MTRC's Novel Infrastructure Financing Model: Rationale Based on a Stackelberg Game of Timing under Uncertainty

Hao Bai

Beijing, China

Benoît Chevalier-Roignant

Cranfield School of Management, Cranfield University,
College Road, Cranfield, Bedford MK43 0AL, UK

Abstract

Because of debt concerns, many cities face challenges in financing their infrastructure. Hong Kong's transit operator designed a model which allows it to exploit the positive externalities of public transport on real estate prices to finance the infrastructure. We develop a Stackelberg leader-follower game of timing under uncertainty to explore this rationale. Our main findings are that internalizing positive externalities provides additional revenue sources for defraying the overall costs of infrastructure investments, thereby accelerating the delivery of infrastructure; in a multi-player stopping game, the equilibrium investment times are not typical first-hitting times. This study provides theoretical insight into infrastructure planning and financing based on compound (growth) options.

Keywords: Stopping game, compound (growth) options, Stackelberg game, positive externalities, infrastructure investments

Preprint submitted to The 23rd Annual International Real Options Conference

1. Introduction

Well-functioning infrastructure is an important driver of a city's attractiveness, competitiveness and economic development in the present globalized investment landscape. New infrastructure provides enhanced connections among places of residence, work and leisure. By so doing, infrastructure investments improve the economic competitiveness of a region and generate a series of compound (growth) options.

Ideally, infrastructure projects should be affordable to users, high-quality and self-sustainable. However, fares are usually set at lower than the full cost recovery level to keep public infrastructure services affordable. Fare revenues alone are rarely sufficient to cover total costs, including capital investments and operations and maintenance expenses, and provide comfortable returns on investments, which calls from subsidizing by the public sector. Due to the sovereign debt concerns, financing is often a bottleneck in the provision of infrastructure. Apart from budget constraints, the public sector often faces trade-offs between investing in infrastructure and other uses of public funds, e.g., in welfare programs. Under such restrictions, transit operators need to actively seek alternative revenue sources for infrastructure delivery.

Although the gap between insufficient funds and great demand for infrastructure can be addressed to some extent by charging users for fees and partnering with private parties, traditional sources of revenue (e.g., general purpose taxes and fees, user charges, and public subsidies) rarely allow transit operators to fully recoup their overall costs, which largely results from the regulation of transit fares (due to their public nature) as well as the huge and sunk expenditures of capital, operations and maintenance. Dixit

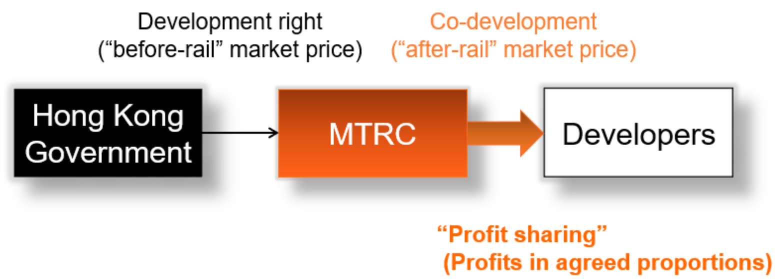

Figure 1: **R+P program: Interplay among the HK government, MTRC and private real-estate developers.** Based on Cervero and Murakami [2009].

and Pindyck [1994] stress three common characteristics among investment projects (including infrastructure): irreversibility, uncertainty and flexibility in timing.[1] In addition to these features, infrastructure projects are normally capital intensive, requiring substantial upfront costs as well as high (fixed) operations and maintenance expenses. The huge capital investments and periodic expenditures may make investors reluctant to enter into the infrastructure market—especially if investors are short-sighted when the benefits from a project are reaped over more than 30 years. Because of these characteristics, investors are more prone to delay a project launch until its value is sufficiently larger than its initial investment costs (in the spirit of real options); in other words, only if the revenue sources for both capital investments and operations and maintenance expenses can be reasonably demonstrated to be stable and secure will the finances be secured. To accelerate the delivery of infrastructure and to bolster the growth of the economy, the search for new revenue sources becomes a top priority.

[1] First, start-up costs are (at least partially) irreversible in the sense that expenditures will not be recovered if the project is reversed. Second, rewards from infrastructure investments are uncertain. Third, investors have some flexibility about the timing of their infrastructure investments.

On the above matters, a great deal can be learned from the successful, Hong Kong's (HK's) recent experience in public transportation. HK is among the few cities in which rail transit operations are highly profitable without government subsidizing. Its financial success rests on the "*Rail Plus Property*" (R+P) financing model developed by the HK's railway operator, the Mass Transit Railway Corporation ("MTRC"). Under the R+P program, the government grants MTRC exclusive property development rights of government-owned land around rail transit stations at a "before-rail" market price. MTRC then makes transit investments and captures the land appreciation created by the R+P model through further granting development rights to private developers at an "after-rail" market price, jointly developing the land and property and sharing profits generated by property development in agreed proportions. The land premiums (i.e., the difference between the before-rail and the after-rail market prices) and the shared profits are used to recoup the capital, operations, and maintenance costs of railway projects.

Incorporating positive externalities into the income stream of a transit operator is an innovative approach to designing, planning and financing capital-intensive infrastructure. Public transport improvements are known to boost the demand for housing nearby. However, under a traditional transit operations model, the operator (e.g., MTRC) rarely benefits from greater housing demand. If an infrastructure project is simply evaluated on the basis of fare revenues, a project which is environmentally and socially favorable but lacks financial profitability is likely to be rejected by private investors straight away, especially if the government does not subsidize or guarantee any minimum rewards. In that case, a springboard investment that would be socially beneficial would not be undertaken until its project value is "deep in the money." In other words, unless the future rewards considerably exceed the sunk costs, the financing issues will not be completely tackled.

It would be advised in the appraisal of an infrastructure investment project to value the follow-on options derived from an early investment. Infrastructure investments open up a series of valuable investment opportunities, also to external parties. The R+P development program presents a new perspective on infrastructure investments because MTRC takes the external economic benefits into consideration, internalizing the externality by granting development rights to private developers as well as sharing the profits with them provided by the subsequent property development in an agreed proportion. MTRC capitalizes on the commercial real estate options derived from the first-stage transit investment opportunity and captures the land value appreciation, viewing the value of property development as a part of the overall value of an infrastructure investment.

Uncertainty is a key driver of infrastructure investments; yet, the MTRC's novel financing model adds an element of strategic uncertainty because of the interactions and synergies with private property developers. This novel design leads naturally to a sequential game situation in which the leader and follower roles are predetermined. The follower is restrained from taking action until the leader has already done so. We model the rationale behind this innovative infrastructure financing scheme, using notions borrowed from real options (in particular compound options) and game theory (in the context of Stackelberg game of timing). We focus on continuous-time models of irreversible infrastructure investment under uncertainty, stress the factors that affect the timely delivery of infrastructure provisions, and derive the equilibrium investment rules by using dynamic programming. In this paper, the problem faced by MTRC is modelled as a Stackelberg leader-follower game of timing under uncertainty.

2. Literature review

Myers [1984] recalls the connection between (a) capital budgeting, which is concerned with project assessment, typically based on the discounted cash flows (DCF), and (b) strategic planning, whose primary objective is to determine the investment decisions that best achieve a long-term objective. Both perspectives are embedded in Real Options Analysis (ROA). Real options are coined in analogy with financial options [Myers, 1977]: a real option gives its holder an opportunity to acquire a real asset (e.g., to invest in a project) at a prespecified cost if conditions turn favorably [Trigeorgis, 1996]. A key benchmark model is the seminal optimal investment timing problem in McDonald and Siegel [1986]. In our model, we deal with a problem where an early investment opens up a chain of further projects; this class of problems is coined "growth options," "options on options," or "compound options" in the literature (Kester [1984], Trigeorgis and Mason [1987]).

There exist some real options papers dealing with infrastructure investments. Smit and Trigeorgis [2009] illustrate practical application of the option games approach in the airport industry context by developing discrete-time models that involve two privatized European airports competing in the times to expand their own lumpy infrastructure capacity. Ukkusuri and Patil [2009] model the demand uncertainty as a scenario tree, analyzing the optimal transportation network investments decision problem over multiple time periods. Besides discrete-time cases, many papers model uncertainty in continuous time as a stochastic process (e.g., a geometric Brownian motion) (see, e.g., Gao and Driouchi [2013], Li et al. [2015]). Most of them specify the value of the investment as a function of the target strategy chosen, determine the optimal transit investment timing given the investment payoff function. The investment timing problem is thus converted into the determination of the optimal investment trigger. Once a stochastic process followed by an underlying asset or factor (e.g., the project value, demand and population) reaches a specified barrier selected by the investor ex ante, the investment project is implemented as well as the investment timing is found. In such cases, the optimal investment timing is precisely the first-hitting time as they address the optimal infrastructure investment issue in a monopoly situation over one decision period. In a multi-player stopping game, however, the equilibrium investment times are not typical first-hitting times. Guo et al. [2018] consider decision makers' time preferences and beliefs by modeling authorities' intertemporal choices on the basis that the planning horizon in public transit investment, such as 20 or 30 years, is excessively longer than the election cycles of government officers (e.g., 4-5 years). While they analyze the impact of intertemporal decisions, they do not consider the impact of positive externalities derived from strategic interactions and the duopoly model of Stackelberg leader-follower game in an irreversible capital-intensive infrastructure investment under uncertainty, which is a main focus of our study.

This paper elaborates on the Stackelberg leader-follower game of timing under uncertainty in the infrastructural industry through a real-world case, providing an economic rationale for a successful infrastructure financing model in HK. Our approach is rooted in the real options analysis (ROA) and leverages on the dynamic programming in the operations research literature as well as the optimal control theory in the mathematics literature by solving variational inequalities [see Bensoussan and Lions, 1982]. We further extend our discussion in depth on the leader's problem that involves more complex continuation/stopping sets based on the research study by Bensoussan et al. [2010]. This is a key novel contribution versus the extant literature.

