Risk neutral demand forecast and real options valuation: the case of crude oil capacity plan.

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Abstract

In this paper, we propose a risk-neutral multi-factors stochastic model for commodity demand. The model estimation is able to incorporate informations from demand historical time series as well as market prices of commodity financial derivatives. Then, the risk neutral model is applied to the case study of crude oil commodity. Inspired from the empirical literature, we consider a two factors model for the demand dynamic. The factors are the commodity spot price and the real per capita gross domestic product. The factors coefficients are estimated using historical series. Furthermore, the risk neutral model of crude oil spot price is calibrated on derivatives prices quoted in the market (Futures and options written on crude oil). Finally, we apply the risk neutral demand dynamic to a real project valuation, e.g. a crude oil production plant. We assess the optimal operating capacity and the maximal expected profit of the production plant. Moreover, we evaluate the expected profit of the plant, considering an option of expansion and its optimal timing. In the real option valuation, we assess the impact of adopting average type options and we compare different risk neutral price/demand dynamics (e.g. diffusive and jump-diffusive processes) and different plausible values of the factors coefficients.[1]

Keywords: Real Options, Demand Forecast, Stochastic Demand, Capacity Planning

1. Introduction

Real projects valuation commonly deals with future demand uncertainty of certain goods. A traditional method to approach these management operations is the real option valuation theory, which is introduced by Pindyck (1988) and discussed among others by Chung (1990), Dixit and Pindyck (1994), Burnetas and Ritchken (2005), Bϕckman et al. (2008), Guthrie (2009), Driouchi et al. (2010), Secomandi and Seppi (2014). In the real option valuation theory, the firm is commonly assumed to be small relative to the markets that trade the relevant goods. Hence, the firm operational policy does not affect market prices that are taken as exogenous stochastic factors. Furthermore, typically, the project valuation is the sum of expected discounted operational cashflows and the expectation is

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taken under the risk neutral probability measure. In order to apply the standard option valuation technique to real projects, we need to determine the appropriate risk-neutral probability measure, or, in other words, the risk-neutral stochastic process describing the evolution of the relevant risk factors over time.

The standard Black-Scholes option pricing model assumes that the risk-neutral stochastic process follows a lognormal process, which is also the common practice for modelling commodity price and demand in real option theory. For example Pindyck (1988) adopts a linear function that relates demand and commodity price, modelled as a stochastic lognormal process. Linear demand functions and lognormal price dynamic are also assumed in Chung (1990), Dixit and Pindyck (1994), Bϕckman et al. (2008). In Pennings and Natter (2001) and Driouchi et al. (2010), the authors model instead the demand dynamic via a normal or lognormal process. However, the assumptions of a normal or lognormal price/demand dynamics are in contrast with empirical evidence in energy, agricultural and other commodity markets (e.g., see Benth et al., 2008 for commodity prices and Cooper, 2003, Hughes et al., 2008, Kilian and Murphy, 2014 for empirical econometric demand function estimations). For these reasons, we adopt structural multi-factor model for the demand dynamic. Inspired from the empirical literature, we consider a two factors model for the crude oil demand dynamic. The factors are the commodity spot price and the real per capita gross domestic product. Moreover, the commodity spot price is modelled using non-Gaussian price dynamics (for instance exponential jump-diffusion models). Hence, our first contribution is the proposal of a novel risk neutral demand forecast procedure. More specifically, we assume a two-factor structural model for the demand dynamic and we incorporate information from demand historical series data as well as commodity financial derivatives (e.g., futures and options) quoted in the market.

Once the stochastic demand model is defined and calibrated, we consider a real option application to the valuation of a capacity planning. Tracking the average historical demand is a long-term capacity strategy aimed at establishing the general level of resources needed by a firm. Average capacity strategies are widespread in the industry and they have a deep financial justification. In fact, the averaging reduces the impact of demand uncertainty in the valuation of capacity operations. In Driouchi et al. (2010), the authors suggest that average real options can help resolve problems of overestimation of flexibility in the capacity strategy. Hence, the second contribution of this paper is the valuation of capacity planning, including the optimal timing feature of basing decisions on average demand. This is possible by considering average type options rather than plain vanilla options.

Finally, we present the case study of a crude oil production plant. The choice of this commodity is motivated by the availability of spot, futures and option prices quoted in the Chicago mercantile exchange (CME) that can be used for calibration of the commodity price model, in addition to an empirical literature on the estimation of structural multi-factor models for the demand (see for instance Cooper, 2003, Hughes et al., 2008, Kilian and Murphy, 2014). For the crude oil demand model, we consider two factors: the commodity spot price and the real per capita GDP (Gross Domestic Product) and we resort on empirical results in literature to test different values of factors coefficients.

The contribution of this paper to the literature of operations management is twofold. Firstly, we propose a novel risk-neutral demand forecast procedure. Secondly, we apply an average type option approach to real project valuation (e.g. capacity planning). In the project valuation, we include optimal timing feature and we compare different spot price models (e.g. diffusive and jump-diffusive processes).

The work is organized as follow. Section 2 describes the risk neutral model for the stochastic demand and discusses its estimation. Section 3 presents the capacity planning valuation problem with the application of average type options. Section 4 describes the numerical schemes used to price the options. Section 5 presents numerical results about calibration and some numerical example of capacity plan valuation. Conclusive remarks are presented in last section.

