A Dynamic Model for Venture Capitalists' Entry–Exit Investment Decisions*

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Abstract

In this paper we develop a dynamic model for guiding Venture Capitalists (VC) on their Entry–Exit Investment Decisions. In our setting, a VC is considering an investment in a start-up which is financed along with the entrepreneur. The model accounts both for homogeneous and heterogeneous beliefs about the growth prospects of the firm, and aims to determine the optimal entry timing and the optimal ownership to be required by the VC. The exiting process is also studied. We start by considering no time restrictions about investment period, but the model is then extended to account for the imposed limited duration of the investment for the VC. The determination of the optimal exit multiple is perhaps the most valuable outcome of the model, specifically when the time restriction is in place therefore showing the value destruction that investors are subject to when limiting their investment exposure. Also, some sensitivity analysis performed allows to assess the impact of important variables on the model outcomes.

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1 Introduction

One of the very first ventures being financed through Venture Capital is as old as 1957, when American Research and Development Corporation invested $70,000 in Digital Equipment Corporation, which went public in 1968 resulting in a return of over 500 times the invested amount, with a stake worth more than $355 million. At the time, the term sheet determining the terms in which the deal would happen would be as simple as to mention the amount to be invested, and the respective equity ownership to be required. Nowadays, term sheets may be many pages long discussing every little detail and account for every imaginable and unimaginable situation that might happen.

The investment decision taken by VCs is very hard and often taken with a conjunction of highly subjective criteria. Some of the most important criteria includes, but are not limited to, the stage of the start-up, competition with other financing sources, previous experience of the team – specially its founders —, the natural entry point of the fund, numerical metrics to determine the financial and operational performance that VC typically use to asses and value of the start-ups to invest –, and the macroeconomic environment. The literature keeps growing, and although many metrics can be found, both financially and/or business driven – such as, for example, the positivity of unit economics [1]–, there are many other concerns that Venture Capitalists take into account that determines the investment decisions. Those concerns might regard the quality of the team, market characteristics – such as how big is the market for the sector in which the product/start-up operates, and the market share that might be taken by the same product/start-up –, competitive hedge of the product or service (such as Intelectual Property rights), the capacity for the project to be scalable, the use of the so-called MVP – minimum viable product –, described as the commercial proof of concept hugely highlighted in the Lean Startup from Ries (2011), and VC specific factors– such as their credibility or network.

Feld and Mendelson (2016) define economics and control as the most important things VCs do look to when negotiating an investment in a start-up firm. In their own words, "Economics refers to the return the investor will ultimately get in a liquidation event, usually either a sale of the company or an initial public offering (IPO), and the terms that have direct impact on this return", while "Control refers to the mechanism that allows the investors to affirmatively exercise control over the business or to veto certain decisions the company can make" (p. 38). Regarding economics, the same authors mention that "it's a mistake to focus only on the valuation when considering the economics of a deal" (p. 39). This is an obvious hot topic – the dark magic versus random science of both pricing and valuing a start-up – which, besides its subjectiveness, is ruled out by long and complex spreadsheets and numbers that might find justified and defensible assumptions in it making it somehow more objective. But again, this is not always the most difficult battlefield VCs and entrepreneurs face in a financing negotiation. According to the authors, a complete approach to the economics of a start-up firm financing deal must include, not being limited to, the pricing, liquidation preferences, important covenants such as the pay-to-play or how to treat the ownership of the stakeholders using, for example, the vesting concept, how to think and negotiate the employee pool, or account for anti-dilution clauses. For the control issues, the biggest portion of negotiation time is spent on issues that regard the board of directors, protective provisions

[1] simply stated as customer lifetime value minus customer acquisition cost.

defined by as "veto rights that investors have on certain actions by the company" (p. 70) on both the VC and key employees side, drag along rights, and conversion.

There is an extreme barrier VCs face on investments, mainly due to its institutional framework[2], which is the fact that the time frame to source, invest and divest in start-ups – thus making a profit for themselves and their limited partners – is extremely limited. The commitment period -– which may also be addressed as the investment period –, usually of a 5 years length, is the time period available for the VCs to source, screen, assess and invest in new start-ups for a given fund. Once this period is over, no further investments in new ventures can be made on behalf of the limited partners of the fund or the fund itself. However, the fund can continue investing in existing ventures started to be financed during the commitment period. Being this particularity of funds a driver for the set up of new funds within the same management company every three to five years, since fresh investment is needed to keep VCs active as investors. As for the total length of time that a fund can remain active, the concept applied is the investment term. While new investments can only happen through the commitment period, follow-on investments can still be made until the end of the investment term. Feld and Mendelson (2016) argue that a typical fund has a 10 year investment term with either two to three one-year extension option or one two-year extension option. Although these results in a twelve to thirteen year length, for very early stage investment funds this maturity may be very short and the authors mention that this type of investment funds can last up to seventeen years. The way VCs can still invest in existing portfolio ventures after the commitment period is over and prior to the expiration of the investment term is through reserves – an amount of the fund locked to invest in follow-on rounds – defined at the moment of the first investment in each venture. These reserves lock the determined amount whose destination will only be the venture it was designed for, therefore reducing the amount of the fund available for new ventures.

To keep its operations running VCs rule their activity by claiming, mainly, three types of income: i) management fees, ii) reimbursement for expenses and iii) carried interest. Let aside the reimbursement expenses, that only represent the repayment – by start-ups who were invested in by VCs – of costs incurred to meet start ups responsibilities like board, clients or suppliers meetings, management fees and carried interest are, respectively, the oxygen and true motivation that feed VCs activities. The management fee is defined as a percentage, ranging from 1.5% to 2.5%, of the total amount raised by the fund, which is claimed annually, but paid through shorter periods of time. Since funds typically last for about 10 years, the amount of management fees can reach up to a total of 15% to 25% of the committed capital, and this amount is used to pay for all the costs a fund might have, from salaries to rents or travels. Worth to mention that, as previously stated, a management company might possess many funds, each of them receiving management fees to feed the management company costs. Interestingly, and once that the management fees decrease the capital amount available to invest in new ventures which is the true purpose of a VC fund, the funds might, in cases where some profitable investment exits occurs earlier in the fund life span, reinvest the amount of management fees in other ventures, thus leveraging the total

[2]There are three main entities within a VC fund structure: i) the management company, typically owned by the most senior partners, which is the entity that employs all the staff and pays for all the fund expenses for a regular operation. ii) The limited partnership, which is a vehicle through which investors (the limited partners) interact with the general partners (VCs) financing them. The final entity, iii) general partnership, is the legal structure behind a general partner or several general partners.

investment capital. Where does the true motivation for setting up a VC fund lies? Carried interest. Although the management fees might boost interesting salaries for partners and staffs and a high quality breathable oxygen for a VC fund to be kept on running, the share of profits returned to the limited partners are the carrot to which all general partners and fund related staffs run to, when performing their day to day tasks. The carried interest is a predetermined proportion of the final profits that limited partners give up on to the general partners to motivate and stimulate their best use of knowledge and expertise. This carried interest is usually of 20%, and might in some cases be defined above a certain predetermined hurdle rate.

The Venture Capital landscape has been growing in relevance for the past decades. As for 2017 alone, PwC and CBInsights (2018) in its Venture Capital Funding Report 2017 states 11,042 deals worth over USD 164,4 Billion, proving how hot and sexy Venture Capital is becoming. Moreover, corporates are looking into it as well, with KPMG (2018) reporting an increase in Corporate Venture Capital investment, as a percentage of the total investments, of over 18.7 percent in the fourth quarter of 2017, proving that even large corporations have a growing appetite for Venture Capital investments. Kortum and Lerner (2001) examine the impact of VC on technological innovation. The authors estimate that Venture Capital accounted for 8% of Industrial Innovation in the decade ending in 1992 in the US, and assuming a constant effectiveness of venture funding, by 1998 Venture Capital has accounted for 14% of US innovative activity. Giat et al. (2007) used a dynamic model to associate entrepreneurs' optimism to projects performance and, consequentially, higher VC returns. In an association of Behaviour Economics, Real Options, and Venture Capital relations with entrepreneurs, the authors derived a qualitative assessment of the influence of optimism in entrepreneurs with the economic value generated, the contract structures signed between VCs and Start-ups, the duration of the relationship, and even how this characteristic might reduce agency costs of risk-sharing. A considered to be "Entrepreneurs Optimism Premium" explains the discrepancy of VC's discount rates of over 40% and the expected returns of VC projects which usually yields around 15%. Cochrane (2005) measured metrics as the mean, standard deviation, alpha and beta of Venture Capital investments, correcting them for selection bias. The author found out that the second, third, and fourth rounds of financing account for less risk, measured by progressively lower volatility. The betas of successive rounds declined from around 1 in the first round to around zero in the fourth.

So far it becomes clear that VCs i) has to profit a lot for their credibility to grow and be able to raise capital for further funds, ii) has to profit as much as possible on ventures exits so their carried interest – after eventual hurdle rates –, can be highly rewarding, iii) have a time restriction preventing them to wait for the right moment to exit from some ventures. Putting together the necessity to maximize profits on a time constraint setting needs some audacity to be successfully accomplished. Among the myriad of possibilities to tackle the problem under sight, there is a methodology that clearly stands out against any other: the use of Real Options. The solution for such a breakthrough in finance as the pricing of financial options was built upon an already mainstream mathematical rational of the ancient science of physics. Immortalized by both Merton (1973) and Black and Scholes (1973), the methodology was revolutionary for the field of finance and the best of its application was yet to be unleashed. Not long after, Myers (1977) has established the beginning of the history of Real Options, in its work entitled "Determinants of Corporate Borrowing". In the paper, the author explores how companies growth opportunities can be seen as call options, and therefore, when dealing with highly uncertain projects, the value of the future option should be added to their Net Present Value.