3. Traditional financing models

Before analyzing the novel R+P model, we set a couple of benchmarks by examining the most common infrastructure financing models, namely "user pays," "government pays," or a combination of these two. We have a greater emphasis on the user-pays model which MTRC uses in conjonction to their novel financing model.

3.1. "User-pays" model

Model setup. Under user-pays transit investments, fare revenues are the sole income stream for the transit operator (e.g., MTRC). For simplicity, we assume these revenues to be deterministic.[2] Specifically, we model the (perpetuity) value of transit operations at time t, Y_t^y, as the solution to a first-order

[2] The assumption is reasonable because the demand for railways, metro and buses services is hardly elastic and less influenced by business and economic industry cycles. In addition, transportation prices are highly regulated because transport is considered a common good; consequently, the metro fares hardly change over time. The fare revenue growth derives primarily from population growth, which is mostly predictable.

ordinary differential equation (ODE), namely

$$Y_0 = y, \tag{1a}$$

$$\mathrm{d}Y_t = \rho Y_t \mathrm{d}t, \tag{1b}$$

where ρ is a constant drift parameter. Equation (1b) implies that the project value grows compoundly at a constant rate of ρ per unit of time. At some future time (to be determined) the operator pays the construction cost I_1 to set up a new metro line in the expectation of future rewards. Because the construction of a large-scale infrastructure project generally requires substantial time from initiation until completion, we consider a time-to-build feature: $h_1 > 0$ is the lag in constructing a new metro line and the operator receives a payoff of $Y^y_{t+h_1}$ if he/she invests at time t. We let r stand for the constant discount rate.

The situation faced by an operator under the user-pays model is a comparatively simple (deterministic) problem of investment timing: the metro operator must determine the time $t \geq 0$ at which to incur a construction cost I_1 in return for a value $Y^y_{t+h_1}$ once construction is completed:

$$v(y) := \sup_{t\geq 0} \quad e^{-rt}\Big\{e^{-rh_1}Y^y_{t+h_1} - I_1\Big\}. \tag{2}$$

We assume that the discount rate $r > \rho$ to ensure that $v(y) < \infty$.

Dynamic programming. The problem (2) can be analyzed using dynamic programming.[3] We let

$$g(y) := e^{-rh_1} Y_{h_1}^{y} - I_1$$

denote a terminal payout received when investing. Here, the rail operator must decide whether to initiate ("stop") or delay the investment ("continue"). If the operator faces such an alternative, then its value–which corresponds to the optimal choice—must be no less than the payoff from either course of action. We now consider each alternative action in turn. Given flexibility in timing, the rail operator cannot be worse off than investing straight away; it must be that $v(y) \geq g(y)$ for all $y \geq 0$. In addition, by Bellman's (1957) "principle of optimality," the value must exceed the payoff:

$$v(y) \geq e^{-r\varepsilon} v\big(ye^{\rho\varepsilon}\big), \quad \varepsilon > 0.$$

If $v(\cdot) \in C^1(\mathbb{R}_+)$, we can let ε go to 0 in the above. At a given point y, one weak inequality must be strict and the other is an equality; this heuristic leads a "complementarity slackness" criterion. In short, the value function $v(y)$ must satisfy

$$\min\big\{rv(y) - \rho y v'(y); v(y) - g(y)\big\} = 0, \quad \forall y > 0. \tag{3a}$$

The dynamic programming equation (3a) is called a variational inequality (VI) following the terminology introduced in Bensoussan and Lions [1982].

[3] Alternatively, we can view it as a static optimization problem. Using dynamic programming allows highlighting similarities when dealing with the stochastic game of timing in a later section.

Economic arguments also lead to two additional conditions. The condition

$$\lim_{y \downarrow 0} \quad v(y) = 0 \tag{3b}$$

asserts that the project is worthless if the users' pool vanishes. We further assume that

$$\lim_{y \uparrow \infty} \frac{v(y)}{g(y)} = 1, \tag{3c}$$

which implies that when the real option is "very deep in the money," the real options value coincides with the net present value $g(y)$.

We solve the problem (3a)–(3c) in Theorem 1 (see proofs in the appendix).

Theorem 1. *The transit operator's value function* (2) *takes the form*

$$v(y) = \begin{cases} \left(\frac{y}{\bar{y}}\right)^{r/\rho} \left(e^{-(r-\rho)h_1}\bar{y} - I_1\right), & y < \bar{y}, \\ e^{-(r-\rho)h_1}y - I_1, & y \geq \bar{y}, \end{cases}$$

where the investment threshold $\bar{y}$ *is given by*

$$\bar{y} = \frac{re^{(r-\rho)h_1}}{r-\rho} I_1 > I_1.$$

Following Theorem 1, the (optimal) decision whether to invest relates to the relative positions of the project value y and of the threshold $\bar{y}$. Alternatively, we can express the optimal stopping rule as $\hat{t}(y) = \frac{1}{\rho} \ln\left(\frac{y \vee \bar{y}}{y}\right)$. This form hightlights a relationship between the optimal time $\hat{t}(y)$ and the threshold $\bar{y}$. The firm should invest straight away [$\hat{t}(y) = 0$] if and only if

$y \geq \bar{y}$. Besides, $\bar{y} > e^{(r-\rho)h_1} I_1$, which is the "zero NPV" threshold. Consequently, the transit operator will defer the investment until the fare revenues are "deep in the money" (not "at the money"). The effect—coined *hysteresis* in the literature—arises even though the transport operator is certain about the growth of its future revenue stream.

Unfortunately, in most cities (Murakami and Gregory [2012]), fares are often regulated and small to keep transit affordable. It is thus difficult to recoup the investment costs, so that many infrastructure projects are postponed, which is not socially desirable quantity of infrastructure.

3.2. "Government-pays" and mixed models

In the *government-pays* model, the sole revenue source of a private investor is governmental transfer payment, i.e., ultimately sourced from the public budget (through taxing or borrowing). Hereby, the public and private parties reach an agreement in which the public party promises to acquire an infrastructure asset from the private party at some specific time and price. It is analogous to a *forward* contract. Government pays delay public expenditures that would appear on the liability side of the government's "balance sheet." Practically, the government engages a private company to develop an infrastructure asset or render infrastructure services, yet the cost of infrastructure is ultimately met from the public purse. Moreover, private funds are not a source of revenues but a way of raising funds, similar to a loan committed to repay the lender. They are still needed to be paid by public budgets in the end. The financial burden on the government entity is likely to raise the sovereign debt level and lead to greater fiscal pressure on households, not necessarily living in the vicinity of the infrastructure.

Besides the (pure) user- and government-pays models, a mix of these two is also widely used in practice. Here, user charges will ultimately be the main revenue sources yet are supplemented by government transfers paid at specific construction milestones. The government guarantee helps securitize the future streams of revenue for the private investor when the prespecified downside events occur, providing strong support for the government to channel more private funds in infrastructure. Further, if the project value is assumed to be stochastic rather than deterministic, a guarantee approximates to a *put option* that protects the private investor from downside risks, at the expense of the public sector (and ultimately at the expense of households). If there is no expiry, such a payment mechanism for the private party can be approximately modeled as the perpetual American put option written on the value of transit operations. While the government guarantee protects the private entity's interests against unfavorable conditions and provides the solid basis for attracting private funds, it implicitly increases the governments fiscal exposures in the form of contingent liabilities. Therefore, the mixed model increases the risk of government bankruptcy particularly when the government is facing high leverage ratios.

In summary, the three widely-used traditional financing models have various constraints to a varying degree. To accelerate infrastructure delivery, more effective financing mechanisms are required. We next formalize MTRC's innovative financing model.

4. MTRC's novel financing model

Beside fare revenues modeled by (1a)–(1b), MTRC derives an income from the trading of the development rights to a property developer (or a consortium of property developers). The decision whether to acquire those rights depends on an average after-rail *property value* $X^x : \Omega \times [0, \infty) \to \mathcal{X} := \mathbb{R}_+$, which is assumed to follow a geometric Brownian motion (GBM) of the form

$$X_0^x = x > 0, \qquad \mathbb{P} - \text{a.s.}, \tag{4a}$$

$$\mathrm{d}X_t^x = \mu X_t^x \,\mathrm{d}t + \sigma X_t^x \,\mathrm{d}Z_t, \qquad t > 0, \tag{4b}$$

where $\mu > \frac{1}{2}\sigma^2$ and $\sigma > 0$ are the constant drift and volatility parameters respectively and Z is a standard Brownian motion.[4]

We model MTRC's novel financing scheme as a Stackelberg game in which the leader and follower roles are set ex ante. We depict the timeline in Figure 2 and use the index i to denote a particular party. MTRC ($i = 1$) starts contruction of the metro line at a (stopping) time τ_1, while the property developer ($i = 2$) acquires the development rights at a price $K > 0$ and starts building the properties around the station at a time $\tau_2 > \tau_1$ for a cost I_2. We consider a time lag $h_2 > 0$ for the planning, building and reselling of the

[4] In HK, land values are strongly driven by rapid urban population expansion and strong economic growth [Hong and Brubaker, 2010]. Beside population, property prices depend on various macroeconomic factors such as housing scarcity, change in housing policies or regulations, the yield differential compared to other asset classes, and foreign-exchange rates. Modeling the feedback effect of population growth on real-estate prices is beyond the scope of this paper; for simplicity, we treat the parameters ρ in (1b) and μ in (4b) independently of one another. For the sake of illustration, a certain dependency is accounted for because we use estimates from real-world projects.

