Production cap and the lumpy flexible investment choice

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Production cap and the lumpy flexible investment choice

Abstract

An upper production limit is incorporated as a cap in a real-option formulation for investigating the conditions favouring a lumpy single-stage investment versus a flexible consecutive two-stage variant. Although having no effect on the investment threshold and timing decision unless the cap is breached, the production cap deflates the investment option value evaluated in the absence of any cap and consequently influences the lumpy-flexible choice. Also, the cap disengages the accepted positive link between volatility and option value, since for volatilities above a certain level, the option value declines with increasing volatility in the presence of a cap. We demonstrate that a flexible as opposed to a lumpy investment policy can be more attractive and sometimes significantly so.

1 Introduction

In formulating analytical real option models, it is customary practice to select geometric Brownian motion to represent the dynamics of the stochastic factor. This assumes the factor can adopt any value along $\mathbb{R}^+$. In practice, the factor may be naturally bounded as illustrated by the physical output generated by most plant and equipment that is limited by their productive capacity even when supplemented with overtime. This capacity constraint can be represented by a cap on the stochastic output, which while having no effect on the investment threshold and timing decision, it does deflate the option value of the opportunity because of the loss of attractiveness. An effective comparison of alternative policy opportunities relies on evaluating their option values so any distortion created by ignoring the capacity limit may lead to erroneous decisions. We examine the choice between an investment that is implemented in a single stage, lumpy, and an equivalent that is implemented in two consecutive stages, flexible, to investigate the impact of a capacity limit on their option values and the resulting policy choice.

2 Analysis

2.1 Fundamental Model

A firm set in a monopolistic situation is considering a project opportunity that is subject to a single source of uncertainty due to demand volume variability, which is described by the geometric Brownian motion process:

$$\mathrm{d}\,\mathrm{q} = \alpha q\,\mathrm{d}t + \sigma q\,\mathrm{d}W\,, \tag{1}$$

where q denotes the periodic demand volume, α the expected drift, σ the volatility, and $\mathrm{d}W$ an increment of the standard Wiener process. The project is subject to a capacity limitation, q_U. Demand is to be fully met when $q \le q_U$. For known unit price p, periodic operating cost f and $q \le q_U$, the active project value V is described by:

$$\tfrac{1}{2}\sigma^2 q^2 \frac{\partial^2 V}{\partial q^2} + \left(r-\delta\right) q \frac{\partial V}{\partial q} + pq - f - rV = 0 \tag{2}$$

where $r > \alpha$ denotes the risk-free rate and $\delta = r - \alpha$ the rate of return shortfall. The idle project value is specified by (2) but with $p,\ f = 0$ while V denotes the project's opportunity value. The generic solution to (2) is:

$$V(q) = \frac{pq}{\delta} - \frac{f}{r} + A_1 q^{\beta_1} + A_2 q^{\beta_2} \tag{3}$$

where A_1, A_2 are to be determined generic constants and β_1, β_2 are the respective positive and negative roots of the characteristic equation:

$$\beta_1, \beta_2 = \left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}\ . \tag{4}$$

In (3), $A_1 q^{\beta_1}, A_2 q^{\beta_2}$ respectively represent the call, Samuelson (1965), and put options, Merton (1973), Merton (1990), for switching the prevailing to an alternative state whenever the relevant threshold is attained. Positive coefficients, $A_1, A_2 > 0$, imply the resulting switch to be value-creating, owned by the firm and economically advantageous. Negative coefficients imply the consequences for the firm to be value-destroying, sacrificial and economically disadvantageous. If demand is not satisfied, $q > q_U$, then the maximum attainable project volume is $\min\{q, q_U\}$ and (3) is amended to become:

$$V(q) = \frac{pq_U}{r} - \frac{f}{r} + A_1 q^{\beta_1} + A_2 q^{\beta_2}\ . \tag{5}$$

2.2 Model I: Lumpy Investment with a Cap

We consider an investment opportunity with known capital cost K to install a project. Since the installed project incurs a known periodic operating cost f, it may operate at a loss in the presumed absence of any abandonment optionality. Further, the capacity of the installed project is capped by the upper limit q_U. The optimal demand threshold for an investing in the project is denoted by $\hat{q}_{(I)01}$, if initially we assume the threshold does not exceed the cap, $\hat{q}_{(I)01} \le q_U$, although this condition is subsequently relaxed. The firm managing the project exists in one of three possible distinct states, see Figure 1a. In the idle pre-investment state-0, the firm is waiting for more favourable information to emerge before making the investment, so $0 < q < \hat{q}_{(I)01}$. In the active post-investment state-1, the project is already installed and actively producing output q provided the production cap is not breached, so $q \le q_U$. In the active state-11, $q > q_U$ and demand is unmet and output is restricted to q_U.

The values for the three distinct states, denoted by $V_{(I)0}(q), V_{(I)1}(q), V_{(I)11}(q)$, respectively, are:

$$V_{(I)0}(q) = A_{(I)01} q^{\beta_1} \quad \text{for } q < \hat{q}_{(I)1}, \tag{6}$$

$$V_{(I)1}(q) = \frac{pq}{\delta} - \frac{f}{r} + A_{(I)11} q^{\beta_1} \quad \text{for } q \le q_U, \tag{7}$$

$$V_{(I)11}(q) = \frac{pq_U}{r} - \frac{f}{r} + A_{(I)112} q^{\beta_2} \quad \text{for } q > q_U. \tag{8}$$

In (6)-(8), $A_{(I)01}$ denotes the investment option coefficient and is expected to be positive, while $A_{(I)11}, A_{(I)112}$ the sacrificial option coefficients incurred when switching between state-1 and -11 are expected to be negative.

