the ability to make strategic incremental investments to regulate the Northern entrepreneur's entry decision as the price of the capital good drops. The more interesting question for economic theory to address here is not then whether the Southern firm gains an advantage, but rather how it manages exactly to use the greater observability along its route to secure a positional rent. Even if much has been written about both strategic commitment and investment in continuous time, this latter question, that of identifying characteristic properties of a leader's strategic investment over time, has not been as extensively addressed. This question is the main subject of this article. Moreover, understanding firm conduct in this setting is of broader relevance than a fictional railroad scenario, as similar conditions arise under imperfect competition whenever one firm has the capability of dividing investment into finer pieces, or stages, than its rivals, an occurrence which we argue further below is not so rare in practice. The following pages establish that under such conditions the conduct of leadership involves a phase of incremental investment during which the leading firm ratchets up its threat to keep on discouraging preemption up until a specific time is reached at which its level of capital jumps to completion so as to capture a positional rent.

This pattern of investment can be understood intuitively by means of Figure 1. Time is on the horizontal axis whereas the vertical axis represents levels of capital in the unit interval for one of the firms in a duopoly, hereafter referred to as firm 1, which correspond to possible fractions of the total route that might be built by the Southern firm. Two loci which result from standard arguments involving optimal timing and preemption are drawn in grey. The curve $T^L(\kappa)$ shows optimal monopoly investment times. In a well-behaved model of investment where a unit of capital is required in order for a firm to start earning a constant flow of profit, there is a negative relationship between a firm's existing capital and its optimal time to invest. This optimal time results from a trade-off between the costs and benefits of waiting, which is in turn sensitive to the firm's capital stock κ. The greater is κ the lesser the additional capital needed to enter the product market, $1-\kappa$, and therefore the lower is the marginal benefit of delay, leading the firm to invest earlier. The other curve $k^P(t)$ is best understood by adopting the perspective of firm 2 (the Northern firm), which itself holds no capital and observes that its rival has accumulated a fraction κ of the necessary amount. Firm 2 can compare the profitability of investing early, so as to enter the market first and enjoy a phase of monopoly profit up until a duopoly phase begins once it is joined in the market by firm 1, with the profitability of refraining from early investment and entering after firm 1 as a duopolist. The greater is κ however, the earlier is firm 1's optimal duopoly investment time for the same reason as in the monopoly situation above, namely that its marginal benefit from waiting is relatively lower. All else equal therefore, the greater is κ, the shorter the span of time that firm 2 can expect a monopoly profit stream if it enters first, and therefore the weaker its incentive to enter first rather than second. A generally positive relationship should therefore be expected to hold between the levels of capital held by firm 1 and the times at which firm 2 is indifferent between either preempting its rival or investing as a follower. In addition to $T^L(\kappa)$ and $k^P(t)$, Figure 1 also shows the preemption time T^P before which entry is undesirable for any firm, the optimal duopoly entry time T^F at which firm 2 invests regardless of firm 1's capital stock and the capital stock $\overline{\kappa}$ above which no entry time before T^F is desirable for firm 2.

With these elements in mind the optimal path of capital accumulation of firm 1, given its singular ability to invest incrementally, can be described. If the cost of the capital input is

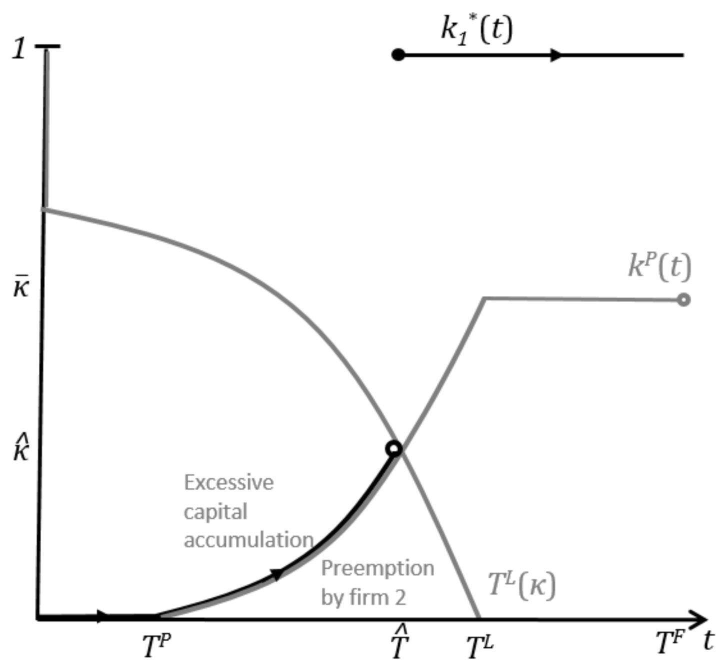

Figure 1: $T^L(\kappa)$ relates optimal monopoly investment times to firm 1's existing capital stock while $k^P(t)$ is the preemptive capital accumulation policy that keeps firm 2 at best indifferent between investing immediately and waiting until T^F. The policy $k_1^*(t)$, in black, has firm 1 accumulating capital incrementally along the schedule $k^P(t)$ from T^P onward up until $\widehat{T}$ when it jumps to completion whereas firm 2 invests as a follower at T^F.

relatively high, it should refrain from any investment. As soon as the price of capital drops sufficiently for market entry to become desirable for its rival (time T^P) it should begin to invest. Accumulating capital above the schedule $k^P(t)$ is excessively costly, whereas accumulating below this schedule would allow the rival to enter the market preemptively. Capital accumulation should therefore follow the path indicated by $k_1^*(t)$ which runs along $k^P(t)$ from T^P onward. Naturally, firm 1 eventually completes its investment and enters the product market. A key insight is that this decision must be an instantaneously optimal choice given its current capital stock. At time $\widehat{T}$ therefore, the remaining mass of capital required to operate in the product market is acquired all at once. This time and the associated capital stock $\widehat{\kappa} = k_1^*\left(\widehat{T}\right)$ are characterized as being the only point at which the firm both efficiently maintains its rival indifferent between entering or not and finds it instantaneously optimal to complete its own investment. The industry then functions as a monopoly up until firm 2 enters (time T^F) and as a duopoly thereafter. A noteworthy feature of these industry dynamics is that the entry time $\widehat{T}$ identified here emerges independently in asymmetric games of preemption, in which the capital stock $\widehat{\kappa}$ represents an exogenous degree of fixed cost asymmetry that delimits preemptive and monopolistic investment outcomes.

To cast these ideas formally, this article models investment in continuous time by two firms competing asymmetrically for leadership in a market. Demand and variable cost are stationary and allow both firms to operate profitably. A unit of capital good is required to operate. Its price is initially high enough for immediate investment to be unprofitable but decreases at a constant rate. Investment is instantaneous and irreversible, capital does not deteriorate and both firms have the same constant discount rate. The economics of the railroad scenario described in the opening paragraph are captured by assuming only firm 1 can invest incrementally. Accordingly its feasible capital accumulation paths are nondecreasing and right-continuous functions from the non-negative reals into a non-trivial subset of the unit interval. Firm 2's investment is binary so its capital accumulation path consists of a single step of unit height.[1]

The strategic investment problem faced by firms in this environment is approached from two complementary directions. First, to get an intuitive understanding of the dynamics of leader investment, a simplified situation is examined where firm 1 chooses its capital accumulation policy to solve a single-firm decision problem under the constraint that it maintains enough capital for preemptive investment to be unprofitable for its rival. If firm 1's capital is infinitely divisible then its optimal policy is represented by the schedule $k_1^*(t)$ in Figure 1. Second, a non-cooperative

[1] Firm 2 might actually have the technical ability to invest incrementally but if firm 1 cannot observe firm 2's capital accumulation prior to product market entry such investment has no strategic effect and firm 2's capital may as well be assumed to be binary.

foundation is obtained for this pattern of investment by studying an asymmetric race in which firm 1 accumulates its unit of capital in finite increments. Formally speaking, this game extends the model of preemption studied by Fudenberg and Tirole [8] to allow for multiple investment rounds. A backward induction argument establishes that there is a unique equilibrium outcome, in which firm 1 follows a discretized capital accumulation policy up until a pivotal stage is reached at which it completes its investment. As the size of increments becomes arbitrarily small, the equilibrium outcome with multistage preemption converges pointwise to $k_1^*(t)$. The last step in the analysis consists in including a noise term in the input price process so as to model the option value of waiting in an uncertain environment. The delay in initial product market entry is then shown to increase with input price volatility, suggesting that option value remains the driving motive of investment even in the presence of competition.

The model rests on several simplifying assumptions. A first assumption is that product market outputs are fixed. The issue of product market shares generally associated with leadership and sequential capacity choice is thus set aside to allow the analysis to focus on dynamic competition. A second assumption is the asymmetry imposed on the feasible capital accumulation strategies of the two firms. As the objective here is to study the dynamic conduct of a leader in a Stackelberg-like framework, it seems natural to postulate a strategic asymmetry analogous to what is assumed in two-period models of capacity choice, even if the grounds for this assumption can reasonably be questioned in other contexts. Moreover a number of real-world circumstances can lead to such asymmetry that warrant mention. First, there are often technological differences between firms at a given point in time. Through chance or history, one firm may develop a singular ability to make a modular use of an input or build it progressively in-house. In the late 1990s for instance, the internet firm Google opted to accumulate computing capacity by linking together commodity personal computers rather than by procuring a high-end system from an external supplier, allowing the clusters used by its search engine to be scaled incrementally with relative ease (Barroso, Dean and Hölzle [2]). A firm may also be better able to stage investment than a competitor because its relationship with a supplier allows it to make nonrefundable deposit payments, because financial asymmetries allow it more frequent access to input markets, or because it otherwise manages more successfully than its rival to render its capital budgeting fractional, irreversible and public.[2]

Broadly viewed, the analysis in this article contributes to the understanding of the strategic

[2]Dixit and Pindyck observe ([6], p. 320) "Even investments that appear to involve only a single decision can turn out to be sequential ... each dollar spent gives the firm an option –which it may or may not exercise– to go ahead and spend the next dollar." Smit and Trigeorgis similarly affirm that strategic investment is best viewed in practice as a compound option, with projects constituting "links in a chain of interrelated investment decisions, the earlier of which set the path for the ones to follow." ([15], p. xxvii)

role of investment in the presence of competition. This role is generally formalized by means of the Stackelberg-Spence-Dixit model of capacity choice, and constitutes a fundamental topic in the study of imperfect competition (Tirole [18]). Accordingly, the central ideas regarding the commitment value of capacity investment are typically derived in the literature in two-stage games even if investment decisions are known to involve richer dynamics.[3]

In order to understand how a firm's capital accumulation is conducted strategically in continuous time, a natural area to turn would seem to be to dynamic capacity accumulation games, such as that of Jun and Vives [10]. In such models however, firms invest in response to both short and long-run incentives which are difficult to disentangle, whereas a distinguishing feature of the introductory railroad scenario is that it clearly separates capital accumulation and dynamic competition on the one hand from product market competition on the other. A dynamic capacity accumulation game that is directly related to the present one can be found in Mills [12], who shows how the realization of first-mover advantage necessitates strategic threats that are costly precisely because they accelerate investment.

The analysis conducted here is therefore closer to game-theoretic models of technology adoption. Technology adoption resembles capacity choice, but typically studies the timing of discrete investment decisions. In the extreme case of infinite information lags, a characteristic pattern known as diffusion equilibrium arises (Reinganum [14]). In such an equilibrium the first firm invests at the optimal monopoly time and earns a positional rent until the second firm invests at the optimal duopoly time, after which the industry operates as a duopoly. Assuming the order of firm investments to be predetermined and noting the analogy between investment timing and output choices, the diffusion equilibrium provides a rough dynamic representation of Stackelberg behavior, in which the positional rent the first firm collects during its phase of monopoly profit substitutes for the first-mover advantage a quantity leader obtains due to its greater market share.

Continuing further within the technology adoption literature, Katz and Shapiro [11] and Huisman [9] study preemption between asymmetric firms in a deterministic and in a stochastic setting respectively. This interaction, though it differs from the introductory example of this article, is closely related to its analysis and it is useful to describe it in some detail. Consider the railroad scenario again, but assume instead that both routes run through unpopulated land and that it is common knowledge that firm 1 has previously built a stretch of track so its remaining route is comparatively shorter. This means that the investment decisions of both firms are binary, with a lower fixed cost of entry for firm 1. If fixed costs are sufficiently asymmetric, firm 1's choice

[3] Dixit is explicit as to the simplifications involved: "It is as if the two players could see through the whole problem and implement the solution immediately." ([4], p. 96)

of investment timing is unconstrained by any rational action of firm 2, and the resulting sequential equilibrium involves the same investment outcomes as the diffusion equilibrium described in the last paragraph. If fixed costs are not too different however, firms may race to enter. In the resulting preemptive equilibrium, the lower cost firm still invests first and obtains a positional rent, but competition affects the timing of its entry. Moreover there exists a critical degree of asymmetry separating those equilibria in which a leader firm invests unconstrained at a monopoly threshold from those equilibria in which its threshold is preemptive. This critical asymmetry corresponds to a level of capital that emerges naturally in the present article, as the degree of asymmetry $\widehat{\kappa}$ that the firm with divisible capital chooses to build up to just before completing its investment.

The analysis of noncooperative choice of capital accumulation policies conducted here rests squarely on existing work on preemption games, whose theory was developed by Fudenberg and Tirole [8] in the deterministic case and has been extended to the stochastic case, as well as along many other dimensions (see Azevedo and Paxson [1] for a survey). In a recent article, Steg [16] gives a careful account of preemption allowing for asymmetric firms. A slight difference between the present model and the existing literature is that the preemption game is formalized here using a reduced form approach to simultaneous investments. This alternative approach, which assumes efficient rationing, does not alter outcomes on the equilibrium path. It is therefore secondary to the main analysis, but simplifies the exposition while providing economically intuitive payoffs that involve differential rents instead of full rent dissipation.

Section 2 presents the main economic assumptions as well as standard terminology and results involving asymmetric preemption. Section 3 discusses the dynamics of leader investment intuitively by studying the optimal capital accumulation policy of a firm that can invest incrementally under the constraint that it regulates its rival's preemption incentive. Section 4 provides a non-cooperative foundation for these investment dynamics by deriving the equilibrium of an investment game in which one firm divides its investment into a finite number of increments. Section 5 incorporates uncertainty by means of a stochastic input price process. Section 6 concludes by discussing an emblematic historical example of strategic commitment in light of the article's analysis.

2 The model

The economic environment in which two firms engage in dynamic competition is described in this section. The main assumptions are presented in Section 2.1, the continuation payoffs that firms obtain when the first market entry occurs in Section 2.2, and Section 2.3 discusses a standard asymmetric preemption game which is referred to throughout the rest of the article.

2.1 Assumptions

Two firms compete to enter a market which can ultimately accommodate both but which is initially profitable for neither. Demand and variable cost are stationary, so active firms earn constant profit flows π^M or π^D depending on whether the product market operates as a monopoly or a duopoly, with $\pi^M \geq 2\pi^D > 0$. Both firms have the same positive and constant discount rate r so capitalized monopoly and duopoly profits are $\Pi^M = \pi^M/r$ and $\Pi^D = \pi^D/r$. A firm must own an entire unit of capital to be active in the product market. The price of a unit of capital $X(t)$ decreases at a constant rate λ, so $X(t) = X(0)e^{-\lambda t}$. Investment is instantaneous and irreversible, there are no adjustment costs, and capital does not depreciate. Assume $X(0) \geq \Pi^M$ so firms initially prefer to wait rather than to enter.