The applications of Real Options to the field of Venture Capital are still reduced, although the range of studies already covers a wide set of important issues. These issues include matters such as valuation, risk, uncertainty, financing needs, earn outs or convertible notes, timings for entrepreneurs to look for investment, moral hazard, managerial replacement, among many others. As Venture Capital entails a large spectrum of company stages and purposes – ranging from early to later stages, for which financing may be used to, for example, prototype ideas, or to growth focus –, many different characteristics might have the chance of being modelled through a Real Options perspective. Regarding staged financing, Koçkesen and Ozerturk (2002) showed that the highest benefit of this financing timing is related to the possibility of investing in a certain start-up in a further round of investment with better conditions than VC competitors by having a competitive advantage related to the lack of information that outsiders might have. The paper also reveals that adverse selection resulting from asymmetric information represents an exit barrier for entrepreneurs with good prospects and creates an endogenous lock-in. Meng (2008) developed a model based on a duopoly patent race showing that patent races cause over investment, value-dissipation, a higher CAPM beta, and a higher return volatility, in excess of 100% sometimes, compared to a joint monopoly. As for the high level of return volatility, it is considered to happen due to technological risks. Tavares-Gärtner et al. (2018b) introduced a taxonomy of contingent payment mechanisms based on the maturity and amount of investment, through which analysed the decision of an entrepreneur – facing a wealth constraint – looking for an external equity provider to back a growth opportunity. The authors concluded that the choice of the optimum mechanism depends on exogenous variables as liquidity preferences or constraints, timing requirements, post-deal integration or overall deal terms. Maya (2004) created an approach named Creative Destruction – Real Options Approach to valuing start-ups when only technological uncertainty is present, when a start-up is in a context of a Creative Destruction process. This, as mentioned by Schumpeter (1942), happens when new consumer goods, new methods of production or transportation, new markets appear to disrupt the economic structure destroying the previous one, in a cyclical way. In this approach the value of the firm results in the sum of the project value without flexibility, and adding the value of the real options the project offers to the firm, which should account for the option to invest conditional on the discovery of a new product considered as a "drastic innovation". The approach explains the high prices investors may be willing to pay for certain growth stocks, proving that overpricing may not be the case, but is instead a recognition of the large growth potential due to innovation. Siller-Pagaza and Otalora (2009) explains that when managers are entrepreneurs contributing with intangible assets to firm – be the expertise, networking or other mean –, the moral hazard when seeking outside equity depends on the value of real options and the percentage he / she receives of free cash flows. The greater the value of the options, the greater the percentage he / she must earn from dividends, to decrease the moral hazard possibility. Leshchinskii and Brisley (2006) studied, in a two-period framework that allow stage financing, how the information available from potential investors determines an entrepreneur choice of financing from a pool of potential investors that includes business angels, Venture Capitalists and traditional investors. The decision of being funded depends on the additional value that investors abilities might bring to solve some problems that can occur along the projects duration and by the actions they can take, such as replacing the manager or cutting the investment. The results show that the entrepreneurs choose either angel or Venture Capital financing with a cost-benefit analysis on the resolution of possible uncertainties, meaning that more value is created than its cost. The list goes on, and the challenges of Venture Capital being tackled through the Real Options methodology keeps increasing. As a short conclusion, the foundations of both Venture Capital and Real Options were addressed, and finally some examples were given on existing frameworks that apply Real Options to Venture Capital. Interestingly, few are the examples of Real Options approaches to Venture Capital from the Venture Capitalist – the investor – point of view, and it is precisely where the present work will be focused on.

This paper unfolds as follows: chapter 2 explains the base case model and has a numerical approach to it. The model is extended in chapter 3 to account for the time restriction VCs face. Chapter 4 has a numerical example to help conceptualizing the time restricted model, since it does not has an analytical solution, and Chapter 5 concludes.

2 Model

The model comprises the investment and posterior exit of a Venture Capitalist in an established entrepreneurial firm, in a dynamic real options approach. This entrepreneurial firm is assumed to be owned by a single Entrepreneur, and hold a growth opportunity, θ ($\theta > 1$), defined by an expansion of its value, V_t, given an investment totalling K.

The limited to no access to debt attached to both the start-up firm and the Entrepreneur pushes the growth opportunity to be financed through the limited financial resources of the Entrepreneur, K_E ($0 < K_E < K$), together with the Venture Capitalist's (VC) funds, $K - K_E$, in a jointly backed equity round. Transaction costs associated to the Venture Capitalist initial investment are considered to be included in $K - K_E$, and those of the Entrepreneur in K_E.

The perspectives on the growth opportunity might differ among the VC and the Entrepreneur as their optimism level diverge. Given these heterogeneous beliefs, there are different growth prospects $\theta, (\theta > 1)$, attached to the Entrepreneur $\theta_E, (\theta_E > 1)$, and Venture Capitalist $\theta_{VC}, (\theta_{VC} > 1)$, for the start up growth option considering the very same investment amount of, respectively, K_E and $K - K_E$[3].

The Venture Capitalist is assumed to not have any funding constraints neither burden any additional opportunity cost from other potential investments in other ventures, or equivalently that the current investment is the best available one. Also, the capital increase (either by the Entrepreneur or Venture Capitalist) is made at no premium or discount.

Post-equity round firm ownership held by the Venture Capitalist is of $0 < Q_{VC} < 1$, while the new Entrepreneur ownership will be of $0 < Q_E < 1$ and $Q_{VC} = 1 - Q_E$.

After investing $K - K_E$ to possess $Q_{VC}\theta_{VC}V_t$ of the start-up firm value, the Venture Capitalist main concern is to dispose the investment at a considerable profit within a predetermined timing[4].

From the moment the deal took effectiveness, the Venture Capitalist is considered to be the seller, V_{VC}, who owns $Q_{VC}\theta_{VC}V_t$ of the start-up firm value. For another player (either another

[3] The case for homogeneous beliefs, whereas both Entrepreneur and VC share the same optimism level on the growth opportunity is considered to be a particular case of the heterogeneous beliefs model, given $\theta = \theta_E = \theta_{VC}$, and is described in appendix A.5.

[4] This time restriction will be addressed latter on, on chapter 3.

Venture Capital fund, a Private Equity fund or another company) – the Buyer – the same target has a higher value of $(1+\gamma)Q_{VC}\theta_{VC}V_t, (\gamma > 0)$. Those synergies, represented by γ, are assumed to be well known by the industry due to Venture Capitalists' experience on past deals used as benchmarks or even deep specific sector knowledge, since γ can be interpreted as a sector specificity. The VC is only willing to dispose the investment if a certain proportion of the synergies possessed by the Buyer, γ, is shared, and so the transaction will be settled at a premium $\phi, (\gamma > \phi > 0)$. It is important to notice that the premium being discussed, ϕ, is not about the value added by the VC to the start-up, but a premium built upon that added value already incorporated in $\theta_{VC}V_t$. Thus, the premium, ϕ, is a proportion of the Buyer synergies, γ, that is simultaneously the minimum proportion of synergies that the VC is willing to accept in order to sell the start-up stake, and the maximum one the bidder is willing to give up on to buy the start-up.

By selling the target from the VC to the Buyer sunk transaction costs of C arises for both players, in a proportion of ϵ and $(1-\epsilon)$ respectively, with $\epsilon \in [0,1]$.

It is assumed that the value of the start-up firm follows a geometric Brownian motion:

$$dV_t = \alpha V_t dt + \sigma V_t dz, \text{ with } V_0 > 0 \tag{1}$$

with σ as the volatility of the start-up value, α as its growth rate – of the start-up – and dz as an increment of a Wiener process with zero mean and variance equal to dt.

It is also assumed that all agents are risk neutral and that the riskless interest rate $r, (r > \alpha)$ controls for the time-value of money. Accordingly, α is a risk neutral drift. In order to achieve the entry-exit option value two different timings will be considered: (i) the entry-option – an option to invest –, determining the optimal ownership that the VC has to request considering a set of variables, and (ii) the exit-option, negotiating the optimal premium to exit the investment in the case of an offer contemplating the right premium. For the determination of the optimal ownership to be required by the VC fot the investment amount of $K-K_E$, the rational in which the model will be built upon is the contribution of Tavares-Gärtner et al. (2018a), whereas the growth opportunity might differ due to the differences regarding who among the VC and Entrepreneur is the most optimist and by how much. As for the determination of the optimal premium upon which the start-up must be disposed by the Venture Capitalist in the exit-option (even if the predetermined investment maturity is not reached[5]) will be done through the contribution of Lambrecht (2004) and Lukas and Welling (2012). Since the entry-option value is dependent on the exit-option value, a backward procedure must be used, and the exit-option has to be the first to be determined.

2.1 VC Exit Option

The exit-option is determined upon the synergies of a certain Buyer, whereas the deal is settled given the conditions placed by the VC – premium, as a proportion of the total buyer's synergies –, for which the Buyer will optimally define the timing.

Once the target is sold, the VC gets the sales price $(1+\phi)Q_{VC}\theta_{VC}V_t, (\phi > 0)$, has to pay the transaction costs of ϵC and transfer the target, of value $Q_{VC}\theta_{VC}V_t$, to the Buyer. It does not incur a loss, if $\phi \geq \frac{\epsilon C}{Q_{VC}\theta_{VC}V_t}$.

[5]The impact of an investment maturity is discussed in chapter 3, through the contribution of Pereira and Rodrigues (2014). The present chapter considers a perpetual setting for the investment exit.

By buying the start-up stake, the Buyer gets $(1+\gamma)Q_{VC}\theta_{VC}V_t$ and in return has to pay the sales price $(1+\phi)Q_{VC}\theta_{VC}V_t$, and the transaction cost $(1-\epsilon)C$. It does not incur a loss if $\gamma \geq \phi + \frac{(1-\epsilon)C}{Q_{VC}\theta_{VC}V_t}$.

Consequently, a sale of the start-up from the VC to the Buyer will create a surplus if and only if $Q_{VC}\theta_{VC}V_t > C$.

The surplus is $\gamma Q_{VC}\theta_{VC}V_t - C$, and partitioning through the negotiation of ϕ.

Therefore, at time t_0, the VC is requiring $\phi > 0$ to the Buyer, which can accept or reject the offer – but does not has to decide immediately, thus having the possibility of postpone the decision.

It is assumed that there is no possibility for further rounds of negotiation or counteroffers. Hence, accepting the offer leads to an acquisition of the VC stake on the start up.

Generalizing, the VC receives $\phi Q_{VC}\theta_{VC}V_t - \epsilon C$ upon closing the deal, while the Buyer receives $\gamma Q_{VC}\theta_{VC}V_t - (1-\epsilon)C$.

Time is continuous, i.e. $t \in (t_0, \infty)$, and the VC sets $\phi \in (0, \infty)$, after which, at every moment in time, the Buyer decides whether to *accept*, *wait*.

The process has a Markovian Perfect Nash Equilibrium path to determine the optimal decision for each parties. Particularly, the VC places the bid defining optimally ϕ in stage one, and the Buyer, conditional on the required premium, ϕ, will choose a threshold value $Q_{VC}\theta_{VC}V_t^*(\phi)$ in stage two at which the offer will be accepted.

This corresponds to an optimal timing decision with $t^* = \min t \geq t_0 | V_t > V_t^*$.