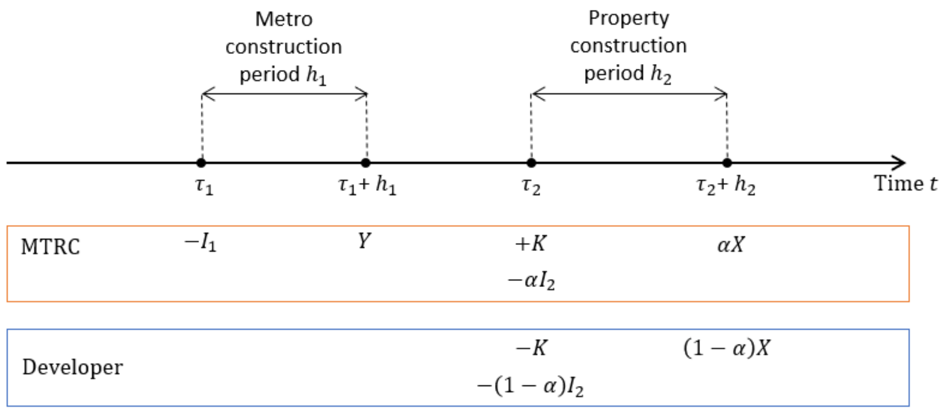

Figure 2: **Model timeline**

properties; the property after completion is worth $X^x_{\tau_2+h_2}$. MTRC's $R+P$ program comprises a profit-sharing rule, whereby a proportion $\alpha \in (0,1)$ of the net proceeds, i.e., after deduction of the construction costs I_2, accrues to MTRC.[5] No party can perfectly forecast the development of the property price and form instead expectations (using the operator $\mathbb{E}$). Both parties are assumed risk-neutral for simplicity and discount at a constant rate r.

We specify now the parties' *objective functionals.* The property developer's objective is

$$J_2^x(\tau_1,\tau_2) := \mathbb{E}\Big[e^{-r\tau_2}\big\{(1-\alpha)\big(e^{-rh_2}X^x_{\tau_2+h_2} - I_2\big) - K\big\}\mathbf{1}_{\{\tau_2>\tau_1\}}\Big], \tag{5}$$

where the indicator $\mathbf{1}_{\{\tau_2>\tau_1\}}$ accounts for the "Stackelberg" constraint $\tau_2 > \tau_1$.

[5]Specifically, the leader receives at time τ_2 the proceeds $\alpha\big(e^{-rh_2}X^x_{h_2} - I_2\big)$, while the follower retains $(1-\alpha)\big(e^{-rh_2}X^x_{h_2} - I_2\big)$. The agreed proportion α may be thought of as the outcome (e.g., the Nash bargaining solution) of a negotiation between the leader and follower taking place at time τ_2. We instead assumed it is a parameter, an assumption which is reasonable if MTRC auctions the development rights because, then, the terms are not negotiated.

MTRC has two income streams by respectively charging: (i) *direct beneficiaries* (i.e., metro passengers) for fare revenues from time $\tau_1 + h_1$ onwards, which are worth $e^{-rh_1} Y^y_{\tau_1+h_1} - I_1$ at time τ_1, and (ii) *indirect beneficiaries* (i.e., the private developer who decides to develop properties at time τ_2) for an income K from the trading of development rights and an agreed proportion of the net property development proceeds. The MTRC's objective is

$$\begin{aligned} J_1^{x,y}(\tau_1, \tau_2) := \mathbb{E}\Big[& e^{-r\tau_1}\Big(e^{-rh_1} Y^y_{\tau_1+h_1} - I_1\Big) \\ & + e^{-r\tau_2}\Big\{\alpha\big(e^{-rh_2} X^x_{\tau_2+h_2} - I_2\big) + K\Big\}\Big]. \end{aligned} \tag{6}$$

We specify the *solution concept* for this Stackelberg game of timing. We assume that players follow Markov strategies and use the superscripts x and y to highlight this ansatz. The leader anticipates that the follower will react optimally to its choice $\tau_1^{x,y}$ with a reaction $\mathcal{T}_2\big(\tau_1^{x,y}\big)$ assumed to be the unique solution to

$$J_2^x\Big(\tau_1^{x,y}, \mathcal{T}_2\big(\tau_1^{x,y}\big)\Big) = \sup_{\tau_2^x \geq \tau_1^{x,y}} \quad J_2^x(\tau_1^{x,y}, \tau_2^x). \tag{7}$$

Given the sequence of decision, the leader faces a decision-theoretic problem:

$$V_1(x,y) := J_1^{x,y}\Big(\hat{\tau}_1^{x,y}, \mathcal{T}_2\big(\hat{\tau}_1^{x,y}\big)\Big) = \sup_{\tau_1^{x,y}} \quad J_1^{x,y}\Big(\tau_1^{x,y}, \mathcal{T}_2\big(\tau_1^{x,y}\big)\Big). \tag{8}$$

We call $\big(\hat{\tau}_1^{x,y}, \mathcal{T}_2^x(\hat{\tau}_1^{x,y})\big)$ the game's Markov Stackelberg equilibrium. To solve for this, we proceed in the reverse investment order, determining the follower's reaction function in (7) first.

5. Property developer's decision

After some computations (see Appendix), one shows that the follower's reaction function in (7) can be written in the form

$$\mathcal{T}_2^x(\tau_1^{x,y}) = \tau_1^{x,y} + \theta_2\big(X^x_{\tau_1^{x,y}}\big); \tag{9}$$

here, $\theta_2(x)$ is the solution to a "*myopic*" problem that becomes relevant once the leader started construction works, namely[6]

$$V_2(x) := \mathbb{E}\big[e^{-r\theta_2(x)}G_2\big(X^x_{\theta_2(x)}\big)\big] = \sup_{\tau_2^x} \quad \mathbb{E}\big[e^{-r\tau_2}G_2\big(X^x_{\tau_2^x}\big)\big], \tag{10}$$

with the function $G_2(\cdot)$ given by

$$G_2(x) := (1-\alpha)\big(e^{-(r-\mu)h_2}x - I_2\big) - K. \tag{11}$$

We now solve for $V_2(x)$ in the myopic problem (10) by using an approach similar to the one used to solve the deterministic problem (2). Given flexibility in timing, the private investor cannot be worse off than investing immediately: $V_2(x) \geq G_2(x)$. Alternatively, the property developer can stay put for a period of time $\varepsilon > 0$ and then pursues the optimal stopping strategy $\theta_2(\cdot)$; this stance leads to the inequality $V_2(x) \geq \mathbb{E}\big[e^{-r\varepsilon}V_2(X^x_\varepsilon)\big]$. As $\varepsilon \to 0$, then it obtains from (a generalized version of) Dynkin's formula [see Bensoussan and Lions, 1982, Theorem 8.5, pp.185-186] that $\mathcal{L}_2 V_2(x) \geq 0$, where $\mathcal{L}_2$ is a

[6]The term $V_2(x)$ in (10) should not be confused with $J_2^x\big(\hat{\tau}_1^{x,y}, \mathcal{T}_2^x(\hat{\tau}_1^{x,y})\big)$.

second-order operator given by

$$\mathcal{L}_2 f := rf - \mu x \frac{\partial f}{\partial x} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}. \tag{12}$$

At a given point x, one weak inequality must be strict and the other is an equality; this heuristic leads a "complementarity slackness" criterion. In short, the value function V_2 must solve the VI given in Lemma 1.

Lemma 1. *The value function V_2 in (10) must satisfy the VI* [7]

$$\min\left\{\mathcal{L}_2 V_2(x); V_2(x) - G_2(x)\right\} = 0, \quad a.e.\ x \in \mathcal{X}, \tag{13a}$$

$$V_2(\cdot) \in \mathsf{C}^1(\mathbb{R}_+) \quad and \quad V_2''(\cdot) \in \mathsf{L}^1_{loc}(\mathbb{R}_+), \tag{13b}$$

$$\lim_{x\to 0} \quad V_2(x) = 0, \tag{13c}$$

$$\lim_{x\to\infty} \quad \frac{V_2(x)}{G_2(x)} = 1. \tag{13d}$$

Before solving for (13a)–(13d) in Theorem 2,[8] we introduce the positive and negative roots, γ_A and γ_B respectively, of the quadratic function

$$\mathcal{Q}(\gamma) := r - \mu\gamma - \frac{1}{2}\sigma^2\gamma(\gamma - 1). \tag{14}$$

[7] Compared to the regularity requirement $V_2(\cdot) \in \mathsf{C}^1(\mathbb{R}_+)$, which we also had in the deterministic case, we introduce another condition $V_2''(\cdot) \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$ in (13b) to ensure that the second-order term in the VI has a mathematical meaning. Smooth fit is a natural consequence of the assumption $V_2(\cdot) \in \mathsf{C}^1(\mathbb{R}_+)$.

[8] We omit a verification theorem. It is known [see, e.g., Bensoussan and Lions, 1982] that a value function of optimal stopping is the probabilistic representation of the solution to a variational inequality.

Theorem 2. *The function $V_2(\cdot)$ given by*

$$V_2(x) = \begin{cases} \left(\frac{x}{\bar{x}_2}\right)^{\gamma_A} \Big[(1-\alpha)\big(e^{-(r-\mu)h_2}\bar{x}_2 - I_2\big) - K\Big], & x < \bar{x}_2, \\ (1-\alpha)\big(e^{-(r-\mu)h_2}x - I_2\big) - K, & x \geq \bar{x}_2, \end{cases}$$

where

$$\bar{x}_2 := \frac{\gamma_A}{\gamma_A - 1}\Big(I_2 + \frac{K}{1-\alpha}\Big)e^{(r-\mu)h_2}$$

solves the VI (13a)–(13d).

