Financially constrained investments: optimal timing and capacity under uncertainty

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Abstract

In this paper we analyze the strategic interplay between a firm and an investor. We consider the investment strategy of an entrant that is financially constrained, i.e., it needs to issue debt. We find the optimal investment moment for the firm in an environment with market uncertainty and bankruptcy risk as well as the optimal production capacity. In addition, the optimal debtholder policy is determined. The effect of debt financing on the investment strategy, total welfare, and the debtholder policy is studied.

1 Model

In our baseline model we are studying the situation of a monopolist that has the option to invest and set capacity I in a market where the inverse demand is given by

$$p(t) = x(t)(1 - \eta I).$$

Here, $\eta > 0$ denotes the price sensitivity parameter. $x(t)$ follows a geometric Brownian motion with trend μ and volatility parameter σ. Discounting is done under rate $r > 0$. We make the usual assumption that $r > \mu > \frac{1}{2}\sigma^2$ to ensure convergence. To finance this project the firm has a sunk cost of δI, for which it needs to acquire debt. Debt holders lend the total sum of δI upon investment to the firm after which the firm will pay the debt holders at rate ρ. We assume that both players are risk neutral value-maximizing firms.

Solving the model, we use a Stackelberg set-up where the firm is Stackelberg follower and the investor is Stackelberg leader. The lender can thereby adjust its policies to attract or deter investment.

We will first discuss a benchmark model where there is no bankruptcy risk. After that we analyze our main model where the firm defaults *à la* Leland if it is not able to fulfill its debt commitments.

In order to describe the optimization problems, let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space of $x(t)$. Here, the filtration associated with the process $x(\cdot)$ is denoted by $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$, collecting the available information up until time t. Moreover, in our optimization problems, operator $\mathbb{E}$ is taken with respect to a measure $\mathbb{P}$ where $x(0) = X$. Stopping times are denoted by τ. If $\mathcal{M}$ is the set of all finite $\mathcal{F}_t$-stopping times, then, in the remainder of the paper, optimal stopping times are such that $\tau \in \mathcal{M}$. For the sake readability we refrain from adding this constraint to the to be formulated optimization problems.

2 The Indefinite Project

As a benchmark case, we will first look at model where there is no bankruptcy risk, i.e., it is assumed that the firm will never default. In such a scenario the firm's optimization problem, for a given coupon rate ρ, is given by

$$V(X) = \sup_{\tau, I \geq 0} \mathbb{E}_0 \left\{ \int_\tau^\infty e^{-rt} x(t)(1-\eta I) I \mathrm{d}t - \int_\tau^\infty e^{-rt} \rho \delta I \mathrm{d}t \right\},$$

where τ is a stopping time. Debt holders solve the optimization problem given by

$$D(X) = \sup_{\rho > 0} \mathbb{E}_0 \left\{ \int_{\tau(\rho)}^\infty e^{-rt} \rho \delta I(\rho) \mathrm{d}t - e^{-r\tau(\rho)} \delta I(\rho) \right\}.$$

For $\tau = 0$, this is $V(X) = \sup_{I \geq 0} \left\{ \frac{X}{r-\mu} I(1-\eta I) - \frac{\rho}{r} \delta I \right\}$ and $D(X) = \sup_{\rho > 0} \left\{ \frac{\rho - r}{r} \delta I(\rho) \right\}$.

2.1 Firm

The proof of Proposition 1 (next page) shows that optimal investment for the firm is given as follows. For a given value of ρ, the firm invests when $X \geq X^*$, where

$$X^*(\rho) = \frac{\beta_1 + 1}{\beta_1 - 1} \frac{\rho \delta (r - \mu)}{r}, \tag{1}$$

with $\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$. In the stopping region the firm sets

$$I^*(X) = \frac{1}{2\eta} \left(1 - \frac{\rho \delta (r-\mu)}{rX} \right).$$

This implies that for $X < X^*$ the firm delays and will eventually set $I^{opt} := I^*(X^*) = \frac{1}{\eta(\beta_1 + 1)}$. One can already note here that the fact that the firm has to find external financing will not influence the investment size I^{opt}. Rather, the firm delays investment to offset the additional cost.