The key assumption of the model is that feasible capital stocks differ for the two firms. Firm 1 has the ability to accumulate capital incrementally, a costly choice since it raises average investment cost while bringing no immediate product market benefit insofar as profit flows do not begin until the firm holds an entire unit, but which has strategic value nevertheless. Firm 2 on the other hand does not have the ability to invest incrementally. The capital accumulation policies that the firms choose are accordingly nondecreasing and right-continuous functions $k_1(t) : \mathbb{R}_+ \to \mathcal{K}$ where $\{0,1\} \subset \mathcal{K} \subseteq [0,1]$ and $k_2(t) : \mathbb{R}_+ \to \{0,1\}$. A policy $k_i(t)$ determines firm i's investment over time in the absence of rival entry and defines a planned completion time $T_i = \inf\{t \in \mathbb{R}_+ | \, k_i(t) = 1\}$ with $T_i = \infty$ (if $k_i(t) < 1$ for all t). If product market entry occurs any firm that has not entered updates its capital accumulation policy in a continuation phase which is a single-firm decision problem. A firm which enters the product market first is said to be the *leader* and a firm which enters second is said to be the *follower*.

2.2 Payoffs

The continuation payoffs that firm i, $i \in \{1,2\}$, obtains as a leader or as a follower in the wake of a first product market entry at time t are denoted $L_i\left(t;\overline{K}_1\right)$ and $F_i\left(t;\overline{K}_1\right)$, and the payoff obtained if both firms enter simultaneously is denoted $M_i\left(t;\overline{K}_1\right)$. These payoffs are measured in initial currency units and defined for a given level of firm 1's capital stock, which is denoted $\overline{K}_1$. They represent the discounted profit streams that forward-looking firms expect to get when their roles as first or second entrant have been determined.

To specify the continuation payoffs, begin with the follower payoffs $F_i\left(t;\overline{K}_1\right)$. These are obtained by studying the decision problem faced by any remaining firm that updates its capital accumulation policy once its rival has entered at a given time t.

Consider firm 2 first, whose problem is simpler as its capital stock before entry is invariably zero. It chooses a time $t_2 \geq t$ to invest so as to maximize the value of entering as a duopolist,

$$\int_{t_2}^{\infty} \pi^D e^{-rs} ds - X(t)e^{-rt_2} = \left(\Pi^D - X(0)e^{-\lambda t_2}\right) e^{-rt_2}$$

in initial currency units. This function is quasiconcave in t_2 over any interval $(t, \infty) \subset \mathbb{R}_+$. Firm 2's optimal policy as a follower is therefore to invest at $t_2^* = \max\left\{t, T^F\right\}$ where

$$T^F = \frac{1}{\lambda} \ln\left(\frac{\lambda + r}{r} \frac{X(0)}{\Pi^D}\right)$$

denotes the unconstrained maximum. The continuation payoff, $F_2\left(t;\overline{K}_1\right) = \left(\Pi^D - X(0)e^{-\lambda t_2^*}\right) e^{-rt_2^*}$, is independent of $\overline{K}_1$. Because this function is a standard payoff in related models with symmetric firms, both $\overline{K}_1$ and the firm subscript are dropped hereafter yielding

$$F\left(t\right) = \begin{cases} \frac{\lambda r^{\frac{r}{\lambda}}}{(\lambda+r)^{\frac{\lambda+r}{\lambda}}} \frac{\left[\Pi^D\right]^{\frac{\lambda+r}{\lambda}}}{[X(0)]^{\frac{r}{\lambda}}}, & t < T^F \\ \left(\Pi^D - X(0)e^{-\lambda t}\right) e^{-rt}, & t \geq T^F \end{cases}.$$

Consider firm 1 next, which faces a more involved problem as its capital stock may be positive when its rival enters, so that it holds $\overline{K}_1 \in [0, 1)$ at the onset of the resulting continuation phase. From the moment firm 2 invests onward, any further incremental investment short of the level required for product market entry only raises firm 1's average investment cost without providing any strategic benefit. Since further incremental accumulation is wasteful, firm 1's optimal policy as a follower is therefore to decide at what time $t_1 \geq t$ to acquire the remaining amount of capital needed for product market entry. Given $\overline{K}_1$, firm 1 thus maximizes the value of this remaining investment,

$$\int_{t_1}^{\infty} \pi^D e^{-rs} ds - \left(1 - \overline{K}_1\right) X(t)e^{-rt_1} = \left(\Pi^D - \left(1 - \overline{K}_1\right) X(0)e^{-\lambda t_1}\right) e^{-rt_1}.$$

Over $\mathbb{R}_+$ this is a quasiconcave function with a unique maximum at

$$T_1^F\left(\overline{K}_1\right) = \begin{cases} \frac{1}{\lambda} \ln\left(\frac{\lambda+r}{r} \frac{\left(1-\overline{K}_1\right)X(0)}{\Pi^D}\right), & \overline{K}_1 \leq 1 - \frac{r}{\lambda+r} \frac{\Pi^D}{X(0)} \\ 0, & \overline{K}_1 > 1 - \frac{r}{\lambda+r} \frac{\Pi^D}{X(0)} \end{cases}.$$

As a follower firm 1's optimal policy is therefore to invest at $t_1^* = \max\left\{t, T_1^F(\overline{K}_1)\right\}$. The continuation payoff is accordingly

$$F_1\left(t;\overline{K}_1\right) = \begin{cases} \frac{\lambda r^{\frac{r}{\lambda}}}{(\lambda+r)^{\frac{\lambda+r}{\lambda}}} \frac{\left[\Pi^D\right]^{\frac{\lambda+r}{\lambda}}}{\left[\left(1-\overline{K}_1\right)X(0)\right]^{\frac{r}{\lambda}}}, & t < T_1^F\left(\overline{K}_1\right) \\ \left(\Pi^D - \left(1-\overline{K}_1\right)X(0)e^{-\lambda t}\right)e^{-rt}, & t \geq T_1^F\left(\overline{K}_1\right) \end{cases}.$$

Observe that the expression of $F_1\left(t;\overline{K}_1\right)$ does not include any investment cost that might be incurred by firm 1 before t, which is sunk when firm roles are determined. This is also the case for the functions $L_1\left(t;\overline{K}_1\right)$ and $M_1\left(t;\overline{K}_1\right)$ defined further below.

Once the follower investment thresholds $T_1^F(\overline{K}_1)$ and T^F have been identified, the leader payoffs $L_i\left(t;\overline{K}_1\right)$, $i \in \{1,2\}$, can be defined. Again consider firm 2 first. Its continuation payoff from leading at time t is

$$\begin{aligned} L_2\left(t;\overline{K}_1\right) &= \int_t^{T_1^F(\overline{K}_1)} \pi^M e^{-rs}ds + \int_{T_1^F(\overline{K}_1)}^{\infty} \pi^D e^{-rs}ds - X(t)e^{-rt} \\ &= \left(\Pi^M - X(0)e^{-\lambda t}\right)e^{-rt} - \left(\Pi^M - \Pi^D\right)e^{-r\max\left\{t,T_1^F(\overline{K}_1)\right\}} \end{aligned}$$

in initial currency units. The second line expresses the continuation payoff as the sum of two terms, the first corresponding to the net present value of a perpetual monopoly and the second to a correction that accounts for the reduction in flow profit generated by firm 1's anticipated entry, which occurs at the follower investment threshold $T_1^F(\overline{K}_1)$ obtained just above.

Observe that $L_2\left(t;\overline{K}_1\right)$ depends on firm 1's capital accumulation through firm 1's follower investment time, with greater capital accumulation decreasing $T_1^F(\overline{K}_1)$ and thus lowering $L_2\left(t;\overline{K}_1\right)$ over $\left[0, T_1^F(\overline{K}_1)\right]$. This effect of firm 1's prior incremental investment on firm 2's leader payoff plays a central role in the analysis of this paper and has an intuitive interpretation. As firm 1 accumulates capital, it decreases the magnitude of the last step required for product market entry. As a result its threat were it to have the follower role becomes more aggressive, since it would find it optimal to enter relatively earlier in a duopoly. This earlier threatened entry in turn reduces the duration of the monopoly phase that firm 2 expects to enjoy by entering first, and thus its payoff from leading.

Consider firm 1 next. If it enters the product market at time t while holding an accumulated capital stock $\overline{K}_1$, the magnitude of its remaining investment step is $1 - \overline{K}_1$. Its continuation payoff from leading is therefore

$$\begin{aligned} L_1\left(t;\overline{K}_1\right) &= \int_t^{T^F} \pi^M e^{-rs}ds + \int_{T^F}^{\infty} \pi^D e^{-rs}ds - \left(1-\overline{K}_1\right)X(t)e^{-rt} \\ &= \left(\Pi^M - \left(1-\overline{K}_1\right)X(0)e^{-\lambda t}\right)e^{-rt} - \left(\Pi^M - \Pi^D\right)e^{-r\max\left\{t,T^F\right\}}. \end{aligned}$$

The difference $L_i\left(t;\overline{K}_1\right)-F_i\left(t;\overline{K}_1\right)$ measures firm i's incentive to lead rather than follow at a given time t. Let $\mathcal{T}_i\left(\overline{K}_1\right)=\left\{t\in\left[0,T^F\right]\middle|\,L_i\left(t;\overline{K}_1\right)-F_i\left(t;\overline{K}_1\right)\geq 0\right\}$, $i\in\{1,2\}$, denote the set of times up until T^F at which leading is individually rational for firm i for a given level of firm 1's capital, $\overline{K}_1$. If $\overline{K}_1>0$, an essential property of the leader and follower payoffs is that over $\left[0,T^F\right)$ firm 1's incentive is greater than firm 2's, that is $L_1\left(t;\overline{K}_1\right)-F_1\left(t;\overline{K}_1\right)>L_2\left(t;\overline{K}_1\right)-F\left(t\right)$ so that $\mathcal{T}_2\left(\overline{K}_1\right)\subset\mathcal{T}_1\left(\overline{K}_1\right)$.[4]

Finally, the continuation payoffs resulting from simultaneous investments are

$$\begin{aligned}M_1\left(t;\overline{K}_1\right) &= \int_t^{\infty}\pi^D e^{-rs}ds-\left(1-\overline{K}_1\right)X(t)e^{-rt}\\ &= \left(\Pi^D-\left(1-\overline{K}_1\right)X(0)e^{-\lambda t}\right)e^{-rt}\end{aligned}$$

and

$$\begin{aligned}M\left(t\right) &= \int_t^{\infty}\pi^D e^{-rs}ds-X(t)e^{-rt}\\ &= \left(\Pi^D-X(0)e^{-\lambda t}\right)e^{-rt}\end{aligned}$$

where the subscript is again omitted for firm 2 because this is a standard payoff in related models.

2.3 Preemption with fixed cost asymmetry

This subsection describes the asymmetric preemption game whose outcomes are referred to throughout the rest of article. Suppose firm 1's investment is constrained to be binary, so its accumulation policy is of the form $\widehat{k}_1\left(t\right):\mathbb{R}_+\rightarrow\{0,1-\kappa\}$ where $\kappa\in(0,1)$ denotes a capital stock it holds initially. Then κ parametrizes the degree of asymmetry between the firms with firm 1 having the comparatively lower fixed cost of entry $1-\kappa$. Firms simultaneously and non-cooperatively choose contingent capital accumulation policies $\widehat{k}_1\left(t\right)$ and $k_2\left(t\right)$ (or to be exact, extended distributions thereof, see Fudenberg and Tirole [8] and Steg [16]) that they can update if rival entry occurs. Because the feasible capital stock levels of both firms are binary, their policies $\widehat{k}_1\left(t\right)$ and $k_2\left(t\right)$ are identified in this section by the completion times T_1 and T_2.

[4] To see intuitively why the inequality holds observe that as $T^F>T_1^F(\overline{K}_1)$, the duopoly phase starts later if firm 1 leads and firm 2 follows than if roles are reversed. Since duopoly lowers the industry's profit flow, industry profit is higher if firm 1 leads and firm 2 follows. Formally,

$$L_1\left(t;\overline{K}_1\right)+F\left(t\right)>L_2\left(t;\overline{K}_1\right)+F_1\left(t;\overline{K}_1\right)$$

and rearranging yields the desired result.

As a benchmark consider the situation of symmetric firms ($\kappa = 0$). Then the conditions $L_1(t;0) - F_1\left(T_1^F(0);0\right) = 0$ and $L_2(t;0) - F\left(T^F\right) = 0$ are identical and have a unique solution in $\left(0, T^F\right)$, denoted T^P and referred to as the preemption time. The preemption time is the moment at which each firm's investment incentive becomes positive, and at which positional rents from any subsequent monopoly phase are fully dissipated. The interval $int\left(\mathcal{T}_i(\kappa)\right) = \left(T^P, T^F\right)$, $i = 1, 2$, over which the firms race to enter just ahead of one another in an interaction that is similar to undercutting in the standard Bertrand model of price competition is referred to as the preemption range. If capital accumulation policies are chosen non-cooperatively, symmetric equilibrium strategies call for both firms to invest at T^P and the resulting outcome has either firm entering with equal probability while its rival follows at T^F.

To describe the equilibrium outcomes of the asymmetric preemption game ($\kappa > 0$), it is first necessary to define the myopically optimal time for monopoly investment. Suppose that roles were predetermined with firm 1 assured of leading, so that it simply chose an entry time t to maximize the payoff $L_1(t;\kappa)$ knowing that firm 2 would subsequently enter at t_2^*. Firm 1's optimal completion time in this case would be

$$T^L(\kappa) = \begin{cases} \frac{1}{\lambda}\ln\left(\frac{\lambda+r}{r}\frac{X(0)(1-\kappa)}{\Pi^M}\right), \ \kappa \leq 1 - \frac{r}{\lambda+r}\frac{\Pi^M}{X(0)} \\ 0, \ \kappa > 1 - \frac{r}{\lambda+r}\frac{\Pi^M}{X(0)} \end{cases}.$$

Observe that $dT^L/d\kappa < 0$ for $\kappa < 1 - \left(r\Pi^M/\left((\lambda + r)X(0)\right)\right)$, and by extension let $T^L = T^L(0)$. In addition to this monopoly time, the characterization of equilibrium involves a critical level of cost asymmetry $\overline{\kappa}$ beyond which firm 2's preemption range vanishes.[5] There are two key payoff configurations and therefore two types of equilibrium, preemptive or sequential, depending upon the level of fixed cost asymmetry.