To interpret Theorem 2, we recall that that one can associate to a first-hitting time $\inf\{t \geq 0 : X_t^x \geq \xi\}$ a discount factor over states given by

$$\mathbb{E}\Big[e^{-r\inf\{t\geq 0: X_t^x \geq \xi\}}\Big] = \Big(\frac{x \wedge \xi}{\xi}\Big)^{\gamma_A}. \tag{15}$$

If the after-rail property price x is above $\bar{x}_2$, the (myopic) property developer acquires the development rights, receiving the amount $(1-\alpha)(e^{-(r-\mu)h_2}x - I_2) - K$. If the property price is below that threshold, then the firm delays receiving the amount $(1-\alpha)(e^{-(r-\mu)h_2}\bar{x}_2 - I_2) - K$ with a delay, at time $\theta_2(x)$; it discounts this amount using the discount factor $(x/\bar{x}_2)^{\gamma_A}$.

In Theorem 2, we proved that the solution $\theta_2(x)$ to the myopic problem (10) is the first-hitting time

$$\theta_2(x) := \inf\big\{t \geq 0 : X_t^x \geq \bar{x}_2\big\}. \tag{16}$$

In reality, the threshold $\bar{x}_2$ is not the threshold above which the private developer develops the property. Indeed, the private investor wants to ensure MTRC has invested in urban rail infrastructure. In other words, we do not claim that the follower's reaction $\mathcal{T}_2^x(\tau_1^{x,y})$ is the first-hitting time in (16), but claim instead—because of the relation (9)—that it is of the form

$$\mathcal{T}_2^x(\tau_1^{x,y}) = \inf\Big\{t \geq \tau_1^{x,y} : X_t^{X^x_{\tau_1^{x,y}}} \geq \bar{x}_2\Big\}. \tag{17}$$

To be able to specify the follower's equilibrium decision, we will need to solve the leader's problem (8) and determine the leader's optimal investment decision $\hat{\tau}_1^{x,y}$.

6. MTRC's problem

We now specify a variational inequality for the leader's problem in Lemma 2. We interpret the term $G_1(x,y)$ given by

$$G_1(x,y) := \begin{cases} \check{G}_1(x,y) := & e^{-(r-\rho)h_1}y - I_1 \\ & +\left(\frac{x}{\bar{x}_2}\right)^{\gamma_A}\Big\{\alpha\big(e^{-(r-\mu)h_2}\bar{x}_2 - I_2\big) + K\Big\}, \quad x < \bar{x}_2, \\ \tilde{G}_1(x,y) := & e^{-(r-\rho)h_1}y + \alpha e^{-(r-\mu)h_2}x \\ & -(I_1 + \alpha I_2 - K), \qquad\qquad\qquad\qquad x \geq \bar{x}_2, \end{cases} \tag{18}$$

as the payoff from investing straight away (or the obstacle in the optimal stopping literature). If MTRC invests when the property value is $x \geq \bar{x}_2$, then it receives both fare revenues and an income from its business relationship with the property developer; otherwise, the trading of development rights is delayed until the property value reaches the threshold $\bar{x}_2$. Beside $\mathcal{L}_2$ in (12), we introduce the operator

$$\mathcal{L}_1 f := \mathcal{L}_2 f - \rho y \frac{\partial f}{\partial y}. \tag{19}$$

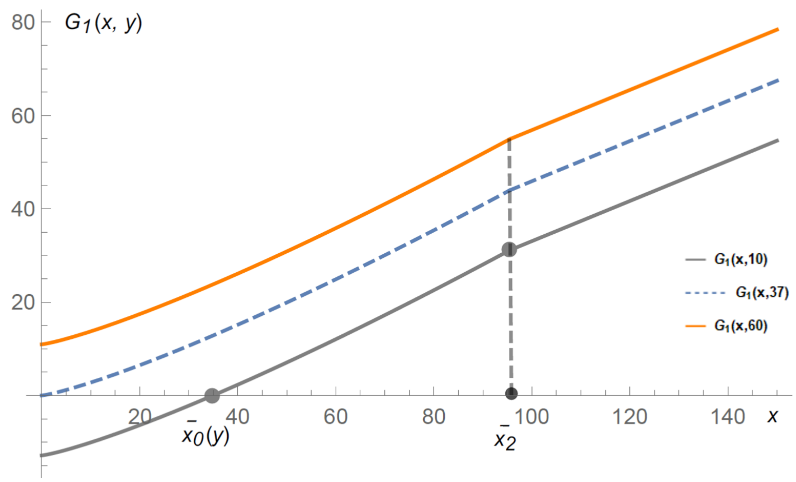

Figure 3: The dashed curve for $x \mapsto G_1(x, y)$ depicts the "real" case with $y = I_1 e^{(r-\rho)h_1} = 37$. The gray ($y = 10$) and orange ($y = 60$) curves are introduced for comparative statics. Cf. parameter values in Footnote 9.

Lemma 2. *The leader's value function V_1 in (8) satisfies the VI*

$$\min\left\{\mathcal{L}_1 V_1(x, y); V_1(x, y) - G_1(x, y)\right\} = 0, \quad a.e.\ x \in \mathcal{X}, \tag{20a}$$

$$V_1 \in \mathsf{C}^1(\mathbb{R}^2_+) \quad and \quad \frac{\partial^2 V_1}{\partial x^2} \in \mathsf{L}^1_{loc}(\mathbb{R}_+). \tag{20b}$$

We recall that if a solution to the VI exists (and the solution to the VI coincides with the value function), then the continuation region is defined implicitly as $\mathcal{C}_1 := \left\{(x, y) \in \mathbb{R}^2_+ : V_1(x, y) > G_1(x, y)\right\}$, while the stopping region is $\mathcal{S}_1 := \mathbb{R}^2_+ \setminus \mathcal{C}_1$.

We start by studying the function G_1 in (18). We depict this function in Figure 3 using as parameter values those of the "SIL(E)" project in HK.[9]

[9]The South Island Line (East) ("SIL(E)") project is a 7-km line costing $I_1 = 17.6$ HKDbn and financed under the R+P model. The construction commenced in May 2011 and completed in December 2016. In December 2017, MTRC awarded the property development package to a consortium for a land premium of $K = 5.2$ HKDbn. The development

We can conclude after some computations that:

Lemma 3. *The obstacle G_1 has the regularly*

$$G_1 \in \mathsf{C}^0(\mathbb{R}_+^2) \quad \textit{and} \quad G_1 \in \mathsf{C}^1(\mathbb{R}_+^2) \textit{ a.e.}. \tag{21}$$

Further, if we assume that

$$I_2\big[(\alpha-1)\gamma_A+1\big]+\frac{K}{1-\alpha}\big(\gamma_A-1+\alpha\big)>0, \tag{22}$$

then $x \mapsto G_1(x,y)$ is monotone increasing on $(0,\infty)$. The obstacle is negative on $(0,\bar{x}_0(y))$ if the firm would not invest solely based on fare revenues, i.e., $y < e^{(r-\rho)h_1} I_1$ and positive otherwise. The curve $\bar{x}_0(\cdot)$ is defined by

$$\bar{x}_0(y) := \left[\frac{(I_1-e^{-(r-\rho)h_1}y)\bar{x}_2^{\gamma_A}}{\alpha\big(e^{-(r-\mu)h_2}\bar{x}_2-I_2\big)+K}\right]^{1/\gamma_A}. \tag{23}$$

Because $x \mapsto G_1(x,y)$ is not continuously differentiable, we need to rule out as continuation region the set $\{(x,y) : \big(0,\bar{x}_1(y)\big)\}$ with $\bar{x}_1(y) < \bar{x}_2$. Indeed, if this set were the continuation region, then the solution to the VI (20a) would not satisfy the regularity (20b) in the candidate stopping region $\{(x,y) : (\bar{x}_1(y),\infty)\}$. We focus first on the case with a threshold policy such that $\bar{x}_1(y) \geq \bar{x}_2$, which cannot be ruled out *a priori*.

project is due for completion by 2023. The metropolitan area's population grew at 1.5% p.a. between 2010 and 2025, while the HK property price grows at 12% p.a. [Suzuki et al., 2015]. Other parameter values are $I_2 = 4.66$ HKDbn, as $\alpha = 0.5$, $r = 0.15$, $\mu = 0.12$, $\rho = 0.015$, $\sigma = 0.15$, $h_1 = 5.5$ and $h_2 = 5$.

6.1. Case A with a threshold policy such that $\bar{x}_1(y) \geq \bar{x}_2$

We introduce the quadratic equation

$$\mathcal{D}(\gamma) := r - \rho - (\mu - \rho)\gamma - \frac{1}{2}\sigma^2\gamma(\gamma - 1), \tag{24}$$

and denote by δ_A and δ_B its positive and negative roots, respectively. Further, we define

$$a := I_2(\delta_A - \gamma_A), \quad b := \gamma_A K(\delta_A - 1), \quad \text{and} \quad c := \delta_A(\gamma_A - 1)\big(I_1 - K\big). \tag{25}$$

We are able to establish the following theorem in Case A:

Theorem 3. *If we make the assumptions*

$$\mu - \rho - \frac{1}{2}\sigma^2 > 0, \tag{26a}$$

$$I_1 + \alpha I_2 > K, \tag{26b}$$

$$0 < \alpha < \alpha_\star := \frac{(a+b+c) - \sqrt{(a+b+c)^2 - 4ac}}{2a} \in (0,1), \tag{26c}$$

then the function V_1 given by

$$V_1(x,y) = \begin{cases} \left(\frac{x}{\bar{x}_1(y)}\right)^{\delta_A} \tilde{G}_1\big(\bar{x}_1(y), y\big), & x < \bar{x}_1(y), \\ \tilde{G}_1(x,y), & x \geq \bar{x}_1(y), \end{cases} \tag{27}$$

with

$$\bar{x}_1(y) := \frac{\delta_A}{\delta_A - 1} \frac{(I_1 + \alpha I_2 - K) - e^{-(r-\rho)h_1} y}{\alpha} e^{(r-\mu)h_2}, \tag{28}$$

and $\tilde{G}_1$ *defined by* (18)*, solves the VI* (20a)–(20b) *whenever*

$$0 < y < y_\star := e^{(r-\rho)h_1}\Big\{-\Big[\frac{\gamma_A}{\gamma_A-1}\frac{\delta_A-1}{\delta_A}\alpha\Big(I_2+\frac{K}{1-\alpha}\Big)\Big]+I_1+(1+\alpha)I_2-K\Big\}. \tag{29}$$

We first interpret the restrictions (26a)–(26c). According to (26a), the excess return $\mu-\rho$ from property investment must be sufficient to compensate for the volatility σ in property prices. Further, from (26b), MTRC must not charge a price K to the property developer that offsets its own (total) investment cost, $I_1 + \alpha I_2$. The restriction (26c) limits the proportion of the net proceeds received by MTRC. Both conditions (26b) and (26c) impose a fair value redistribution between the two parties. This also relates to the notion of *co-opetition* (cf. Brandenburger and Stuart [2007], Trigeorgis and Reuer [2017]) because it mixes elements of cooperation and competition.