2.2 Debt holder

The investor providing funds to the firm sets ρ. Notice from equation (1) that if ρ is set such that $X^*(\rho) \leq X$, the firm is induced to invest immediately, whereas for ρ such that $X^*(\rho) > X$, investment is delayed. The former, as $X^*(\rho)$ is linear in ρ, happens when $\rho \leq \hat{\rho}(X)$, where

$$\hat{\rho}(X) = \frac{\beta_1 - 1}{\beta_1 + 1} \frac{rX}{\delta(r - \mu)}.$$

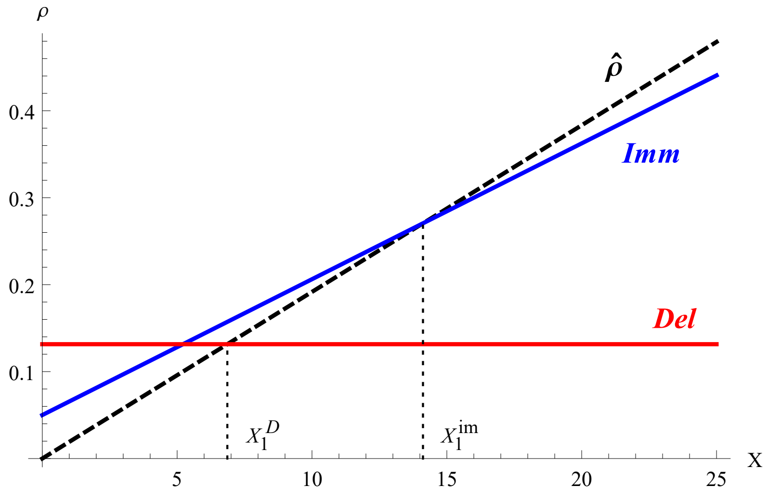

Figure 1: Illustration of $\hat{\rho}(X)$, $\rho^{imm}(X)$, and ρ^{del}.

The latter happens for $\rho > \hat{\rho}(X)$. As a result, the investor faces optimization problem

$$\begin{aligned} D(X) &= \sup_{\rho>0} \; \mathbb{E}_0 \left\{ \int_{\tau(\rho)}^{\infty} e^{-rt} \rho \delta I^*(X_{\tau(\rho)}) \mathrm{d}t - e^{-r\tau(\rho)} \delta I^*(X_{\tau(\rho)}) \right\} \\ &= \sup_{\rho>0} \begin{cases} \frac{\rho - r}{r} \delta I^*(X) & \text{if } \rho < \hat{\rho}(X), \\ \\ \left(\frac{X}{X^*(\rho)} \right)^{\beta_1} \frac{\rho - r}{r} \delta I^{opt} & \text{if } \rho \geq \hat{\rho}(X). \end{cases} \end{aligned}$$

The following proposition gives the solution to the optimization problem. Thereto, let us define $X_1^{del} = \frac{\beta_1(\beta_1+1)}{(\beta_1-1)^2}\delta(r-\mu)$ and $X_1^{imm} = \frac{\beta_1+1}{\beta_1-3}\delta(r-\mu)$ if $\beta_1 > 3$ and $X_1^{imm} = \infty$ otherwise. In the following proposition, $\rho^E(X)$ denotes the coupon rate that maximizes the investor's profits, given as a function of the state variable.

Proposition 1 *For $X < X_1^{del}$ investment is delayed until X_1^{del}. Upon investment, the firm starts paying coupon rate $\rho^{del} = \frac{r\beta_1}{\beta_1-1}$. For $X \geq X_1^{del}$ the firm undertakes investment immediately. Either by setting the coupon rate equal to $\hat{\rho}(X)$ or by setting $\rho^{imm}(X) = \frac{r(X+\delta(r-\mu))}{2\delta(r-\mu)} = \frac{r}{2}\left(1 + \frac{X}{\delta(r-\mu)}\right)$:*

$$\rho^E(X) = \begin{cases} \rho^{del} & \text{if } X < X_1^{del}, \\ \hat{\rho}(X) & \text{if } X_1^{del} \leq X < X_1^{imm}, \\ \rho^{imm}(X) & \text{if } X \geq X_1^{imm}. \end{cases}$$

Figure 1 illustrates the regions graphically. Note that $\rho^{imm}(X) < \hat{\rho}(X)$ for $X > X_1^{imm}$. The observation that for sufficiently large X the investor sets a coupon rate below $\hat{\rho}(X)$ follows from the fact that the firm can choose its investment size. When the investor increases ρ, there are two effects. The direct effect would be an increase in the investor's instantaneous profits as the firm is paying a higher coupon rate. However, since the firm needs to pay more, it will set a lower capacity, which in turn will lower the investor's instantaneous profits. As a result, for sufficiently large X, the last effect outweighs the first so that setting $\rho = \hat{\rho}(X)$ is too costly for the investor. This is only the case, though, for $\beta_1 > 3$. For small β_1, i.e. high σ, the investor will always set $\hat{\rho}(X)$ for $X > X_1^{del}$.