The values $A_{(I)11}, A_{(I)112}$ are obtained from the value-matching relationship and smooth-pasting condition ruling at the boundary between state-1 and -11. The complete derivation for Model I is relegated to Appendix A. From there we find:

$$A_{(I)11} = \frac{pq_U^{1-\beta_1}\left(-r + r\beta_2 - \delta\beta_2\right)}{r\left(\beta_1 - \beta_2\right)\delta} < 0, \; A_{(I)112} = -\frac{pq_U^{1-\beta_2}\left(-r + r\beta_1 - \delta\beta_1\right)}{r\left(\beta_1 - \beta_2\right)\delta} < 0. \tag{9}$$

since $r - \delta > 0$. Further, $A_{(I)11}$ declines in absolute terms as q_U increases, so any relaxation of the upper limit enhances the project value in state-1.

The threshold $\hat{q}_{(I)1}$ and coefficient $A_{(I)01}$ are obtained from the value-matching relationship and associated smooth-pasting condition ruling at the boundary of state-0 and -1. This yields:

$$\hat{q}_{(I)1} = \frac{\beta_1}{\beta_1 - 1}\left(f + rK\right)\frac{\delta}{rp}, \tag{10}$$

$$A_{(I)01} = \frac{p\hat{q}_{(I)1}^{1-\beta_1}}{\beta_1\delta} + A_{(I)11}. \tag{11}$$

While $\hat{q}_{(I)} \leq q_U$, the cap has no effect on the threshold and the investment timing. In contrast, the cap makes the investment opportunity less attractive. In (11), the option coefficient is specified as the sum of two elements, the first equalling the coefficient in the absence of any cap, and the second reflecting the sacrifice levied on the state-1.

2.2.1 *Breached Capacity*

If the threshold $\hat{q}_{(I)01}$ exceeds the cap, then exercising the investment option entails switching from state-0 to -11, see Figure 1b. This entails amending the value-matching and smooth pasting expressions accordingly, from which the revised threshold $\hat{q}_{(I)011}$ and coefficient $A_{(I)011}$ are obtained as:

$$\hat{q}_{(I)11} = q_U \left(\frac{\left(pq_U - f - rK \right) \beta_1 \delta}{pq_U \left(r - r\beta_1 + \delta\beta_1 \right)} \right)^{\frac{1}{\beta_2}}, \tag{12}$$

$$A_{(I)011} = \frac{-\left(pq_U - f - rK \right) \beta_2}{\left(\beta_1 - \beta_2 \right) r} \hat{q}_{(I)01}^{-\beta_1}. \tag{13}$$

Provided $\hat{q}_{(I)01} > q_U$, $\hat{q}_{(I)011} > q_U$. If $q_U = \hat{q}_{(I)01}$, then (12) simplifies to (10) and (13) to (11). Further, if viability is assured with $pq_U > f + rK$, then $\partial \hat{q}_{(I)11} / \partial q_U < 0$ and $\partial A_{(I)011} / \partial q_U > 0$. As q_U falls in value and the bite of the cap intensifies, the threshold increase is necessary to compensate the downside risk and mitigate the possibility of the project making a loss, while the decline in the investment option value reflects the project's increasing unattractiveness.

2.3 *Model II: Flexible Investment with a Cap*

Model I represents a lumpy opportunity exercisable as a single project. An alternative strategy is Model II, which divides the project into two constituent stages, stage-1 and -2, both of which require completion for the project to be complete. The two constituent stages are managed consecutively such that the second can only be activated provided the first is operational and operating at full capacity. This inherent flexibility has the merit of initially exposing less capital to the misfortunes of downside risk while offering the opportunity to gain from upside potential by upscaling. But this flexibility is attended with additional capital costs, otherwise all divisible capital projects would always be established in a piecewise fashion. We impose the condition $K_1 + K_2 > K$ to ensure completing the project in a single stage is less expensive than completing it in two consecutive stages, where K_1 and K_2 denote the capital costs for the two stages, respectively. Despite this, the two stage framework is not further disadvantaged. When its second stage is attained, both Model I and II share identical caps and periodic operating costs. Model II stage-1 cap is denoted by $q_u < q_U$ and stage-2 by q_U, which means the two models are directly comparable since they both attain identical caps. The known additional periodic operating costs for stage-1 and -2 are denoted by f_1 and f_2, respectively, where $f_1 + f_2 = f$. In this way, the distinction between two models is attributable to capital costs only and not to any operating cost differences. Again, we exclude the possibility of abandonment optionality.

The capacity expansion option is presumed to become available only if the installed capacity is fully utilized. Expanding an under-utilized installed capacity is seen as uneconomic because of the accompanying additional capital and operating costs engendered by the expansion. When the current output level is below capacity $q < q_u$, a small output increase accompanied with no capacity change incurs no additional costs, but a similar increase with a capacity change entails both a capital expense and an increase in operating costs. We conceptualize the expansion option to be triggered only if full capacity at stage-1 is attained. This does not necessarily entail that the stage-2 investment threshold occurs at the stage-1 cap q_u, but rather that the market volume trigger is at least equal to q_u.

The firm managing the project exists in one of five possible distinct states, see Figure 2a. In the idle state-0, the firm is waiting for more propitious conditions to emerge before making the investment, so $0 < q < \hat{q}_{(II)01}$ where $\hat{q}_{(II)01}$ denotes the optimal threshold for investing K_1 to engender stage-1. In the active state-1, the firm has already invested in the project and is actively producing output q provided the cap is not breached, so $q \le q_u$. In active state-11, the market demand exceeds the stage-1 cap so output is constrained to q_u, but the firm now owns the expansion option and is deliberating on raising the output cap to q_U, so $q_u < q < \hat{q}_{(II)12}$ where

$\hat{q}_{(II)12}$ denotes the optimal threshold for investing K_1 to engender stage-2. In the active state-2, the firm has already invested in additional capacity and is producing an output q provided the q_U cap is not breached. $q \le q_U$. In the active state-21, market demand exceeds the cap and the firm is producing at full capacity q_U. Again, we initially assume the two thresholds do not exceed the caps for the respective stage, $\hat{q}_{(II)01} \le q_u$, $\hat{q}_{(II)12} \le q_U$, although this condition is subsequently relaxed.