From Theorem 3, we characterize MTRC's optimal investment rule in Case A as the first-hitting time

$$\hat{\tau}_1^{x,y} = \inf\big\{t \geq 0 : X_t^x \geq \bar{x}_1(y)\big\}, \quad \text{if } 0 < y < y_\star. \tag{30}$$

In other words, MTRC should invest if the property value exceeds the level $\bar{x}_1(y) > \bar{x}_2$. This case illustrated in Figure 4 yields interesting economic insights. First, when MTRC invests, the property developer immediately follows suit.[10] Such a pattern is often observed in reality. A case in point is the property development plan around Canary Wharf, one of the main financial centers in London, and the urban planning scheme following the

[10]Indeed, according to (16), $\theta_2\big(X^x_{\tau_1^{x,y}}\big) = 0$ if $x \geq \bar{x}_1(y) \geq \bar{x}_2$.

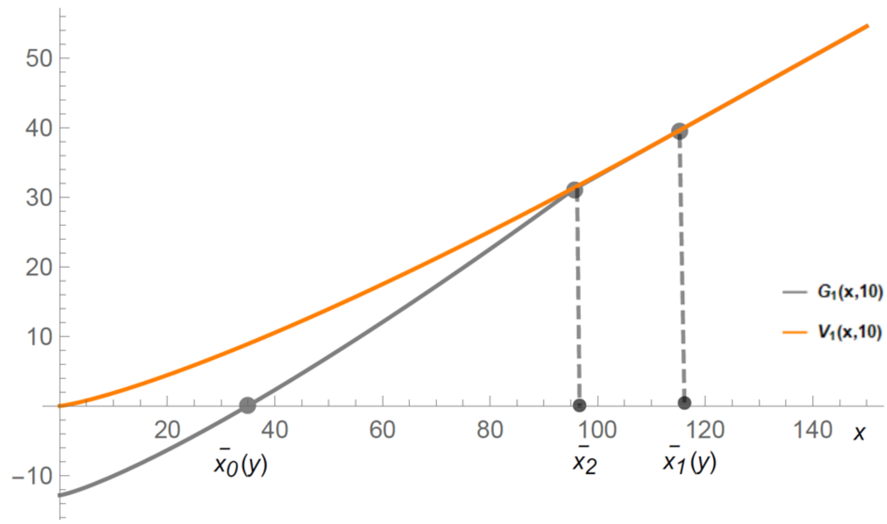

Figure 4: Graph of the MTRC's value function $x \mapsto V_1(x, y)$ for a given $y = 10$.

Jubilee line's extension.[11] Second, because of the difference between the thresholds $\bar{x}_1(y)$ and $\bar{x}_2$, the private developer will not invest when the real option is "deep in the money" (i.e, $x \geq \bar{x}_2$) as a myopic investor would, but when it is even "deeper in the money" (i.e., $\bar{x}_1(y) > \bar{x}_2$). This delay is due to the Stackelberg nature of this timing game, with the leader investing only when its own real option is "deep in the money."

7. Policy implications

This article stresses implications for public policies on infrastructure investments. Conventionally, the upfront capital investment and the periodical operations and maintenance costs should be recovered by fare revenues

[11] An eastward extension of the Jubilee Line was proposed in the 1970s, yet not finalized until the late 1990s when the developers of Canary Wharf agreed to pay GBP500m to complete the extension.

and/or by government subsidies. Yet, transit fares are usually regulated and rarely sufficient to allow recovering costs. Government subsidies and guarantees would explicitly or implicitly increase the government's financial burden and risk. These conditions eventually lead to a delayed infrastructure delivery.

MTRC's success rests not on government subsidizing but on its real estate business. MTRC capitalizes on complimentary sources of revenues (i.e., fare revenues, land premiums, and the profit sharing with private developers in real estate sale and lease) to cover the overall costs of capital investment, operations, and maintenance. This novel financing approach is based on the "*beneficiary pays*" principle, which states that the beneficiaries of the infrastructure services or improvements that increase land values should partly bear public investment costs or return their benefits to the public (Suzuki et al. [2015]; Ardila-Gomez and Ortegon-Sanchez [2016]). In the R+P model, charging *indirect beneficiaries* (i.e., the private developer) allows MTRC to recoup the huge capital investments at the outset; charging the *direct beneficiaries* (i.e., metro passengers) helps MTRC cover the operations and maintenance costs that are relatively small-scale and periodic or ongoing. By blending the multiple financing instruments that respond to the time variations of expenditures, MTRC operates without government subsidizing and runs a highly profitable and efficient business with financial sustainability. The comprehensive and sustainable urban transport financing scheme also releases the HK government from the financial pressure to provide the public support for the rail operator.

In addition to economic efficiency and financial sustainability, the R+P model is socially fairer than the government pays and government guarantees/subsidies. In both cases, the cost of infrastructure is ultimately met from the public purse, indicating that everyone pays for the infrastructure investment (including the herdsmen in Tibet). Instead of taxing the general public, MTRC only "taxes" those who benefit from transit improvements (i.e., users and the private developer) with social equity. Moreover, the R+P model produces a network effect on a city's public transport development, helping HK achieve long-term transport sustainability goals. A well-developed transit system connects other transport modes with existing metro lines, which helps increase the percentage of public transportation use and provides great convenience for people's life. The newly constructed metro lines can create spillovers and synergies, benefiting not only the newly developed corridors but the existing ones as well. A mass transit system can also reduce traffic congestion and save passengers' costs and time. Hence, the larger the network, the greater the environmental and social benefits.

While the MTRC's novel financing model confers various positive benefits in many aspects, there is an ongoing debate over land value capture and housing affordability. HK is one of the most densely populated cities in the world, and thus lands are scarce resources. Land scarcity increases the financial viability of property development. Some critics state that the R+P approach exacerbates the issues of the HK's high housing and rental prices. To enhance the bankability of transit investments, MTRC charges indirect beneficiaries (i.e., private developers) by granting exclusive development rights around the transit stations. The private developer then continues to transfer this portion of costs (i.e., land premiums) to end customers (e.g., house buyers and renters), which increases property prices. However, the high-end development concept of R+P is not applied to all transit stations, and a number of residential properties in HK are within 500 meters of many transit stations [Suzuki et al., 2015]. While high property prices have been an issue and land resources in HK has been scare, instead of charging high income tax or other general taxes to all residents, the R+P financing model helps MTRC to finance a metro line from those who benefit from a transport service (including passengers and private developers), which enables the government to increase revenue sources at the outset without the accumulation of public debt and evenly balances economic efficiency and social fairness.

8. Conclusion

Our paper presents a new perspective on infrastructure investment under uncertainty, viewing it as a springboard that can generate follow-on growth opportunities in a multi-player stopping game. We examine the underlying rationale behind the two stakeholders' investment timing decisions in the context of HK's R+P program by using notions borrowed from compound (growth) options and Stackelberg games. We model MTRC's problem as a duopoly case of Stackelberg leader-follower game of timing in comparison with traditional infrastructure financing models. Much can be learned from the real-world novel financing model.

First, the infrastructure financing gap stems from insufficient sources of revenue. Only if a project can demonstrate favorable and sufficient revenue sources for both upfront capital investments and periodic operations and maintenance expenditures can issues of financing and delivery be addressed successfully. In other words, unless the adequacy of revenue streams is addressed, there remains a financing gap in infrastructure investments. To accelerate the delivery of infrastructure, one should account for positive externalities that derive from the prior infrastructure investment and turn these revenue streams into capital that can be used today to initiate a capital-intensive project.

Second, in a multi-agent stopping game, the equilibrium investment times are not typical first-hitting times. Most extant real options literature suggests that the exercising times are precisely first-hitting times (from above or below depending on the context); once the process reaches the critical threshold, the decision maker takes action. However, in a sequential Stackelberg stopping game, the follower's reaction is contingent upon the leader's action, and the leader must consider the effect of follower's anticipated entry. Therefore, the equilibrium stopping time are not necessarily the first-hitting times.

Our article contributes to extant real options literature by analyzing the rationale based on the option games approach via a case study of the infrastructural industry. Our approach is novel, rigorous, general and can be readily applied to other infrastructure-oriented development schemes and capital-intensive projects, yet we believe that extending it further forward could enhance our understanding of investment decisions under uncertainty. Future research could further explore independencies among projects over time by considering portfolios of real options in combination with game-theoretic thinking, or extend our lumpy entry investment decisions to incremental capacity expansion options in the context of the Stackelberg leader-follower game of timing in dynamic and uncertain environments.

References

A. Ardila-Gomez and A. Ortegon-Sanchez. *Sustainable urban transport financing from the sidewalk to the subway: Capital, operations, and maintenance financing*. The World Bank, 2016.

R. Bellman. *Dynamic programming*. Princeton University Press., 1957.

A. Bensoussan and J.-L. Lions. *Applications of variational inequalities in stochastic control*. North-Holland, New York, 1982.