2. Risk neutral model for stochastic demand

Typically, the real options literature assumes very simple risk neutral demand dynamic, i.e. a single factor normal or log-normal stochastic process. In this work, instead, we propose a structural multi-factors model, with non-Gaussian stochastic processes. The relevant demand factors are specific to each commodity, e.g. the real per capita GDP (Gross Domestic Product) of the firm's country for crude oil or the temperature for natural gas. Clearly, the spot price should be a relevant factor for the demand dynamic of any commodity. To make concrete our analysis, this work considers the case of crude oil commodity; and following the empirical literature concerning demand function estimation (see for instance Cooper, 2003 and Hughes et al., 2008, Kilian and Murphy, 2014), we assume a two-factors model in a discrete time setting with step Δ, described by the following equation

$$Q_n = K\, S_n^{\eta}\, G_n^{\beta}. \tag{1}$$

Here, Q_n, S_n and G_n represent the demand, the commodity spot price and the real per capita GDP at time $n\Delta$. Moreover, K is a positive constant, $\eta < 0$ is the spot price coefficient (elasticity parameter) and $\beta \in \mathbb{R}$ is the GDP coefficient. In the limit of $\eta \to 0$, the demand faced by the firm is unrelated to commodity market price but it depends only on the GDP factor, while in the limit of $\beta \to 0$, there is a perfect dependence between price and demand of the commodity. The parameter K is a normalization constant, i.e. $K = \frac{Q_0}{S_0^\eta \, G_0^\beta}$. Assuming that the GDP factor is independent from the price, the forward demand curve is calculated as

$$\mathbb{E}\left[Q_n\right] = K \, \mathbb{E}\left[S_n^\eta\right] \, \mathbb{E}\left[G_n^\beta\right] \; = Q_0 \, \mathbb{E}\left[\left(\frac{S_n}{S_0}\right)^\eta\right] \, \mathbb{E}\left[\left(\frac{G_n}{G_0}\right)^\beta\right] \tag{2}$$

where the expectations are under the risk neutral measure, i.e. $\mathbb{E}\left[S_n\right] = S_0 \, e^{r\, n\, \Delta}$ and r is the constant risk free rate, net of the convenience yield (here to be taken zero). Without loss of generality, we can normalize the commodity price and the GDP factor, i.e. $S_0 = G_0 = 1$.

The estimation of the demand dynamic is performed in two steps. Firstly, the coefficients η and β are estimated, using historical series of the demand faced by the firm, market commodity prices and real per capita GDP, through the following linear regression

$$\ln Q_n = \ln K + \eta \, \ln S_n + \beta \ln G_n + \epsilon_n, \tag{3}$$

where ϵ_n is the homoscedastic error term . Since historical data on a specific firm demand are not available, we estimate the parameters η and β using the aggregate market demand and we suppose that firms are homogeneous, i.e. we do not consider firm specific factors.

The spot price dynamic is defined under the risk neutral measure as

$$S_n = S_0 e^{\sum_{j=1}^{n} \xi_j}, \tag{4}$$

where ξ_j are independent variables. In particular, we consider two specifications for the ξ_j: the binomial diffusive model of Cox et al., 1979 (CRR model) and the multinomial jump-diffusive model proposed by Hilliard and Schwartz, 2005 (HS model). Then, the risk neutral dynamic of the spot price S_n is estimated using quoted derivatives in the market (for instance Futures and options). Finally the risk neutral dynamic of the process G_n is considered. Unfortunately, no liquid financial instruments written on GDP are available in the market[2]. Hence, we have to make some simplifying assumptions on the risk neutral dynamic of the GDP factor. Firstly, we suppose that under the risk neutral

[2] In recent years, several central banks and other policy institutions have expressed an interest in GDP-linked sovereign bonds as a potential instrument to promote financial stability (see for instance the European Commission discussion paper at `https://ec.europa.eu/info/sites/info/files/economy-finance/dp073_en.pdf`). However, no GDP-linked bonds currently exist in financial market.

measure G_n is a martingale process, i.e. $\mathbb{E}_m[G_n] = u_m$ for all $0 < m < n$ and we adopt for it a CRR binomial model. The volatility can be estimated using historical data on real per capita GDP.

3. The capacity planning case

In the first part of this section, we develop a real option model to set the optimal operating capacity under demand uncertainty. Moreover, we present also the problem of the optimal time to expand the plant capacity.

3.1. The optimal operating capacity problem

In this section, we recall the static capacity optimization problem presented in Driouchi et al. (2010) and we explain how the introduction of average type options is natural and intuitive in this contest. A firm aims to set at time zero the operating capacity M of a plant in a time horizon T, so that the expected profit of the plant is optimized. The expected plant revenue is given by the product between the contribution margin $m > 0$ per unit demand and the produced quantity. As in Pennings and Natter (2001) and Driouchi et al. (2010), we assume that the contribution margin m is fixed and that there is no inventory. The commodity amount that can be produced is capped by the operating capacity M. Hence, the expected plant revenue at a fixed time horizon T is

$$\begin{aligned} V(M) &= \mathbb{E}\left[e^{-r\,T}\, m\, \min(Q(0,T), M)\right], \\ &= e^{-r\,T}\, m\, \left(M - \mathbb{E}\left[(M - Q(0,T))^+\right]\right), \end{aligned} \tag{5}$$

where $(\cdots)^+$ denotes the positive part, r is the discounting risk free rate and $Q(0,T)$ is the cumulated stochastic demand in the interval $[0,T]$. We work in a discrete time framework, i.e. $T = N\,\Delta$, where N is the number of time steps and Δ is the length of a step and $Q(0,T)$ is modelled as

$$Q(0,T) = \sum_{n=0}^{N} Q_n. \tag{6}$$

Here, Q_n is the demand in a single time step and it is defined in Section 2, equation (1). The revenue can be rewritten using the capacity in a single time step $\tilde{M} = \frac{M}{N+1}$ and the time average of the demand in the following way

$$V(M) \;=\; e^{-r\,T}\, m\, \left(M - (N+1)\,\mathbb{E}\left[\left(\tilde{M} - \frac{1}{N+1}\sum_{n=0}^{N} Q_n\right)^+\right]\right), \tag{7}$$

Hence, the option embedded in the plant revenue is of average type. The plant expected profit is defined as revenue minus production cost as

$$C(M) = V(M) - c\,M, \tag{8}$$

where c denotes the fixed cost per unit capacity. We define M_I as the installed capacity, i.e. the greatest possible operating capacity of the plant. The optimal operating capacity M^* is set in the interval $[0, M_I]$, in order to maximise the expected profit, i.e.

$$M^* = \text{argmax}_{0 \leq M \leq M_I} C(M). \tag{9}$$

The numerical method used to evaluate the real option embedded in the expected plant revenue and hence to solve the optimization (9) is explained in details in Appendix A.