The respective project values for the five distinct states, $V_{(II)0}(q), V_{(II)1}(q), V_{(II)11}(q), V_{(II)2}(q)$, $V_{(II)21}$ can now be specified:

$$V_{(II)0}(q) = A_{(II)01}q^{\beta_1} \quad \text{for } q < \hat{q}_{(II)1}, \tag{14}$$

$$V_{(II)1}(q) = \frac{pq}{\delta} - \frac{f_1}{r} + A_{(II)11}q^{\beta_1} \quad \text{for } q \le q_u, \tag{15}$$

$$V_{(II)11}(q) = \frac{pq_u}{r} - \frac{f_1}{r} + A_{(II)111}q^{\beta_1} + A_{(II)112}q^{\beta_2} \quad \text{for } q > q_u, \tag{16}$$

$$V_{(II)2}(q) = \frac{pq}{\delta} - \frac{f_1}{r} - \frac{f_2}{r} + A_{(II)21}q^{\beta_1} \quad \text{for } q \le q_U, \tag{17}$$

$$V_{(II)21}(q) = \frac{pq_U}{r} - \frac{f_1}{r} - \frac{f_2}{r} + A_{(II)212}q^{\beta_2} \quad \text{for } q > q_U. \tag{18}$$

In (14)-(18), $A_{(II)01}, A_{(II)111}$ denote the investment option coefficients associated with investing the amounts K_1, K_2 to raise the cap to q_u, q_U, respectively, and are expected to be positive. The option coefficients $A_{(II)11}, A_{(II)112}$ and $A_{(II)21}, A_{(II)212}$ reflect the sacrificial value imposed by the two caps q_u, q_U, respectively, and are all negative.

$A_{(II)21}, A_{(II)212}$ are obtainable from the value matching relationship and smooth pasting condition ruling at the boundary between stage-2 and -21. The complete derivation for Model II is relegated to Appendix B. From there we find: respectively:

$$A_{(II)21} = \frac{pq_U^{1-\beta_1}\left(-r + r\beta_2 - \delta\beta_2\right)}{r\left(\beta_1 - \beta_2\right)\delta} < 0, \; A_{(II)212} = -\frac{pq_U^{1-\beta_2}\left(-r + r\beta_1 - \delta\beta_1\right)}{r\left(\beta_1 - \beta_2\right)\delta} < 0 \tag{19}$$

From (9) and (19), $A_{(II)21} = A_{(I)11}, A_{(II)212} = A_{(I)112}$. The sacrificed values due to the mismatch between market demand and productive capacity are identical for Model I and II, since they depend on only the capacity upper limit q_U and not the cost structure.

$A_{(II)111}, A_{(II)112}$ are obtainable from the value matching relationship and smooth pasting condition ruling at the boundary between stage-11 and -2. This yields:

$$A_{(II)111} = A_{(II)21} + \frac{\hat{q}_{(II)12}^{-\beta_1}}{\beta_1 - \beta_2}\left[\frac{(1-\beta_2)\,p\hat{q}_{(II)12}}{\delta} + \beta_2\left(K_2 + \frac{f_2}{r} + \frac{pq_u}{r}\right)\right], \tag{20}$$

$$A_{(II)112} = \frac{\hat{q}_{(II)12}^{-\beta_2}}{\beta_1 - \beta_2}\left[\frac{(\beta_1 - 1)\,p\hat{q}_{(II)12}}{\delta} - \beta_1\left(K_2 + \frac{f_2}{r} + \frac{pq_u}{r}\right)\right], \tag{21}$$

where $\hat{q}_{(II)12}$ denotes the to-be-determined stage-2 investment threshold. This is obtained from the value matching relationship and smooth pasting condition ruling at the boundary between stage-1 and -11, which yields the non-linear equation:

$$\hat{q}_{(II)12}^{-\beta_2}\left[r(\beta_1 - 1)\,p\hat{q}_{(II)12} - \delta\beta_1(rK_2 + f_2 + pq_u)\right] = q_u^{-\beta_2}\left(r(\beta_1 - 1) - \delta\beta_1\right)pq_u. \tag{22}$$

It also yields the coefficient $A_{(II)11}$:

$$\begin{aligned} A_{(II)11} = {} & \frac{\beta_2(r-\delta) - r}{r(\beta_1 - \beta_2)\delta}\left(pq_U^{1-\beta_1} + pq_u^{1-\beta_1}\right) \\ & + \frac{\hat{q}_{(II)12}^{-\beta_1}}{r(\beta_1 - \beta_2)\delta}\left[p\hat{q}_{(II)12}(1-\beta_2)r + (pq_u + f_2 + K_2)\delta\beta_2\right]. \end{aligned} \tag{23}$$

Finally, $\hat{q}_{(II)01}$ and $A_{(II)01}$ are obtainable from the value matching relationship and smooth pasting condition ruling at the boundary between stage-0 and -1. This yields:

$$\hat{q}_{(II)01} = \frac{\beta_1(f_1 + rK_1)\delta}{p(\beta_1 - 1)r}, \tag{24}$$

$$A_{(II)01} = \frac{p\hat{q}_{(II)01}}{\beta_1\delta\hat{q}_{(II)01}^{\beta_1 - 1}} + A_{(II)11}. \tag{25}$$

While $\hat{q}_{(II)01} \le q_u$, the cap does not affect the stage-1 threshold or the investment timing. In contrast, the cap makes the opportunity less attractive since $A_{(II)11} < 0$. In (25), $A_{(II)01}$ is the sum of two constituent elements, namely the coefficient in the absence of stage-1 cap and the coefficient reflecting the sacrifice levied due to the cap during state-1.