If $\kappa \in (0, \overline{\kappa})$, then the indifference condition $L_2(t;\kappa) - F(t) = 0$ has two solutions $\underline{t}_\kappa$ and $\overline{t}_\kappa$ in $\left(T^P, T_1^F(\kappa)\right)$, with $\underline{t}_\kappa < \overline{t}_\kappa$, that delimit firm 2's preemption range. The first of these is firm 2's preemption time, which is a function of its rival's capital stock that is hereafter denoted $T^P(\kappa)$. Implicit differentiation of the condition $L_2\left(T^P;\kappa\right) - F\left(T^P\right) = 0$ establishes that $dT^P/d\kappa > 0$ whereas similarly $d\overline{t}_\kappa/d\kappa < 0$, so firm 2's preemption range shrinks with κ. In a (preemptive)

[5] The preemption range vanishes at at T^L for $\kappa = \overline{\kappa}$, where

$$\overline{\kappa} = 1 - \left(\frac{\lambda + r}{\lambda}\frac{\frac{\Pi^M}{\Pi^D} - 1}{\left(\frac{\Pi^M}{\Pi^D}\right)^{\frac{\lambda+r}{\lambda}} - 1}\right)^{\frac{\lambda}{r}}$$

(see Appendix A.1).

equilibrium, the outcome resulting from non-cooperative choice of accumulation policies involves investment by firm 1 at time $\min\left\{T^L(\kappa), T^P(\kappa)\right\}$ and by firm 2 at T^F.

If $\kappa \in [\overline{\kappa}, 1)$, then either $\underline{t}_\kappa = T^L = \overline{t}_\kappa$ (if $\kappa = \overline{\kappa}$) or $L_2(t;\kappa) < F(t)$ for all $t < T^F$ (if $\kappa > \overline{\kappa}$). As firm 2's preemption range is empty, the game reduces to a single firm decision problem. In a (sequential) equilibrium, the outcome resulting from non-cooperative choice of accumulation policies consists of investment by firm 1 at $T^L(\kappa)$ and by firm 2 at T^F.

To simplify exposition throughout the rest of the article, let $T^P(0) = T^P$ and $T^P(\overline{\kappa}) = T^L$ so $T^P(\kappa)$ is continuous over $[0, \overline{\kappa}]$. Furthermore, set $T^P(\kappa) = T^F$ for $\kappa > \overline{\kappa}$ so that for all κ, $T^P(\kappa)$ denotes the first time at which leading becomes profitable for firm 2. Then, in equilibrium firm 1 invests at $\min\left\{T^L(\kappa), T^P(\kappa)\right\}$ for any κ. Because $dT^L/d\kappa < 0$ whereas $dT^P/d\kappa > 0$, the difference $T^L(\kappa) - T^P(\kappa)$ is strictly decreasing in κ and there exists a unique level of asymmetry $\widehat{\kappa} > 0$ such that $T^L(\widehat{\kappa}) = T^P(\widehat{\kappa})$. This asymmetry and the associated investment time play a central role in the analysis so it is useful to define $\widehat{T}$ as the critical time for which $T^L(\widehat{\kappa}) = T^P(\widehat{\kappa})$.

To summarize:

Proposition 1 *(Katz and Shapiro [11]) If firm* 1*'s relative fixed cost is* $1-\kappa$, $\kappa \in (0,1)$, *it invests at* $\min\left\{T^L(\kappa), T^P(\kappa)\right\}$ *and earns a positional rent in equilibrium whereas firm* 2 *invests at* T^F.

(see Appendix A.2 for a complete description of the game and proof).

2.3.1 Preemptive capital accumulation

Based on Proposition 1 a specific capital accumulation policy that plays a key role in the rest of the analysis, the *preemptive capital accumulation policy*, can be defined. This policy is denoted $k^P(t)$ and describes the path firm 1's capital accumulation should follow over $t \in \left[0, T^F\right)$ if its capital is sufficiently divisible in order to suppress firm 2's investment incentive.

Observe first that at any $t \leq T^P$ firm 2 is necessarily outside or on the boundary of its preemption range and no capital is required to discourage its entry. Over $\left(T^P, T^L\right)$ firm 2's investment incentive can be positive however, and the preemptive capital accumulation policy is derived by viewing firm 2's indifference condition as an identity. In this range, firm 2's preemption time is a monotonic function $T^P(\kappa)$ whose inversion yields an explicit form for the level of firm 1's capital stock that leaves firm 2 indifferent between obtaining leader and follower payoffs. For $\kappa = \overline{\kappa}$ finally, firm 2's investment incentive vanishes at T^L and is negative at all other $t < T^F$, so by holding this amount of capital firm 1 effectively shuts out investment by its rival up until the time T^F at which firm 2 enters regardless of firm 1's capital stock. The preemptive capital accumulation policy thus has the specific form

$$k^P(t) = \begin{cases} 0, & t < T^P \\ 1 - \frac{r}{\lambda+r}\frac{\Pi^D}{X(0)}\left(\frac{\Pi^M - \Pi^D}{\left(\Pi^M - X(0)e^{-\lambda t}\right)e^{-rt} - \frac{\lambda r^{\frac{r}{\lambda}}}{(\lambda+r)^{\frac{\lambda+r}{\lambda}}}\frac{\left[\Pi^D\right]^{\frac{\lambda+r}{\lambda}}}{\left[X(0)\right]^{\frac{r}{\lambda}}}}\right)^{\frac{\lambda}{r}}, & T^P \leq t < T^L \\ \overline{\kappa}, & T^L \leq t < T^F \end{cases}.$$

Recall that firm 1's capital accumulation makes its follower entry threat more aggressive by lowering its duopoly investment time T_1^F and hence firm 2's leader payoff $L_2(t;\kappa)$. Over $\left(T^P, T^L\right)$ the intuition underlying $k^P(t)$ is that as the input price decreases, leadership becomes relatively more attractive for firm 2 all else equal, and by following this policy firm 1 offsets the decrease in input price by raising its capital stock just enough to keep its threat sufficiently potent enough to discourage its rival's entry. More generally, any policy $k_1(t)$ that firm 1 chooses which satisfies the no-preemption constraint $k_1(t) \geq k^P(t)$ keeps its rival at bay, allowing it to postpone its own entry beyond the inefficiently early time T^P that preemption otherwise induces.

3 The dynamics of gradual leadership: main insights

This section describes the dynamic pattern of leader investment discussed in the introduction more precisely, leaving the task of explaining how this outcome arises from non-cooperative choices of capital accumulation policies by the firms up to the next section. To simplify matters suppose that capital is perfectly divisible for firm 1 ($\mathcal{K} = [0,1]$) and moreover that it has a first-mover advantage in the sense that it can choose when to lead product market entry, provided that firm 2 does not have an incentive to preempt before then. In other words firm 1 must maintain a sufficient capital stock to discourage firm 2's investment up until entering the product market, but can select any entry time within the range $\left(T^P, T^F\right)$ over which leadership is desirable.[6] Solving the constrained maximization problem corresponding to these assumptions yields a unique optimal capital accumulation path for firm 1, $k_1^*(t)$, which is described in the next proposition (where $\mathbf{1}_A$ denotes the indicator function which takes the value 1 if the condition A is true and 0 otherwise).

[6] In fact firm 1 can ensure itself unilaterally a payoff arbitrarily close to the leader value $V^P(t)$ defined further below in the proof of Proposition 2, by following a policy for which investment is never individually rational for firm 2 such as $k_1^\delta(t) = k_1^*(t+\delta)$ for small enough δ.

Proposition 2 *Under an entry threat firm* 1*'s optimal capital accumulation policy is* $k_1^*(t) = k^P(t)\mathbf{1}_{t<\widehat{T}} + \mathbf{1}_{t\geq\widehat{T}}$.

Proof Observe first that up until firm 1 enters the product market, its optimal capital accumulation policy must satisfy the no-preemption constraint $k_1(t) \geq k^P(t)$ with equality, since additional capital accumulation is costly and unnecessary to discourage entry by firm 2. Firm 1's optimal policy must therefore be of the form $k_1(t) = k^P(t)\mathbf{1}_{s<t} + \mathbf{1}_{s\geq t}$. Such a policy yields it a leader continuation payoff $L_1\left(t; k^P(t)\right)$ provided that $t < T^F$, so that firm 1's net present value measured in initial currency units is

$$V^P(t) = L_1\left(t; k^P(t)\right) - \int_0^t X(s)e^{-rs}dk^P(s). \tag{1}$$

Because $L_1\left(t; \overline{K}_1\right)$ decreases beyond $T^L\left(\overline{K}_1\right)$ and $T^L\left(\overline{K}_1\right) < T^L$ for $\overline{K}_1 > 0$, firm 1's optimal entry time must lie in $\left(T^P, T^L\right)$. The optimal capital accumulation policy therefore solves $\max_{t\in(T^P,T^L)} V^P(t)$. Over this range, $k^P(t)$ and hence $V^P(t)$ are differentiable, yielding the first-order condition

$$\begin{aligned}
&\frac{\partial L_1\left(t^*; k^P(t^*)\right)}{\partial t} + \frac{\partial L_1\left(t^*; k^P(t^*)\right)}{\partial \overline{K}_1}\left[k^P(t^*)\right]' - X(t^*)e^{-rt^*}\left[k^P(t^*)\right]' \\
&= \frac{\partial L_1\left(t^*; k^P(t^*)\right)}{\partial t} = 0,
\end{aligned}$$

which has the specific form

$$-r\Pi^M e^{-rt^*} + (\lambda + r)\left(1 - k^P(t^*)\right)X(0)e^{-(\lambda+r)t^*} = 0. \tag{2}$$

The second-order condition is

$$r^2\Pi^M e^{-rt} - (\lambda + r)^2\left(1 - k^P(t)\right)X(0)e^{-(\lambda+r)t} - (\lambda + r)\left[k^P(t)\right]' X(0)e^{-(\lambda+r)t} < 0.$$

Using the first-order condition to substitute for $r\Pi^M e^{-rt^*}$ in the left-hand side of the above inequality gives

$$-(\lambda + r)\left(\lambda\left(1 - k^P(t^*)\right) + \left[k^P(t^*)\right]'\right)X(0)e^{-(\lambda+r)t^*}$$

which is negative as $\left[k^P(t)\right]' > 0$ over $\left(T^P, T^L\right)$, so the objective is strictly quasiconcave.

The first-order condition (2) admits a unique solution t^*, satisfying

$$t^* = \frac{1}{\lambda}\ln\left(\frac{\lambda + r}{r}\frac{X(0)\left(1 - k^P(t^*)\right)}{\Pi^M}\right).$$

Comparing this expression with the definition of $T^L(\kappa)$ in Section 2.3 establishes that t^* is an optimal time to invest for a firm holding a capital stock $\overline{K}_1 = k^P(t^*)$. Because the no-preemption constraint is satisfied with equality, it is also the case that $t^* = T^P\left(k^P(t^*)\right)$. Therefore t^* is just the critical time $\widehat{T}$ defined in Section 2.3.$\square$

To interpret the optimal policy $k_1^*(t)$ observe that it involves a gradual form of leadership which has firm 1 using preemptive capital accumulation to progressively escalate its threat up until its desired investment time is reached. Because at the margin the strategic investment that firm 1 must undertake to delay its lead entry is exactly offset by a corresponding reduction in the investment required for completion, the first-order condition determining the timing of firm 1's market entry just equates the instantaneous marginal cost of delay $r\Pi^M$ with the instantaneous marginal benefit of delay given $k^P(t^*)$, $(\lambda + r)\left(1 - k^P(t^*)\right)X(t^*)$. At the moment $\widehat{T}$ that firm 1 enters the product market, its investment is therefore myopically optimal. Firm 2 subsequently enters the product market as a follower at T^F. Figure 1 illustrates the locus of optimal monopoly investment times, the preemptive capital accumulation policy, and the optimal capital accumulation policy. The leading entry time and corresponding sunk capital $\left(\widehat{T}, \lim_{t\to\widehat{T}^-} k_1^*(t)\right)$ are the only point that lies both on the optimal monopoly investment locus $T^L(\kappa)$ and on the no-preemption constraint $k^P(t)$.

Consider an increase in the level of monopoly profit π^M which raises the incentive to lead while leaving the baseline follower payoff unchanged. One would expect a greater first-mover advantage to induce earlier entry by firm 1 in order to lengthen the monopoly phase, and therefore to result in more rapid capital accumulation. This intuition must be verified though, since an increase in monopoly profit also raises the cost of preemptive capital accumulation, as firm 1 must compensate for firm 2's own greater preemption incentive. Dividing (2) by $re^{-r\widehat{T}}$ and substituting in the expression of $k^P(t)$, the condition becomes

$$-\Pi^M + \Pi^D X\left(\widehat{T}\right)\left(\frac{\Pi^M - \Pi^D}{\left(\Pi^M - X\left(\widehat{T}\right)\right)\left[X\left(\widehat{T}\right)\right]^{\frac{r}{\lambda}} - \frac{\lambda r^{\frac{r}{\lambda}}}{(\lambda+r)^{\frac{\lambda+r}{\lambda}}}\left[\Pi^D\right]^{\frac{\lambda+r}{\lambda}}}\right)^{\frac{\lambda}{r}} = 0$$

after cancelling $X(0)$ terms inside and outside the brackets. Rearranging then yields

$$\frac{X\left(\widehat{T}\right)}{\Pi^D} + \frac{\lambda r^{\frac{r}{\lambda}}}{(\lambda+r)^{\frac{\lambda+r}{\lambda}}}\left[\frac{X\left(\widehat{T}\right)}{\Pi^D}\right]^{-\frac{r}{\lambda}} = \frac{\Pi^M}{\Pi^D} - \left(\frac{\Pi^M}{\Pi^D}\right)^{1-\frac{r}{\lambda}} + \left(\frac{\Pi^M}{\Pi^D}\right)^{-\frac{r}{\lambda}}.$$

The right-hand side is an increasing function of Π^M,[7] whereas the derivative of the left-hand side with respect to X is $\left(1 - \left(r\Pi^D / (\lambda + r) X\left(\widehat{T}\right)\right)^{(\lambda+r)/\lambda}\right) / \Pi^D$. This expression is positive, as $X\left(\widehat{T}\right) > X^F$ so $\Pi^D / X\left(\widehat{T}\right) < (\lambda + r) / r$. Because $dX/d\Pi^M > 0$, it follows that $d\widehat{T}/d\Pi^M < 0$ so greater monopoly profit does indeed result in accelerated capital accumulation and earlier product market entry by firm 1. This comparative static can also be obtained geometrically. Greater monopoly profit raises the marginal cost of waiting for a monopoly firm, lowering its optimal investment threshold at any given level of pre-existing capital so that the locus of optimal investment times $T_1^L(\kappa)$ shifts left. However greater monopoly profit also raises firm 2's incentive to lead, so firm 1 must hold more capital in order to maintain firm 2 indifferent between leading and following, shifting the locus $k^P(t)$ upward. As a result the time $\widehat{T}$ at which product market entry occurs decreases.

The limiting case $\Pi^D = 0$ where neither firm enters as a follower in the continuation phase is of intrinsic interest. In this case the preemptive capital accumulation policy reduces to $k^P(t) = \mathbf{1}_{t \geq T^P}$ so firm 1 cannot scale its follower entry threat through incremental investment. The gradual leadership dynamics described above unravel and both firms seek to enter at the break-even time $(1/\lambda)\ln\left(X(0)/\Pi^M\right)$, earning zero profits. This situation is of broader relevance because the commitment value of strategic investment is often related to an incumbent firm's ability to deter entry. Suppose that the industry environment allows the first entrant to take a complementary technical or regulatory action, which is not prohibitively costly, that renders subsequent entry unprofitable to its rival. Then if firms cannot commit beforehand not to deter subsequent entry upon investment the follower investment times become $T_1^F = T_2^F = \infty$, and preemptive capital accumulation cannot take place. The role of incremental investment and the leadership dynamics described here therefore hinge on the inability of firm 2 to deter firm 1 as a an incumbent in the product market.