3.2. The optimal time to expand

We consider the case of a plant with a suboptimal installed capacity, for instance a plant that operates at the installed capacity M_I, which means that the expected profit in equation (8) is an increasing function in the range $[0, M_I]$. The firm has the possibility to expand the plant at predetermined future decision dates $\{T_1, T_2, ..., T_{R-1}\}$, with $T_R = T$ (see chapter 5 in Guthrie (2009) for the case of a single decision date). The plant can be expanded of an additional capacity M_e, with a cost I. We define n_e as the number of time steps between two consecutive decision dates, i.e. $T_i = i\,n_e\,\Delta$ for all $i = 1, ..R-1$. The level of flexibility of the firm is represented by the frequency of the decision making process. The maximal level of flexibility is $n_e = 1$, i.e. each time step is a decision date, while the minimal level is $n_e = N$, i.e. there are no decision dates and the expansion is not possible. Greater flexibility means a larger value of the expansion real option. We define x_i as the operational state of the plant at time T_i. It can assume two values: $x_i = 0$ if the plant is not expanded and $x_i = 1$ if the plant is expanded. Moreover, we define π_i as the action at time T_i and it can also assume two values: $\pi_i = 0$ (do nothing) or $\pi_i = 1$ (expand). The plant cash-flows at times T_i with $i = 1, .., R-1$ are

$$Y_i(x_i, \pi_i) = \begin{cases} m\, \min(M, Q(T_{i-1}, T_i)) - c\,M, & \text{if } (x_i, \pi_i) = (0,0), \\ m\, \min(M, Q(T_{i-1}, T_i)) - c\,M - I, & \text{if } (x_i, \pi_i) = (0,1), \\ m\, \min((M + M_e), Q(T_{i-1}, T_i)) - c\,(M + M_e), & \text{if } (x_i, \pi_i) = (1,0), \end{cases}$$

where $Q(T_{i-1}, T_i)$ is the cumulated demand in the time interval $[T_{i-1}, T_i)$, $M = \frac{M_I}{R}$ is the operating capacity in $[T_{i-1}, T_i)$ and M_e is a fraction of M. To highlight the optionality features of the problem, the cash-flows can be rewritten as

$$Y_i(x_i, \pi_i) \quad = \quad \begin{cases} (m-c)\, M - m\, \left(M - Q(T_{i-1}, T_i)\right)^+, & \text{if } (x_i, \pi_i) = (0,0), \\ (m-c)\, M - m\, \left(M - Q(T_{i-1}, T_i)\right)^+ - I, & \text{if } (x_i, \pi_i) = (0,1), \\ (m-c)\, (M + M_e) - m\, \left(M + M_e - Q(T_{i-1}, T_i)\right)^+, & \text{if } (x_i, \pi_i) = (1,0). \end{cases}$$

Hence, the previous equation makes clearer that the cash-flows contain two optionality features. The first is a negative put option with strike price M or $M + M_e$, due to the fact that the produced quantity is capped by the operating capacity. The second is the expansion option, i.e. the possibility to increase the operating capacity from M to $M + M_e$ at a cost I. The cumulated demand in the interval $[T_{i-1}, T_i)$ is modelled as

$$Q(T_{i-1}, T_i) = \sum_{n=(i-1)\cdot n_e}^{i\cdot n_e - 1} Q_n, \tag{10}$$

where Q_n is the demand in a single time step and it is defined in Section 2. The cash-flows can be rewritten as functions of the average demand in the interval $[T_{i-1}, T_i)$, using the capacity in a time step $\tilde{M} = \frac{M}{n_e}$ and $\tilde{M}_e = \frac{M_e}{n_e}$,

$$Y_i(x_i, \pi_i) \quad = \quad \begin{cases} (m-c)\, M - n_e\, m\, \left(\tilde{M} - \frac{1}{n_e} \sum_{n=(i-1)\cdot n_e}^{i\cdot n_e - 1} Q_n\right)^+, & \text{if } (x_i, \pi_i) = (0,0), \\ (m-c)\, M - n_e\, m\, \left(\tilde{M} - \frac{1}{n_e} \sum_{n=(i-1)\cdot n_e}^{i\cdot n_e - 1} Q_n\right)^+ - I, & \text{if } (x_i, \pi_i) = (0,1), \\ (m-c)\, (M + M_e) - n_e\, m\, \left((\tilde{M} + \tilde{M}_e) - \frac{1}{n_e} \sum_{n=(i-1)\cdot n_e}^{i\cdot n_e - 1} Q_n\right)^+, & \text{if } (x_i, \pi_i) = (1,0). \end{cases}$$

The expected firm profit is obtained solving the following stochastic optimization problem

$$C = \max_{\pi \in \Pi} \sum_{i=1}^{R} e^{-rT_i}\, \mathbb{E}\left[Y_i(x_i(x_{i-1}, \pi_i), \pi_i)\right], \tag{11}$$

where $\pi = \{\pi_0, \pi_1, .., \pi_R\}$ is an operational policy and Π is the set of non anticipative policies, such that if $\pi_i = 1$, then $\pi_j = 0$ for $j \neq i$. To solve the optimization problem, we fix the boundary conditions: the plant is not expanded at time zero, i.e. $x_0 = 0$ and the last date $T_R = T$ is not a decision date, i.e. $\pi_R = 0$.

The optimization (11) is equivalent to a stochastic dynamic programming equation using a two dimensional Markovian state variable. The detailed derivation of the stochastic dynamic programming equation is given in Appendix B.

4. Model applications

In this section, we present a real application of the model to the valuation of the capacity planning of a crude oil production plant. Firstly, we calibrate the risk neutral parameters of the spot price dynamic using Futures and options written on crude oil. Secondly, we build the risk neutral demand forward curve. Using the estimated demand dynamic, we evaluate the real options embedded in the capacity plan problems presented in Section 3.

4.1. Calibration of spot price dynamic

We calibrate the CRR and the HS jump-diffusion model with piecewise constant parameters to American plain vanilla options written on light sweet crude oil Futures at April 7^{th} 2017 with maturities 10, 40, 69, 101, 132, 161, 193, 222, 405 and 586 days.

The CRR model with piecewise constant volatility perfectly fits the at-the money prices of American plain vanilla options, while the HS jump-diffusion model captures also the volatility smile term structure. Table 1 and 2 report the calibrated parameter values and the root mean square errors (RMSE) for both models. Figure 1 reports examples of the calibration results for the HS jump-diffusion model.