2.3.1 Stage-2 Breached Capacity

If the threshold $\hat{q}_{(II)12}$, (22), exceeds the stage-2 cap q_U, then exercising the option entails switching from state-11 to -21. This entails amending the value-matching and smooth pasting expressions accordingly, from which we obtain the revised threshold $\hat{q}_{(II)121}$ and the coefficients $A_{(II)111}, A_{(II)112}, A_{(II)11}$:

$$\hat{q}_{(II)121}^{-\beta_2} = \frac{\left(pq_U^{1-\beta_2} - pq_u^{1-\beta_2}\right)\left(\delta\beta_1 - r\left(\beta_1 - 1\right)\right)}{\left(pq_U - pq_u - f_2 - rK_2\right)\beta_1\delta}, \tag{26}$$

$$A_{(II)111} = \frac{-\hat{q}_{(II)121}^{-\beta_1}}{r\left(\beta_1 - \beta_2\right)}\left(pq_U - pq_u - f_2 - rK_2\right)\beta_2 \tag{27}$$

$$A_{(II)112} = \frac{q_U^{-\beta_2}\left(-r + r\beta_1 - \delta\beta_1\right)pq_U}{r\left(\beta_1 - \beta_2\right)\delta} + \frac{\hat{q}_{(II)121}^{-\beta_2}}{r\left(\beta_1 - \beta_2\right)\delta}\left(pq_U - pq_u - f_2 - rK_2\right)\beta_1\delta \tag{28}$$

$$A_{(II)11} = \frac{-\hat{q}_{(II)121}^{-\beta_1}\left(pq_U - pq_u - f_2 - rK_2\right)\beta_2\delta}{r\left(\beta_1 - \beta_2\right)\delta} + \frac{pq_u^{1-\beta_1}\left(r\left(\beta_2 - 1\right) - \beta_2\delta\right)}{r\left(\beta_1 - \beta_2\right)\delta}. \tag{29}$$

Although $A_{(II)21}, A_{(II)212}$ and $\hat{q}_{(II)01}$ remain unchanged, the revisions (26)-(29) only affect $A_{(II)01}$ through (25). Consequently, the threshold exceeding the stage-2 cap has a bearing on the investment option.

2.4 Decamps Model

3 Numerical Illustrations

Further insights concerning the behaviours of the models presented in the previous section are obtained through numerical illustrations. Initially, we assume the values for the parameters underpinning the stochastic process are $r = 0.04$, $\delta = 0.04$ and $\sigma = 0.2$, which implies that $\beta_1 = 2$ and $\beta_2 = -1$.

3.1 Model I

The behaviour of Model I investigated for a product price $p = 1$, periodic operating cost $f = 1.6$ and capital expenditure $K = 100$. The solution values for various capacity upper limits q_U are illustrated in Table 1. The evaluated investment thresholds are less than the corresponding upper capacity limit, $\hat{q}_{(I)01} \leq q_U$, so for these data sets the cap is not breached. As expected, the thresholds are equal. However, as the cap increases in value and becomes less stringent, the investment option value as reflected by $A_{(I)01}$ becomes more valuable. The two coefficients, $A_{(I)11}, A_{(I)112}$ corresponding to the respective sacrificial option value for $q \leq q_U$ in state-1 and $q > q_U$ in state-11 are negative as expected. Their magnitudes are affected by the capacity cap, but in different ways. As q_U decreases and the cap becomes increasingly more stringent, the sacrifice incurred in switch from state-1 to -11 as reflected by the magnitude of $A_{(I)11}$ becomes increasingly onerous as more is foregone. In contrast, the sacrifice incurred in switch from state-1 to -11 as reflected by the magnitude of $A_{(I)112}$ becomes increasingly less onerous as less is foregone.

Table 1

Model I Solution Values with Cap Variations

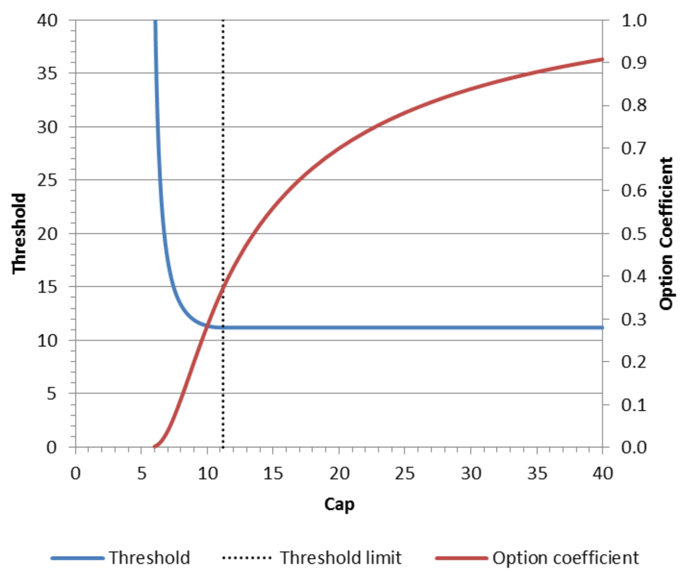
Cap q_U	15	20	25
Option coefficient $A_{(I)11}$	-0.5556	-0.4167	-0.3333
Option coefficient $A_{(I)112}$	-1875.0	-3333.3	-5208.3
Option coefficient $A_{(I)01}$	0.5605	0.6994	0.7827
Investment threshold $\hat{q}_{(I)01}$	11.20	11.20	11.20

This table is evaluated from (10) and (11) for $r = 0.04$, $\delta = 0.04$, $\sigma = 0.2$, $p = 1$, $f = 1.6$, $K = 100$.

If the cap is breached, $\hat{q}_{(I)01} > q_U$, the solutions for the investment threshold and coefficient are revised as specified in §2.2.1 while the values of $A_{(I)11}, A_{(I)112}$ remain unchanged. In figure 3, we illustrate the profiles of the threshold and option coefficient for variations in the cap that are greater than and less than the threshold limit defined in the absence of a cap. If $\hat{q}_{(I)01} \leq q_U$, the threshold remains constant, but if q_U is decreased increasingly below the threshold limit, then the threshold as specified by $\hat{q}_{(I)011}$ increases without bound. A more stringent cap necessitates an investment only at ever increasing threshold levels in order to mitigate the possibility of future losses. Figure 3 also reveals the decreasing behaviour of the option value as the cap tightens, indicating a decreasing attractiveness of the project as the cap declines.