4 Preemption with endogenous asymmetry

This section studies an entry race in which one of the participants can divide its investment into finitely many increments so as to progressively lower its entry cost. The gradual leadership policy

[7] Multiplying by $\left[\Pi^M\right]^{(r-\lambda)/\lambda}\left[\Pi^D\right]^{-r/\lambda}$, the derivative of this expression is of the same sign as

$$\left(\frac{\Pi^M}{\Pi^D}\right)^{\frac{\lambda+r}{\lambda}} - \frac{\lambda - r}{\lambda}\frac{\Pi^M}{\Pi^D} - \frac{r}{\lambda},$$

which is zero at $\Pi^M = \Pi^D$ and increasing in Π^M.

$k_1^*(t)$ in Figure 1 and Proposition 2 is thereby obtained as the limit of equilibrium outcomes when increments become arbitrarily small.

Suppose that firm 1's feasible capital stock levels are $\mathcal{K} = \{0, 1/N, ..., 1\}$, N being a large integer. Dynamic competition between the two firms determines first entry into the product market. This competition is modeled as a sequential game whose outcomes unfold over time. Each stage consists of a race in which firms choose at what time to invest next. The race is an asymmetric one, as the magnitude of any investment is $1/N$ for firm 1, and 1 for firm 2. Together these stages determine the subgame perfect pattern of investments in the industry. At any moment at which neither firm has entered the product market, firm 1 thus holds a non-negative stock of $\xi \leq N-1$ capital increments which, along with the initial input price, identifies the current industry state. Firms choose investment times non-cooperatively, and these choices determine when the next investment occurs and any subsequent stage of the game. If the firms choose identical times, an efficient rationing rule determines the outcome of their decisions. Once an investment occurs, either the game ends if one of the firms has accumulated an entire unit of capital and any firm that has yet not accumulated a unit of capital updates the timing of its next investment, or the game moves to a subsequent stage if only firm 1 invests and it has not accumulated an entire unit.

4.1 The stage game

In a given stage ξ, firm 1's capital stock is ξ/N. Let t_ξ denote the time at which stage ξ is reached. If $t_\xi \geq T^F$ equilibrium decisions are straightforward, consisting of immediate investments for both firms that result in terminal payoffs $(M_1(t_\xi; \xi/N), M(t_\xi))$. The analysis therefore focuses on starting times $t_\xi < T^F$ for $\xi > 0$. For $\xi = 0$ moreover, $t_0 < T^P$ by assumption.

Each firm i, $i \in \{1, 2\}$, chooses at what time $T_i \in [t_\xi, T^F]$ to invest next. The next investment in the industry therefore occurs at $\min\{T_1, T_2\}$. The key assumption is that the magnitude of each investment is $1/N$ for firm 1, and 1 for firm 2, as stated in the section introduction.[8]

The continuation payoffs firms obtain if an investment occurs at a time $t \geq t_\xi$ depend on both the stage and the pattern of investments. If firm 1 invests in stage $\xi = N-1$ or firm 2 invests the game ends with firms obtaining the appropriate terminal payoffs, whereas if only firm 1 invests in a stage $\xi < N-1$ the game shifts to stage $\xi + 1$. If only firm 1 invests, the payoff profile is either $(L_1(t; 1-(1/N)), F(t))$ (if $\xi = N-1$) or $\left(\widehat{V}_1^{\xi+1}(t) - (1/N)X(t)e^{-rt}, \widehat{V}_2^{\xi+1}(t)\right)$

[8] Firm 1 could be allowed to choose a magnitude i when it invests, $i \in \{1, ..., N-\xi\}$ representing a number of increments, without altering the main conclusions of the analysis.

(if $\xi < N-1$) where $\widehat{V}_i^{\eta}(t)$ denotes firm i's equilibrium value if stage η is reached at time t.[9] If only firm 2 invests, the payoff profile is $(F_1(t;\xi/N), L_2(t;\xi/N))$. The payoff profile if both firms invest simultaneously is either $(M_1(t;1-(1/N)), M(t))$ (if $\xi = N-1$) or $(F_1(t;(\xi+1)/N) - (1/N)X(t)e^{-rt}$, $L_2(t;(\xi+1)/N)$ (if $\xi < N-1$).

To complete the specification of the stage game, the strategies T_1 and T_2 must be mapped into outcomes. If $T_1 \neq T_2$ the outcomes are straightforward as firm i is a leader (follower) if and only if it has a strictly lower (higher) planned investment time than its rival. In the subcase $T_1 < T_2$ where firm 1 leads, the payoff profile, denoted $\left(L_1^{\xi}(T_1), F_2^{\xi}(T_1)\right)$, is equal to $(L_1(T_1;\xi/N), F(T_1))$ (if $\xi = N-1$) or $\left(\widehat{V}_1^{\xi+1}(T_1) - (1/N)X(t)e^{-rt},\ \widehat{V}_2^{\xi+1}(T_1)\right)$ (if $\xi < N-1$). In the subcase $T_1 > T_2$ where firm 2 leads, the payoff profile is denoted $\left(F_1^{\xi}(T_2), L_2^{\xi}(T_2)\right)$ and equal to $(F_1(T_2;\xi/N), L_2(T_2;\xi/N))$. Finally, if $T_1 = T_2 = T$ the payoff profile $\left(S_1^{\xi}(T), S_2^{\xi}(T)\right)$ is assumed to be $\left(L_1^{\xi}(T) - \max\left\{L_2(T;\xi/N) - F_2^{\xi}(T), 0\right\},\ F_2^{\xi}(T)\right)$.

To motivate this last of these payoff profiles, observe first that if $L_2(T;\xi/N) < F_2^{\xi}(T)$, simultaneous investments cannot constitute equilibrium choices and the specific form of $\left(S_1^{\xi}(T), S_2^{\xi}(T)\right)$ is of secondary importance. The more interesting situation is therefore if $L_2(T;\xi/N) - F_2^{\xi}(T) \geq 0$. In this case, both firms have non-negative investment incentives. This is because firm 1's leader payoff in any stage ξ satisfies $L_1^{\xi}(t) \geq L_1(t;\xi/N)$ by revealed preference, whereas in subsequent stages firm 1 holds firm 2 to a follower payoff in equilibrium so that $F_2^{\xi}(t) = F(t)$.[10] Therefore, using a standard property of asymmetric preemption for the second inequality (see footnote 4), it must be that

$$
\begin{aligned}
L_1^{\xi}(t) - F_1\left(t;\frac{\xi}{N}\right) &\geq L_1\left(t;\frac{\xi}{N}\right) - F_1\left(t;\frac{\xi}{N}\right) \\
&\geq L_2\left(t;\frac{\xi}{N}\right) - F(t) = L_2\left(t;\frac{\xi}{N}\right) - F_2^{\xi}(t).
\end{aligned} \tag{3}
$$

Letting $\mathcal{T}_i^{\xi}(\xi/N) = \left\{t \in \left[t_{\xi}, T^F\right] \middle| L_i^{\xi}(t) - F_i^{\xi}(t) \geq 0\right\}, i \in \{1,2\}$, then $\mathcal{T}_2^{\xi}(\xi/N) \subseteq \mathcal{T}_1^{\xi}(\xi/N)$ with strict inclusion if $\xi > 0$. The simultaneous investment profile given above reflects these asymmetric investment incentives by attributing a weakly higher payoff to firm 1, which incorporates a variable degree of rent dissipation. Specifically, for $\xi = 0$ investment incentives are

[9] In fact $\widehat{V}_1^{\eta}(t)$ generally depends on the strategies of both firms, as subsequent stages may have multiple equilibria (see footnote 14 for more on this point). However it is argued further below in the text that along the equilibrium path, strategies can be restricted to a subset allowing $\widehat{V}_1^{\eta}(t)$ to be treated as a function of t here for simplicity of exposition.

[10] Firm 1 can always obtain its equilibrium asymmetric preemption payoff (with $\kappa = \xi/N$) by setting $T_1^{\xi'} = \max\left\{t_{\xi}, \min\left\{T^L(\xi/N), T^P(\xi/N)\right\}\right\}$ for all $\xi' > \xi$ (see Appendix A.2 for a derivation of these strategies).

identical for both firms, and the profile attributes follower payoffs consistently with the complete dissipation of rents that can be expected to occur between evenly matched competitors. Otherwise the second inequality in (3) is strict and $\mathcal{T}_2^{\xi}(\xi/N) \subset \mathcal{T}_1^{\xi}(\xi/N)$. On the boundary of $\mathcal{T}_2^{\xi}(\xi/N)$ only firm 1 has a positive investment incentive, and it obtains the whole positional rent. In the interior of $\mathcal{T}_2^{\xi}(\xi/N)$ both firms have a positive investment incentive, but firm 1's is strictly larger and it earns a differential rent above its follower payoff amounting to the excess of its incentive over firm 2's, $\left(L_1^{\xi}(t) - F_1(t;\xi/N)\right) - \left(L_2(t;\xi/N) - F_2^{\xi}(t)\right)$. Firm 1's rent in this last case is analogous to the profit of the low cost firm in asymmetric Bertrand competition.

The simultaneous investment profile given above applies explicitly in industries where the provision of the input is subject to an instantaneous unit capacity constraint and there is free entry upstream. If the downstream firms invest simultaneously, a static allocative mechanism such as competitive bidding should ration their demands efficiently.[11] In an input market equilibrium, the supplier accordingly appropriates any common positional rent downstream, which by (3) corresponds to $L_2(t;\xi/N) - F_2^{\xi}(t)$. If $L_1^{\xi}(t) - F_1(t;\xi/N) = L_2(t;\xi/N) - F_2^{\xi}(t)$ the supplier is indifferent between allocating the input to either firm and there is full downstream rent dissipation, whereas if $L_1^{\xi}(t) - F_1(t;\xi/N) > L_2(t;\xi/N) - F_2^{\xi}(t)$ the input is assumed to be attributed to firm 1, which earns the differential rent over its follower payoff. Finally, it should be noted that the specific form of $\left(S_1^{\xi}(T), S_2^{\xi}(T)\right)$ is not essential in fact for the main result of the section, and another tie-breaking specification such as random assignment could be used instead so long as firm 1 is able to realize its first-mover advantage along the equilibrium path.[12]

[11] To incorporate bidding explicitly in the model one would let B_i denote firm i's bid for the input so that in a given stage firm i chooses $(B_i, T_i) \in \mathbb{R}_+ \times \left[t_\xi, T^F\right]$ (supplier participation requires that $B_i \geq X(T_i)$), modifying payoff expressions accordingly.

[12] It would seem simpler to just posit that if $T_1 = T_2 = T$ the firms invest simultaneously and invariably obtain $\left(M_1^{\xi}(T), M_2^{\xi}(T)\right) = (M_1(t; 1-(1/N)), M(t))$ (if $\xi = N-1$) or $(F_1(t;(\xi+1)/N) - (1/N)X(t)e^{-rt}, L_2(t;(\xi+1)/N))$ (if $\xi < N-1$). However this outcome is not an accurate representation of their economic interaction. There exist payoff configurations where each firm prefers to invest first but simultaneous investments are jointly suboptimal (if $L_i^{\xi}(t) \geq F_i^{\xi}(t) > M_i^{\xi}(t)$, $i \in \{1,2\}$) and the timing of investments would in effect be coordinated in a mixed strategy equilibrium of the discrete time game that the continuous time formulation is generally thought to approximate. For example if $\xi = N-1$ the continuation payoffs are those of asymmetric preemption with $\kappa = 1-(1/N)$, and if moreover $1-(1/N) < \overline{\kappa}$ then for any $t \in \mathcal{T}_2(1-(1/N))$ with $t < T^F$, $L_i^{\xi}(t) \geq F_i^{\xi}(t) > M_i^{\xi}(t), i \in \{1,2\}$. In preemption games this issue is generally resolved either by expanding the strategy space to allow a continuous time representation of such strategies (Fudenberg and Tirole [8]) or by positing a random assignment of leader and follower roles (Dutta and Rustichini [7]). The former approach is parsimonious in its assumptions but involves notationally costly strategies. Like the latter approach, the simultaneous investment profile in the text requires making an assumption about the economic environment but results in a single outcome in each stage which is supported by pure strategies.

Given the continuation payoffs, stage payoffs are

$$V_1^{\xi}(T_1,T_2)=\begin{cases} L_1^{\xi}(T_1), & T_1<T_2 \\ S_1^{\xi}(T), & T_1=T_2=T \\ F_1\left(T_2;\frac{\xi}{N}\right), & T_1>T_2 \end{cases} \text{ and } V_2^{\xi}(T_1,T_2)=\begin{cases} L_2\left(T_2;\frac{\xi}{N}\right), & T_2<T_1 \\ S_2^{\xi}(T), & T_1=T_2=T \\ F_2^{\xi}(T_1), & T_2>T_1 \end{cases},$$

and the normal form of the stage game is

$$\left(\{1,2\},\left[t_{\xi},T^F\right]\times\left[t_{\xi},T^F\right],\left\{V_1^{\xi}(T_1,T_2),V_2^{\xi}(T_1,T_2)\right\}\right).$$

4.2 Equilibrium

The main result in this section identifies the equilibrium path of firm 1's capital accumulation, $k_1^N(t)$. It establishes that in a subgame perfect equilibrium, firm 1 invests at a succession of preemptive times $T^P(\xi/N)$, $\xi=0,1,...,\lfloor N\widehat{\kappa}\rfloor$ up until it enters the product market at time $\widehat{T}^N=\max\left\{T^L\left(\left(\lfloor N\widehat{\kappa}\rfloor+1\right)/N\right),T^P\left(\lfloor N\widehat{\kappa}\rfloor/N\right)\right\}$, and firm 2 enters the product market subsequently at T^F.

Proposition 3 *The subgame perfect investment times* $\left\{\left(\widehat{T}_1^{\xi},\widehat{T}_2^{\xi}\right)\right\}_{\xi\in\{0,\dots,N-1\}}$ *satisfy*

$$\left(\widehat{T}_1^{\xi},\widehat{T}_2^{\xi}\right)=\begin{cases} \left(T^P\left(\frac{\xi}{N}\right),T^P\left(\frac{\xi}{N}\right)\right), & \xi\in\{0,1,...,\lfloor N\widehat{\kappa}\rfloor\} \\ \left(\widehat{T}^N,T_2^*\right) \text{ with } T_2^*\in\left(\widehat{T}^N,T^F\right], & \xi\in\{\lfloor N\widehat{\kappa}\rfloor+1,...,N-1\} \end{cases}.$$

Proposition 3 implies that firm 1's capital accumulation path in equilibrium is

$$k_1^N(t)=\sum_{j=0}^{\lfloor N\widehat{\kappa}\rfloor}\frac{1}{N}\mathbf{1}_{t\geq T^P\left(\frac{j}{N}\right)}+\frac{N-1-\lfloor N\widehat{\kappa}\rfloor}{N}\mathbf{1}_{t\geq\widehat{T}^N},$$

whereas firm 2's is $k_2^N(t)=\mathbf{1}_{t\geq T^F}$. To relate this result to the rest of the paper, suppose that the divisibility of firm 1's capital (N) becomes arbitrarily large. Then the size of firm 1's investment steps becomes arbitrarily small and its entry time, $\max\left\{T^P\left(\lfloor N\widehat{\kappa}\rfloor/N\right),T^L\left(\left(\lfloor N\widehat{\kappa}\rfloor+1\right)/N\right)\right\}$, approaches $\widehat{T}$. Therefore $\lim_{N\to\infty}k_1^N(t)=k_1^*(t)$ pointwise, and the equilibrium outcome of the N-stage preemption game converges to the optimal policy in a Stackelberg choice of capital accumulation policies discussed in Section 3.