CRR model		
maturity (days)	volatility	RMSE (%)
10	0.2457	2%
40	0.2637	5%
69	0.3121	8%
101	0.2636	12%
132	0.2894	18%
161	0.1892	16%
193	0.3330	18%
222	0.2700	21%
405	0.2515	34%
586	0.2012	41%

Table 1: Calibration results of the CRR model on American plain vanilla options written on Light Sweet Crude Oil Futures and quoted in CME market at April 7^{th} 2017.

Finally, to test the goodness of calibration, we reproduce prices of average options quoted in the market. In particular, we consider European Asian[3] options written on light sweet crude oil Futures at April 7^{th} 2017. To price the Asian option, we use the tree method presented in Gambaro et al.

[3]In financial markets, the options written on the time average of an asset are called Asian options.

HS jump-diffusion model					
maturity (days)	volatility	jump intensity	jump mean	jump vol.	RMSE (%)
10	0.2127	1.3983	0.0064	0.1409	1%
40	0.2383	1.2237	-0.1077	0.0678	3%
69	0.1653	3.9785	-0.1420	0.0957	2%
101	0.1072	1.9228	-0.1895	0.1119	2%
132	0.1425	1.5725	-0.2120	0.1402	3%
161	0.1379	0.7594	-0.2813	0.1305	1%
193	0.0145	0.8370	-0.3967	0.0167	1%
222	0.0064	0.7398	-0.2082	0.4430	3%
405	0.0061	0.5121	-0.2098	0.3790	2%
586	0.1148	0.1262	-0.5419	0.5186	2%

Table 2: Calibration results of the HS jump-diffusion model on American plain vanilla options written on Light Sweet Crude Oil Futures and quoted in CME market at April 7^{th} 2017.

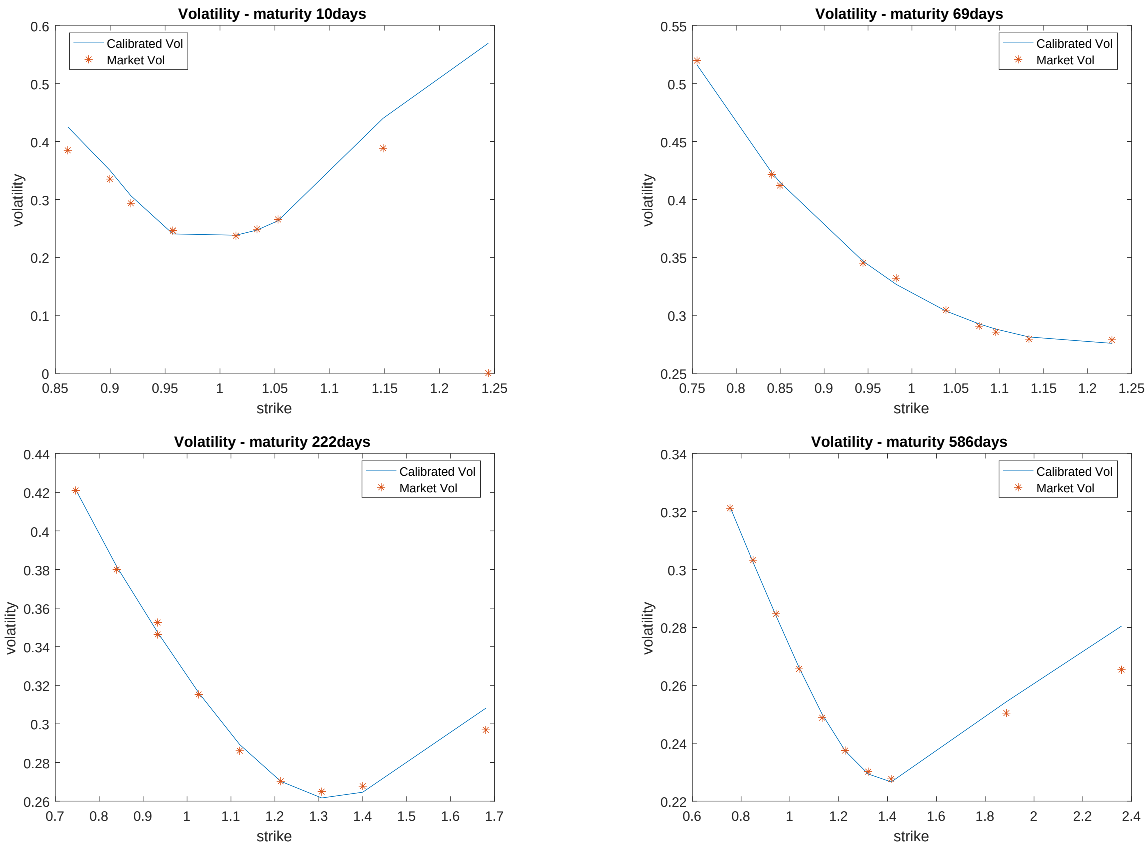

Figure 1: Market versus model implied volatility for the HS jump-diffusion model calibration (see Table 2).

(2019), described in details in Appendix C. In Table 3, we compare quoted prices with the average option prices obtained using the calibrated parameters. As we expect, the HS jump-diffusion model produces much better results.

4.2. Risk neutral demand forecasts

In this section, we illustrate how the proposed model can be used to build risk neutral demand forecast, i.e. market implied expectation of demand at different time horizons. This is made possible

maturity (days)	relative strike (%)	option type	market price	CRR	Abs. Rel. Error (%)	HS jump-diff.	Abs. Rel. Error (%)
175	93.50%	Put	2.56	2.09	18%	2.52	2%
207	93.40%	Put	2.89	2.42	16%	2.88	0%
237	93.30%	Put	3.16	2.73	13%	3.32	5%
266	93.30%	Put	3.43	2.99	13%	3.63	6%
175	84.18%	Put	1.19	0.73	39%	1.27	6%
207	84.05%	Put	1.44	0.95	34%	1.55	8%
237	83.97%	Put	1.67	1.17	30%	1.92	15%
266	83.96%	Put	1.89	1.35	29%	2.19	16%
			AAPRE		24%		7%
			Max. APRE		39%		16%