The resulting investment size equals

$$I^*(X) = \begin{cases} \frac{1}{\eta(\beta_1+1)} & \text{if } X < X_1^{imm}, \\ \frac{1}{4\eta}\left(1 - \frac{\delta(r-\mu)}{X}\right) & \text{if } X > X_1^{imm}. \end{cases}$$

Interestingly, if you compare these values to the scenario where the firm does not need lending, we obtain

$$I^*(X) = \begin{cases} \frac{1}{\eta(\beta_1+1)} & \text{if } X < \tilde{X}^*, \\ \frac{1}{2\eta}\left(1 - \frac{\delta(r-\mu)}{X}\right) & \text{if } X > \tilde{X}^*, \end{cases}$$

with $\tilde{X}^* = \frac{\beta_1+1}{\beta_1-1}\delta(r-\mu)$. This means that the traditional underinvestment only applies to the scenario where investment is undertaken for scenarios with relatively high demand.

The proof of the proposition shows that $\hat{X}_1^D < \hat{X}_1^{Im} \Leftrightarrow \beta_1 + 1 > 0$ so that ρ^E is always as defined as in the proposition.

2.3 Fixed capacity

2.4 Effect of uncertainty

2.5 Sarkar

In Sarkar (?), a similar scenario is studied, where the optimization problem is such the company value is maximized, which is equal to $V(X) + D(X)$. As we study the strategic interaction between the players, our set-up is different. We make a short comparison. For $X = X^*$ we find

$$V(X^*) = \frac{\delta\beta_1}{\eta(\beta_1+1)(\beta_1-1)^2}$$
$$D(X^*) = \frac{\delta}{\eta(\beta_1-1)^2}$$

so that $D + V = \frac{\delta}{\eta(\beta_1-1)^2}\frac{2\beta_1+1}{\beta_1+1}$. In Sarkar this was

$$D + V = \frac{X^*}{r-\mu}I^*(1-\eta I^*) - \frac{\rho}{r}\delta I^* + \frac{\rho - r}{r}\delta I^* = \frac{\delta}{\eta(\beta_1+1)(\beta_1-1)}.$$

3 Bankruptcy Risk

3.1 Firm's Investment Decision

Assume that the firm defaults *à la* Leland. That means that the firm's bankruptcy threshold is an optimal stopping such that

$$\tau_B = \text{argsup}_{\tau_B}\mathbb{E}_0\left[\int_0^{\tau_B}\left(\frac{X}{r-\mu}I(1-\eta I) - \rho\delta I\right)e^{-rt}\mathrm{d}t\right].$$

Solving this optimization problem gives $X_B(\rho) = \frac{\beta_2}{\beta_2-1}\frac{r-\mu}{r}\frac{\rho\delta}{1-\eta I(\rho)}$, with $\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$.

The firm's investment problem becomes

$$\begin{aligned} V(X) &= \sup_{\tau, I \geq 0} \mathbb{E}_0 \left\{ \int_{\tau}^{\tau_B(\rho)} e^{-rt} x(t)(1-\eta I) I \mathrm{d}t - \int_{\tau}^{\tau_B(\rho)} e^{-rt} \rho\delta I \mathrm{d}t \right\} \\ &= \sup_{I \geq 0} \begin{cases} \frac{X}{r-\mu} I(1-\eta I) - \frac{\rho}{r}\delta I + \frac{\rho}{r(1-\beta_2)}\delta I \left(\frac{X}{X_B(\rho)}\right)^{\beta_2} & \text{for } X \geq X^*(\rho), \\ AX^{\beta_1} & \text{for } X < X^*(\rho), \end{cases} \end{aligned}$$

where $X^*(\rho)$ is the investment threshold so that τ^* is the first hitting time.

The optimal investment size I as a function of X follows from

$$\left(\frac{X}{X_B(I)}\right)^{\beta_2} = \frac{r(\beta_2-1)}{\rho\delta} \frac{1-\eta I}{1-(\beta_2+1)\eta I} \left(\frac{X}{r-\mu}(1-2\eta I) - \frac{\rho}{r}\delta\right). \tag{2}$$

Applying the smooth pasting and value matching conditions we find that, for $x(0)$ sufficiently small, the firm's investment size, for a given ρ, is

$$I^* = \frac{1}{\eta(\beta_1+1)},$$

and the optimal investment threshold $X^*(\rho)$ satisfies

$$\frac{(\beta_1-\beta_2)\rho\delta}{r(1-\beta_2)} \left(\frac{X}{X_B(\rho)}\right)^{\beta_2} - \frac{\beta_1\rho\delta}{r} + \frac{X\beta_1(\beta_1-1)}{(\beta_1+1)(r-\mu)} = 0. \tag{3}$$

Otherwise, the firm invests instantly at X and the corresponding investment capacity $I(X)$ is a solution of equation (2).