The proof of the proposition runs as follows. First, the set of possible equilibrium strategies is narrowed substantially by observing that firm 1 can assure itself of obtaining a specific leader payoff while maintaining a follower payoff for firm 2 provided that it follows a capital accumulation policy which is slightly above $k_1^N(t)$. Next it is shown using a backward induction argument that firm 1's capital accumulation path $k_1^N(t)$ is supported by equilibrium play in each stage. Since the last stage of the game corresponds to the asymmetric preemption game of Section 2.3, its equilibrium is straightforward. The analysis of the preceding stages is simplified by observing that their continuation payoffs similarly involve asymmetric preemption if they are reached along the equilibrium path. In each of these stages therefore, firm 1's leader payoff inherits the essential properties of the low cost firm's leader payoff under preemption so that it has a non-empty and connected quasi-preemption range, whereas firm 2 has standard preemption payoffs and a preemption range which is a subset of firm 1's. The equilibrium choices in stages $\xi < N-1$ therefore resemble those described in Section 2.3, and the outcome described in Proposition 3 results.

Proof First observe that firm 1 can assure itself unilaterally of leading arbitrarily near $\widehat{T}^N$ by accumulating capital along the path $k_1^\delta(t) = k_1^N(t+\delta)$ for small δ, as $k_1^\delta(t) > k^P(t)$ for $t < \widehat{T}^N$ and investment at any given time t is not individually rational for firm 2 if $\lim_{s\to t^-} k_1(s) > k^P(t)$. By setting investment times according to the policy $k_1^\delta(t)$ firm 1 can obtain a payoff arbitrarily close to

$$V_1^0(N) = L_1\left(\widehat{T}^N; \frac{\lfloor N\widehat{\kappa}\rfloor + 1}{N}\right) - \sum_{j=0}^{\lfloor N\widehat{\kappa}\rfloor} \frac{1}{N} X\left(T^P\left(\frac{j}{N}\right)\right) e^{-rT^P\left(\frac{j}{N}\right)}.$$

A subgame perfect equilibrium of the investment game must yield firm 1 a payoff that is at least as large as $V_1^0(N) - f(\delta)$, where $\lim_{\delta\to 0} f(\delta) = 0$. Letting investment increments become arbitrarily small, $\lim_{N\to\infty} V_1^0(N) = V^P\left(\widehat{T}\right)$ as $\lim_{N\to\infty} \widehat{T}^N = \widehat{T}$ and $\lim_{N\to\infty} k_1^N(t) = k_1^*(t)$. With large N, firm 1 therefore cannot earn measurably less in equilibrium than the payoff it gets by following the optimal policy $k_1^*(t)$ (the reader may refer to Section 3 for the definitions of $V^P(t)$ and $k_1^*(t)$). Firm 1 cannot obtain more than $V^P\left(\widehat{T}\right)$ either, since that would imply violating the no-preemption constraint. Moreover, similar reasoning applies in any stage ξ provided that it is reached at a time $t_\xi \leq \widehat{T}_1^\xi$. Finally, no strategy yields firm 2 more than the follower payoff in the equilibrium of any subgame regardless of starting time, since firm 1 can always get a leader payoff replicating the asymmetric preemption outcome (see footnote 10 above). The rest of the proof consists in identifying equilibrium strategies that support the candidate path of capital accumulation in the industry $(k_1(t), k_2(t)) = \left(k_1^N(t), \mathbf{1}_{t\geq T^F}\right)$ defined by the equilibrium play described in the proposition.

To identify the equilibrium strategies supporting the outcome described above note first that the equilibrium in each stage is sensitive both to the starting time and the magnitude of firm 1's capital stock. If $\xi/N > \overline{\kappa}$ (case A), it is straightforward to establish a stronger result than needed for the argument by characterizing equilibrium completely. In this case firm 2 invariably follows and the stage equilibrium is sequential. If $\widehat{\kappa} < \xi/N \leq \overline{\kappa}$ (case B), firm 2 has a nonempty quasi-preemption range but does not constrain firm 1's equilibrium play. Finally if $\xi/N \leq \widehat{\kappa}$ (case C), the equilibrium is preemptive and firm 2 constrains firm 1's equilibrium play if $\xi/N < \widehat{\kappa}$ ($\xi/N = \widehat{\kappa}$ constitutes a limiting case). These three cases are examined successively below.

case A :

In an equilibrium of the investment game, in any stage $\xi > N\overline{\kappa}$, the firms choose $\widehat{T}_1^{\xi} = \max\left\{t_{\xi}, T^L(\xi/N)\right\}$ and $\widehat{T}_2^{\xi} \in \left(\max\left\{t_{\xi}, T^L(\xi/N)\right\}, T^F\right]$. This is established by the following induction argument.

Consider stage $\xi = N-1$ first. In this last stage, the stage game is one of asymmetric preemption with $\kappa = 1-(1/N)$. Provided that $N > 1/(1-\overline{\kappa})$ this game reduces to single firm decision problem and the equilibrium stage strategies, $\widehat{T}_1^{N-1} = \max\{t_{N-1}, T^L(1-(1/N))\}$ and $\widehat{T}_2^{N-1} \in \left(\max\left\{t_{N-1}, T^L(1-(1/N))\right\}, T^F\right]$ (see case a in Appendix A.2), have the form of $\widehat{T}_1^{\xi}$ and $\widehat{T}_2^{\xi}$ given above.

Next, assume that for all stages $\xi' \in \{\xi+1, ..., N\}$ the equilibrium choices $\widehat{T}_1^{\xi'}$ and $\widehat{T}_2^{\xi'}$ are of the above form. If firm 1 leads in a stage ξ with $N\overline{\kappa} < \xi < N-1$, firm 2's continuation payoff is $F_2^{\xi}(t) = \widehat{V}_2^{\xi+1}(t) = F(t)$ given the subsequent play specified in the induction hypothesis. Moreover since $\xi > N\overline{\kappa}$, $L_2(t;\xi/N) < F(t)$ for all $t < T^F$ so $\mathcal{T}_2^{\xi}(\xi/N) = \left\{T^F\right\}$. As firm 2 invariably prefers to follow, its best response to any $T_1 < T^F$ chosen by firm 1 is $T_2 \in \left(T_1, T^F\right]$ and the stage game reduces to an individual decision problem for firm 1. Setting a time T_1 yields firm 1 a payoff $L_1^{\xi}(T_1) = \widehat{V}_1^{\xi+1}(T_1) - (1/N)X(T_1)e^{-rT_1}$. By the induction hypothesis, in stage $\xi+1$ firm 1 leads at $\max\left\{t_{\xi+1}, T^L((\xi+1)/N)\right\}$ and since $T^L(\kappa)$ is a decreasing function, firm 1's subsequent investments and product market entry also occur at $\max\left\{t_{\xi+1}, T^L((\xi+1)/N)\right\}$. Therefore $\widehat{V}_1^{\xi+1}(T_1) = L_1\left(\max\left\{T_1, T^L((\xi+1)/N)\right)\right\}; (\xi+1)/N\right)$. Firm 1 cannot obtain a better payoff by investing at two separate times,[13] so it chooses $\widehat{T}_1^{\xi} = \max\left\{t_{\xi}, T^L(\xi/N))\right\}$ and firm 2 chooses $\widehat{T}_2^{\xi} \in \left(\max\left\{t_{\xi}, T^L(\xi/N)\right\}, T^F\right]$.

[13]Setting $T_1 < \widehat{T}_1^{\xi}$ so as to stagger its investments (if $t_{\xi} < T^L(\xi/N)$), firm 1 obtains

$$
\begin{aligned}
& L_1\left(\max\left\{T_1, T^L\left(\frac{\xi+1}{N}\right)\right\}; \frac{\xi+1}{N}\right) - \frac{1}{N}X(T_1)e^{-rT_1} \\
&= L_1\left(\max\left\{T_1, T^L\left(\frac{\xi+1}{N}\right)\right\}; \frac{\xi}{N}\right) - \frac{1}{N}\left(X(T_1)e^{-rT_1} - X\left(\max\left\{T_1, T^L\left(\frac{\xi+1}{N}\right)\right\}\right)e^{-r\max\left\{T_1, T^L\left(\frac{\xi+1}{N}\right)\right\}}\right) \\
&< L_1\left(\max\left\{t_{\xi}, T^L\left(\frac{\xi}{N}\right)\right\}; \frac{\xi}{N}\right)
\end{aligned}
$$

where the inequality holds because the bracketed term in the second line is non-negative as $X(t)e^{-rt}$ is decreasing in t, and time $\max\left\{t_{\xi}, T^L(\xi/N)\right\}$ maximizes $L_1(t;\xi/N)$ over $\left[t_{\xi}, T^F\right]$.

Hence for all $\xi > N\overline{\kappa}$, in an equilibrium of the investment game the firms invest at times $\widehat{T}_1^{\xi}$ and $\widehat{T}_2^{\xi}$ as specified above.

case B :

On a candidate equilibrium path of the investment game, stage ξ with $N\widehat{\kappa} < \xi \leq N\overline{\kappa}$ is reached at $t_\xi = T^P(\lfloor N\widehat{\kappa}\rfloor/N)$ (if $\xi = \lfloor N\widehat{\kappa}\rfloor + 1$ or $T^P(\lfloor N\widehat{\kappa}\rfloor/N) \geq T^L((\lfloor N\widehat{\kappa}\rfloor + 1)/N))$ or $T^L((\lfloor N\widehat{\kappa}\rfloor + 1)/N)$ (otherwise). Then $\mathcal{T}_2^{\xi}(\xi/N) = \left[T^P(\xi/N), \bar{t}_{\xi/N}\right] \cup \left\{T^F\right\}$ so firm 2 has a non-empty preemption range if $\xi < N\overline{\kappa}$ (whereas $\xi = N\overline{\kappa}$ is the limiting case $\mathcal{T}_2^{\xi}(\xi/N) = \left\{T^L, T^F\right\}$). In an equilibrium of the stage game, firms invest at $\widehat{T}_1^{\xi} = \widehat{T}^N$ and $\widehat{T}_2^{\xi} \in \left(\widehat{T}^N, T^F\right]$. This is established by the following induction argument, where $\overline{\xi} = \lfloor N\overline{\kappa}\rfloor$ denotes the last stage belonging to this case.

Consider stage $\overline{\xi}$ first. If firm 1 invests, firm 2's continuation payoff is $F_2^{\overline{\xi}}(t) = \widehat{V}_2^{\overline{\xi}+1}(t) = F(t)$ given the subsequent play in case A, so $\mathcal{T}_2^{\overline{\xi}}\left(\overline{\xi}/N\right) = \left[T^P\left(\overline{\xi}/N\right), \bar{t}_{\overline{\xi}/N}\right] \cup \left\{T^F\right\}$. Next, observe that if firm 1 leads at time T_1 it obtains $L_1^{\overline{\xi}}(T_1) = \widehat{V}_1^{\overline{\xi}+1}(T_1) - (1/N)X(T_1)e^{-rT_1}$ where $\widehat{V}_1^{\overline{\xi}+1}(T_1) = L_1\left(\max\left\{T_1, T^L\left((\overline{\xi}+1)/N\right)\right\};\ (\overline{\xi}+1)/N\right) = L_1\left(T_1;\ (\overline{\xi}+1)/N\right)$ given the subsequent play in case A and inequality $t_{\overline{\xi}} \geq T^L\left((\overline{\xi}+1)/N\right)$. Firm 1 cannot obtain a better payoff by investing at separate times (by the same argument as in footnote 13), so it prefers to lead at $T_1 = \max\left\{t_{\overline{\xi}}, T^L\left(\overline{\xi}/N\right)\right\} = \widehat{T}^N$. Since $\widehat{T}^N < T^P\left(\overline{\xi}/N\right)$, setting $T_2 \leq \widehat{T}^N$ is unprofitable for firm 2. Therefore $\widehat{T}_1^{\overline{\xi}} = \widehat{T}^N$ and $\widehat{T}_2^{\overline{\xi}} \in \left(\widehat{T}^N, T^F\right]$ so that the equilibrium times in stage $\overline{\xi}$ have the form given above.

Next, assume that for all stages $\xi' \in \left\{\xi+1, ..., \overline{\xi}\right\}$ the equilibrium times $\widehat{T}_1^{\xi'}$ and $\widehat{T}_2^{\xi'}$ are of the above form. Given the subsequent play specified by the induction hypothesis, firm 2's continuation payoff if firm 1 invests is $F_2^{\xi}(t) = \widehat{V}_2^{\xi+1}(t) = F(t)$, so that $\mathcal{T}_2^{\xi}(\xi/N) = \left[T^P(\xi/N), \bar{t}_{\xi/N}\right] \cup \left\{T^F\right\}$. Next, if firm 1 leads at time T_1, its continuation payoff is $L_1^{\xi}(T_1) = \widehat{V}_1^{\xi+1}(T_1) - (1/N)X(T_1)e^{-rT_1}$ where $\widehat{V}_1^{\xi+1}(T_1) = \widehat{V}_1^{0}(T_1) - (1 - (\xi/N))\left(X(T_1)e^{-rT_1} - X\left(\widehat{T}^N\right)e^{-r\widehat{T}^N}\right)$ for $T_1 \leq \widehat{T}^N$. Firm 1 cannot obtain a better payoff by investing at separate times, so it prefers to lead at $T_1 = \max\left\{t_\xi, T^L(\xi/N)\right\} = \widehat{T}^N$. As $\widehat{T}^N < T^P(\xi/N)$, setting $T_2 \leq T_1$ is unprofitable for firm 2. Therefore $\widehat{T}_1^{\xi} = \widehat{T}^N$ and $\widehat{T}_2^{\xi} \in \left(\widehat{T}^N, T^F\right]$.[14]

[14] Off the equilibrium path, firm 2 could "game" the tie-breaking rule by credibly threatening firm 1 with a lower rent (differential instead of positional) in stages $\xi < \overline{\xi}$ if it did not delay investment by some ε. Firm 2 would do this playing a strategy with $T_2^{\xi} = t_\xi + \varepsilon$, and $T_2^{\xi'} = t_\xi$ (if $T_1^{\xi} < t_\xi + \varepsilon$) or $> T_2^{\xi}$ (if $T_1^{\xi} \geq t_\xi + \varepsilon$) for some ξ', $\xi < \xi' \leq \overline{\xi}$, where $T_2^{\xi'} \in \left[t_{\xi'}, T^F\right]$ is an equilibrium of the stage game if stage ξ' is reached at $t_{\xi'} \in int\left(\mathcal{T}_2(\xi'/N)\right)$. However, as argued at the beginning of the proof, firm 2 cannot obtain more than ther follower payoff in the investment game, so its strategies may just as well be restricted to those that consist in always investing along the equilibrium path

Hence if stage ξ, $N\widehat{\kappa} \leq \xi \leq N\overline{\kappa}$, is reached on the equilibrium path the firms invest at times $\widehat{T}_1^{\xi}$ and $\widehat{T}_2^{\xi}$ as specified above.

case C :

Any stage ξ with $\xi \leq N\widehat{\kappa}$ is reached at $t_\xi = T^P(\xi/N)$ on a candidate equilibrium path of the investment game if $\xi > 0$ or at $t_0 < T^P$ if $\xi = 0$. Then $\mathcal{T}_2^{\xi}(\xi/N) = \left[T^P(\xi/N), \bar{t}_{\xi/N}\right] \cup \left\{T^F\right\}$ so firm 2 has a non-empty preemption range, and $t_\xi \in \mathcal{T}_2^{\xi}(\xi/N)$ if $\xi > 0$ or $t_0 < T^P$ if $\xi = 0$. The firms invest at times $\widehat{T}_1^{\xi} = \widehat{T}_2^{\xi} = T^P(\xi/N)$. Observe that firm 2 constrains firm 1's equilibrium choice provided that $\xi < N\widehat{\kappa}$, since $T^P(\xi/N) < T^L(\xi/N)$. This is established by the following induction argument, where $\widehat{\xi} = \lfloor N\widehat{\kappa} \rfloor$ denotes the last stage belonging to this case.