The exercising of the option manifests itself on accepting the offer. At this stage, it is assumed that this flexibility is not limited to a fixed maturity, and therefore the possibility to accept the offer is a perpetual real option.

Consequently, the value of the option to acquire the VC stake in the start-up held by the Buyer is the solution of the following maximization problem in stage two:

$$F_{Buyer}(V_t) = \max_t E[((\gamma-\phi)Q_{VC}\theta_{VC}V_t - (1-\epsilon)C)e^{-rt}] \tag{2}$$

Where $E[.]$ denotes the expectation operator.

Solving the previous equation yields [6]:

$$F_{Buyer}(V_t) = ((\gamma-\phi)Q_{VC}\theta_{VC}V^*_{Buyer} - (1-\epsilon)C)\left(\frac{V_t}{V^*_{Buyer}}\right)^{\beta} \tag{3}$$

With $\beta = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1$, and

$$V^*_{Buyer} = \frac{\beta}{(\beta-1)}\frac{(1-\epsilon)C}{Q_{VC}\theta_{VC}(\gamma-\phi)} \tag{4}$$

In contrast, the VC will choose ϕ in stage one such that it maximizes:

$$F_{VC_{Exit}}(\phi) = \max_{\phi} E[(\phi Q_{VC}\theta_{VC}V^*_{Buyer} - \epsilon C)e^{(-rt^*)}] \tag{5}$$

Which can also be presented as follows: [7]:

[6] Please refer to Appendix A.1 for the solution through the use of contingent claims.
[7] Please refer to Appendix A.2 for the solution through the use of contingent claims.

$$F_{VC_{Exit}}(\phi) = \max_{\phi} E\left[\left(\phi Q_{VC}\theta_{VC}\left(\frac{\beta}{\beta-1}\frac{(1-\epsilon)C}{Q_{VC}\theta_{VC}(\gamma-\phi)}\right) - \epsilon C\right)\left(\frac{V_{VC_{Exit}}}{(\frac{\beta}{\beta-1}\frac{(1-\epsilon)C}{Q_{VC}\theta_{VC}(\gamma-\phi)})}\right)^{\beta}\right] \tag{6}$$

Or equivalently:

$$F_{VC_{Exit}}(\phi) = \left(\left(\frac{\beta}{\beta-1}\frac{(1-\epsilon)C\phi}{(\gamma-\phi)}\right) - \epsilon C\right)\left(\frac{V_{VC_{Exit}}}{\frac{\beta}{\beta-1}\frac{(1-\epsilon)C}{Q_{VC}\theta_{VC}(\gamma-\phi)}}\right)^{\beta} \tag{7}$$

Proposition 1. *The optimal required premium for the VC is as follows:*

$$\phi^* = \frac{\gamma(1+(\beta-2)\epsilon)}{(\beta-\epsilon)} \tag{8}$$

Proposition 2. *The optimal timing threshold V^*_{Buyer}, becomes:*

$$V^*_{Buyer}(\phi^*) = \frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta_{VC}} \tag{9}$$

The triggers determined by Propositions 1 and 2 define the optimal premium and timing for the transaction to happen, and are crucial pieces for the entry decision to be taken by the Venture Capitalist, whose entry-option value is to be determined next.

2.2 VC Entry-Exit Option

As previously mentioned, the entry-option aim is to optimally define the ownership the Venture Capitalist should require considering an investment amount, $K - K_E$, and the growth prospects, θ_{VC}, on the start-up value regarding that investment amount. The setting herein presented considers the case of different levels of optimism[8] regarding the growth prospect from either the Entrepreneur and the VC, and is introduced through the contribution of Tavares-Gärtner et al. (2018a).

It is important to emphasize the assumption that both the Entrepreneur and Venture Capitalist truthfully share their optimism levels – their beliefs – on the growth prospects with each other[9].

Given so, with a growth prospect $\theta_{VC}, (\theta_{VC} > 1)$, disclosed by the Venture Capitalist, the optimal ownership for the latter, who ,again, has to immediately account for its next step of an exit of that venture – due to its business nature –, considering the investment $K - K_E$, can be seen as follows:

[8]Again, the case for equal perspectives on the growth prospects, regarded as homogeneous beliefs, whereas both Entrepreneur and VC share the same optimism level on the growth opportunity is considered to be a particular case of the heterogeneous beliefs model, given $\theta = \theta_E = \theta_{VC}$, and is described in appendix A.5.

[9]thus the Entrepreneur believes and shares his/her growth prospects for the start-up, θ_E, and so does the Venture Capitalist regarding his/her growth prospects, θ_{VC}.

$$F_{VC_{Entry}}(V_t) =$$

$$\max_t E\left[\left(Q_{VC}\theta_{VC}V_t + \left(\frac{(\beta-\epsilon)C}{(\beta-1)^2}\left(\frac{V_t}{\frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta_{VC}}}\right)^{\beta}\right) - (K-K_E)\right)e^{-rt}\right] \quad (10)$$

Solving the previous equation yields[10] the following proposition:

Proposition 3. *The optimal timing to entry in the start-up capital is:*

$$V^*_{VC_{Entry}} = \frac{\beta}{\beta-1}\frac{(K-K_E)}{Q_{VC}\theta_{VC}} \quad (11)$$

The value function for the Entrepreneur, considering $Q_E = 1 - Q_{VC}$, comes as follows:

$$F_E(V_t) = \max_t E[((1-Q_{VC})\theta_E V - V - K_E)e^{-rt}] \quad (12)$$

Solving the previous equation yields[11]:

$$V^*_E = \frac{\beta}{(\beta-1)}\frac{K_E}{((1-Q_{VC})\theta_E - 1)} \quad (13)$$

Aligning the optimum investment timing through optimal ownership results in the following proposition:

Proposition 4. *Venture Capitalist's post-money optimum ownership in the heterogeneous beliefs setting:*

$$Q^*_{VC} = \frac{(K-K_E)(\theta_E - 1)}{\theta_E K + K_E(\theta_{VC} - \theta_E)} \quad (14)$$

Consequently, the optimal Entrepreneur's post-money ownership is:

$$Q^*_E = \frac{K + K_E(\theta_{VC} - 1)}{\theta_E K + K_E(\theta_{VC} - \theta_E)} \quad (15)$$

Once the exit and entry triggers are well defined, as above, it is possible to extrapolate an exit multiple, and assess its behaviour as well as understand to which variables the multiple is sensible the most. This multiple is the result of the exit trigger, as per defined in Eq. (9), divided by the entry trigger, as per defined in Eq. (11).

Proposition 5. *The exit multiple is as follows:*

$$\text{Exit multiple} = \frac{\beta}{\beta-1}\frac{(1-\frac{\epsilon}{\beta})C}{(K-K_E)\gamma} \quad (16)$$

[10]Please refer to Appendix A.3 for the solution through the use of contingent claims.
[11]Please refer to Appendix A.4 for the solution through the use of contingent claims.

Interestingly, neither the value triggers nor optimal ownership in the entry-exit options are affected by the exit option. This result makes the present model yielding triggers very similar to the ones derived by Tavares-Gärtner et al. (2018a) whose differences lies on the model methodology[12].

3 The impact of the Time Restriction

Since VC funds typically last for a predetermined maturity – averaging 10 years, as stated by Feld and Mendelson (2016) –, and the above models considers a perpetual entry-exit option, a time constrain must be put in place in order to fully adapt the model to reality. In order to do so, the time restriction is incorporated through the contribution of Pereira and Rodrigues (2014), considering a short position in a Forward Start Option (FSO) conceptualized upon the one used for certain-lived monopolies under preemption. Considering a limited maturity project available for T years, this is equivalent to consider a short position in a Forward Start Call Option on the project, i.e. an option that ceases to be available after T, or as soon as the trigger is achieved. Under the risk neutral expectation the value of the option is:

$$FSO_{VC_{Exit}}(V_t) = E[V_{VC_{Exit_{(t)}}}]e^{-r(T-t)} \tag{17}$$

Where $FSO_{VC_{Exit}}(V_t)$ is the present value of the option to invest in a limited maturity project and $V_{VC_{Exit_{(t)}}}$ is the value of the VC option to invest at time T. At that moment, the state variable $V(t)$ at T can be either below or above the trigger $V^*_{Buyer}(\phi^*)$ given in Eq. (9). For the latter case, it will sell the start-up stake in exchange of the present value of future cash flows, plus the premium,ϕ, which is similar to an European call option with maturity T exercised if $(V > V^*_{Buyer}(\phi^*, T))$. However, if the firm does not sell at time T, the option expires losing its value.

Proposition 6. [13] *The value of the Forward Start Option considering homogeneous beliefs is given by:*

$$FSO_{VC_{Exit}}(V_t) = \frac{(\beta-\epsilon)C}{(\beta-1)^2}\left(\frac{V_t}{\frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta_{VC}}}\right)^{\beta} N(-d3(V_{VC_{Exit}})) \tag{18}$$

Where $N(.)$ is the cumulative normal integral and

$$d3(V_{VC_{Exit}}) = d1(V_{VC_{Exit}}) + (\beta-1)\sigma\sqrt{T} \tag{19}$$

where[14]:

$$d1(V_{VC_{Exit}}) = \frac{\ln\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta_{VC}}\right)}\right) + (\alpha + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{20}$$

The probability distribution $N(-d3(V_{VC_{Exit}}))$captures the value of exercising the option to sell the start-up in a later stage (after T) if the trigger $V^*_{Buyer}(\phi^*)$ is not reached.

[12]since their model use cash-flows other than the start-up value as the stochastic variable to assess.
[13]Please refer to Appendix A.6 for the proofs of Proposition 6.
[14]For further details please refer toPereira and Rodrigues (2014).

It is extremely important to notice that, for the purposes of the theoretical challenge embraced, the rational must be inverted, and a short position on the Forward Start Option should be considered in order to limit the Venture Capitalist exposure to a certain investment in a certain start-up for the predetermined maturity of the investment.

The current setting includes the purchase of a stake in a given start-up to be entered at time t_0, with a planned exit to happen at T, or prior to that if the right premium, ϕ, is payed. For this to happen, it is necessary to combine the option to exit, together with a short position in the Forward Start Call Option and the model that determines the optimal ownership structure of the investment considering differences on the optimism level among the VC and Entrepreneur.