3.2 Debt Holder's Optimal Decision

The debt holder's problem changes in two ways. First, coupon payments are now only received until bankruptcy. Secondly, if the firm defaults, the assets are transferred to the debt holder. The liquidation value upon investment is the value of the firm corrected for bankruptcy cost, which will be denoted by fraction α. In line with the literature (see, e.g.,), we then obtain liquidation value

$$(1-\alpha)\frac{X_B(\rho)}{r-\mu} I(\rho)(1-\eta I(\rho)) = (1-\alpha)\frac{\beta_2}{\beta_2-1}\frac{\rho\delta I(\rho)}{r}.$$

It follows,

$$\begin{aligned} D(X) &= \sup_{\rho > 0} \mathbb{E}_0 \left\{ \int_{\tau(\rho)}^{\tau_B} e^{-rt}\rho\delta I(\rho)\mathrm{d}t - e^{-r\tau(\rho)}\delta I(\rho) + e^{-r\tau_B(\rho)}(1-\alpha)\frac{\beta_2}{\beta_2-1}\frac{\rho\delta I(\rho)}{r} \right\} \\ &= \sup_{\rho > 0} \begin{cases} \left(\frac{X}{X^*(\rho)}\right)^{\beta_1} \frac{\delta\rho}{r(\beta_1+1)} \left(\frac{\rho-r}{\rho} - \left(\frac{X^*(\rho)}{X_B(\rho)}\right)^{\beta_2} \frac{1-\alpha\beta_2}{1-\beta_2}\right) & \text{if } X < X^*(\rho), \\ \delta I^*(\rho)\frac{\rho-r}{r} - \frac{\rho\delta I^*(\rho)}{r}\left(\frac{X}{X_B(\rho)}\right)^{\beta_2}\frac{1-\alpha\beta_2}{1-\beta_2} & \text{if } X \geq X^*(\rho). \end{cases} \end{aligned} \tag{4}$$

Unfortunately, these equations do not allow for closed form solutions. Without formally showing it yet, we state the solution is as follows. Similar to the scenario without bankruptcy, one can define $\hat{\rho}(X)$ such that $\rho > \hat{\rho}(X)$ is equivalent to $X^*(\rho) > X$ and vice versa. $\hat{\rho}(X)$ follows from equation (3). Finding the argsup for the respective regions will lead to a qualitatively similar policy. There exist X_1^{del}, $X_1^{imm} \in \mathbb{R}$, $0 < X_1^{del} < X_1^{imm} \leq \infty$, such that

$$\rho^E(X) = \begin{cases} \rho^{del} & \text{if } X < X_1^{del}, \\ \hat{\rho}(X) & \text{if } X_1^{del} \leq X < X_1^{imm}, \\ \rho^{imm}(X) & \text{if } X \geq X_1^{imm}, \end{cases}$$

where ρ^{del} and $\rho^{imm}(X)$ are the solutions of the first and second case of (4), respectively.

3.3 Effect of uncertainty

Appendix A

Short version of proof of Proposition 1:

Imm. inv. $\rho^{Im} = \text{argsup}_{\rho>0}\left\{\frac{\delta}{r}\frac{\rho-r}{2\eta}\left(1-\frac{\rho\delta(r-\mu)}{rX}\right)\right\} = \frac{r(X+\delta(r-\mu))}{2\delta(r-\mu)} \leq \hat{\rho}(X) \Leftrightarrow X > \frac{\beta_1+1}{\beta_1-3}\delta(r-\mu) \equiv \hat{X}_1^{Im}$.
If $\beta_1 < 3$, then $\hat{X}_1^{Im} = \infty$.

Del. inv. $\rho^{D} = \text{argsup}_{\rho>0}\left\{\left(\frac{Xr}{\delta\rho(r-\mu)}\frac{\beta_1-1}{\beta_1+1}\right)^{\beta_1}\frac{\delta(\rho-r)}{\eta r(\beta_1+1)}\right\} = \frac{r\beta_1}{\beta_1-1} > \hat{\rho}(X) \Leftrightarrow X < \frac{\beta_1(\beta_1+1)}{(\beta_1-1)^2}\delta(r-\mu) \equiv \hat{X}_1^{D}$.
Lemma: $\hat{X}_1^{D} < \hat{X}_1^{Im} \Leftrightarrow \beta_1 + 1 > 0$. □