A. Bensoussan, J. D. Diltz, and S. Hoe. Real options games in complete and incomplete markets with several decision makers. *SIAM Journal on Financial Mathematics*, 1(1):666–728, 2010.

A. Brandenburger and H. Stuart. Biform games. *Management Science*, 53 (4):537–549, 2007.

R. Cervero and J. Murakami. Rail and property development in Hong Kong: Experiences and extensions. *Urban Studies*, 46(10):2019–2043, 2009.

A. K. Dixit and R. S. Pindyck. *Investment under uncertainty*. Princeton University Press, 1994.

Y. Gao and T. Driouchi. Incorporating Knightian uncertainty into real options analysis: Using multiple-priors in the case of rail transit investment. *Transportation Research Part B: Methodological*, 55:23–40, 2013.

Q.-W. Guo, S. Chen, P. Schonfeld, and Z. Li. How time-inconsistent preferences affect investment timing for rail transit. *Transportation Research Part B: Methodological*, 118:172–192, 2018.

Y.-H. Hong and D. Brubaker. Integrating the proposed property tax with the public leasehold system. *China's Local Public Finance in Transition*, pages 165–90, 2010.

W. C. Kester. Today's options for tomorrow's growth. *Harvard Business Review*, 62:153–160, 1984.

Z.-C. Li, Q.-W. Guo, W. H. Lam, and S. Wong. Transit technology investment and selection under urban population volatility: A real option perspective. *Transportation Research Part B: Methodological*, 78:318–340, 2015.

R. McDonald and D. Siegel. The value of waiting to invest. *The Quarterly Journal of Economics*, 101(4):707–727, 1986.

J. Murakami and K. I. Gregory. Transit value capture: New town codevelopment models and land market updates in Tokyo and Hong Kong. *Value Capture and Land Policies*, pages 285–320, 2012.

S. C. Myers. Determinants of corporate borrowing. *Journal of Financial Economics*, 5(2):147–175, 1977.

S. C. Myers. Finance theory and financial strategy. *Interfaces*, pages 126–137, 1984.

H. T. Smit and L. Trigeorgis. Valuing infrastructure investment: An option games approach. *California Management Review*, 51(2):79–100, 2009.

H. Suzuki, J. Murakami, Y.-H. Hong, and B. Tamayose. *Financing transit-*

oriented development with land values: Adapting land value capture in developing countries. The World Bank, 2015.

L. Trigeorgis. *Real options: Managerial flexibility and strategy in resource allocation*. MIT press, 1996.

L. Trigeorgis and S. P. Mason. Valuing managerial flexibility. *Midland Corporate Finance Journal*, 5(1):14–21, 1987.

L. Trigeorgis and J. J. Reuer. Real options theory in strategic management. *Strategic Management Journal*, 38(1):42–63, 2017.

S. V. Ukkusuri and G. Patil. Multi-period transportation network design under demand uncertainty. *Transportation Research Part B: Methodological*, 43(6):625–642, 2009.

Appendices

Appendix A. Proof of Theorem 1

Dynamic programming equation. The function $t \mapsto Y_t^y := ye^{\rho t}$ solves (1a) and (1b) and is thus a solution to the ODE (1a)–(1b). It is immediate that

$$v(y) \geq g(y) := ye^{-(r-\rho)h_1} - I_1. \quad \forall y \geq 0. \tag{A.1}$$

Besides, by the "principle of optimality,"

$$v(y) \geq e^{-r\varepsilon} v\big(ye^{\rho\varepsilon}\big), \quad \varepsilon > 0.$$

Noting that if $v(\cdot) \in C^1(\mathbb{R}_+)$, we then have by the fundamental law of calculus

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(e^{-r\varepsilon} v(y^{\rho\varepsilon})\Big)\Big|_{\varepsilon=0} := \lim_{\varepsilon \downarrow 0} \quad \frac{e^{-r\varepsilon} v\big(ye^{\rho\varepsilon}\big) - v(y)}{\varepsilon} = -rv(y) + \rho y v'(y).$$

We thus conclude that

$$rv(y) - \rho y v'(y) \geq 0, \quad \text{a.e. } y \in \mathbb{R}_+.$$

Because the firm will either invest straight away or wait, then we conclude that the value function (2) satisfies

$$0 = \min\big\{v(y) - g(y); rv(y) - \rho y v'(y)\big\} \tag{A.2}$$

provided the function $v(\cdot) \in C^1(\mathbb{R}_+)$ and $\lim_{y\downarrow 0} v(y) = 0$.

Boundary problem. We conjecture that if (A.2) admits a solution $v(y)$, then this solution solves the problem

$$rv(y) - \rho y v'(y) = 0, \qquad y < \bar{y}, \tag{A.3a}$$

$$v(y) = g(y), \qquad y \geq \bar{y}. \tag{A.3b}$$

where the scalar $\bar{y}$ is an unknown.

We conjecture that (A.3a) admits a solution of the form $y \mapsto y^{\gamma}$, which holds true if $\gamma = r/\rho$. For $\lim_{y \downarrow 0} v(y) = 0$ we need to assume $r > \rho$. We have

$$v(y) = \begin{cases} c \times y^{\frac{r}{\rho}}, & y < \bar{y}, \\ y e^{-(r-\rho)h_1} - I_1, & y \geq \bar{y}. \end{cases}$$

Yet, we postulate that $v(\cdot)$ is $C^1(\mathbb{R}_+)$; it follows that

$$c = \left(\frac{1}{\bar{y}}\right)^{\frac{r}{\rho}} \left\{ \bar{y} e^{-(r-\rho)h_1} - I_1 \right\} \quad \text{and} \quad \bar{y} = \frac{r}{r-\rho} I_1 e^{(r-\rho)h_1}. \tag{A.4}$$

We thus obtained the function in Theorem 1.

Verification of the DP equation. For a solution to (A.3a)–(A.3b) to solve (A.2) it also needs to satisfy

$$v(y) \geq g(y), \qquad y < \bar{y}, \tag{A.5a}$$

$$rv(y) - \rho y v'(y) \geq 0, \qquad y \geq \bar{y}. \tag{A.5b}$$

From Theorem 1, the inequality (A.5a) is equivalent to

$$\left(\frac{y}{\bar{y}}\right)^{\frac{r}{\rho}}\Big\{\bar{y}e^{-(r-\rho)h_1} - I_1\Big\} \geq ye^{-(r-\rho)h_1} - I_1, \tag{A.6}$$

$$\bar{y}^{-\frac{r}{\rho}}g(\bar{y}) \geq y^{-\frac{r}{\rho}}g(y), \quad y < \bar{y}. \tag{A.7}$$

which is satisfied if $\upsilon \mapsto \upsilon^{-\frac{r}{\rho}}g(\upsilon)$ is monotone increasing on $(y, \bar{y})$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\upsilon}\big(\upsilon^{-\frac{r}{\rho}}g(\upsilon)\big) &= \upsilon^{-\frac{r}{\rho}-1}\Big[\upsilon g'(\upsilon) - \frac{r}{\rho}g(\upsilon)\Big] \\ &= \upsilon^{-\frac{r}{\rho}-1}\Big[-\upsilon e^{-(r-\rho)h_1}\frac{r-\rho}{\rho} + \frac{r}{\rho}I_1\Big].\end{aligned}$$

From (A.4),

$$\frac{\mathrm{d}}{\mathrm{d}\upsilon}\big(\upsilon^{-\frac{r}{\rho}}g(\upsilon)\big) = \upsilon^{-\frac{r}{\rho}-1}e^{-(r-\rho)h_1}\frac{r-\rho}{\rho}\Big[-\upsilon + \bar{y}\Big].$$

It is then immediate that $\frac{\mathrm{d}}{\mathrm{d}\upsilon}\big(\upsilon^{-\frac{r}{\rho}}g(\upsilon)\big) > 0$ if $\upsilon \in (0, \bar{y})$, which proves the inequality (A.5a).

We now consider the inequality (A.5b), which from Theorem 1 is equivalent to proving that $\big[r-\rho\big]ye^{-(r-\rho)h_1} \geq rI_1$; this is immediate by definition of $\bar{y}$ in (A.4). This completes the proof of Theorem 1.

Appendix B. Proof of Theorem 2

We first want to establish the relation (9). Thanks to the law of iterated expectations, we can rewrite (5) as

$$J_2^x(\tau_1^{x,y}, \tau_2^x) = \mathbb{E}\Big[e^{-r\tau_2}\Big\{(1-\alpha)\Big(e^{-rh_2}\mathbb{E}\big[X_{h_2}^{X_{\tau_2}^x}\big] - I_2\Big) - K\Big\}\mathbf{1}_{\{\tau_2 > \tau_1\}}\Big].$$

Besides, X_t^x is lognormally distributed, so $\mathbb{E}\big[X_{h_2}^x\big] = xe^{\mu h_2}$. It follows from the strong Markov property that

$$J_2^x(\tau_1^{x,y}, \tau_2^x) = \mathbb{E}\big[e^{-r\tau_2} G\big(X_{\tau_2}^x\big)\mathbf{1}_{\{\tau_2 > \tau_1\}}\big],$$

where the function $G_2(\cdot)$ is defined in (11). The relation (9) immediately follows.

We now want to solve the VI (13a)–(13d). We conjecture that the continuation set $\mathcal{C}_1 = \{x > 0 : V_2(x) > G_2(x)\}$ is of the form $(0, \bar{x}_2)$. The conjecture about the form of the continuation set allows us to relate the solution to the VI (13a) to a free-boundary problem (FBP), namely

$$\mathcal{L}_2 V_2(x) = 0, \qquad x < \bar{x}_2, \tag{B.1a}$$

$$V_2(x) = G_2(x), \qquad x \geq \bar{x}_2, \tag{B.1b}$$

where $\bar{x}_2$ is free boundary. Given the conjectured regularity (13b), we also consider the smooth-fit conditions:

$$V_2(\bar{x}_2) = (1-\alpha)\Big(e^{-(r-\mu)h_2}\bar{x}_2 - I_2\Big) - K, \tag{B.1c}$$

$$V_2'(\bar{x}_2) = (1-\alpha)e^{-(r-\mu)h_2}. \tag{B.1d}$$

The boundary conditions (13c) and (13d) are also supposed to be satisfied by the solution to the FBP (B.1a)–(B.1d).