Figure 3

Model I Investment Threshold and Option Coefficient for Cap Variations

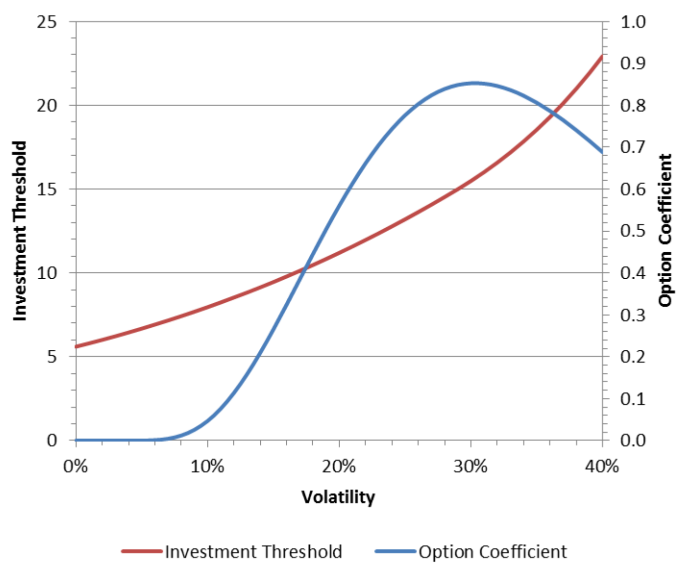


This figure is evaluated using (10) and (11) if $\hat{q}_{(I)01} \leq q_U$ and (12) and (13) if $\hat{q}_{(I)01} > q_U$, for $r = 0.04$, $\delta = 0.04$, $\sigma = 0.2$, $p = 1$, $f = 1.6$, $K = 100$. Threshold limit is defined by the threshold in the absence of a cap, and is 11.2.

Although volatility is generally recognised as to exert a positive influence on the threshold and option value, this seems to only partially hold for the option value in the presence of a cap. In figure 4, their profiles are illustrated for volatility variations and $q_U = 15$. While the threshold continuously increases with volatility over the illustrated range, the option value increases until the volatility attains a sufficiently high level for the threshold to equal the cap, which occurs for $\sigma = 0.3$. For greater volatility levels, the cap is breached and the option value declines. In the absence of a cap, the option value and the project attractiveness increase with volatility owing to the expectation that higher volatilities bring attends a greater likelihood of higher values for the underlying. However, when the cap is present and limits the underlying value, the beneficial consequences of a high volatility do not exist and the project declines in value. Volatility has a positive impact on the option value unless the cap is breached, in which case the impact is negative.

Figure 4

Model I Investment Threshold and Option Coefficient for Volatility Variations



This figure is evaluated using (10) and (11) if $\hat{q}_{(I)01} \leq q_U$ and (12) and (13) if $\hat{q}_{(I)01} > q_U$, for $r = 0.04, \delta = 0.04$, , $p = 1$, $f = 1.6$, $K = 100, q_U = 15$. The threshold equals the cap for $\sigma = 0.3$.

$$p = q \tag{30}$$

Appendix A: Derivations for Model I

The coefficients $A_{(I)11}, A_{(I)112}$ are obtained from the value-matching relationship and associated smooth-pasting condition ruling at the boundary between state-1 and -11, namely:

$$V_{(I)1}(q)\Big|_{q=q_U} = V_{(I)11}(q)\Big|_{q=q_U}, \frac{\partial V_{(I)1}(q)}{\partial q}\Bigg|_{q=q_U} = \frac{\partial V_{(I)11}(q)}{\partial q}\Bigg|_{q=q_U}, \tag{A.1}$$

or respectively as:

$$A_{(I)11}q_U^{\beta_1} + pq_U/\delta - f/r = A_{(I)112}q_U^{\beta_2} + pq_U/\delta - f/r, \tag{A.2}$$

$$\beta_1 A_{(I)11}q_U^{\beta_1-1} + p/\delta = \beta_2 A_{(I)112}q_U^{\beta_2-1}. \tag{A.3}$$

Substituting $A_{(I)112} = q_U^{-\beta_2}\left(pq_U/\delta + \beta_1 A_{(I)11}q_U^{\beta_1}\right)\Big/\beta_2$ from (A.3) into (A.2) and simplifying yields:

$$A_{(I)11} = -\frac{pq_U^{1-\beta_1}\left(r - r\beta_2 + \delta\beta_2\right)}{r\left(\beta_1 - \beta_2\right)\delta}, \tag{A.4}$$

$$A_{(I)112} = \frac{pq_U^{1-\beta_2}\left(r - r\beta_1 + \delta\beta_1\right)}{r\left(\beta_1 - \beta_2\right)\delta}. \tag{A.5}$$

The investment threshold $\hat{q}_{(I)}$ and option coefficient $A_{(I)01}$ are obtained from the value-matching relationship and associated smooth-pasting condition ruling at the boundary between state-0 and -1:

$$V_{(I)0}(q)\Big|_{q=\hat{q}_{(I)1}} = V_{(I)1}(q) - K\Big|_{q=\hat{q}_{(I)1}}, \frac{\partial V_{(I)0}(q)}{\partial q}\Bigg|_{q=\hat{q}_{(I)1}} = \frac{\partial V_{(I)1}(q)}{\partial q}\Bigg|_{q=\hat{q}_{(I)1}} \tag{A.6}$$

or respectively as:

$$A_{(I)01}\hat{q}_{(I)1}^{\beta_1} = A_{(I)11}\hat{q}_{(I)1}^{\beta_1} + p\hat{q}_{(I)1}/\delta - f/r - K\,, \tag{A.7}$$

$$\beta_1 A_{(I)01}\hat{q}_{(I)1}^{\beta_1-1} = \beta_1 A_{(I)11}\hat{q}_{(I)1}^{\beta_1-1} + p/\delta\,. \tag{A.8}$$