By combinig those models, for the regions where $t < T$, the result is as follows:

$$F_{VC_{Exit}}(V_t) = Q_{VC}\theta_{VC}V_t + \text{Exit Option}_{VC} - (K - K_E) - FSO(V_t) \tag{21}$$

After rearranging, the equation becomes:

$$F_{VC_{Exit}}(V_t) = Q_{VC}\theta_{VC}V_t + \frac{(\beta-\epsilon)C}{(\beta-1)^2}\left(\frac{V_t}{\left(\frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta_{VC}}\right)}\right)^{\beta} N(d3(V_{VC_{Exit}})) - (K - K_E) \tag{22}$$

For the entire rational to be complete, it is still needed to account for the value of the option once the time matures without being exercised, when the option becomes worthless. Thus, the Venture Capitalist is pushed to dispose the start-up at T at its market value. When this happens, a net loss arise from the transaction costs, of ϵC, since the Venture Capitalist has no other options available.

At T, it is assumed that the bargaining power belongs entirely to the Buyer and therefore the VC does not benefit from any of the synergies that might exist.

As described by Nielsen (1992), the risk-adjusted probability that the option will be exercised is captured by $N(d2(V_{VC_{Exit}}))$, similar to what happens within the Black and Scholes formula. Given that, by multiplying the present value of the transaction costs by $N(-d2(V_{VC_{Exit}}))$ the function[15] will capture the probability of the option to expire worthless, thus generating only the net loss whose amount are the transaction costs.

Hence, considering this scenario of reaching T without selling at the premium $(1 + \phi)$, and accounting for the probability of this to happens, comes:

Proposition 7. *The value function of the option to enter in the start-up capital considering the time restricted setting is given by*[16]*:*

[15] The present value of the transaction costs are of $E[\epsilon C]e^{-rT} = \epsilon C e^{-rT} N(-d2(V_{VC_{Exit}}))$

[16] The case for homogeneous beliefs – equal perspectives on the growth prospects –, given $\theta = \theta_E = \theta_{VC}$, within the time restricted setting, is described in appendix A.7.

$$F_{VC_{Exit}}(V_t) = Q_{VC}\theta_{VC}V_t + \frac{(\beta-\epsilon)C}{(\beta-1)^2}\left(\frac{V_t}{\left(\frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta_{VC}}\right)}\right)^{\beta} N(d3(V_{VC_{Exit}})) -$$

$$(K-K_E) - \epsilon C e^{-rT} N(-d2(V_V C_{Exit})) \tag{23}$$

where:

$$d2(V_{VC_{Exit}}) = d1(V_{VC_{Exit}}) - \sigma\sqrt{T} \tag{24}$$

As for the value function for the Entrepreneur, it keeps unchanged since the time restriction only applies to the VC's decision to exit the investment:

$$F_E(V_t) = (1-Q_{VC})\theta_E V_t - V_t - K_E \tag{25}$$

4 Numerical Example

The model, when time restricted, do not yield an analytical result, and therefore its outputs can only be shown and proved through a numerical approach.

Following Tavares-Gärtner et al. (2018a) rational, the assumptions taken were based on De Meza and Southey 1996, Manove and Padilla (1999), Koellinger et al. (2007), Hmieleski and Baron (2009) or Landier and Thesmar (2008) on the perspective that Entrepreneurs' growth prospects for a certain investment are more optimistic than the ones of the Venture Capitalists, and so $\theta_E > \theta_{VC}$. It is shown the particular case of similar levels of optimism along VC and Entrepreneur, stated as θ, explained in appendix A.5 and A.7. The volatility measure, σ, also follows Tavares-Gärtner et al. (2018a), being based on Liu and Yang (2015), whereas the riskless interest rate meets FED and ECB long term inflation targets. Returns of the start-up is regarded as more conservative, $\delta = 5\%$, the investment maturity, T, was considered to be of 7 years[17]. The base case parameters are shown in table 1.

The numerical example below compares the perpetual and time-restricted models, in order to assess the impact time has in the Venture Capitalist's decisions. The point is the alignment of Entrepreneur and VC, determining the entry-exit value triggers as well as the optimum VC ownership – which can be approached in two different ways –, and the behaviour of the exit multiple stressing the investment maturity.

An important aspect to be interpreted using the previous assumptions, in table 1, is the behaviour of the optimal premium the VC must request in order to sell the start-up stake it has acquired. It is observable in figure 1 that a straight line with a slope of 0.64 derived from Proposition 1, $\frac{(1+(\beta-2)\epsilon)}{(\beta-\epsilon)}$, yields an optimal premium of 9.65% when the synergies, γ, are of 15%. This means that the VC, for the given assumptions, will always bear 64.43% of the total synergies the Buyer is expecting.

Figure 2 shows the behaviour of the time-restricted entry-exit trigger against the perpetual one.

[17]This maturity was chosen due to the fact that it is an intermediate value standing above the 5 years of the typical commitment period and below the 10 years of the typical investment term as per described in Feld and Mendelson (2016) and already discussed in section 1.

Variable	Value	Description
γ	15%	Synergies of the Buyer
ϵ	50%	Proportion of exit transaction costs supported by the VC
C	80	Transaction costs of the exit
K	800	Total investment needed to finance the growth opportunity
K_E	250	Entrepreneur's own funds
α	-3%	Growth rate of the start-up
r	2%	Riskless interest rate
σ	25%	Volatility
θ	2	Growth prospects - homogeneous beliefs setting
θ_{VC}	2	Growth prospects of the VC
θ_E	3	Growth prospects of the Entrepreneur
T	7	Investment maturity

Table 1: Key numerical assumptions, time-restricted model.

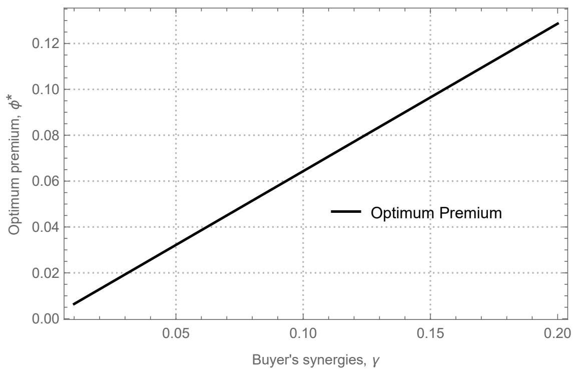

Figure 1: Computing the optimal premium, σ, the VC is willing to accept in order to sell the start-up stake, for different values of the synergies, γ, possessed by the Buyer. Keeping all the assumptions taken in table 1, except for the stressed ones.

The time-restricted entry-exit triggers have the particularity of being calculated keeping constant the ownership required by the VC, to be considered the static one, for the considered invest cost – and this is the one resulting from the perpetual model, respectively equations (71) and (14) for homogeneous and heterogeneous beliefs.

As intuitively expected, keeping static the ownership required by the VC increases the value trigger in the time-restricted setting when compared to the perpetual model. The shorter the investment maturity considered the more pronounced the difference among the triggers becomes. On the other hand, both models become closer from each other while the investment maturity increases, which is consistent with the economic intuition behind it. The functions explicit in figure 3 has a different approach, dynamically determining the optimal VC ownership that align both the perpetual and time-restricted models entry-exit value triggers for any given investment maturity within the time-restricted approach.

Those optimal VC ownerships, determined using the dynamic approach to the time restricted model allowing both models to be align for any investment maturity considered, are shown in figure 4.

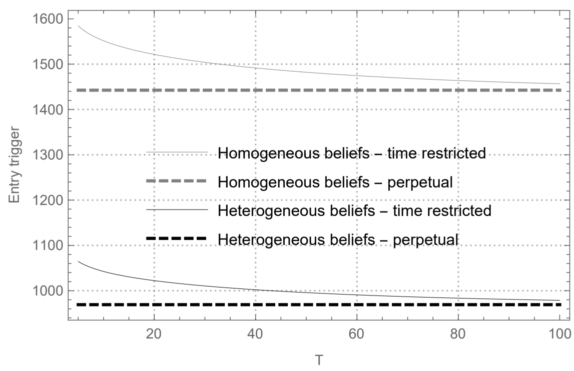

Figure 2: Computing the optimal VC entry trigger, V^*_{VC-hom} and V^*_{VC-het}, in a setting of homogeneous and heterogeneous beliefs, applied to both the perpetual and time restricted models, for different maturities, T, considering the static VC post-money optimum ownership, Q^*_{VC}, within the time-restricted model. Keeping all the assumptions taken in table 1, except for the stressed ones.

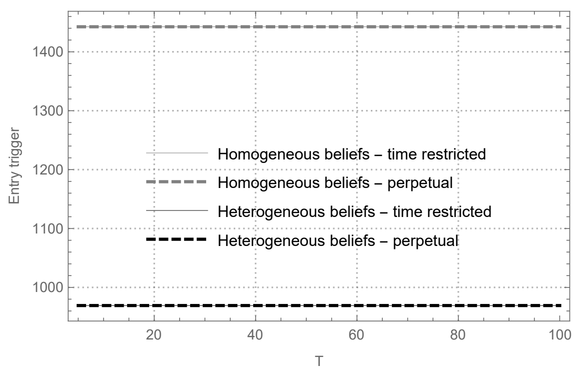

Figure 3: Computing the optimal VC entry trigger,V^*_{VC-hom} and V^*_{VC-het}, in a setting of homogeneous and heterogeneous beliefs, applied to both the perpetual and time restricted models, for different maturities, T, considering a dynamic approach for the optimal ownership structure within the time-restricted model. Keeping all the assumptions taken in table 1, except for the stressed ones.

The economic intuition behind this results are quite similar to the one reached within the analysis to figure 2 whereas it is clear that for an investment which is time-restricted, either the entry-exit value trigger increases while the maturity shortens for a static VC growth prospect ownership, or the entry-exit value trigger might be kept constant and aligned with the perpetual model, but as time shortens the optimal ownership has to increase. Figure 5 shows only the convergence of the time-restricted model to the perpetual one, using a maturity up until 100 years, which, although unrealistic, proves how impactful a time restriction might be in such an industry as Venture Capital.

As for the impact of different growth prospects, θ, θ_{VC} and θ_E, on the entry-exit value triggers for both homogeneous and heterogeneous beliefs, figure 6 shows that each of the lines cross each other for an entry-exit trigger, V^*, of 1442.5, when $\theta_E = 2$ and of 721.3 when $\theta_{VC} = 3$. The present results, as figure 3 helps to determine, are the same as if no time-restriction was in place, since both

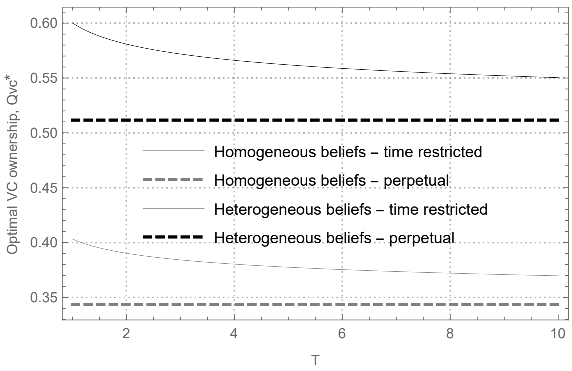

Figure 4: Computing the VC post-money optimum ownership, in a setting of homogeneous and heterogeneous beliefs, applied to both the perpetual (static) and time restricted (dynamic) models, for different maturities, T. Keeping all the assumptions taken in table 1, except for the stressed ones.