We can easily show that $x \mapsto x^{\gamma_A}$ and $x \mapsto x^{\gamma_B}$ are independent solutions to the ODE (B.1a). More generally, any linear combination of these two functions are solutions to this ODE. Because of the boundary condition (13d), we focus on solutions of the form $V_2(x) = A_2 x^{\gamma_A}$ on $(0, \bar{x}_2)$. We re-write the smooth-fit conditions (B.1c)–(B.1d) as:

$$\begin{aligned} A_2\big(\bar{x}_2\big)^{\gamma_A} &= (1-\alpha)\Big(e^{-(r-\mu)h_2}\bar{x}_2 - I_2\Big) - K, \\ \gamma_A A_2\big(\bar{x}_2\big)^{\gamma_A - 1} &= (1-\alpha)e^{-(r-\mu)h_2}. \end{aligned}$$

These two conditions are sufficient to determine the two unknowns $\bar{x}_2$ and A_2. The free boundary $\bar{x}_2$ is given in Theorem 2, while

$$A_2 = \Big(\frac{1}{\bar{x}_2}\Big)^{\gamma_A}\Big[(1-\alpha)(e^{-(r-\mu)h_2}\bar{x}_2 - I_2) - K\Big].$$

It remains to check the inequalities

$$V_2(x) \geq G_2(x), \qquad x < \bar{x}_2, \tag{B.2a}$$

$$\mathcal{L}_2 V_2(x) \geq 0, \qquad x \geq \bar{x}_2 \tag{B.2b}$$

to establish that the function $V_2(\cdot)$ in Theorem 2 solves the VI (13a)–(13d). We re-write (B.2a) as

$$\bar{x}_2^{-\gamma_A} G\big(\bar{x}_2\big) \geq x^{-\gamma_A} G_2(x), \quad x < \bar{x}_2,$$

which holds true if $x \mapsto x^{-\gamma_A} G_2(x)$ is monotone increasing on $(0, \bar{x}_2)$. We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(x^{-\gamma_A} G_2(x)\Big) = (1-\alpha)x^{-\gamma_A - 1}\Big\{(1-\gamma_A)e^{-(r-\mu)h_2}x + \gamma_A\big[I_2 + \frac{K}{1-\alpha}\big]\Big\}.$$

From Theorem 2, we have

$$\gamma_A\big[I_2 + \frac{K}{1-\alpha}\big] = (\gamma_A - 1)\bar{x}_2 e^{-(r-\mu)h_2}.$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\Big(x^{-\gamma_A}G_2(x)\Big) &= (1-\alpha)x^{\gamma_A - 1}(\gamma_A - 1)e^{-(r-\mu)h_2}\big[\bar{x}_2 - x\big] \\ &> 0 \text{ if } x \in (0, \bar{x}_2).\end{aligned}$$

The inequality (B.2b) is equivalent to

$$x \geq \frac{r}{r-\mu}e^{(r-\mu)h_2}\big[I_2 + \frac{K}{1-\alpha}\big], \quad x \geq \bar{x}_2.$$

For the above to hold, it suffices that

$$\frac{\gamma_A}{\gamma_A - 1} \geq \frac{\frac{r}{\mu}}{\frac{r}{\mu} - 1}. \tag{B.3}$$

From (14),

$$\mathcal{Q}\Big(\frac{r}{\mu}\Big) = -\frac{1}{2}\sigma^2\frac{r}{\mu}\frac{r}{r-\mu} < 0 \leq \mathcal{Q}(\gamma_A).$$

Besides,

$$\mathcal{Q}'(\gamma) = -\big(\mu - \frac{1}{2}\sigma^2\big) - \gamma\sigma^2,$$

which is negative under the (sufficient) assumption that $\mu > \frac{1}{2}\sigma^2$. We thus conclude that $\frac{r}{\mu} > \gamma_A$. Because the function $x \mapsto \frac{x}{x-1}$ is monotone decreasing, we have established (B.3) and completed the proof of Theorem 2.

If we use the notations $a \wedge b := \min\{a; b\}$ and $a \vee b := \max\{a; b\}$, we can re-write Theorem 2 as

$$V_2(x) = \Big(\frac{x \wedge \bar{x}_2}{\bar{x}_2}\Big)^{\gamma_A} G_2\big(x \vee \bar{x}_2\big). \tag{B.4}$$

Appendix C. Proof of Lemma 2

By the law of iterated expectations, we can re-write (6) as

$$J_1^{x,y}\big(\tau_1, \tau_2\big) = \mathbb{E}\Big[e^{-r\tau_1}\Big\{e^{-rh_1} Y_{h_1}^{Y_{\tau_1}^y} - I_1\Big\} + e^{-r\tau_2}\Big\{\alpha\Big(e^{-rh_2}\mathbb{E}\big[X_{h_2}^{X_{\tau_2}^x}\big] - I_2\Big) + K\Big\}\Big].$$

Now given the semigroup nature of $Y_\cdot^y$ and $X_\cdot^x$, we can write

$$J_1^{x,y}\big(\tau_1, \tau_2\big) = \mathbb{E}\Big[e^{-r\tau_1}\Big\{e^{-(r-\rho)h_1} Y_{\tau_1}^y - I_1\Big\} + e^{-r\tau_2}\Big\{\alpha\Big(e^{-(r-\mu)h_2} X_{\tau_2}^x - I_2\Big) + K\Big\}\Big].$$

Given the reaction function (17), we obtain

$$\begin{aligned} J_1^{x,y}\Big(\tau_1^{x,y}, \mathcal{T}_2^x(\tau_1^{x,y})\Big) = \mathbb{E}\Big[& e^{-r\tau_1^{x,y}}\Big\{e^{-(r-\rho)h_1} Y_{\tau_1^{x,y}}^y - I_1\Big\} \\ &+ \Big(\frac{X_{\tau_1^{x,y}}^x \wedge \bar{x}_2}{\bar{x}_2}\Big)^{\gamma_A}\Big\{\alpha(e^{-(r-\mu)h_2}(X_{\tau_1^{x,y}}^x \vee \bar{x}_2) - I_2) + K\Big\}\Big]. \end{aligned}$$

It is immediate that the function G_1 in (18) is a floor function for the value function V_1 in (8). Besides, by the principle of dynamic programming,

$$V_1(x,y) \geq \mathbb{E}\Big[e^{-r\varepsilon} V_1\big(X_\varepsilon^x, Y_\varepsilon^y\big)\Big]. \tag{C.1}$$

If V_1 is regular (in a sense that we shall specify next), then we can use Dynkin's formula (in a generalized form) to obtain

$$\mathbb{E}\Big[e^{-r\varepsilon}V_1\big(X_\varepsilon^x, Y_\varepsilon^y\big)\Big] = V_1(x,y) - \mathbb{E}\int_0^\varepsilon \mathcal{L}_1 V_1\big(X_s^x, Y_s^y\big)\mathrm{d}s,$$

where the operators $\mathcal{L}_1$ are given respectively by (19).

From (C.1) we then obtain as $\varepsilon \downarrow 0$ the inequality $\mathcal{L}_1 V_1(x,y) \geq 0$ almost every $x \in \mathcal{X}$. We also have a complementary slackness condition. The VI in this case reads (20a)–(20b).

Appendix D. Proof of Lemma 3

First, it is immediate that $x \mapsto G_1(x,y)$ is continuous at $\bar{x}_2$. Yet, because

$$\frac{\partial G_1}{\partial x}\big(\bar{x}_2-, y\big) = \frac{\partial \check{G}_1}{\partial x}\big(\bar{x}_2, y\big) = \gamma_A \frac{\alpha\big(e^{-(r-\mu)h_2}\bar{x}_2 - I_2\big) + K}{\bar{x}_2},$$

while

$$\frac{\partial G_1}{\partial x}\big(\bar{x}_2+, y\big) = \frac{\partial \check{G}_1}{\partial x}\big(\bar{x}_2, y\big) = \alpha e^{-(r-\mu)h_2},$$

we can conclude that $x \mapsto G_1(x,y)$ is not continuously differentiable, but satisfies (21).

Further, it follows from (18) that

$$\begin{aligned}\frac{\partial \check{G}_1}{\partial x}(x,y) &= \gamma_A x^{\gamma_A - 1}\bar{x}_2^{-\gamma_A}\Big\{\alpha\big(e^{-(r-\mu)h_2}\bar{x}_2 - I_2\big) + K\Big\},\\ \frac{\partial^2 \check{G}_1}{\partial x^2}(x,y) &= \gamma_A(\gamma_A - 1)x^{\gamma_A - 2}\bar{x}_2^{-\gamma_A}\Big\{\alpha\big(e^{-(r-\mu)h_2}\bar{x}_2 - I_2\big) + K\Big\}.\end{aligned}$$

If we assume Equation (22), then we can claim that the obstacle $x \mapsto \check{G}_1(x,y)$

in (18) is monotone increasing on $(0, \bar{x}_2)$ from $e^{-(r-\rho)h_1}y - I_1$ to $\tilde{G}_1(\bar{x}_2-, y)$ and concave. If $y < e^{(r-\rho)h_1}I_1$, then the obstacle $x \mapsto \check{G}_1(x, y)$ is negative on $\big(0, \bar{x}_0(y)\big)$ with $\bar{x}_0(y)$ given by (23) and positive on $\big(\bar{x}_0(y), \bar{x}_2\big)$; otherwise (i.e., if $y > e^{(r-\rho)h_1}I_1$), then $x \mapsto \check{G}_1(x, y)$ is positive on $(0, \bar{x}_2)$. Further, from (18), we have

$$\frac{\partial \tilde{G}_1}{\partial x}(x, y) = \alpha e^{-(r-\mu)h_2} > 0.$$

This completes the proof of the Lemma 3.