Consider stage $\widehat{\xi}$ first. If firm 1 invests, firm 2's continuation payoff is $F_2^{\widehat{\xi}}(t) = \widehat{V}_2^{\widehat{\xi}+1}(t) = F(t)$ given the subsequent play in cases A and B, so $\mathcal{T}_2^{\xi}\left(\widehat{\xi}/N\right) = \left[T^P\left(\widehat{\xi}/N\right), \bar{t}_{\widehat{\xi}/N}\right] \cup \left\{T^F\right\}$. To identify the equilibrium play, observe that the firms both have non-negative investment incentives at times which are attainable since $t_{\widehat{\xi}} \leq T^P\left(\widehat{\xi}/N\right)$. By a similar argument to that in the asymmetric preemption game (see case c in Appendix A.2),[15] it must be that $\widehat{T}_1^{\widehat{\xi}} = \widehat{T}_2^{\widehat{\xi}} = T^P\left(\widehat{\xi}/N\right)$. In stage $\widehat{\xi}$ therefore, the equilibrium choices $\widehat{T}_1^{\widehat{\xi}}$ and $\widehat{T}_2^{\widehat{\xi}}$ have the form given above.

Next, assume that for all stages $\xi' \in \left\{\xi+1, ..., \widehat{\xi}\right\}$, the equilibrium choices $\widehat{T}_1^{\xi'}$ and $\widehat{T}_2^{\xi'}$ have this form. Similarly to the previous cases, given the subsequent equilibrium play, $F_2^{\xi}(t) = F(t)$ so $\mathcal{T}_2^{\xi} = \left[T^P(\xi/N), \bar{t}_{\xi/N}\right] \cup \left\{T^F\right\}$. Both firms have non-negative investment incentives at times which are attainable since $t_\xi \leq T^P(\xi/N)$, and again by a preemption argument it must be that $\widehat{T}_1^{\xi} = \widehat{T}_2^{\xi} = T^P(\xi/N)$. In stage ξ therefore, the equilibrium choices $\widehat{T}_1^{\xi}$ and $\widehat{T}_2^{\xi}$ have the desired form.

Hence if stage ξ, $\xi \leq N\widehat{\kappa}$, is reached on the candidate equilibrium path the firms invest at $\widehat{T}_1^{\xi}$ and $\widehat{T}_2^{\xi}$ as specified above. As $F_2^{\xi}(t) = F(t)$ and by definition $L_2\left(T^P(\xi/N); \xi/N\right) = F(T^P(\xi/N))$, $S_1^{\xi}\left(T^P(\xi/N)\right) = L_1^{\xi}\left(T^P(\xi/N)\right)$ so firm 1's payoff in case C is unaffected by firm 2's simultaneous investment. The stage ξ payoff to leading in this case incorporates firm 1's subsequent incremental investments, so that

$$L_1^{\xi}\left(T^P(\xi/N)\right) = L_1\left(\widehat{T}^N; \frac{\widehat{\xi}+1}{N}\right) - \sum_{j=\xi}^{\widehat{\xi}} \frac{1}{N} X\left(T^P\left(\frac{j}{N}\right)\right) e^{-rT^P\left(\frac{j}{N}\right)}.$$

at the first profitable opportunity.

[15] Because firm 1 prefers to delay if it can and firm 2 does not invest until doing so is individually rational, it must be that $T_1^{\widehat{\xi}}, T_2^{\widehat{\xi}} \geq T^P\left(\widehat{\xi}/N\right)$. Moreover $T_1^{\widehat{\xi}} \neq T_2^{\widehat{\xi}}$ cannot arise in equilibrium as either one firm prefers to delay or the other prefers to preempt, and finally $T_1^{\widehat{\xi}} = T_2^{\widehat{\xi}} > T^P\left(\widehat{\xi}/N\right)$ cannot be an equilibrium either because firms prefer to preempt.

The equilibrium stage payoffs are accordingly $\widehat{V}_1^{\xi}(t_\xi) = L_1^{\xi}(t_\xi)$ and $\widehat{V}_2^{\xi}(t_\xi) = F(t_\xi)$ for $\xi > 0$ and $\widehat{V}_1^0(t_0) = V_1^0(N)$ for $\xi = 0$.[16] $\square$

5 Input price uncertainty and leadership dynamics

The dynamics industry investments are generally governed by market uncertainty, which gives rise to a value of waiting for a more opportune moment to invest. Firms must then trade off two opposing incentives, as on the one hand the threat of competition creates a strategic incentive to act quickly and commit, but on the other hand option value due to uncertainty calls for flexibility and delay (Chevalier-Roignant and Trigeorgis [3]). A natural extension of the present analysis is therefore to inquire into the effect of uncertainty on the capital accumulation policy of the firm whose capital is relatively divisible.

Suppose that a noise term is added so the unit price of capital follows a geometric Brownian motion $dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$ where $W(t)$ is a standard Wiener process and $X(0) = x > \Pi^M$.[17] Firm 1 is assumed to have the capability of implementing threshold policies allowing it to invest incrementally in this setting. Suppose that its capital is perfectly divisible, so firm 1's feasible accumulation policies are represented by nondecreasing and right-continuous functions $k_1(x) : \mathbb{R}_+ \to [0,1]$ that define its capital stock process absent rival investment as a function of the input price path, $K_1(t) = \sup_{s\in[0,t]} \{k_1(X(s))\}$. Assume that firm 2 stands ready to enter whenever profitable and that firm 1's problem therefore consists in regulating its rival's entry threat up until its own desired entry threshold is reached. As in Section 3, it determines an optimal investment threshold under the constraint that firm 2's entry incentive is maintained at zero. Despite the greater underlying complexity of the payoff functions, the main economic difference that the additional stochastic term in the input price brings to the model is that choices and hence payoffs are defined over price thresholds which determine stochastic investment times, but the broad economic intuitions are otherwise similar.

Let X^P, X^L and X^F denote the respective input price thresholds in symmetric preemption equilibrium, for monopoly and for duopoly. These thresholds are analogs to T^P, T^L and T^F in the deterministic case where firms set investment times are directly. Like T^P, X^P does not

[16] The stage equilibrium outcomes derived under the efficient rationing assumption are also equilibrium outcomes if firms play extended mixed strategies (Fudenberg and Tirole [8], Steg [16]), so the equilibrium path $k_1^N(t)$ is supported by closed-loop strategies as well.

[17] The model in Section 2 corresponds to $\mu = -\lambda$ and $\sigma = 0$ whereas if $\sigma > 0$ firms have a motive to wait even if $\mu > 0$.

have a closed form whereas $X^L = (\beta/(\beta+1))\,\Pi^M$ and $X^F = (\beta/(\beta+1))\,\Pi^D$, where β is a parameter function that reflects expected discounting in the stochastic case.[18] The preemptive capital accumulation policy is defined over $(X^F, x]$ and has the specific form

$$k^P(x) = \begin{cases} \overline{\kappa}_X, & X^F < x \leq X^L \\ 1 - \left(\dfrac{(\beta+1)\left(\frac{\Pi^M}{\Pi^D}-1\right)}{\frac{(\beta+1)^{\beta+1}}{\beta^\beta}\left(\frac{\Pi^M}{\Pi^D}-\frac{x}{\Pi^D}\right)\left(\frac{x}{\Pi^D}\right)^\beta - 1} \right)^{\frac{1}{\beta}}, & X^P \leq x < X^L \\ 0, & x > X^P \end{cases}$$

where $\overline{\kappa}_X$ is a constant analogous to $\overline{\kappa}$ in the deterministic case (See Appendix A.3).

By a similar argument to Section 3, firm 1's optimal capital accumulation policy is $k_1^*(x) = k^P(x)\mathbf{1}_{x>\widehat{X}} + \mathbf{1}_{x\leq\widehat{X}}$. This policy is preemptive up until a critical threshold $\widehat{X}$ is reached, which is implicitly defined by firm 1's first-order condition

$$\widehat{X} = \frac{\beta}{\beta+1}\frac{\Pi^M}{1-k^P\left(\widehat{X}\right)}.$$

This threshold is represented in (x, K_1)-space by the intersection of $k^P(x)$ with the locus of monopoly investment thresholds $X_1^L\left(\overline{K}_1\right) = (\beta/(\beta+1))\left(\Pi^M/\left(1-\overline{K}_1\right)\right)$ (see Figure 2). Because of the volatility of the input price, the time at which the threshold $\widehat{X}$ is first hit and firm 1 enters is stochastic. Up until then, were it to have the follower role, its probability of completing its investment in a given future time interval is an increasing function of its capital stock. Firm 1's optimal capital accumulation policy thus involves a form of brinkmanship insofar as along the path of its capital accumulation, its follower threat presents firm 2 with a gradually increasing risk of losing its monopoly position.

Incorporating a stochastic input price allows the effect of uncertainty on investment decisions to be studied by varying the value of the volatility parameter. An increase in σ, which reduces β, has several effects. First, it raises option value, which leads firms to delay product market entry. For a given capital stock level, firm 1 thus lowers its monopoly threshold $X_1^L(\kappa)$. The effect of greater volatility on preemption is generally complex, but with geometric Brownian motion greater volatility attenuates preemption, shifting the preemptive capital accumulation policy $k^P(x)$ down

[18] Specifically

$$\beta(\mu,\rho,\sigma) = \frac{\mu}{\sigma^2} - \frac{1}{2} + \sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

satisfies $\beta > 0$ and $\partial\beta/\partial\sigma < 0$. Letting $\tau(X) = \inf\{t \in \mathbb{R}_+ |\, X(t) \leq X\}$ denote the first hitting time for threshold $X \leq x = X(0)$, $\mathbb{E}_x e^{-r\tau(X)} = (X/x)^\beta$.

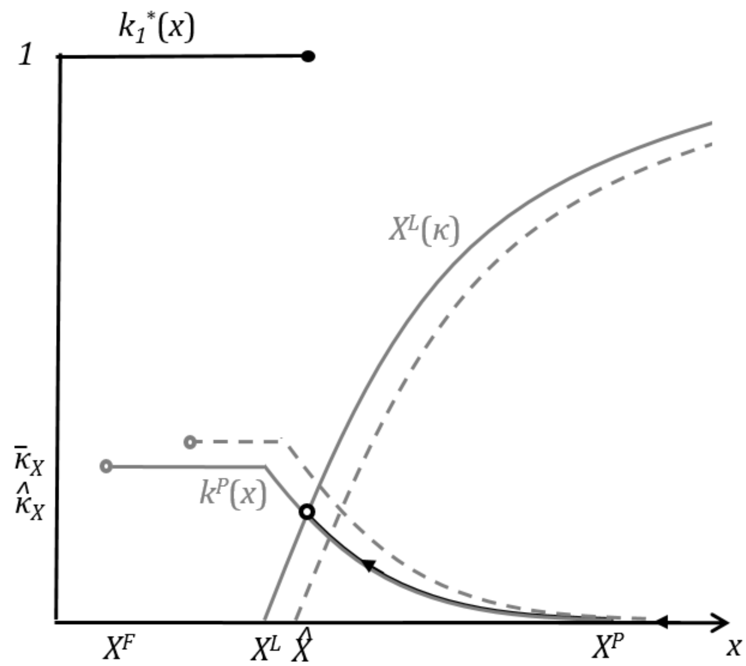

Figure 2: In the stochastic case greater volatility shifts $X^L(\kappa)$ and $k^P(x)$ leftward, lowering the threshold $\widehat{X}$ and delaying the jump to completion.

and to the left. Therefore $\partial\widehat{X}/\partial\sigma < 0$, which can be checked by direct calculation (see Appendix A.3.2). As the limited and asymmetric form of competition in the present model allows the leader to invest at a (myopically) optimal threshold despite the presence of a rival, the effect of greater volatility thus turns out to mirror the role this parameter has on investment decisions in the absence of a competitive threat. Finally, the effect of greater volatility on the size of firm 1's last investment step will generally depend on the relative magnitude of the effects on monopoly investment and preemption.

6 Conclusion

This article has studied dynamic competition in an industry in which one firm subdivides its investment in a capital input to make a gradual strategic commitment. Provided that the price of this input is initially high enough for firms to prefer delaying, such a firm follows a policy that involves accumulating capital preemptively and leading product market entry at a threshold that is both an instantaneous monopoly optimum and a preemptive equilibrium. Moreover if the input price follows a stochastic process, the timing of this firm's market entry is positively related to a measure of market uncertainty.

The emblematic example of strategic commitment, with capital having been sunk literally in order to gain a military rather than a business advantage, is the scuttling of the conquistador Hernando Cortés' ships before marching on the city of Mexico. Dixit and Nalebuff [5] explain that this action both signalled the Spaniards' determination to their Aztec adversary and compelled Cortés' men to fight by physically barring the possibility of retreat. Taking the perspective sketched in the present article, this historical event can also be related with an emphasis on more incremental steps during the first months of the conquistador's expedition.

According to Prescott [13] the destruction of the fleet occurred about six months after Cortés and his men first reached Mexican shores. The Spaniards had already had several exchanges with local populations and with the Aztec emperor Montezuma's emissaries over the course of the spring and early summer, and had begun to build a colony near the present day city of Veracruz. Several key new figures now numbered among their party, among whom Cortés' charismatic mistress La Malinche. Finally, a few of the men had just conspired to escape back to Cuba. Reinstated into its context, the sinking of their ships is better understood therefore as the culmination of a lengthier process during which the conquistadors familiarized themselves with their surroundings and made various types of commitments before the subsequent leap in their military engagement.