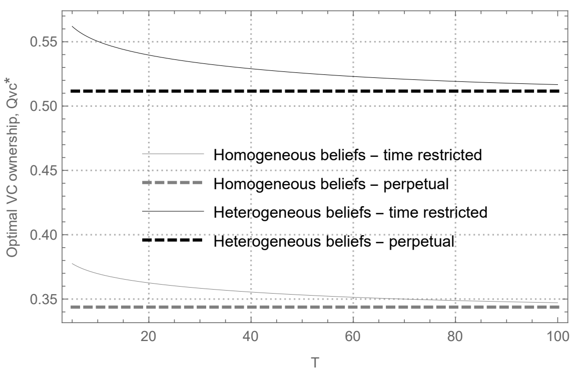

Figure 5: Computing the VC post-money optimum ownership, in a setting of homogeneous and heterogeneous beliefs, applied to both the perpetual (static) and time restricted (dynamic) models, for different and higher maturities, T, than those of figure 4. Keeping all the assumptions taken in table 1, except for the stressed ones.

triggers are aligned through the increase in the required optimal ownership. Keeping constant the optimal ownership determined in the perpetual model, the entry-exit trigger, V^*, would become 1568.0, when $\theta_E = 2$ and 784.0 when $\theta_{VC} = 3$, representing an increase of respectively 125.5 or 8,7% and 62.7 or 8.7% for $\theta_E = 2$ and $\theta_{VC} = 3$.

With the same behaviour as when the perpetual model applies, the derivatives of the value function for the VC in regard to every growth prospects are negative, $\frac{\partial V^*_{VC-hom}}{\partial \theta} < 0$, $\frac{\partial V^*_{VC-hom}}{\partial \theta_{VC}} < 0$ and $\frac{\partial V^*_{VC-hom}}{\partial \theta_E < 0}$, on the right economic intuition that the smaller the growth prospects, the bigger opportunity cost, and therefore the higher the entry trigger, since the VC's upside potential is reduced. Again, the chart behaviour is strongly influenced by the base case assumptions on the growth prospects and that is why, as explained on the analysis of figure 6, the crossing points are $\theta_E = 2$ and $\theta_{VC} = 3$, and the homogeneous beliefs curve yields a higher trigger when compared to any of the heterogeneous beliefs setting when $\theta < 2$, $\theta_{VC} < 2$ and $\theta_E < 2$, and a lower trigger when $\theta > 3$, $\theta_{VC} > 3$ and $\theta_E > 3$.

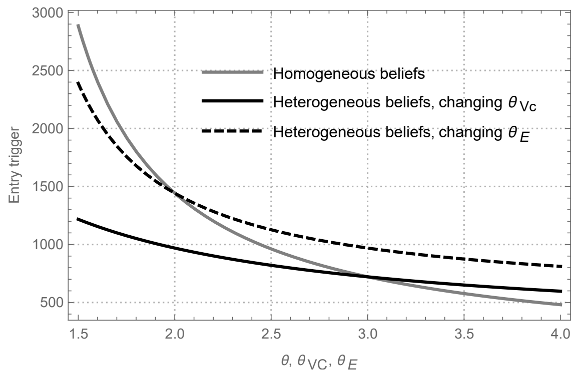

Figure 6: Computing the optimal VC entry trigger,V^*_{VC-hom} and V^*_{VC-het}, in a setting of homogeneous and heterogeneous beliefs, applied to both the perpetual and time restricted models, for different growth prospects, θ, θ_{VC} and θ_E, considering a dynamic approach for the optimal ownership structure within the time-restricted model. Keeping all the assumptions taken in table 1, except for the stressed ones.

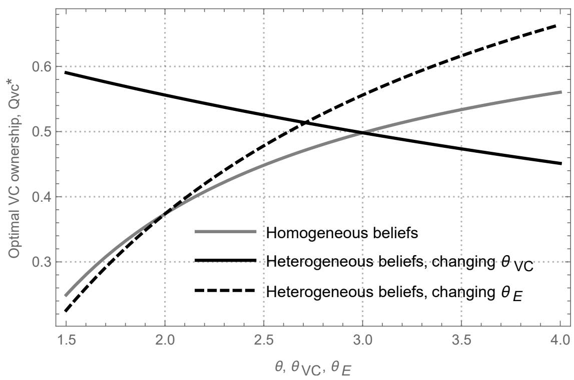

Figure 7: Computing the VC post-money optimum ownership, in a setting of homogeneous and heterogeneous beliefs, applied to both the perpetual and time-restricted models, for different growth prospects, θ, θ_{VC} and θ_E. Keeping all the assumptions taken in table 1, except for the stressed ones.

Figure 7 presents the behaviour of the ownership triggers for both homogeneous and heterogeneous beliefs, considering different growth prospects, θ, θ_{VC} and θ_E, for the investment maturity of 7 years.

Homogeneous and heterogeneous beliefs cross each other lines for an optimal ownership, Q^*_{VC}, of 37.4%, when $\theta_E = 2$ and of 49.8% when $\theta_{VC} = 3$. These results, when compared with the perpetual model numerical analysis, given an optimal ownership, Q^*_{VC}, of 34.4%, when $\theta_E = 2$ and of 45.8% when $\theta_{VC} = 3$, represents an increase of the required VC ownership of 3.0 p.p., when $\theta_E = 2$ and of 4.0 p.p. when $\theta_{VC} = 3$. Heterogeneous beliefs curves cross each other for $\theta = \theta_E = \theta_{VC} = 2.7$, at a Q^*_{VC} of 51.4%, which is the exact same combination of growth prospects where heterogeneous lines cross each other within the perpetual model, but for a Q^*_{VC} of 47.3%, representing an increase of 4.1 p.p. As economically intuitive – and already explained within the analysis of the perpetual model –, the optimal ownership required by the VC – for

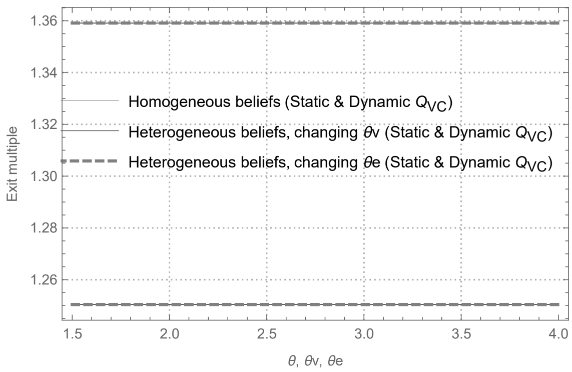

Figure 8: Computing the implied exit multiple considering the optimal VC triggers for both entering, V^*_{VC-hom} and V^*_{VC-het}, and exiting, $V^*_{Buyer}(\phi^*)$, the investment, applied to both homogeneous and heterogeneous beliefs, for changes in the growth prospects, θ, θ_{VC} and θ_E, considering a dynamic approach for the optimal ownership structure within the time-restricted model in the upper line and a static approach for the optimal ownership structure within the time-restricted model in the lower line. Keeping all the assumptions taken in table 1, except for the stressed ones.

the same investment cost – decreases as its growth prospects increases against the Entrepreneurs', $\frac{\partial V^*_{VC-het}}{\partial \theta_{VC}} < 0$. The opposite happens as the Entrepreneurs' growth prospects increases against the ones of the VC, $\frac{\partial V^*_{VC-het}}{\partial \theta_E} > 0$, and within the homogeneous beliefs setting, $\frac{\partial V^*_{VC-het}}{\partial \theta} > 0$.

For the determination of the exit multiple for the VC in the time restricted setting, the exit perpetual option – because it does not rely on time to be determined – has to be divided by the time restricted entry-exit option. Since the time restricted entry-exit option can be determined holding static Q^*_{VC} thus adjusting V^*, or through a dynamic Q^*_{VC} for which V^* yields the same result as in the perpetual setting, both scenarios are considered in figure 8.

The multiple, as in the perpetual setting, does not change for different growth prospects neither for homogeneous nor heterogeneous beliefs considering different growth prospects for either the Entrepreneur or Venture Capitalist. However, the case is different when considering a static or dynamic approach for Q^*_{VC}. As intuitively extrapolated, since the dynamic Q^*_{VC} is designed for V^* to match the perpetual results, the multiple resulting from this approach yields the precise same multiple as the perpetual model, of 1,36. As for the multiple arising from the static Q^*_{VC} – the one resulting from the perpetual model – the multiple for an investment maturity of 7 years is reduced to 1,2504. Knowing from the analysis performed to the perpetual model that volatility is the parameter to which the multiple is sensible the most, figure 9 shows how sensitive to volatility both multiples are – considering a static and dynamic approach for Q^*_{VC}.

From figure 9 it is possible to already have a hint on what behaviour the multiple will have when stressing the investment maturity. The dynamic approach to Q^*_{VC} is the one yielding the biggest multiple for every level of volatility – which is the same as the perpetual model, once again –, and the gap for the static Q^*_{VC} gets bigger as the levels of volatility increase.

As figure 9 left behind the curtain, the exit multiple yielded when used the dynamic approach to Q^*_{VC} is the same as the perpetual model thus being static for every investment maturity. This happens due to the fact that Q^*_{VC} is the variable that changes in order for V^* to be kept at the perpetual model level.

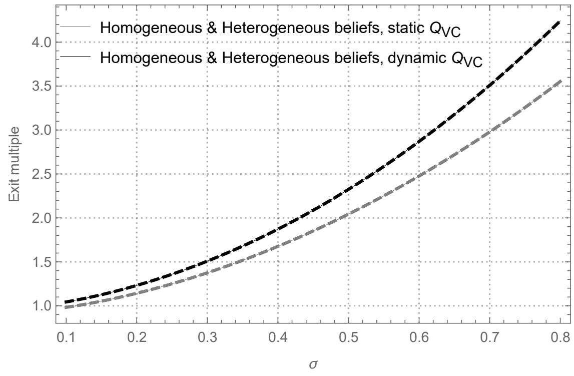

Figure 9: Computing the implied exit multiple considering the optimal VC triggers for both entering,V^*_{VC-hom} and V^*_{VC-het}, and exiting, $V^*_{Buyer}(\phi^*)$, the investment, applied to both homogeneous and heterogeneous beliefs, for different volatility, σ, levels, considering a dynamic approach for the optimal ownership structure within the time restricted model in the upper line and a static approach for the optimal ownership structure within the time-restricted model in the lower line. Keeping all the assumptions taken in table 1, except for the stressed ones.