From (A.8):

$$A_{(I)01} = \frac{p\hat{q}_{(I)1}^{1-\beta_1}}{\beta_1\delta} + A_{(I)11}\,, \tag{A.9}$$

which when substituted in (A.7) yields:

$$\hat{q}_{(I)1} = \frac{\beta_1}{\beta_1 - 1}(f + rK)\frac{\delta}{rp}. \tag{A.10}$$

If $f/r + K > (p\,q_U/\delta)(\beta_1 - 1)/\beta_1$, then $\hat{q}_{(I)01} > q_U$ and the value-matching relationship and smooth-pasting conditions require amending accordingly. From (A.6), those ruling at the boundary between state-0 and -11 are:

$$V_{(I)0}(q)\Big|_{q=\hat{q}_{(I)11}} = V_{(I)11}(q) - K\Big|_{q=\hat{q}_{(I)11}}, \frac{\partial V_{(I)0}(q)}{\partial q}\Bigg|_{q=\hat{q}_{(I)11}} = \frac{\partial V_{(I)11}(q)}{\partial q}\Bigg|_{q=\hat{q}_{(I)11}}. \tag{A.11}$$

The respective value-matching relationship and smooth-pasting condition are:

$$A_{(I)011}\hat{q}_{(I)011}^{\beta_1} = A_{(I)112}\hat{q}_{(I)011}^{\beta_2} + pq_U/r - f/r - K\,, \tag{A.12}$$

$$\beta_1 A_{(I)011}\hat{q}_{(I)011}^{\beta_1-1} = \beta_2 A_{(I)112}\hat{q}_{(I)011}^{\beta_2-1} \tag{A.13}$$

From (A.13):

$$A_{(I)011} = \frac{\beta_2}{\beta_1}A_{(I)112}\hat{q}_{(I)011}^{\beta_2-\beta_1} = \frac{-(pq_U - f - rK)\beta_2}{(\beta_1 - \beta_2)r}\hat{q}_{(I)11}^{-\beta_1} \tag{A.14}$$

which when substituted in (A.13) yields:

$$\hat{q}_{(I)11} = \left(\frac{\beta_1(pq_U - f - rK)}{(\beta_1 - \beta_2)r\left(-A_{(I)22}\right)}\right)^{\frac{1}{\beta_2}} = q_U\left(\frac{(pq_U - f - rK)\beta_1\delta}{pq_U(r - r\beta_1 + \delta\beta_1)}\right)^{\frac{1}{\beta_2}}. \tag{A.15}$$

From (A.15):

$$\frac{\partial \hat{q}_{(I)11}}{\hat{q}_{(I)11}\partial q_U} = \frac{\beta_2 p + f + rK}{\beta_2 \left(pq_U - f - rK \right)} < 0\,, \tag{A.16}$$

provided $\beta_2 p + f + rK > 0$. From (A.14):

$$\frac{\partial A_{(I)01}}{A_{(I)01}\partial q_U} = \frac{p}{pq_U - f - rK} - \frac{\beta_1}{\hat{q}_{(I)11}} \frac{\partial \hat{q}_{(I)11}}{\partial q_U} > 0\,. \tag{A.17}$$

Appendix B: Derivations for Model II

The coefficients $A_{(II)21}, A_{(II)212}$ are obtained from the value-matching relationship and associated smooth-pasting condition ruling at the boundary between state-2 and -21, namely:

$$V_{(II)2}\left(q\right)\Big|_{q=q_U} = V_{(II)21}\left(q\right)\Big|_{q=q_U}\,, \frac{\partial V_{(II)2}\left(q\right)}{\partial q}\Bigg|_{q=q_U} = \frac{\partial V_{(II)21}\left(q\right)}{\partial q}\Bigg|_{q=q_U} \tag{B.1}$$

or, respectively, as:

$$A_{(II)21} q_U^{\beta_1} + \frac{pq_U}{\delta} - \frac{f_1 + f_2}{r} = A_{(II)212} q_U^{\beta_2} + \frac{pq_U}{r} - \frac{f_1 + f_2}{r}\,, \tag{B.2}$$

$$\beta_1 A_{(II)21} q_U^{\beta_1 - 1} + \frac{q_U}{\delta} = \beta_2 A_{(II)212} q_U^{\beta_2 - 1}\,. \tag{B.3}$$

Substituting $A_{(II)212} = q_U^{-\beta_2} \left(pq_U / \delta + \beta_1 A_{(II)21} q_U^{\beta_1 - 1} \right) \Big/ \beta_2$ from (B.3) into (B.4) yields:

$$A_{(II)21} = \frac{\beta_2 \left(r - \delta \right) - r}{r \left(\beta_1 - \beta_2 \right) \delta}\, p q_U^{1-\beta_1}\,, \tag{B.5}$$

so:

$$A_{(II)212} = \frac{\beta_1 \left(r - \delta \right) - r}{r \left(\beta_1 - \beta_2 \right) \delta}\, p q_U^{1-\beta_2}\,. \tag{B.6}$$

The coefficients $A_{(II)111}, A_{(II)112}$ are obtained from the value matching relationship and smooth pasting condition ruling at the boundary between state-11 and -2:

$$V_{(II)11}(q)\Big|_{q=\hat{q}_{(II)12}} = V_{(II)2}(q)\Big|_{q=\hat{q}_{(II)12}} - K_2, \frac{\partial V_{(II)11}(q)}{\partial q}\Bigg|_{q=\hat{q}_{(II)12}} = \frac{\partial V_{(II)2}(q)}{\partial q}\Bigg|_{q=\hat{q}_{(II)12}}, \tag{B.7}$$

where $\hat{q}_{(II)12}$ denotes the stage-2 investment opportunity threshold. The value matching relationship and smooth pasting condition are, respectively:

$$A_{(II)111}\hat{q}_{(II)12}^{\beta_1} + A_{(II)112}\hat{q}_{(II)12}^{\beta_2} + \frac{pq_u}{r} - \frac{f_1}{r} = A_{(II)21}\hat{q}_{(II)12}^{\beta_1} + \frac{p\hat{q}_{(II)12}}{\delta} - \frac{f_1 + f_2}{r} - K_2, \tag{B.8}$$

$$\beta_1 A_{(II)111}\hat{q}_{(II)12}^{\beta_1 - 1} + \beta_2 A_{(II)112}\hat{q}_{(II)12}^{\beta_2 - 1} = \beta_1 A_{(II)21}\hat{q}_{(II)12}^{\beta_1 - 1} + \frac{p}{\delta}. \tag{B.9}$$