The piece of evidence that best supports viewing the destruction of the fleet as a part of a broader plan is that the Spaniards did not actually sink every single one of their ships strategically (but one, to be exact), or at least not literally so. In fact, the first ship to be sunk in a game-theoretic sense was sent not to lie on the Caribbean seabed but rather across the Atlantic, for a motive complementary to the one that governed the subsequent destruction of the fleet. Upon setting off from Cuba, Cortés had defied the orders of the island's governor Velasquez, and he needed to make a provision for the continuation game that would arise in the event that the conquest of Mexico succeeded. Cortés therefore cemented his *pronunciamento* by sending several men away on his best vessel as emissaries bearing the expedition's accumulated treasure, to pledge allegiance directly to the Spain's monarch.[19] It is after this, when a few of his men had nearly commandeered one of the remaining ships in order to return to Cuba, that Cortés forged and carried out the more fabled scheme to wreck the remainder of his fleet.

Interpreting this last act as a link in a chain rather than an isolated move is consistent with the view that strategic investment involves exercising a compound option, to which this article has sought to contribute. The formal analysis predicts that whereas higher monopoly rents accelerate capital accumulation, greater uncertainty delays the jump to completion, all else equal. In an economic reading of Cortés' conquest therefore, one should conclude that incremental steps over the first months of the expedition, though less individually striking, together played as considerable a role as the ten or so ships the conquistadors eventually scuttled, in the conduct of their highly lucrative but also most improbable undertaking.

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[19] See Prescott [13], p. 362:

> "Cortés ... knew that all the late acts of the colony, as well as his own authority, would fall to the ground without royal sanction. He knew, too, that the interest of Velasquez, which was great at court, would, so soon as he was acquainted with his secession, be wholly employed to circumvent and crush him. He resolved to anticipate his movements, and to send a vessel to Spain, with despatches addressed to the emperor himself, announcing the nature and extent of his discoveries, and to obtain, if possible, the confirmation of his proceedings."

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A Appendix

A.1 Asymmetric preemption threshold $T^P(\kappa)$ and upper bound $\overline{\kappa}$

The preemption time $\underline{t}_\kappa$ and the bound $\overline{\kappa}$ are both derived from the condition $L_2(t,\kappa) - F(t) = 0$, which has the specific form

$$\left(\Pi^M - X(0)e^{-\lambda t}\right)e^{-rt} - \left(\Pi^M - \Pi^D\right)\left(\frac{r}{\lambda + r}\frac{\Pi^D}{(1-\kappa)\,X(0)}\right)^{\frac{r}{\lambda}} - \frac{\lambda r^{\frac{r}{\lambda}}}{(\lambda + r)^{\frac{\lambda + r}{\lambda}}}\frac{\left[\Pi^D\right]^{\frac{\lambda + r}{\lambda}}}{[X(0)]^{\frac{r}{\lambda}}} = 0$$

provided that $t < T^F$. The left-hand side is a quasi-concave function of t with a maximum at T^L, which is shifted downward as κ increases. For $\kappa \in (0, \overline{\kappa})$, $\underline{t}_\kappa$ (and hence $T^P(\kappa)$) is well-defined and lies in $\left(T^P, T^L\right)$. In this interval an increase in κ, as it shifts the left-hand-side function downward, increases the lower root and decreases the upper root $\overline{t}_\kappa$, so that $d\underline{t}_\kappa/d\kappa > 0$ whereas $d\overline{t}_\kappa/d\kappa < 0$.

The level $\overline{\kappa}$ is the capital stock for which firm 2's investment incentive vanishes (at T^L), so that it satisfies $L_2\left(T^L;\overline{\kappa}\right) = F\left(T^L\right)$. Substituting for $t = T^L$ in the above equation (note that $e^{-rT^L} = \left(r\Pi^M/\left(\lambda + r\right)X(0)\right)^{r/\lambda}$) and dividing by $\left(r/\left(\left(\lambda + r\right)X(0)\right)\right)^{r/\lambda}$ gives

$$\frac{\lambda}{\lambda + r}\left[\Pi^M\right]^{\frac{\lambda + r}{\lambda}} - \left(\Pi^M - \Pi^D\right)\left(\frac{\Pi^D}{1 - \overline{\kappa}}\right)^{\frac{r}{\lambda}} = \frac{\lambda}{\lambda + r}\left[\Pi^D\right]^{\frac{\lambda + r}{\lambda}}$$

which after rearrangement yields the expression in the text.

A.2 Equilibrium of asymmetric preemption game

A slightly stronger result than Proposition 1 is established here, allowing an arbitrary starting time t_0. The first step is to describe the game formally.

If $t_0 \geq T^F$ immediate investment is weakly dominant for both firms as $L_i(t;\kappa) = F_i(t;\kappa)$ is decreasing over $\left(T^F, \infty\right)$, $i \in \{1, 2\}$, so any $\left(\widehat{T}_1, \widehat{T}_2\right) \in [t_0, \infty)^2$ with $\min\left\{\widehat{T}_1, \widehat{T}_2\right\} = t_0$ is an equilibrium whose outcome consists of simultaneous investments at t_0. The analysis therefore focuses on starting times $t_0 < T^F$. As $L_i(t;\kappa)$ is decreasing over $\left(T^F,\infty\right)$, firms are assumed to choose investment times $T_i \in \left[t_0, T^F\right]$, $i \in \{1,2\}$. Efficient tie-breaking is assumed to prevail if both firms want to invest at the same time (see Section 4.1). If $T_1 = T_2 = T$, the payoff profile is accordingly $(S_1(T;\kappa), S_2(T;\kappa)) = (L_1(T;\kappa) - \max\{L_2(T;\kappa) - F(T), 0\},\ F(T))$.[20] The payoffs of the game are

$$V_1(T_1,T_2) = \begin{cases} L_1(T_1;\kappa), & T_1 < T_2 \\ S_1(T;\kappa), & T_1 = T_2 = T \\ F_1(T_2;\kappa), & T_1 > T_2 \end{cases} \text{ and } V_2(T_1,T_2) = \begin{cases} L_2(T_2;\kappa), & T_2 < T_1 \\ S_2(T;\kappa), & T_2 = T_1 = T \\ F(T_1), & T_2 > T_1 \end{cases},$$

and the normal form is

$$\left(\{1,2\}, \left[t_0, T^F\right] \times \left[t_0, T^F\right], \{V_1(T_1,T_2), V_2(T_1,T_2)\}\right).$$

Then

Proposition 1' *In equilibrium,* $\widehat{T}_1 = \max\{t_0, \min\left\{T^L(\kappa), T^P(\kappa)\right\}\}$ *and*

$$\widehat{T}_2 \in$$

$$\begin{cases} \left\{\max\left\{t_0, T^P(\kappa)\right\}\right\} \ (\textit{if } \kappa < \widehat{\kappa}) \ \textit{or} \\ \left(\max\left\{t_0, T^L(\kappa)\right\}, T^F\right] \ (\textit{if } \kappa \geq \widehat{\kappa}), & t_0 < \max\left\{T^L(\kappa), T^P(\kappa)\right\} \\ \left[t_0, T^F\right], & \max\left\{T^L(\kappa), T^P(\kappa)\right\} \leq t_0 \leq \bar{t}_\kappa \\ \left(t_0, T^F\right], & t_0 > \bar{t}_\kappa \end{cases} \quad (\textit{if } \kappa \leq \overline{\kappa}),$$

or

$$\left(\max\left\{t_0, T^L(\kappa)\right\}, T^F\right] \ (\textit{if } \kappa > \overline{\kappa}).$$

Proof The level of κ determines whether the high cost firm's preemption threshold is defined or not and, when it is defined, its position relative to the low cost firm's monopoly threshold. The equilibrium characterization therefore involves three cases which are examined successively. If $\kappa > \overline{\kappa}$ (case a), firm 2 invariably follows and the equilibrium is sequential. If $\kappa \in [\widehat{\kappa}, \overline{\kappa}]$ (case b), firm 2's preemption range is non-empty (for $\kappa < \overline{\kappa}$, $\kappa = \overline{\kappa}$ being included as a limiting instance of this case) but although the equilibrium is preemptive firm 2's equilibrium choice does not constrain firm 1's. Finally if $\kappa \in (0, \widehat{\kappa})$ (case c), firm 2's preemption range is non-empty, the equilibrium is preemptive and firm 2's equilibrium choice constrains firm 1's.

[20] Efficient rationing allocates the input to firm 1 since $L_1(T;\kappa) - F_1(T;\kappa) > L_2(T;\kappa) - F(T)$ for $\kappa > 0$, but outside of $\mathcal{T}_2(\kappa)$ the rationing rule is of secondary importance in any case because as leading is not individually rational for firm 2, simultaneous investments cannot constitute an equilibrium.

case a :

If $\kappa > \overline{\kappa}$, then $\mathcal{T}_2(\kappa) = \{T^F\}$ so firm 2 does not attempt to lead at any $t < T^F$. The asymmetric preemption game then reduces to an individual decision problem for firm 1. Its optimum is $\widehat{T}_1 = \max\{t_0, T^L(\kappa)\}$ and firm 2 chooses any $\widehat{T}_2$ in $(\max\{t_0, T^L(\kappa)\}, T^F]$.

case b :

If $\kappa \in [\widehat{\kappa}, \overline{\kappa}]$, then $T^L(\kappa) \leq T^P(\kappa) \leq \bar{t}_\kappa < T^F$. Moreover, $\mathcal{T}_2(\kappa) = [T^P(\kappa), \bar{t}_\kappa] \cup \{T^F\}$ and firm 2's preemption range is non-empty for $\kappa < \overline{\kappa}$, $\kappa = \overline{\kappa}$ being the limiting case in which $\mathcal{T}_2(\kappa) = \{T^L, T^F\}$ has empty interior. Since $T^P(\kappa) \geq T^L(\kappa)$ however, firm 2's rational behavior does not constrain firm 1's preferred choice. Firm 1 can play its optimum $\widehat{T}_1 = \max\{t_0, T^L(\kappa)\}$ and firm 2 chooses $\widehat{T}_2 \in (\max\{t_0, T^L(\kappa)\}, T^F]$ (if $t_0 \notin \mathcal{T}_2(\kappa)$) or $[t_0, T^F]$ (if $t_0 \in \mathcal{T}_2(\kappa)$).

To verify that such $\widehat{T}_1$ and $\widehat{T}_2$ constitute the equilibrium strategies for any t_0, consider first starting times $t_0 < T^P(\kappa)$. Leading is unprofitable for firm 2 up until $T^P(\kappa)$, so it must choose a T_2 such that $T_2 \geq T^P(\kappa) \geq T^L(\kappa)$. There are two subcases with respect to κ to examine. First, if $\kappa \in (\widehat{\kappa}, \overline{\kappa}]$ then $T^P(\kappa) > T^L(\kappa)$ and firm 1 just sets its myopically optimal time $\widehat{T}_1 = \max\{t_0, T^L(\kappa)\}$ while firm 2 chooses any $\widehat{T}_2 \in (\max\{t_0, T^L(\kappa)\}, T^F]$. Otherwise $\kappa = \widehat{\kappa}$ in which case $T^P(\widehat{\kappa}) = T^L(\widehat{\kappa})$. Whereas firm 1's myopically optimal policy is $T_1 = \max\{t_0, T^L(\widehat{\kappa})\} = T^L(\widehat{\kappa})$, firm 2 is indifferent between any $T_2 \in [T^L(\widehat{\kappa}), T^F]$ because $S_2(T^L(\widehat{\kappa}); \widehat{\kappa}) = F(t)$ if it sets $T_2 = T^L(\widehat{\kappa}) = T_1$. Whether firm 2 invests simultaneously or not does not affect firm 1's payoff as $L_2(T^L(\widehat{\kappa}); \widehat{\kappa}) - F(T^L(\widehat{\kappa})) = 0$, so firm 1 sets $\widehat{T}_1 = \max\{t_0, T^L(\kappa)\}$ while firm 2 chooses $\widehat{T}_2 \in [\max\{t_0, T^L(\kappa)\}, T^F]$.

Next consider starting times t_0 such that $T^P(\kappa) \leq t_0 \leq \bar{t}_\kappa$. As $t_0 \geq T^L(\kappa)$ then holds, $L_1(t;\kappa)$ is decreasing over (t_0, T^F) and moreover $L_1(t;\kappa) \geq S_1(t;\kappa) > F_1(t;\kappa)$. Immediate investment is therefore a strictly dominant strategy for firm 1, so $\widehat{T}_1 = t_0$. Because $S_2(t_0;\kappa) = F(t_0)$ in this range, firm 2 is indifferent between simultaneous investment and following and hence between all $\widehat{T}_2 \in [t_0, T^F]$. Note that here firm 1's payoff is a correspondence whose value depends on firm 2's equilibrium choice. It is either $L_1(t_0;\kappa) - (L_2(t_0;\kappa) - F(t_0))$ (if $\widehat{T}_2 = t_0 \in (T^P(\kappa), \bar{t}_\kappa)$, in which case $L_2(t_0;\kappa) - F(t_0) > 0$) or $L_1(t_0;\kappa)$ (otherwise).

Finally, consider starting times $t_0 > \bar{t}_\kappa$. Then $L_1(t;\kappa)$ is decreasing and firm 1, preferring immediate investment, sets $\widehat{T}_1 = t_0$ while firm 2 prefers to follow and sets $\widehat{T}_2 \in (t_0, T^F]$.

case c :

If $\kappa \in (0, \widehat{\kappa})$ then $T^P(\kappa) < T^L(\kappa) < \bar{t}_\kappa$, so $\mathcal{T}_2(\kappa) = [T^P(\kappa), \bar{t}_\kappa] \cup \{T^F\}$ and firm 2's preemption range is non-empty. As $T^P(\kappa) < T^L(\kappa)$, firm 2 constrains firm 1's equilibrium choice if $t_0 < T^L(\kappa)$. Such a situation arises when firm 1 prefers delaying until $T^L(\kappa)$ while firm 2 has a non-negative investment incentive, so preemption occurs. Otherwise over the rest of the preemption range firm 2 is indifferent between simultaneous investment and following as described in case b, and thereafter it prefers follow. The equilibrium strategies are $\widehat{T}_1 = \max\left\{t_0, T^P(\kappa)\right\}$ and $\widehat{T}_2 = \max\left\{t_0, T^P(\kappa)\right\}$ (if $t_0 < T^L(\kappa)$), $\widehat{T}_2 \in \left[t_0, T^F\right]$ (if $T^L(\kappa) \leq t_0 \leq \bar{t}_\kappa$) or $\left(t_0, T^F\right]$ (if $t_0 > \bar{t}_\kappa$).

To verify that these are the equilibrium strategies, consider first starting times $t_0 < T^L(\kappa)$. The firms then both have positive investment incentives over times in $(T^P(\kappa), T^L(\kappa))$. For $t_0 \leq T^P(\kappa)$, firm 2 must choose $T_2 \geq T^P(\kappa)$ since investing earlier is not individually rational, and therefore $T_1 \geq T^P(\kappa)$ since firm 1's leader payoff is increasing over $(T^P(\kappa), T^L(\kappa))$. Moreover $T_i < T_j$ with $T_i \geq \max\left\{t_0, T^P(\kappa)\right\}$ cannot be an equilibrium. If $T_i > \max\left\{t_0, T^P(\kappa)\right\}$ then firm j profitably preempts by setting some $T_j < T_i$, whereas if $T_i = \max\left\{t_0, T^P(\kappa)\right\} < T_j$ then firm i prefers to delay by setting $T_i' = T_i + \varepsilon$ for small ε. Therefore in equilibrium, $\widehat{T}_1 = \widehat{T}_2$, and as $\widehat{T}_1 = \widehat{T}_2 > \max\left\{t_0, T^P(\kappa)\right\}$ cannot constitute an equilibrium since either firm has an incentive to preempt, it must be that $\widehat{T}_1 = \widehat{T}_2 = \max\left\{t_0, T^P(\kappa)\right\}$. The resulting payoffs for firm 1 and firm 2 are $L_1\left(\max\left\{t_0, T^P(\kappa)\right\}; \kappa\right) - \left[L_2\left(\max\left\{t_0, T^P(\kappa)\right\}; \kappa\right) - F\left(\max\left\{t_0, T^P(\kappa)\right\}\right)\right]$ and $F(t_0)$ respectively.