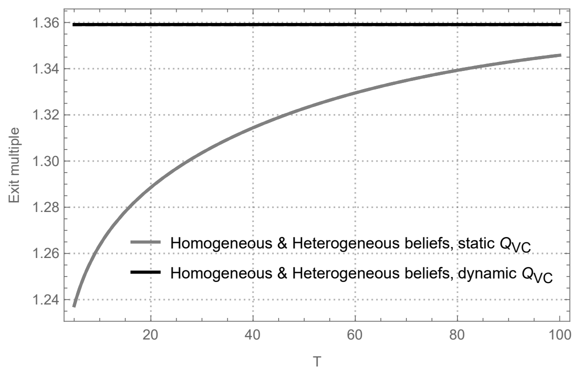

Figure 10: Computing the implied exit multiple considering the optimal VC triggers for both entering, V^*_{VC-hom} and V^*_{VC-het}, and exiting, $V^*_{Buyer}(\phi^*)$, the investment, applied to both homogeneous and heterogeneous beliefs, for different investment maturities, T, levels, considering a dynamic approach for the optimal ownership structure within the time-restricted model in the upper line and a static approach for the optimal ownership structure within the time-restricted model in the lower line. Keeping all the assumptions taken in table 1, except for the stressed ones.

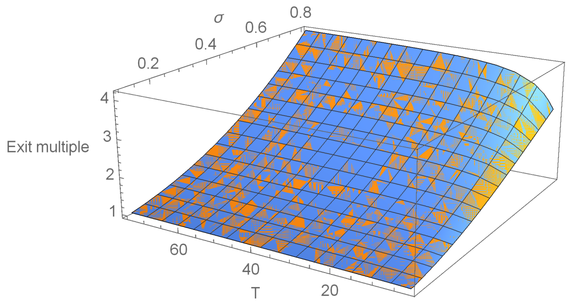

Figure 11: Computing the implied exit multiple considering the optimal VC triggers for both entering,V^*_{VC-hom} and V^*_{VC-het}, and exiting, $V^*_{Buyer}(\phi^*)$, the investment, applied to both homogeneous and heterogeneous beliefs, for different volatility, σ, and investment maturity, T, levels, considering a static approach for the optimal ownership structure within the time-restricted model. Keeping all the assumptions taken in table 1, except for the stressed ones.

The exit multiple computed using the perpetual model Q^*_{VC} – the static one –, yields, for increasing values of the investment maturity, results which are closer from the multiple calculated in the perpetual model. The results shows that the shorter the investment maturity, the smaller the exit multiple. This is consistent with its economic intuition, since the entry triggers are higher for lower investment maturities, thus reducing the exit multiple since the exit trigger is hold constant due to its independency from the investment maturity – being driven by the synergies of the Buyer, and the share of those synergies with the Venture Capitalist. Figure 11 allows to observe both behaviours simultaneously with an exit multiple exponentially bigger for higher measures for volatility, σ, and lower for smaller investment maturities in a converging path to the perpetual model results keeping everything else constant.

5 Conclusions

The present work envisage the entire investment decision of a Venture Capitalist when considering to invest in a star-up firm. Since VCs face high risk when investing, and the maturity of this type of investment are typically short, it is important to conceptualize, sooner than latter, which exit multiple to require, determining the optimal amount to pay and optimal ownership, so then the start-up can be disposed for the optimal share of the synergies a given Buyer is expected to have. This implemented setting allowed for some interesting conclusions on the economic intuitions behind the decisions a VC usually takes.

Firstly, it is important to mention that although the model assumes one only Venture Capitalist investing in a seed round of a certain star-up, this assumption, even if relaxed, does not impact the results since the typical behaviour when more than a Venture Capitalist is to invest is to recognize one as the lead investor, which centralize the decisions and negotiation process thus smoothing the investment decision.

Curiously, the entry decision the Venture Capitalist takes when assessing a start-up is not affected by its exit in a perpetual setting. The economic intuition attached to this conclusion is that without a time restriction to enter in the venture the Venture Capitalist can wait until the optimum timing for the exit to arrive and it does not matter how far in time that optimum will arrive. This means that the perpetual model yield results very similar to the ones achieved by Tavares-Gärtner et al. (2018a), whose only differences lie on the modelling methodology used since the stochastic variable used by the authors was the start-up cash flows whereas the present model considers the start-up value.

The previous results change a lot when the time restriction is put in place, where the exit decision does impact the entrance level. Generically, the intuition would be an increasing in the entry trigger as shorter the investment maturity becomes, but this only happens keeping constant the optimal post-money ownership. For the model also considers the possibility of keeping the entry triggers unchanged thus increasing the required optimal post-money ownership for decreasing investment maturities, which is consistent with its economic intuition.

When stressing the growth prospects either for the perpetual and for the time restricted models, the intuition is both a decreasing entry triggers and exit triggers for increasing growth prospects and the other way around. For the optimal VC post-money ownership, it increases for higher growth prospects within the homogeneous beliefs setting, and also for increasing growth prospects of the Entrepreneur against a static VC's growth prospects. On the other hand, for increasing VC's growth prospects against static Entrepreneur's growth prospects – typically less common –, the VC post-money ownership decreases.

As for the exit multiple analysis, the variable to which this indicator is most sensible to is the volatility, with a positive correlation. A statement which is true in both the perpetual and time restricted models. The exit multiple increases exponentially with higher levels of volatility, and that is also truth for the growth rates of the start-up. An interesting result comes with the time restriction setting, since the multiple, as economically intuitive, decreases with shorter maturities, which can be explained by a decrease in the opportunity cost due to an expected shorter investment exposure, and also due to the fact that the entry trigger is also bigger therefore reducing the upside potential since the exit trigger is hold constant.

The current model, although producing very interesting results and confirming economic intuitions and dynamics associated to the Venture Capitalist's investment decision, has to be continuously upgraded in order to increase its fit to reality and be fully regarded as a hands on tool for investment professionals. The case for taking into account further financing rounds within the investment maturity – therefore accounting for the dilution effect which is one of the most likely events both VCs and Entrepreneurs face – would be an helpful tool. Also, relax the assumption about the perpetual entry time, thus applying a time-restriction to the investment decision would bring meaningful conclusions to the table, since the current setting considers an infinite time to wait for the optimum timing to invest and Venture Capital firms face severe competition to find the best start ups to invest as sooner as possible in order to to maximize their upside potential.

References

Black, F. and Scholes, M.: 1973, The pricing of options and corporate liabilities, *Journal of political economy* **81**(3), 637–654.

Cochrane, J. H.: 2005, The risk and return of venture capital, *Journal of financial economics* **75**(1), 3–52.

De Meza, D. and Southey, C.: 1996, The borrower's curse: optimism, finance and entrepreneurship, *The Economic Journal* pp. 375–386.

Feld, B. and Mendelson, J.: 2016, *Venture deals: Be smarter than your lawyer and venture capitalist*, John Wiley & Sons.

Giat, Y., Subramanian, A. and Hackman, S.: 2007, Risk, uncertainty and optimism in venture capital relationships.

Hmieleski, K. M. and Baron, R. A.: 2009, Entrepreneurs' optimism and new venture performance: A social cognitive perspective, *Academy of management Journal* **52**(3), 473–488.

Koçkesen, L. and Ozerturk, S.: 2002, Staged financing and endogenous lock-in: A model of start-up finance, *Technical report*, Citeseer.

Koellinger, P., Minniti, M. and Schade, C.: 2007, "i think i can, i think i can": Overconfidence and entrepreneurial behavior, *Journal of economic psychology* **28**(4), 502–527.

Kortum, S. and Lerner, J.: 2001, Does venture capital spur innovation?, *Entrepreneurial inputs and outcomes: New studies of entrepreneurship in the United States*, Emerald Group Publishing Limited, pp. 1–44.

KPMG: 2018, Venture pulse q4 2017, *Technical report*.

Lambrecht, B. M.: 2004, The timing and terms of mergers motivated by economies of scale, *Journal of Financial Economics* **72**(1), 41–62.

Landier, A. and Thesmar, D.: 2008, Financial contracting with optimistic entrepreneurs, *The Review of Financial Studies* **22**(1), 117–150.

Leshchinskii, D. and Brisley, N.: 2006, Stage financing, vc short-termism and managerial replacement as a real option.

Liu, Y. and Yang, J.: 2015, Optimal investment of private equity, *Finance Research Letters* **14**, 76–86.

Lukas, E. and Welling, A.: 2012, Negotiating m&as under uncertainty: The influence of managerial flexibility on the first-mover advantage, *Finance Research Letters* **9**(1), 29–35.

Manove, M. and Padilla, A. J.: 1999, Banking (conservatively) with optimists, *The RAND Journal of Economics* pp. 324–350.

Maya, C.: 2004, In search of the true value of a start-up firm: creative destruction and real options approach (cd-rom), *Proceedings of the 8th Annual International Conference on Real Options, Montréal Canada*, pp. 200–268.

Meng, R.: 2008, A patent race in a real options setting: Investment strategy, valuation, capm beta, and return volatility, *Journal of Economic Dynamics and Control* **32**(10), 3192–3217.

Merton, R. C.: 1973, Theory of rational option pricing, *The Bell Journal of economics and management science* pp. 141–183.

Myers, S. C.: 1977, Determinants of corporate borrowing, *Journal of financial economics* **5**(2), 147–175.

Nielsen, L. T.: 1992, *Understanding N (d1) and N (d2): Risk-adjusted probabilities in the Black-Scholes model*, INSEAD.

Pereira, P. J. and Rodrigues, A.: 2014, Investment decisions in finite-lived monopolies, *Journal of Economic Dynamics and Control* **46**, 219–236.

PwC and CBInsights: 2018, Moneytree report q4 2017, *Technical report*.

Ries, E.: 2011, *The lean startup: How today's entrepreneurs use continuous innovation to create radically successful businesses*, Crown Books.

Schumpeter, J.: 1942, Creative destruction, *Capitalism, socialism and democracy* **825**, 82–85.

Siller-Pagaza, G. and Otalora, G.: 2009, Moral hazard when the entrepreneur seeks outside equity, *Journal of business and entrepreneurship* **21**(1), 80.