Table 3: Comparison between prices average options written on Light Sweet Crude Oil Futures and quoted in CME market at April 7^{th} 2017 and prices obtain with model and parameter values calibrated in Tables 1 and 2.

via equation (2), where we combine the structural model introduced in equation (1) and the risk neutral model calibrated to market data as described in previous section. In particular, the elasticity parameter η for crude oil is taken from the empirical literature (see Cooper (2003), Hughes et al. (2008), Kilian and Murphy (2014), Knittel and Pindyck (2016) and the references therein) and the plausible interval is $-0.1 \leq \eta \leq -0.3$. Cooper (2003), Hughes et al. (2008) also present estimates of the coefficient β of the GDP factor. A plausible range is $0.5 \leq \beta \leq 1$. Moreover, in the linear regression (see equation (3)), to a larger value of $\eta = -0.1$ corresponds a lower value of $\beta = 0.5$ and viceversa. For the crude oil price S_n, we use the previously calibrated parameters and reported in Tables 1 and 2. Figure 2 reports the risk neutral forward curve of the demand, see equation (2). Figure 2 on the left shows the forward curves in the limit $\beta \to 0$ for different values of the elasticity parameter η. Figure 2 on the right shows the forward curves for different values of η and β fixing a volatility of the GDP factor σ_u equal to 0.3. In the realistic case, we have $\eta \neq 1$ and $\beta \neq 0$, hence the demand forward curve shape depends in a non linear way on the variance, the skewness and the kurtosis of the two factors. In the limit of $\beta \to 0$ and for $\eta = 1$, the demand forward curve is equal to the rescaled curve of crude oil forward prices ($S_0 = Q_0 = 100$), and it is independent on the higher moments of the log-price dynamic.

4.3. Real option valuation

In this section, we discuss a concrete example of the expected profit maximization as in equation (9). Moreover, we evaluate also the optimal time to expand (see equations (11) and (B.4)). We assume a two years time horizon T, an installed capacity $MI = 160/(N+1)$ units, a contribution margin $m = 1$, a unit cost $c = 0.35$, a current demand $Q_0 = 100$ units. For the demand dynamic, we adopt the model developed in Section 2. Moreover, to evaluate the average real options, we use the numerical schemes presented in Appendix A and Appendix B.

Figure 3 shows the expected profit of the plant using the demand value at time T (NO AVERAGE) or the demand average value in $[0, T]$ (AVERAGE). Figure 4 points out that jumps in the price dynamic push up the expected profit. This effect is due to the higher value of the expected forward demand curve shown in Figure 2. Table 4 reports the optimal level of operating capacity and the relative maximal expected profit. Adopting the time average to forecast the demand, the firm increases the expected profit of the plant and save resources using a lower optimal operating capacity. This is due to the fact that the put option embedded in equation (7) has a lower value, tracking the average demand. Moreover, adding jumps, both the optimal capacity level and the maximal profit grow.

For the valuation of the expected plant profit including the expansion option (see equation (11)), we use an installed capacity of the plant equal to 80 units, that is lower than optimal operating levels reported in Table 4. This means that the installed capacity is not sufficient to satisfy the demand and an expansion of the plant is needed. The operating capacity in the period between two decision dates is $M = \frac{80}{R}$ units. The considered values of R are $2, 4, 8$ and 504, which correspond to an annual, semi-annual, quarterly and daily frequency of the decision dates. The plant capacity can be expanded of a quantity $Me = 0.3\,M$ at a cost $I = 5$. Table 5 reports the expected profit of the plant for two demand dynamics (CRR or HS with jumps), different values of demand parameters η and β, and annual or daily frequency of the decision dates. The raise in decision-making frequency has two opposite effects. In fact, more flexibility increases the value of the expansion option and therefore the profit. However, the time average of the demand is calculated over shorter time intervals, so a more volatile underlying pushes up the (negative) value of the put option embedded in the operational cash-flows and the profit decreases. The overall effect depends on model parameters. This is evident in Table 5 and Figure 5. In fact, in Table 5 the profit decreases from annually to daily frequency for $(\eta, \beta) = (-0.3, 1)$, while for $(\eta, \beta) = (-0.1, 0.5)$ the profit increases. Moreover, Figure 5 shows that the trend of the expected profit with respect to the decision frequency depends on both the parameters (η, β) and on the expansion cost I. As for the case without expansion option, jumps push up the expected profit values.

5. Conclusions

In this paper, we present a structural multi-factor model for stochastic demand under the risk neutral measure. The model is applied to the valuation of a capacity planning, through a real option approach, tracking the time average demand. In the case study of a crude oil plant, we confirm that an average option approach reduces the impact of demand uncertainty in the evaluation of the expected

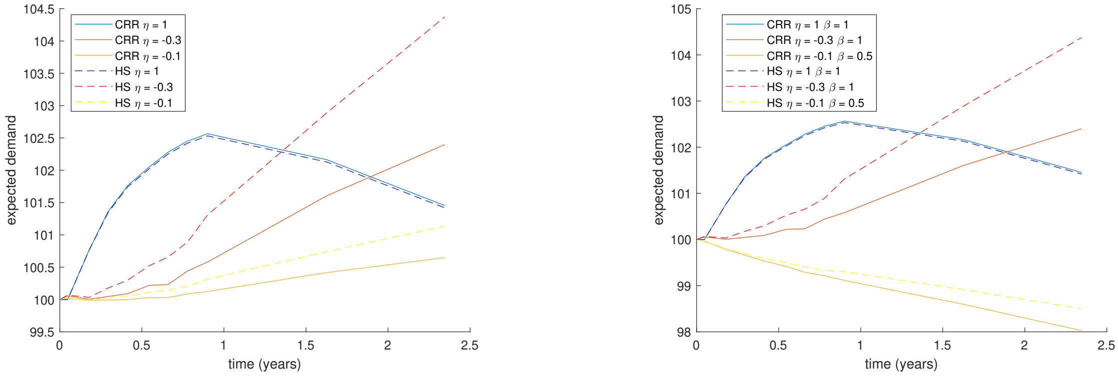

Figure 2: Figure on the left shows the forward curve of the demand in the limit of $\beta \to 0$ for different values of η. Figure on the right shows the curve for different values of η and β fixing $\sigma_u = 0.3$. In the two figures $Q_0 = 100$, and the parameters of the spot price dynamic are reported in Table 1 for CRR model and reported Table 2 for the HS model.