Appendix E. Proof of Theorem 3

In Case A, we conjecture that $\mathcal{C}_1$ is of the form $\{(x, y) : (0, \bar{x}_1(y))\}$ with $\bar{x}_1(y) \geq \bar{x}_2$: MTRC should start contruction if the property price $x \geq \bar{x}_1(y)$. If this ansatz holds, then the solution to (20a) also solves the free-boundary problem (FBP):

$$\mathcal{L}_1 V_1(x, y) = 0, \qquad x < \bar{x}_1(y), \tag{E.1a}$$

$$V_1(x, y) = \tilde{G}_1(x, y), \qquad x \geq \bar{x}_1(y), \tag{E.1b}$$

where $\tilde{G}_1$ is defined in (18).

We conjecture that the ODE (E.1a) has a solution of the form $f(x, y) = x^\gamma y^{1-\gamma}$. We have then $\mathcal{L}_1 f(x, y) = \mathcal{D}(\gamma) x^\gamma y^{1-\gamma}$, where $\mathcal{D}(\cdot)$ is given in (24). We study the quadratic function $\mathcal{D}(\cdot)$. We have $\mathcal{D}(-\infty) = \mathcal{D}(+\infty) = -\infty$ and

$$\mathcal{D}'(\gamma) = -\big[\mu - \rho - \frac{1}{2}\sigma^2\big] - \gamma\sigma^2 \lesseqgtr 0 \iff \gamma \gtreqless \gamma^\star := -\frac{\mu - \rho - \frac{1}{2}\sigma^2}{\sigma^2}.$$

We assume that the equation (26a) is satisfied. So the function $\mathcal{D}(\cdot)$ is increasing on $(-\infty, \gamma^\star)$ from $-\infty$ to $\mathcal{D}(\gamma^\star)$ and decreasing on $(\gamma^\star, \infty)$ from $\mathcal{D}(\gamma^\star)$ to $-\infty$. After some calculations it obtains

$$\mathcal{D}(\gamma^\star) = r - \rho + \frac{\left[\mu - \rho - \frac{1}{2}\sigma^2\right]^2}{2\sigma^2} > 0.$$

So the function $\mathcal{D}(\cdot)$ admits two roots, one positive noted δ_A and one negative noted δ_B. Further, $\mathcal{D}(1) = r - \mu > 0$ so that $1 \in \big(\delta_B, \delta_A\big)$. We have $\mathcal{D}(\gamma_A) = (\gamma_A - 1)\rho > 0$ so $\gamma_A \in (1, \delta_A)$. So we have:

$$\delta_B < 0 < 1 < \gamma_A < \delta_A.$$

We can now write the solution to the FBP (E.1a)–(E.1b) as

$$V_1(x,y) = \begin{cases} A x^{\delta_A} y^{1-\delta_A} + B x^{\delta_B} y^{1-\delta_B}, & x < \bar{x}_1(y), \\ \tilde{G}_1(x,y), & x \geq \bar{x}_1(y). \end{cases}$$

There remains to find three unknowns A, B and $\bar{x}_1(y)$. For $V_1(x,y)$ to be finite as $x \downarrow 0$, we set $B = 0$. We want to find a function V_1 that is regular in the sense of (20b). The smooth-fit conditions in x read

$$A\bar{x}_1(y)^{\delta_A} y^{1-\delta_A} = e^{-(r-\rho)h_1} y + \alpha e^{-(r-\mu)h_2} \bar{x}_1 - (I_1 + \alpha I_2 - K), \quad \text{(E.2a)}$$

$$\delta_A A \bar{x}_1(y)^{\delta_A - 1} y^{1-\delta_A} = \alpha e^{-(r-\mu)h_2}. \quad \text{(E.2b)}$$

The expression for $\bar{x}_1(y)$ in (28) follows from (E.2a)–(E.2b) after some computations; it also follows that

$$A = \frac{\tilde{G}_1\big(\bar{x}_1(y), y\big)}{\bar{x}_1(y)^{\delta_A} y^{1-\delta_A}}. \tag{E.3}$$

It is immediate that the function $\bar{x}_1(\cdot)$ given in (28) is monotone decreasing on $(0, \infty)$ from

$$\bar{x}_1(0) = \frac{\delta_A}{\delta_A - 1} \frac{(I_1 + \alpha I_2 - K)}{\alpha} e^{(r-\mu)h_2},$$

to $-\infty$. We now assume that (26b) is satisfied, so $\bar{x}_1(0) > 0$. It obtains after some (tedious) calculations that

$$\bar{x}_1(0) - \bar{x}_2 = \frac{e^{(r-\mu)h_2}}{(\delta_A - 1)(\gamma_A - 1)} \frac{f(\alpha)}{\alpha(1-\alpha)},$$

where

$$f(\alpha) := a\alpha^2 - \big(a + b + c\big)\alpha + c,$$

with a, b and c positive and defined by (25) and postive.

It is immediate that $\bar{x}_2 < \bar{x}_1(0)$ iff $f(\alpha) < 0$. We study the quadratic function $f(\cdot)$. This function is concave because $f''(\alpha) = 2a > 0$. This function is decreasing on $(-\infty, \frac{a+b+c}{2a})$ from $f(-\infty) = +\infty$ to $f\big(\frac{a+b+c}{2a}\big)$ and increasing on $(\frac{a+b+c}{2a}, +\infty)$ from $f\big(\frac{a+b+c}{2a}\big)$ to $f(+\infty) = +\infty$. We have $f(0) = c > 0$ and $f(1) = -b$. So necessarily the function $f(\cdot)$ admits a unique root in $(0, 1)$, $\alpha_\star$ defined in (26c).

We note that

$$\bar{x}_1(y) > \bar{x}_2 \iff 0 < y < y_\star,$$

with $y_\star$ defined in (29). We can now conclude that if $0 < y < y_\star$ with $y_\star$ given in (29), then the function V_1 given in (27) is a solution to the FBP (E.1a) with the regularity (20b).

We now want to prove that the function V_1 in (27) solves the VI (20a). It remains to check that

$$\bar{x}_1(y)^{-\delta_A}\tilde{G}_1(\bar{x}_1(y), y) \geq x^{-\delta_A}G_1(x, y), \qquad x < \bar{x}_1(y), \tag{E.4a}$$

$$\mathcal{L}_1\tilde{G}_1(x, y) \geq 0, \qquad x \geq \bar{x}_1(y). \tag{E.4b}$$

To establish the inequality (E.4a) for $x \in (\bar{x}_2, \bar{x}_1(y))$, it suffices to verify that $x \mapsto x^{-\delta_A}\tilde{G}_1(x, y)$ is monotone increasing on $\big(\bar{x}_2, \bar{x}_1(y)\big)$ for $y \in (0, y_\star)$. From the definition of $\tilde{G}_1$ in (18), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\Big(x^{-\delta_A}\tilde{G}_1(x, y)\Big) =& - x^{-\delta_A}\alpha e^{-(r-\mu)h_2}(\delta_A - 1)\\ &+ \delta_A x^{-\delta_A - 1}\big[(I_1 + \alpha I_2 - K) - e^{-(r-\rho)h_1}y\big]\\ =&(\delta_A - 1)x^{-\delta_A - 1}\alpha e^{-(r-\mu)h_2}\big(\bar{x}_1(y) - x\big),\end{aligned}$$

where the second line comes from (28). The derivative is obviously positive if $x \in (0, \bar{x}_1(y)) \supset (\bar{x}_2, \bar{x}_1(y))$. To establish (E.4a) it now remains to prove that for $x \in (0, \bar{x}_2)$

$$\bar{x}_1(y)^{-\delta_A}\tilde{G}_1(\bar{x}_1(y), y) \geq x^{-\delta_A}\check{G}_1(x, y).$$

Because $\bar{x}_1(y)^{-\delta_A}\tilde{G}_1(\bar{x}_1(y), y) \geq x^{-\delta_A}\tilde{G}_1(x, y)$ for $x \leq x_1(y)$, it is sufficient to prove that $\tilde{G}_1(x, y) \geq \check{G}_1(x, y)$ or

$$0 \geq \alpha e^{-(r-\mu)h_2} x \underbrace{\left[\left(\frac{x}{\bar{x}_2}\right)^{\gamma_A - 1} - 1\right]}_{\leq 0} + \underbrace{\left[\left(\frac{x}{\bar{x}_2}\right)^{\gamma_A} - 1\right]}_{\leq 0} \big(K - \alpha I_2\big),$$

which is satisfied if $K > \alpha I_2$.

We now want to prove the inequality (E.4b), which is equivalent to proving that

$$\mathcal{L}_1\tilde{G}_1(x, y) = (r-\rho)e^{-(r-\rho)h_1}y + (r-\mu)\alpha e^{-(r-\mu)h_2}x - r(I_1 + \alpha I_2 - K) \geq 0, \quad x \geq \bar{x}_1(y),$$

or that

$$x \geq \left[\frac{r}{r-\mu}(I_1 + \alpha I_2 - K) - \frac{r-\rho}{r-\mu}e^{-(r-\rho)h_1}y\right]\frac{e^{(r-\mu)h_2}}{\alpha}, \quad x \geq \bar{x}_1(y).$$

By definition of $\bar{x}_1(y)$ in (28), it appears that the above inequality is satisfied because $\frac{\delta_A}{\delta_A - 1} > \frac{r}{r-\mu}$. So we prove the Theorem 3 in Case A.