Tavares-Gärtner, M., Pereira, P. J. and Brandão, E.: 2018a, Heterogeneous beliefs and optimal ownership in entrepreneurial financing decisions, *Quantitative Finance* **18**(11), 1947–1958.

Tavares-Gärtner, M., Pereira, P. J. and Brandão, E.: 2018b, Optimal contingent payment mechanisms and entrepreneurial financing decisions, *European Journal of Operational Research* **270**(3), 1182–1194.

A Appendices

A.1 Solution of Eq. (2) through the contingent-claim approach

The following expression, Eq. (2), can also be solved through the contingent-claim approach:

$$F_{Buyer}(V_t) = \max_t E\left[((\gamma-\phi)Q_{VC}\theta_{VC}V_t - (1-\epsilon)C)e^{-rt}\right] \tag{26}$$

Based on Dixit and Pindyck (1994), where the value of the option held by the Buyer to invest in the start-up owned by the VC, must satisfy the following ordinary differential equation (ODE):

$$\frac{1}{2}\sigma^2 V_t^2 F^{''}(V_t) - (r-\delta)V_t F^{'}(V_t) - rF(V_t) = 0 \tag{27}$$

Subject to some conditions that must be imposed, in order to obtain the appropriate solution. The general solution for this ODE is well known and takes the form:

$$F_{Buyer}(V_t) = A_1^{\beta_1} + A_2^{\beta_2} \tag{28}$$

where A_1 and A_2 are arbitrary constants that need to be determined, and β_1 and β_2 are the roots of the fundamental quadratic:

$$Q(\beta) = \frac{1}{2}\sigma^2\beta(\beta-1) + (r-\delta)\beta - r \tag{29}$$

i.e.:

$$\beta_1 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{30}$$

$$\beta_2 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{31}$$

In addition, $F_{Buyer}(V_t)$ must satisfy the following boundary conditions:

$$\lim_{V_{Buyer}\to 0} F_{Buyer}(V_t) = 0 \tag{32}$$

$$\lim_{V_{Buyer}\to V^*_{Buyer}} F_{Buyer}(V_t) = (\gamma-\phi)Q_{VC}\theta V_t - (1-\epsilon)C \tag{33}$$

$$\lim_{V_{Buyer}\to V^*_{Buyer}} F^{'}_{Buyer}(V_t) = (\gamma-\phi)Q_{VC}\theta \tag{34}$$

Condition $F(0) = 0$ arises from the observation that if V_{Buyer} goes to zero, it will stay at zero (this is an implication of the stochastic process for V_{Buyer}). Therefore, the option will be of no value when V=0. The other two conditions come from consideration of optimal investment. V^*_{Buyer} is the price at which it is optimal to invest, then $F_{Buyer}(V_t^*) = (\gamma-\phi)Q_{VC}\theta V_t^*(t) - (1-\epsilon)C$ is the value-matching condition: it just says that upon investing, the firm receives a net payoff $(\gamma-\phi)Q_{VC}\theta V_t^*(t) - (1-\epsilon)C$. Finally, condition $F^{'}_{Buyer}(V_t) = (\gamma-\phi)Q_{VC}\theta$ is the "smooth-pasting" condition. If $F_{Buyer}(V_t)$ were not continuous and smooth at the critical exercise point,V^*_{Buyer}, one could do better by exercising at a different point.

To find $F(V_{Buyer})$, we must solve equation $\frac{1}{2}\sigma^2 V_t^2 F^{''}(V_t) - (r-\delta)VF^{'}(V_t) - rF(V_t) = 0$ subject to the boundary conditions. In order to respect the first boundary condition, and realizing that $\lim_{V \to 0} A_2^{\beta_2} = \infty$ (because $\beta_2 < 0$), A_2 must be set equal to zero and the solution must take the form: $F(V_t) = A_1^{\beta_1}$. Where A_1 is a constant that is yet to be determined, and $\beta_1 > 1$ is a known constant whose value depends on the parameters δ, σ and r of the differential equation.

$$F_{Buyer}(V_t) = \begin{cases} ((\gamma - \phi)Q_{VC}\theta V_t^* - (1-\epsilon)C)) \left(\frac{V_t}{V_{Buyer}^*}\right)^{\beta}, \\ \text{for } V_t < V_{Buyer}^* \\ (\gamma - \phi)Q_{VC}\theta V_t^* - (1-\epsilon)C), \\ \text{for } V_t \geq V_{Buyer}^* \end{cases} \tag{35}$$

A.2 Solution of Eq. (5) through the contingent-claim approach

The following expression, Eq. (5), can also be solved through the contingent-claim approach:

$$F_{VC_{Exit}}(V_t) = \max_t E\left[(\phi Q_{VC}\theta V_{Buyer}^* - \epsilon C)e^{-rt}\right], V_{Buyer}^* = \frac{\beta}{(\beta-1)} \frac{(1-\epsilon)C}{Q_{VC}\theta(\gamma - \phi)} \tag{36}$$

Based on Dixit and Pindyck (1994), where the value of the option held by the Buyer to invest in the start-up owned by the VC, must satisfy the following ordinary differential equation (ODE) :

$$\frac{1}{2}\sigma^2 V_t^2 F^{''}(V_t) - (r - \delta)V_t F^{'}(V_t) - rF(V_t) = 0 \tag{37}$$

Subject to some conditions that must be imposed, in order to obtain the appropriate solution. The general solution for this ODE is well known and takes the form:

$$F_{VC_{Exit}}(V_t) = A_1^{\beta_1} + A_2^{\beta_2} \tag{38}$$

where A_1 and A_2 are arbitrary constants that need to be determined, and β_1 and β_2 are the roots of the fundamental quadratic:

$$Q(\beta) = \frac{1}{2}\sigma^2\beta(\beta - 1) + (r - \delta)\beta - r \tag{39}$$

i.e.:

$$\beta_1 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{40}$$

$$\beta_2 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{41}$$

In addition, $F_{VC_{Exit}}(V_t)$ must satisfy the following boundary conditions:

$$\lim_{V_{VC_{Exit}} \to 0} F_{VC_{Exit}}(V_t) = 0 \tag{42}$$

$$\lim_{V_{VC_{Exit}} \to V_{VC_{Exit}}^*} F_{VC_{Exit}}(V_t) = \phi Q_{VC}\theta \left(\frac{\beta}{(\beta-1)} \frac{(1-\epsilon)C}{Q_{VC}\theta(\gamma-\phi)}\right) - \epsilon C \tag{43}$$

$$\lim_{V_{VC_{Exit}} \to V^*_{VC_{Exit}}} F^{'}_{VC_{Exit}}(V_t) = \frac{\partial F_{VC_{Exit}}(V_t)}{\partial V_t}\Big|_{V_t = V^*_{VC_{Exit}}} \tag{44}$$

Following the standard procedures, the function $F(V_{VC_{Exit}})$ comes as follows:

$$F_{VC_{Exit}}(V_t) = \begin{cases} \left(\phi Q_{VC}\theta\left(\frac{\beta}{(\beta-1)}\frac{(1-\epsilon)C}{Q_{VC}\theta(\gamma-\phi)}\right) - \epsilon C\right)\left(\frac{V_t}{\left(\frac{\beta}{(\beta-1)}\frac{(1-\epsilon)C}{Q_{VC}\theta(\gamma-\phi)}\right)}\right)^{\beta}, \\ \text{for } V_t < V^*_{VC_{Exit}} \\ \phi Q_{VC}\theta\left(\frac{\beta}{(\beta-1)}\frac{(1-\epsilon)C}{Q_{VC}\theta(\gamma-\phi)}\right) - \epsilon C, \\ \text{for } V_t \geq V^*_{VC_{Exit}} \end{cases} \tag{45}$$

A.3 Solution of Eq. (10) through the contingent-claim approach

The following expression, Eq. (10), can also be solved through the contingent-claim approach:

$$F_{VC_{Entry}}(V_t) = Q_{VC}\theta_{VC}V_t + \left(\frac{(\beta-\epsilon)C}{(\beta-1)^2}\left(\frac{V_t}{\frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta_{VC}}}\right)^{\beta}\right) - (K - K_E) \tag{46}$$

Based on Dixit and Pindyck (1994), where the value of the option held by the Buyer to invest in the start-up owned by the VC, must satisfy the following ordinary differential equation (ODE) :

$$\frac{1}{2}\sigma^2 V_t^2 F^{''}(V_t) - (r-\delta)V_t F^{'}(V_t) - rF(V_t) = 0 \tag{47}$$

Subject to some conditions that must be imposed, in order to obtain the appropriate solution. The general solution for this ODE is well known and takes the form:

$$F_{VC_{Entry}}(V_t) = A_1^{\beta_1} + A_2^{\beta_2} \tag{48}$$

where A_1 and A_2 are arbitrary constants that need to be determined, and β_1 and β_2 are the roots of the fundamental quadratic:

$$Q(\beta) = \frac{1}{2}\sigma^2\beta(\beta-1) + (r-\delta)\beta - r \tag{49}$$

i.e.:

$$\beta_1 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{50}$$

$$\beta_2 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{51}$$

In addition, $F_{VC_{Entry}}(V_t)$ must satisfy the following boundary conditions:

$$\lim_{V_{VC_{Entry}} \to 0} F_{VC_{Entry}}(V_t) = 0 \tag{52}$$

$$\lim_{V_{VC_{Entry}} \to V^*_{VC_{Entry}}} F_{VC_{Entry}}(V_t) = Q_{VC}\theta_{VC}V_t + \left(\frac{(\beta-\epsilon)C}{(\beta-1)^2} \left(\frac{V}{\frac{\beta}{(\beta-1)^2} \frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta_{VC}}} \right)^{\beta} \right) - (K - K_E) \tag{53}$$

$$\lim_{V_{VC_{Entry}} \to V^*_{VC_{Entry}}} F'_{VC_{Entry}}(V_t) = \frac{\partial F_{VC_{Entry}}(V_t)}{\partial V_t} \Big|_{V_t = V^*_{VC_{Entry}}} \tag{54}$$

Following the standard procedures, the function $F_{VC_{Entry}}(V_t)$ comes as follows:

$$F_{VC_{Entry}}(V_t) = \begin{cases} Q_{VC}\theta_{VC}V_t + \left(\frac{(\beta-\epsilon)C}{(\beta-1)^2} \left(\frac{V_t}{\frac{\beta}{(\beta-1)^2} \frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta_{VC}}} \right)^{\beta} \right) - (K - K_E) \left(\frac{V_t}{V_t^*} \right)^{\beta}, \\ \text{for } V_t < V^*_{VC_{Entry}} \\ Q_{VC}\theta_{VC}V_t + \left(\frac{(\beta-\epsilon)C}{(\beta-1)^2} \left(\frac{V}{\frac{\beta}{(\beta-1)^2} \frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta_{VC}}} \right)^{\beta} \right) - (K - K_E), \\ \text{for } V_t \geq V^*_{VC_{Entry}} \end{cases} \tag{55}$$