Substituting $A_{(II)112} = \hat{q}_{(II)12}^{-\beta_2}\left(\beta_1 A_{(II)21}\hat{q}_{(II)12}^{\beta_1} - \beta_1 A_{(II)111}\hat{q}_{(II)12}^{\beta_1} + p\hat{q}_{(II)12}/\delta\right)\Big/\beta_2$ from (B.9) into (B.8) yields:

$$A_{(II)111} = A_{(II)21} + \frac{\hat{q}_{(II)12}^{-\beta_1}}{\beta_1 - \beta_2}\left[\frac{(1-\beta_2)\, p\hat{q}_{(II)12}}{\delta} + \beta_2\left(K_2 + \frac{f_2}{r} + \frac{pq_u}{r}\right)\right], \tag{B.10}$$

so:

$$A_{(II)112} = \frac{\hat{q}_{(II)12}^{-\beta_2}}{\beta_1 - \beta_2}\left[\frac{(\beta_1 - 1)\, p\hat{q}_{(II)12}}{\delta} - \beta_1\left(K_2 + \frac{f_2}{r} + \frac{pq_u}{r}\right)\right]. \tag{B.11}$$

The coefficient $A_{(II)11}$ and threshold $\hat{q}_{(II)12}$ are obtained from the value matching relationship and smooth pasting condition ruling at the boundary between state-1 and -11:

$$V_{(II)1}(q)\Big|_{q=q_u} = V_{(II)11}(q)\Big|_{q=q_u}, \frac{\partial V_{(II)1}(q)}{\partial q}\Bigg|_{q=q_u} = \frac{\partial V_{(II)11}(q)}{\partial q}\Bigg|_{q=q_u}. \tag{B.12}$$

or, respectively, as:

$$A_{(II)11}q_u^{\beta_1} + \frac{pq_u}{\delta} - \frac{f_1}{r} = A_{(II)111}q_u^{\beta_1} + A_{(II)112}q_u^{\beta_2} + \frac{pq_u}{r} - \frac{f_1}{r}, \tag{B.13}$$

$$\beta_1 A_{(II)11}q_u^{\beta_1 - 1} + \frac{p}{\delta} = \beta_1 A_{(II)111}q_u^{\beta_1 - 1} + \beta_2 A_{(II)112}q_u^{\beta_2 - 1}. \tag{B.14}$$

Substituting $A_{(II)11} = q_u^{-\beta_1}\left(\beta_1 A_{(II)111}q_u^{\beta_1} + \beta_2 A_{(II)112}q_u^{\beta_2} - pq_u/\delta\right)\Big/\beta_1$ from (B.14) into (B.13), and simplifying yields:

$$A_{(II)112} = \frac{\hat{q}_u^{-\beta_2}\left(r\left(\beta_1 - 1\right) - \delta\beta_1\right) p q_u}{r\left(\beta_1 - \beta_2\right)\delta}. \tag{B.15}$$

Eliminating $A_{(II)112}$ from (B.11) and (B.15) yields:

$$\hat{q}_{(II)12}^{-\beta_2}\left[r\left(\beta_1 - 1\right) p\hat{q}_{(II)12} - \delta\beta_1\left(rK_2 + f_2 + pq_u\right)\right] = q_u^{-\beta_2}\left(r\left(\beta_1 - 1\right) - \delta\beta_1\right) p q_u. \tag{B.16}$$

The solution for $\hat{q}_{(II)12}$ is obtained from (B.16) by using numerical methods with

$$\frac{\delta\beta_1\left(rK_2 + f_2 + pq_u\right)}{r\left(\beta_1 - 1\right) p}$$

as a possible initial estimate. Also from (B.14), and substituting (B.5), (B.10) and (B.11) to yield after simplification:

$$\begin{aligned} A_{(II)11} = {} & \frac{\beta_2\left(r - \delta\right) - r}{r\left(\beta_1 - \beta_2\right)\delta}\left(pq_U^{1-\beta_1} + pq_u^{1-\beta_1}\right) \\ & + \frac{\hat{q}_{(II)12}^{-\beta_1}}{r\left(\beta_1 - \beta_2\right)\delta}\left[p\hat{q}_{(II)12}\left(1 - \beta_2\right) r + \left(pq_u + f_2 + K_2\right)\delta\beta_2\right] \end{aligned} \tag{B.17}$$

The coefficient $A_{(II)01}$ and the stage-1 investment threshold $\hat{q}_{(II)01}$ are obtained from the value matching relationship and smooth pasting condition ruling at the boundary between state-0 and -1:

$$V_{(II)0}\left(q\right)\Big|_{q=\hat{q}_{(II)01}} = V_{(II)1}\left(q\right)\Big|_{q=\hat{q}_{(II)01}} - K_1, \quad \frac{\partial V_{(II)0}\left(q\right)}{\partial q}\bigg|_{q=\hat{q}_{(II)01}} = \frac{\partial V_{(II)1}\left(q\right)}{\partial q}\bigg|_{q=\hat{q}_{(II)01}}, \tag{B.18}$$