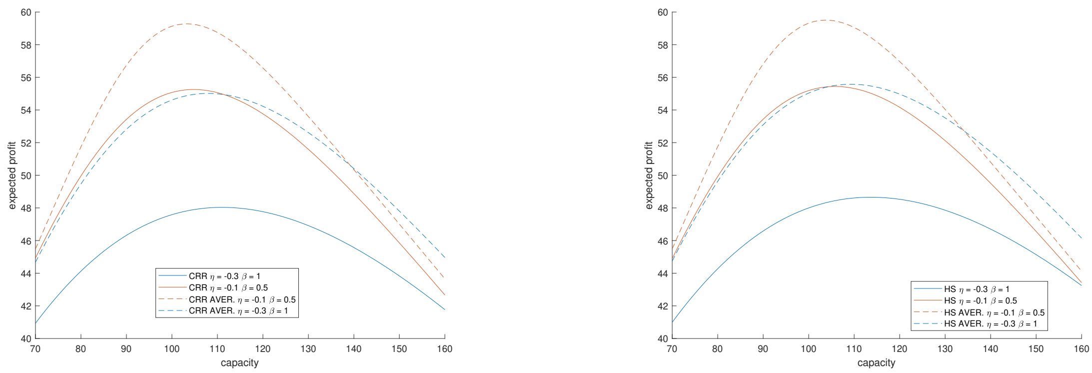

Figure 3: Figure on the left shows the expected profit of the plant using the point or the average estimation of the demand and the CRR model for the spot price dynamic. Figure on the right shows the expected revenue using the HS jump-diffusion model for the spot price dynamic.

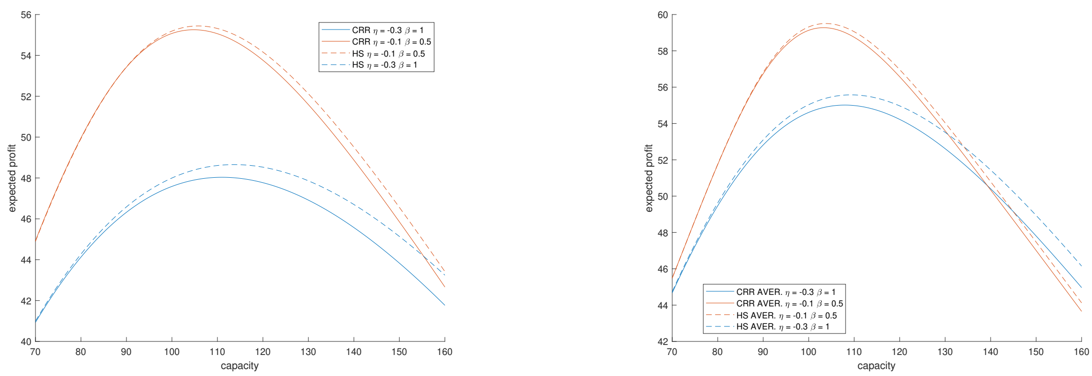

Figure 4: The two figures compare the expected profit obtained using a model without or with jumps, i.e. the CRR and the HS models. Figure on the left is obtained without time average of the demand, while figure on the right with the time average.

NO AVERAGE	CRR		HS jump-diff.	
(η, β)	$\mathbf{M}^*$	$\mathbf{C}(\mathbf{M}^*)$	$\mathbf{M}^*$	$\mathbf{C}(\mathbf{M}^*)$
(-0.3,1)	111.03	48.03	113.67	48.65
(-0.1,0.5)	104.80	55.25	105.77	55.44
AVERAGE	**CRR**		**HS jump-diff.**	
(η, β)	$\mathbf{M}^*$	$\mathbf{C}(\mathbf{M}^*)$	$\mathbf{M}^*$	$\mathbf{C}(\mathbf{M}^*)$
(-0.3,1)	107.95	55.01	109.32	55.57
(-0.1,0.5)	103.26	59.27	103.74	59.50

Table 4: The table reports the optimal operating capacity and the maximum expected profit (see equation (9)). The results are obtained using $M_i = 160$ units, $\sigma_u = 0.3$ and spot price parameters reported in Tables 1 and 2 for the CRR and the HS model, respectively.

Frequency	(η, β)	CRR	HS jump-diff.
Annually	(-0.3,1)	49.78	49.98
	(-0.1,0.5)	51.49	51.68
Daily	(-0.3,1)	48.91	49.11
	(-0.1,0.5)	52.87	52.97

Table 5: The table reports the expected profit including the expansion option (see equation (11)). The results are obtained using $M = \frac{80}{R}$ units, $R = 2$ and 504, $Me = 0.3\, M$, $I = 5$, $\sigma_u = 0.3$ and the spot price parameters reported in Tables 1 and 2 for the CRR and the HS model, respectively.

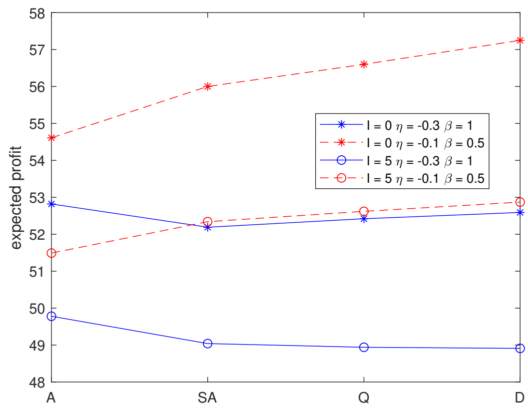

Figure 5: The figure shows the trend of the expected profit with respect to the frequency of the decision dates: annually (A), semi-annually (SA), quarterly (Q), daily (D). The results are obtained using $M = \frac{80}{R}$ units, $R = 2, 4, 8$ and 504, $Me = 0.3\, M$, $\sigma_u = 0.3$ and the spot price parameters reported in Tables 1 for the CRR model.

profit and in the estimation of the optimal capacity level. Furthermore, the introduction of jumps in the demand dynamic impacts sensibly the expected forward demand, the optimal capacity level and the maximal expected profit. Moreover, we study the expected profit including an expansion option and we compare different frequencies of the decision-making process. However, numerical tests reveal that the elasticity and the GDP coefficient have the most important role in the estimation of the optimal expected profit. In the literature, the difference between estimates of the demand function parameters is due to two different time regimes, i.e. short run or long run demand. Hence, a possible extension of the model presented in Section 2 is the adoption of time dependent parameters η and β, that assume two different values in the short and in the long time regime. Moreover, also the price dynamic can switch between the two regimes (for instance a mean reverting process in the short regime and an exponential jump-diffusion process in the long run).