A.4 Solution of Eq. (12) through the contingent-claim approach

The following expression, Eq. (12), can also be solved through the contingent-claim approach:

$$F_E(V_t) = (1 - Q_{VC})\theta_E V_t - V_t - K_E \tag{56}$$

Based on Dixit and Pindyck (1994), where the value of the option held by the Buyer to invest in the start-up owned by the VC, must satisfy the following ordinary differential equation (ODE) :

$$\frac{1}{2}\sigma^2 V_t^2 F^{''}(V_t) - (r - \delta)V_t F^{'}(V_t) - rF(V_t) = 0 \tag{57}$$

Subject to some conditions that must be imposed, in order to obtain the appropriate solution. The general solution for this ODE is well known and takes the form:

$$F_E(V_t) = A_1^{\beta_1} + A_2^{\beta_2} \tag{58}$$

where A_1 and A_2 are arbitrary constants that need to be determined, and β_1 and β_2 are the roots of the fundamental quadratic:

$$Q(\beta) = \frac{1}{2}\sigma^2\beta(\beta - 1) + (r - \delta)\beta - r \tag{59}$$

i.e.:

$$\beta_1 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}} \tag{60}$$

$$\beta_2 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}} \tag{61}$$

In addition, $F_E(V_t)$ must satisfy the following boundary conditions:

$$\lim_{V_E \to 0} F_E(V_t) = 0 \tag{62}$$

$$\lim_{V_E \to V_E^*} F_E(V_t) = (1 - Q_{VC})\theta_E V_t - V_t - K_E \tag{63}$$

$$\lim_{V_E \to V_E^*} F_E^{'}(V_t) = \frac{\partial F_E(V_t)}{\partial V_t}\Big|_{V_t = V_E^*} \tag{64}$$

Following the standard procedures, the function $F_E(V_t)$ comes as follows:

$$F_E(V_t) = \begin{cases} (1 - Q_{VC})\theta_E V_t - V_t - K_E \left(\frac{V_t}{V_E^*}\right)^\beta, \\ \text{for } V_t < V_E^* \\ (1 - Q_{VC})\theta_E V_t - V_t - K_E, \\ \text{for } V_t \geq V_E^* \end{cases} \tag{65}$$

A.5 The particular case for similar optimism levels – Homogeneous Beliefs

This setting assumes that both the Entrepreneur and Venture Capitalist possess the same optimism level regarding the start-up growth prospects – homogeneous beliefs –, $\theta_E = \theta_{VC} = \theta > 1$, whereby the players individually assess their decision whether to invest or not in the growth opportunity, and jointly determine the ownership each will optimally take by undertaking the investment.

Since both players held an option to wait for the optimal ownership trigger, and so flexibility, uncertainty and cost irreversibility – at least at certain extent – are in place, the problem deserves a real options approach in order to be glamorously solved.

Given the homogeneous beliefs in place, with a growth prospect θ agreed by both players, the optimal ownership for the Venture Capitalist, who has to immediately account for its next step of an exit of that venture – due to its business nature –, considering the investment $K - K_E$, can be seen as follows:

$$F_{VC_{Entry(\text{hom})}}(V_t) = Q_{VC}\theta V_t + ExitOption_{VC} - (K - K_E) \tag{66}$$

Consequently[18], the Venture Capitalist's option to enter in the venture is given by:

$$F_{VC_{Entry(\text{hom})}}(V_t) =$$
$$\max_t E\left[\left(Q_V C\theta V_t + \left(\frac{(\beta-\epsilon)C}{(\beta-1)^2}\left(\frac{V_t}{\left(\frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta}\right)}\right)^\beta\right) - (K - K_E)\right)e^{-rt}\right] \tag{67}$$

Solving the previous equation yields [19] the following proposition.

The optimal timing to entry in the start-up capital in the homogeneous beliefs setting is:

$$V^*_{VC_{Entry(\text{hom})}} = \frac{\beta}{\beta - 1}\frac{(K - K_E)}{Q_{VC}\theta} \tag{68}$$

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[18]The simplified version of the VC Exit Option is: $F_{VC_{Entry(\text{hom})}}(V_{(\phi^*,t)}) = \frac{(\beta-\epsilon)C}{(\beta-1)^2}\left(\frac{V_t}{\left(\frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta}\right)}\right)^\beta$

[19]Please refer to Appendix A.7 for the solution through the use of contingent claims.

The value function for the Entrepreneur, considering $Q_E = 1 - Q_{VC}$, comes as follows:

$$F_{E_{Entry(\text{hom})}}(V_t) = \max_t E[((1 - Q_{VC})\theta V_t - V_t - K_E)e^{-rt}] \tag{69}$$

Solving the previous equation yields:

$$V^*_{E_{Entry(\text{hom})}} = \frac{\beta}{(\beta - 1)} \frac{K_E}{((1 - Q_{VC})\theta - 1)} \tag{70}$$

Aligning the optimum investment timing through the optimal ownership yields the following proposition.

Venture Capitalist's post-money optimum ownership in the homogeneous beliefs setting:

$$Q^*_{VC} = \frac{(K - K_E)(\theta - 1)}{\theta K} \tag{71}$$

Consequently, the optimal Entrepreneur's post-money ownership is:

$$Q^*_E = \frac{(K + K_E)(\theta - 1)}{\theta K} \tag{72}$$

A.6 Proof of Proposition 6

In order to derive the value of the Forward Start Option it is needed to discount the expected risk neutral value of the VC:

$$FSO_{VC_{Exit}}(V_t) = E[V_{VC_{Exit_{(t)}}}]e^{-r(T-t)} \tag{73}$$

Where $FSO_{VC_{Exit}}(V_t)$ is the present value of the option to invest in a limited maturity project and $V_{VC_{Exit_{(t)}}}$ is the value of the VC's option to invest at time T.

This Forward Start Option follows the rational of Pereira and Rodrigues (2014), being the $FSO_{VC_{Exit}}(V_t)$ an asset-or-nothing call option on $\frac{(\beta-\epsilon)C}{(\beta-1)^2}\left(\frac{V_t}{\left(\frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta}\right)}\right)^{\beta}$, with trigger $V^*_{Buyer}(\phi^*)$ and maturity T:

$$E[V_{VC_{Exit_{(t)}}}]e^{-r(T-t)}1_{V_t \geq V^*_{Buyer}(\phi^*)} = \frac{(\beta - \epsilon)C}{(\beta - 1)^2}\left(\frac{V_t}{\left(\frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta}\right)}\right)^{\beta} N(d3(V_{VC_{Exit}})) \tag{74}$$

Where: $1_{condition}$ equals 1 if the condition is met, and 0 otherwise

$$d1(V_{VC_{Exit}}) = \frac{\ln\left(\frac{V_t}{\left(\frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta}\right)}\right) + \left(\alpha + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \tag{75}$$

$$d3(V_{VC_{Exit}}) = d1(V_{VC_{Exit}}) + (\beta - 1)\sigma\sqrt{T} \tag{76}$$

For further details, please refer to Pereira and Rodrigues (2014).

A.7 Time Restricted Entry-Exit Option, considering similar optimism levels – Homogeneous Beliefs

The time restricted Entry-Exit Option which considers homogeneous beliefs can be seen as follows[20]:

$$F_{VC_{Entry(\text{hom})}}(V_t) = Q_{VC}\theta V_t + ExitOption_{VC} - (K - K_E) - FSO_{VC_{Exit}}(V_t) \tag{77}$$

After rearranging, the equation becomes:

$$F_{VC_{Entry(\text{hom})}}(V_t) = Q_{VC}\theta V_t +$$

$$\frac{(\beta-\epsilon)C}{(\beta-1)^2}\left(\frac{V_t}{(\frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta})}\right)^{\beta} N(d3(V_{VC_{Exit}})) - (K - K_E), \text{ for } t < T \tag{78}$$

Again, for the value of the option considering the possibility of the maturity to expire without the option being exercised, when the Venture Capitalist is pushed to dispose the start-up after T at its market value, only transaction costs, of ϵC, arise.

The value function of the option to enter in the start-up capital considering the time restricted homogeneous beliefs setting is given by T:

$$F_{VC_{Entry(\text{hom})}}(V_t) = Q_{VC}\theta V_t + \frac{(\beta-\epsilon)C}{(\beta-1)^2}\left(\frac{V_t}{(\frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta})}\right)^{\beta} N(d3(V_{VC_{Exit}})) -$$

$$(K - K_E) - \epsilon C e^{-rT} N(-d2(V_{VC_{Exit}})) \tag{79}$$

where:

$$d2(V_{VC_{Exit}}) = d1(V_{VC_{Exit}}) - \sigma\sqrt{T} \tag{80}$$

The first term of Eq. (79) represents the value the VC gets, $Q_{VC}\theta V_t$, in exchange for $(K - K_E)$. The second term represents the time restricted exit option of the Venture Capitalist, exercised if $V_t > V^*_{Buyer}(\phi^*, (T-t))$. The third term represents the initial investment of the Venture Capitalist, of $(K - K_E)$. The last one accounts for the transaction costs of the Venture Capitalist, which is the only cash-outflow in the case it does not completes the transaction prior to T. This happens since the option lose its value and the VC is forced to dispose the start-up at its market value, V, that represents an inflow, which is offset by the loss of the ownership – representing an outflow of V –, thus having a net loss attached to the transaction due to the fact that has to bear the transaction costs.

Here, the value function for the Entrepreneur keeps unchanged, since the time restriction only apply for the VC's decision to exit the investment:

[20] Again, the simplified version of the VC Exit Option in homogeneous beliefs setting is as follows: $F(V_{VC_{Entry(\text{hom})}}(\phi^*, t)) = \frac{(\beta-\epsilon)C}{(\beta-1)^2}\left(\frac{V_t}{\left(\frac{\beta}{(\beta-1)^2}\frac{(\beta-\epsilon)C}{Q_{VC}\gamma\theta}\right)}\right)^{\beta}$

$$F(V_{VC_{Entry(\mathrm{hom})}}(t)) = (1 - Q_{VC})\theta V_t - V_t - K_E \tag{81}$$