Next, consider $t_0 \geq T^L(\kappa)$. Then a similar argument to the one in case b establishes that firm 1 invests immediately whereas firm 2 is either indifferent between simultaneous investment and following (if $t_0 \leq \bar{t}_\kappa$) or prefers to follow (if $t_0 > \bar{t}_\kappa$). In equilibrium therefore, $\widehat{T}_1 = t_0$ and $\widehat{T}_2 \in \left[t_0, T^F\right]$ (if $t_0 \leq \bar{t}_\kappa$) or $\left(t_0, T^F\right]$ (if $t_0 > \bar{t}_\kappa$). $\square$

In all equilibria, either firm 1 leads (either directly because $\widehat{T}_1 < \widehat{T}_2$ or due to efficient rationing if $\widehat{T}_1 = \widehat{T}_2$) or payoffs are identical, and in all cases firm 2 obtains a follower payoff $\widehat{V}_2(t_0) = F(t_0)$. When the game starts in the interior of firm 2's preemption range, firm 1's payoff is a correspondence whose multiple values, when they arise, are the sum of its follower payoff and either a differential rent or the full positional rent depending upon whether firm 2 invests simultaneously or as a follower:

$$
\widehat{V}_1(t_0) = \begin{cases} L_1\left(\max\left\{t_0, T^P(\kappa)\right\}; \kappa\right) - \left[L_2\left(\max\left\{t_0, T^P(\kappa)\right\}; \kappa\right) - F(t_0)\right], & t_0 \leq T^L(\kappa) \\ L_1(t_0; \kappa) - \mathbf{1}_{\widehat{T}_1 = \widehat{T}_2}\left[L_2(t_0; \kappa) - F(t_0)\right], & T^L(\kappa) < t_0 < \bar{t}_\kappa \quad (\text{if } \kappa < \widehat{\kappa}), \\ L_1(t_0; \kappa), & t_0 \geq \bar{t}_\kappa \end{cases}
$$

or

$$\begin{cases} L_1\left(\max\left\{t_0, T^L(\kappa)\right\};\kappa\right), & t_0 \leq T^P(\kappa) \\ L_1(t_0;\kappa) - \mathbf{1}_{\widehat{T}_1=\widehat{T}_2}\left[L_2(t_0;\kappa) - F(t_0)\right], & T^P(\kappa) < t_0 < \bar{t}_\kappa \quad (\text{if } \widehat{\kappa} \leq \kappa \leq \overline{\kappa}), \\ L_1(t_0;\kappa), & t_0 \geq \bar{t}_\kappa \end{cases}$$

or

$$L_1\left(\max\left\{t_0, T^L(\kappa)\right\};\kappa\right) \ (\text{if } \kappa > \overline{\kappa}).$$

A.3 Stochastic case

Fudenberg and Tirole [8]'s analysis can be applied to asymmetric firms and extended to the stochastic case with investment modeled as in Dixit and Pindyck [6] (Steg [16]). The main steps involved in defining the loci $X_1^L(\kappa)$ and $k^P(X)$ and obtaining the comparative static $\partial k^P\left(\widehat{X}\right)/\partial\sigma$ are the following.

A.3.1 Continuation payoffs

For a given level of the current price $X(t) = x$ of the capital good, let $V^F(x)$ denote the value of the option on the duopoly profit stream with capitalized value Π^D. In the continuation region, it satisfies

$$rV^F(x)\,dt = \mathbb{E}dV^F(x).$$

Developing the right hand side using Itô's lemma, taking the expectation and rearranging yields

$$\frac{\sigma^2}{2}x^2\left[V^F(x)\right]'' + \mu x\left[V^F(x)\right]' - rV^F(x) = 0$$

along with boundary and smooth pasting conditions $\lim_{x\to\infty} V^F(x) = 0$, $V^F\left(X^F\right) = \Pi^D - X^F$, and $\left[V^F\right]'\left(X^F\right) = -1$. For $\sigma > 0$ the fundamental quadratic $(\sigma^2/2)b(b-1) + b\mu - r = 0$ has two roots of which only the negative root $b' = -\left(\left(\mu/\sigma^2\right) - (1/2)\right) - \sqrt{\left((1/2) - (\mu/\sigma^2)\right)^2 + (2r/\sigma^2)}$ is consistent with the first boundary condition. The solution is therefore of the form $V^F(x) = Ax^{b'}$. Setting $\beta = -b'$, the exercise threshold of the duopoly growth option is $X^F = (\beta/(\beta+1))\,\Pi^D$, $A = \left[X^F\right]^{\beta+1}/\beta$ and

$$V^F(x) = \begin{cases} \Pi^D - x, & x \leq X^F \\ Ax^\beta, & x > X^F. \end{cases}$$

The value in initial currency units of obtaining the follower option at the stochastic time when the input price first hits a threshold $X \leq x = X(0)$, $\tau(X) = \inf\{t \geq 0 |\, X(t) \leq X\}$ is therefore

$$F(X) = \mathbb{E}_x\left[V^F(X)\,e^{-r\tau(X)}\right] = V^F(X)\left(\frac{X}{x}\right)^\beta.$$

The leader payoffs are

$$
\begin{aligned}
L_1\left(X;\overline{K}_1\right) &= \mathbb{E}_x\left[\int_{\tau(X)}^{\tau\left(\min\left\{X,X^F\right\}\right)} \pi^M e^{-rs}ds + \int_{\tau\left(\min\left\{X,X^F\right\}\right)}^{\infty} \pi^D e^{-rs}ds - \left(1-\overline{K}_1\right)X(\tau(X))e^{-r\tau(X)}\right] \\
&= \left(\Pi^M - \left(1-\overline{K}_1\right)X\right)\left(\frac{X}{x}\right)^{\beta} - \left(\Pi^M - \Pi^D\right)\left(\frac{\min\left\{X,X^F\right\}}{x}\right)^{\beta}
\end{aligned}
$$

and

$$
\begin{aligned}
L_2\left(X;\overline{K}_1\right) &= \mathbb{E}_x\left[\int_{\tau(X)}^{\tau\left(\min\left\{X,X_1^F\left(\overline{K}_1\right)\right\}\right)} \pi^M e^{-rs}ds + \int_{\tau\left(\min\left\{X,X_1^F\left(\overline{K}_1\right)\right\}\right)}^{\infty} \pi^D e^{-rs}ds - X(\tau(X))e^{-r\tau(X)}\right] \\
&= \left(\Pi^M - X\right)\left(\frac{X}{x}\right)^{\beta} - \left(\Pi^M - \Pi^D\right)\left(\frac{\min\left\{X,X_1^F\left(\overline{K}_1\right)\right\}}{x}\right)^{\beta}
\end{aligned}
$$

where $X_1^F\left(\overline{K}_1\right) = (\beta/(\beta+1))\left(\Pi^D/\left(1-\overline{K}_1\right)\right)$ denotes firm 1's duopoly investment threshold. Maximization of $L_1\left(X;\overline{K}_1\right)$ with respect to X results in the monopoly threshold $X_1^L\left(\overline{K}_1\right)$ given in the text.

A.3.2 Preemptive capital accumulation policy $k^P(x)$

Firm 2's preemption threshold $X^P(\kappa)$ is the upper root of the condition $L_2(X;\kappa) = F\left(X^F\right)$, $X \in \left(X^F, x\right)$, which is well-defined so long as κ does not exceed the upper bound $\overline{\kappa}_X$ which is derived below. After normalization by $x^{-\beta}$ this condition is

$$
\left(\Pi^M - X^P(\kappa)\right)\left[X^P(\kappa)\right]^{\beta} - \left(\Pi^M - \Pi^D\right)\left(\frac{\beta}{\beta+1}\frac{\Pi^D}{1-\kappa}\right)^{\beta} - \frac{\beta^{\beta}\left[\Pi^D\right]^{\beta+1}}{(\beta+1)^{\beta+1}} = 0.
$$

Similarly to Appendix A.1, setting $X^P(\kappa) = X^L = (\beta/(\beta+1))\,\Pi^M$ yields the upper bound beyond firm 2's preemption range vanishes,

$$
\overline{\kappa}_X = 1 - \left(\frac{(\beta+1)\left(\frac{\Pi^M}{\Pi^D}-1\right)}{\left(\frac{\Pi^M}{\Pi^D}\right)^{\beta+1}-1}\right)^{\frac{1}{\beta}}.
$$

For $\kappa \in (0,\overline{\kappa}_X)$, the upper root $X^P(\kappa) \in \left(X^L, X^P\right)$ is well-defined and rearranging yields the expression of $k^P(x)$ given in the text.

A.3.3 Equilibrium and sensitivity analysis

In (X, K_1)-space, the locus

$$X_1^L\left(\overline{K}_1\right) = \frac{\beta}{\beta+1}\frac{\Pi^M}{1-\overline{K}_1}$$

is defined for $\overline{K}_1 \in [0,1)$ and increasing and convex over its range with $X_1^L(0) = X^L$ and $\lim_{\overline{K}_1 \to 1} X_1^L\left(\overline{K}_1\right) = \infty$. The locus $k^P(X)$ defined in the text is decreasing for $X \in \left(X^L, X^P\right)$ with $k^P(X^L) = \overline{\kappa}_X$ and $k^P(X^P) = 0$. There is therefore a unique intersection, which is $\left(\widehat{X}, k^P(\widehat{X})\right)$.

As $\partial X_1^L/\partial\beta = X_1^L/\left(\beta\left(\beta+1\right)\right) > 0$ and $\partial\beta/\partial\sigma < 0$, an increase in volatility shifts $X_1^L\left(\overline{K}_1\right)$ leftward. The effect of greater volatility on $k^P(x)$ is more complex, but for geometric Brownian motion a standard result is that $\partial X^P\left(\kappa\right)/\partial\beta > 0$ so that an increase in volatility shifts $k^P(x)$ down and to the left.

To sign $\partial\widehat{X}/\partial\beta$, raise the equilibrium condition defining $\widehat{X}$ in the text to the power β and substitute for $\left(1 - k^P(\widehat{X})\right)^\beta$ to get

$$\frac{\left(\beta+1\right)\left(\frac{\Pi^M}{\Pi^D} - 1\right)}{\frac{(\beta+1)^{\beta+1}}{\beta^\beta}\left(\frac{\Pi^M}{\Pi^D} - \frac{\widehat{X}}{\Pi^D}\right)\left(\frac{\widehat{X}}{\Pi^D}\right)^\beta - 1}\widehat{X}^\beta = \left(\frac{\beta}{\beta+1}\right)^\beta\left[\Pi^M\right]^\beta.$$

To express this condition more compactly define $\widehat{x} = \widehat{X}/\Pi^D$ and $m = \Pi^M/\Pi^D$, then divide by $\left[\Pi^D\right]^\beta$. Cross-multiplying and rearranging yields an implicit definition of the equilibrium entry threshold,

$$\Gamma\left(\widehat{x},\beta\right) := m^\beta\widehat{x}^{\beta+1} - \left(m^{\beta+1} - m + 1\right)\widehat{x}^\beta + \frac{m^\beta\beta^\beta}{\left(\beta+1\right)^{\beta+1}} = 0.$$

The sign of $d\widehat{x}/d\beta = -\left(\partial\Gamma/\partial\beta\right)/\left(\partial\Gamma/\partial\widehat{x}\right)$ must then be determined. Since $\widehat{X} \geq X^L$, $\widehat{x} \geq \left(\beta/\left(\beta+1\right)\right)m$ and hence $\left(\beta+1\right)m^\beta\widehat{x} \geq \beta m^{\beta+1}$. Therefore

$$\frac{\partial\Gamma}{\partial\widehat{x}} = \left(\left(\beta+1\right)m^\beta\widehat{x} - \beta\left(m^{\beta+1} - m + 1\right)\right)\widehat{x}^{\beta-1} \geq \beta(m-1)\widehat{x}^{\beta-1} > 0.$$

To sign $\partial\Gamma/\partial\beta$, note that $\partial\left(m^\beta\beta^\beta/\left(\beta+1\right)^{\beta+1}\right)/\partial\beta = \left(\ln\left(m\beta/\left(\beta+1\right)\right)\right)\left(m^\beta\beta^\beta/\left(\beta+1\right)^{\beta+1}\right)$. Differentiation therefore yields

$$\frac{\partial\Gamma}{\partial\beta} = m^\beta\widehat{x}^{\beta+1}\ln\left(m\widehat{x}\right) - m^{\beta+1}\widehat{x}^\beta\ln m - \left(m^{\beta+1} - m + 1\right)\widehat{x}^\beta\ln\widehat{x} + \frac{m^\beta\beta^\beta}{\left(\beta+1\right)^{\beta+1}}\ln\left(\frac{m\beta}{\beta+1}\right).$$

Using the definition of $\Gamma\left(\widehat{x},\beta\right)$ to substitute for $m^\beta\beta^\beta/\left(\beta+1\right)^{\beta+1} = \left(m^{\beta+1} - m + 1\right)\widehat{x}^\beta - m^\beta\widehat{x}^{\beta+1}$,

$$\frac{\partial\Gamma}{\partial\beta} = m^\beta\widehat{x}^{\beta+1}\ln\left(\frac{\left(\beta+1\right)\widehat{x}}{\beta}\right) - m^{\beta+1}\widehat{x}^\beta\ln m + \left(m^{\beta+1} - m + 1\right)\widehat{x}^\beta\ln\left(\frac{\beta m}{\left(\beta+1\right)\widehat{x}}\right).$$

Since $m > 1$ and $\widehat{x} > m\beta/(\beta+1)$, $\widehat{x} > \beta/(\beta+1)$ so the first summand is positive whereas the last two summands are negative. Since by definition $\Gamma(\widehat{x},\beta) = 0$, $m^{\beta}\widehat{x}^{\beta+1} < \left(m^{\beta+1} - m + 1\right)\widehat{x}^{\beta}$, the first two summands are bounded above by

$$\left(m^{\beta+1} - m + 1\right)\widehat{x}^{\beta}\ln\left(\frac{(\beta+1)\,\widehat{x}}{\beta}\right) - m^{\beta+1}\widehat{x}^{\beta}\ln m = m^{\beta+1}\widehat{x}^{\beta}\ln\left(\frac{(\beta+1)\,\widehat{x}}{\beta m}\right) - (m-1)\,\widehat{x}^{\beta}\ln\left(\frac{(\beta+1)\,\widehat{x}}{\beta}\right).$$

Applying this bound and simplifying, $\partial\Gamma/\partial\beta < -(m-1)\,\widehat{x}^{\beta}\ln m < 0$. This establishes $\partial\widehat{x}/\partial\beta > 0$ and hence $\partial\widehat{X}/\partial\sigma < 0$.