This is a working paper, future researches include: extending the commodity spot price dynamic to other Markovian processes (e.g. mean reverting processes and stochastic volatility processes) and considering a dependence between the spot price and the GDP factor .

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Appendix A. Evaluation of the expected profit through average options

In this section, we specify the numerical methods used to evaluate the real options presented in Section 3.1 with the demand stochastic process of Section 2.

If we work in discrete time setting with $T = N\Delta$, then the average demand is defined as

$$Q_A(T) = \frac{1}{N+1}\sum_{n=0}^{N} Q_n$$

Using the demand model defined in Section 2, the payoff of an average option can be rewritten as

$$(M - Q_A(T))^+ = \left(M - K\frac{1}{N+1}\sum_{n=0}^{N} S_n^\eta\, G_n^\beta\right)^+ = K\, S_N^\eta\, G_n^\beta\, (Z_N)^+$$

where the auxiliary variable Z_N is defined as

$$Z_N = \frac{\tilde{M}\,(N+1) - \sum_{n=0}^{N} G_n\, S_n^\eta}{G_n\, S_N^\eta},$$

and $\tilde{M} = \frac{M}{K}$. The above set-up leads to the following recursion for the state variable Z,

$$Z_j = \frac{\tilde{M}\,(N+1) - \sum_{n=0}^{j} G_n\, S_n^{\eta}}{G_j\, S_j^{\eta}} = \frac{u_{j-1}}{G_j} \left(\frac{1}{e^{\xi_j}}\right)^{\eta} Z_{j-1} - 1, \quad \text{for } j = 1, 2, ..., N. \tag{A.1}$$

Define the processes

$$\begin{aligned} \tilde{B}_n &= \frac{M_{n,N}}{M_{0,N}\, e^{rn\Delta}}, \qquad\qquad (A.2)\\ M_{n,N} &= \mathbb{E}_{n\Delta}\left[\frac{S_N^{\eta}\, G_n^{\beta}}{e^{r(N-n)\Delta}}\right]. \end{aligned}$$

$\tilde{B}_n$ is a martingale under the risk neutral measure, i.e.

$$\mathbb{E}_{m\Delta}\left[\tilde{B}_n\right] = \tilde{B}_m \quad \text{and} \quad \mathbb{E}\left[\tilde{B}_n\right] = 1,$$

and it can be interpreted as a discrete Radon-Nikodym derivatives. Then, we obtain

$$\mathbb{E}\left[S_N^{\eta}\, G_n^{\beta}\, Z_N^{+}\right] = M_{0,N}\, e^{rT}\, \tilde{\mathbb{E}}\left[Z_N^{+}\right], \tag{A.3}$$

where the expected value $\tilde{\mathbb{E}}[\ ]$ is under the measure $\tilde{\mathbb{Q}}$ defined by $\tilde{B}_n$, i.e. using $M_{n,N}$ as numéraire. Hence, the price of the average option is calculated as

$$\mathbb{E}\left[\max(M - Q_A(T), 0)\right] = \mathbb{E}[Q_N]\, \tilde{\mathbb{E}}\left[Z_N^{+}\right], \tag{A.4}$$

where the risk neutral expected demand $\mathbb{E}[Q_N]$ is calculated as in equation (2). Under the hypothesis of independence of increments for $\ln S_j$ and $\ln G_j$, the process Z_j is Markovian. Hence, average options can be priced recursively on a one-dimensional tree for the process Z_j, while, average options with early exercise need a recursion on a two dimensional three for the process $(Q_j,\ Z_j)$ (see Gambaro et al. (2019) for details).

To exemplify the change of measure used previously, we consider a simplified setting. We take the limit $\beta \to 0$, and we suppose that ξ_j is a binomial random variable (CRR model) with law under the risk neutral measure

$$\begin{aligned} \xi_j &= \begin{cases} \sigma\sqrt{\Delta} & p \\ -\sigma\sqrt{\Delta} & 1-p \end{cases}, \\ p &= \frac{e^{r\Delta} - x_2}{x_1 - x_2} \text{ and } x_1 = e^{\sigma\sqrt{\Delta}},\ x_2 = e^{-\sigma\sqrt{\Delta}}. \end{aligned}$$

Using the martingale property of numéraire

$$
\begin{aligned}
\tilde{\mathbb{E}}\left[\frac{e^{r\Delta}}{M_{1,1}}\right] &= \frac{1}{M_{0,1}}, \\
\frac{e^{r\Delta}}{S_0^\eta}\left(\tilde{p}\frac{1}{x_1^\eta} + (1-\tilde{p})\frac{1}{x_2^\eta}\right) &= \frac{e^{r\Delta}}{S_0^\eta}\frac{1}{(px_1^\eta + (1-p)x_2^\eta)},
\end{aligned}
$$

we obtain the probability distribution of ξ_j under the new measure

$$
\tilde{p} = p\,\frac{x_1^\eta}{px_1^\eta + (1-p)x_2^\eta}. \tag{A.5}
$$

Then, we assume CRR model for both S_n and G_n, with volatilities σ_S and σ_u, respectively and probabilities of an up movement equal to

$$
\begin{aligned}
p_S &= \frac{e^{r\Delta} - x_2}{x_1 - x_2} \text{ and } x_1 = e^{\sigma_S\sqrt{\Delta}},\ \ x_2 = e^{-\sigma_S\sqrt{\Delta}}, \\
p_u &= \frac{1 - y_2}{y_1 - y_2} \text{ and } y_1 = e^{\sigma_u\sqrt{\Delta}},\ \ y_2 = e^{-\sigma_u\sqrt{\Delta}}.
\end{aligned}
$$

Moreover, S_n and G_n are independent under the risk neutral measure. Using the martingale property of numéraire we obtain

$$
\begin{aligned}
\tilde{\mathbb{E}}\left[\frac{e^{r\Delta}}{M_{1,1}}\right] &= \frac{1}{M_{0,1}}, \\
\tilde{\mathbb{E}}\left[\frac{e^{r\Delta}}{S_1^\eta u_1^\beta}\right] &= \frac{e^{r\Delta}}{\mathbb{E}[S_1^\eta]\,\mathbb{E}[u_1^\beta]}
\end{aligned}
$$

The equivalent measure $\tilde{Q}$ is not uniquely defined. However, if we choose the new equivalent measure $\tilde{Q}$ such that S_n and G_n are independent, then we obtain

$$
\begin{aligned}
\tilde{p}_S &= p_S\,\frac{x_1^\eta}{p_S x_1^\eta + (1-p_S)x_2^\eta}, \\
\tilde{p}_u &= p_u\,\frac{y_1^\beta}{p_u y_1^\beta + (1-p_u)y_2^\beta}.
\end{aligned}
$$

Appendix B. Evaluation of the expansion option

To solve the optimization (11), we need to formulate the problem using Markovian state variables. Similarly to the method used in Appendix A, we define the process $X_n = (Q_n, Z_n)$, with

$$Z_n = \frac{\sum_{j=(i-1)n_e}^{n} Q_j}{Q_n}, \tag{B.1}$$

$$\tag{B.2}$$

for $(i-1)\,n_e \le n \le i\,n_e - 1$ and $i = 1, ..., R$. For the bidimensional process $X_n = (Q_n, Z_n)$, the following iterations hold

$$\begin{aligned} Z_n = Z_{n-1}\,\frac{Q_n}{Q_{n-1}} + 1 \qquad & \text{for } n \neq (i-1)\,n_e, \\ Z_n = 1 \qquad & \text{for } n = (i-1)\,n_e. \end{aligned} \tag{B.3}$$

Since the process Q_n is Markovian, then the process X_n is Markovian as well. If $n_e = 1$, then $X_n = Q_n$ and the dynamic programming problem becomes unidimensional. If $n_e = 1$, we have the maximum level of flexibility for the firm, i.e. each time step is a decision date and the demand is evaluated at each time step. If n_e is greater than one, the decision to expand is taken based on the average value of the demand between two successive decision dates. The minimum level of flexibility is $n_e = N$, in this case there is no expansion and the profit is calculated based on the average demand over the period $[0, T]$, as in Appendix A. The cash-flows $Y_i(x_i, \pi_i)$ can be rewritten as a function of the Markovian state variable X_n in the following way

$$Y_i(x_i, \pi_i; X_{i\,n_e-1}) = \begin{cases} (m-c)\,M - m\,(M - Q_{i\,n_e-1}\,Z_{i\,n_e-1})^+, & \text{if } (x_i, \pi_i) = (0,0), \\ (m-c)\,M - m\,(M - Q_{i\,n_e-1}\,Z_{i\,n_e-1})^+ - I, & \text{if } (x_i, \pi_i) = (0,1), \\ (m-c)\,(M - M_e) - m\,(M + M_e - Q_{i\,n_e-1}\,Z_{i\,n_e-1})^+, & \text{if } (x_i, \pi_i) = (1,0), \end{cases}$$

where the superscript $(...)^+$ denotes the positive part. Hence, the optimization (11) is equivalent to the following stochastic dynamic programming for $n = 0, 1, ..., N$ and $i = 0, .., R-1$

$$\begin{aligned} C_n(x_i, X_n) &= e^{-r\,\Delta}\,\mathbb{E}\left[C_{n+1}(x_i, X_{n+1}) \mid X_n\right] \qquad \text{for } n \neq i\,n_e - 1, \\ C_{i\,n_e-1}(x_i, X_{i\,n_e-1}) &= \max_{\pi_i \in \mathcal{A}(x_i)} Y_i(x_i, \pi_i; X_{i\,n_e-1}) + e^{-r\,\Delta}\,\mathbb{E}\left[C_{i\,n_e}(x_{i+1}, X_{i\,n_e}) \mid X_{i\,n_e-1}\right], \end{aligned} \tag{B.4}$$

where $\mathcal{A}(x_i)$ is the set of feasible actions. The initial condition is

$$C_{R\,n_e-1}(x_R, X_{R\,n_e-1}) = Y_R(x_R, 0; X_{R\,n_e-1}),$$

and the expected profit is $C = C_0(0, X_0)$.

Appendix C. Asian option written on Light Sweet Crude Oil Futures

The underlying asset price of an average option contract on commodities is the average of settlement prices from the first business day to the last business day of a calendar month. We consider a standard type of contract that is mainly traded in over the counter oil markets (e.g. WTI). It refers to the futures price of the light sweet crude oil every business day. As the expiration of futures contract is about a week before the end of the calendar month, the futures contracts with two consecutive maturities become the underlying assets of the average option (see for instance Shiraya and Takahashi (2011)). By equation 4, the price of a forward (or futures) contract at time $n\Delta$ with maturity $N\Delta$ $(N > n)$ is

$$F_{n,N} \quad = \quad \mathbb{E}_n\left[S_N\right] = S_n\,\mathbb{E}\left[e^{\sum_{j=n+1}^{N}\xi_j}\right] = S_n \prod_{j=n+1}^{N} m_j,$$

with $m_j = \mathbb{E}\left[e^{\xi_j}\right]$ and $F_{N,N} = S_N$. The payoff of the previously described Asian option written on oil futures has the form

$$\left(\frac{1}{N+1}\left(\sum_{n=0}^{N_1} F_{n,N_1} + \sum_{n=N_1+1}^{N} F_{n,N_2}\right) - K\right)^+. \tag{C.1}$$

Hence, it is a fixed strike European Asian option with strike price K. The three numbers of time steps $0 \le N_1 \le N \le N_2$ represent the maturity of the first future, of the option and of the second future, respectively. The price of the Asian option written on crude oil futures can be estimated using numerical method presented in Gambaro et al. (2019).