The respective value matching relationship and smooth pasting condition are:

$$A_{(II)01}\hat{q}_{(II)01}^{\beta_1} = A_{(II)11}\hat{q}_{(II)01}^{\beta_1} + \frac{p\hat{q}_{(II)01}}{\delta} - \frac{f_1}{r} - K_1, \tag{B.19}$$

$$\beta_1 A_{(II)01}\hat{q}_{(II)01}^{\beta_1 - 1} = \beta_1 A_{(II)11}\hat{q}_{(II)01}^{\beta_1 - 1} + \frac{p}{\delta} \tag{B.20}$$

Substituting $A_{(II)01} = A_{(II)11} + \hat{q}_{(II)01}^{-\beta_1} p\hat{q}_{(II)01} / \beta_1\delta$ from (B.20) into (B.19), and simplifying yields:

$$\hat{q}_{(II)01} = \frac{\beta_1\left(f_1 + rK_1\right)\delta}{p\left(\beta_1 - 1\right) r}. \tag{B.21}$$

Also from (B.20) and substituting (B.21) yields:

$$A_{(II)01} = \frac{p\hat{q}_{(II)01}}{\beta_1 \delta \hat{q}_{(II)01}^{\beta_1 - 1}} + A_{(II)11}. \tag{B.22}$$

If $\hat{q}_{(II)12} > q_U$, then the value matching relationship and smooth pasting condition ruling at the boundary between state-11 and -2, (B.7), is replaced by those between state-11 and -21, while the others remain intact. The revised expressions are:

$$V_{(II)11}(q)\Big|_{q=\hat{q}_{(II)121}} = V_{(II)21}(q)\Big|_{q=\hat{q}_{(II)121}} - K_2, \left.\frac{\partial V_{(II)11}(q)}{\partial q}\right|_{q=\hat{q}_{(II)121}} = \left.\frac{\partial V_{(II)21}(q)}{\partial q}\right|_{q=\hat{q}_{(II)121}}, \tag{B.23}$$

or respectively as:

$$A_{(II)111}\hat{q}_{(II)121}^{\beta_1} + A_{(II)112}\hat{q}_{(II)121}^{\beta_2} + \frac{pq_u}{r} - \frac{f_1}{r} = A_{(II)212}\hat{q}_{(II)121}^{\beta_2} + \frac{p\hat{q}_U}{r} - \frac{f_1 + f_2}{r} - K_2, \tag{B.24}$$

$$\beta_1 A_{(II)111}\hat{q}_{(II)121}^{\beta_1 - 1} + \beta_2 A_{(II)112}\hat{q}_{(II)121}^{\beta_2 - 1} = \beta_2 A_{(II)212}\hat{q}_{(II)121}^{\beta_2 - 1}. \tag{B.25}$$

From (B.25), $A_{(II)112} = A_{(II)212} - \beta_1 A_{(II)111}\hat{q}_{(II)121}^{\beta_1 - \beta_2} / \beta_2$, which when substituted in (B.24) yields:

$$A_{(II)111} = \frac{-\hat{q}_{(II)121}^{-\beta_1}}{r(\beta_1 - \beta_2)}\left(pq_U - pq_u - f_2 - rK_2\right)\beta_2, \tag{B.26}$$

so:

$$A_{(II)112} = \frac{q_U^{-\beta_2}\left(-r + r\beta_1 - \delta\beta_1\right)pq_U}{r(\beta_1 - \beta_2)\delta} + \frac{\hat{q}_{(II)121}^{-\beta_2}}{r(\beta_1 - \beta_2)\delta}\left(pq_U - pq_u - f_2 - rK_2\right)\beta_1\delta. \tag{B.27}$$

Since the expressions pertaining to stage-11 are unaffected, then by combining (B.27) with (B.15) yields:

$$\left(pq_U^{1-\beta_2} - pq_u^{1-\beta_2}\right)\left(r(\beta_1 - 1) - \delta\beta_1\right) + \hat{q}_{(II)121}^{-\beta_2}\left(pq_U - pq_u - f_2 - rK_2\right)\beta_1\delta = 0, \tag{B.28}$$

so:

$$\hat{q}_{(II)121}^{-\beta_2} = \frac{\left(pq_U^{1-\beta_2} - pq_u^{1-\beta_2}\right)\left(\delta\beta_1 - r(\beta_1 - 1)\right)}{\left(pq_U - pq_u - f_2 - rK_2\right)\beta_1\delta}. \tag{B.29}$$

Combining (B.27) and (B.28) with (B.14) and simplifying yields:

$$A_{(II)11} = \frac{-\hat{q}_{(II)121}^{-\beta_1}\left(pq_U - pq_u - f_2 - rK_2\right)\beta_2\delta}{r(\beta_1 - \beta_2)\delta} + \frac{pq_u^{1-\beta_1}\left(r(\beta_2 - 1) - \beta_2\delta\right)}{r(\beta_1 - \beta_2)\delta}. \tag{B.30}$$

Since the expressions pertaining to stage-1 are unaffected, then the investment opportunity threshold $\hat{q}_{(II)01}$ is found from (B.21) and the option coefficient $A_{(II)01}$ from:

$$A_{(II)01} = A_{(II)11} + \hat{q}_{(II)01}^{-\beta_1} \, p\hat{q}_{(II)01} \big/ \beta_1 \delta , \tag{B.31}$$

where $A_{(II)11}$ is given by (B.30).

It is straightforward to demonstrate that $\hat{q}_{(II)12} = \hat{q}_{(II)121}$ when $\hat{q}_{(II)12}$ is determined from (B.16) given $\hat{q}_{(II)12} = q_U$ and $\hat{q}_{(II)121}$ is determined from (B.29) given $\hat{q}_{(II)121} = q_U$.

If $\hat{q}_{(II)01} > q_u$, then the value matching relationship and smooth pasting condition ruling at the boundary between state-0 and -1, (B.18), are replaced by those between state-0 and -11. The revised expressions are:

$$V_{(II)0}\left(q\right)\Big|_{q=\hat{q}_{(II)011}} = V_{(II)11}\left(q\right)\Big|_{q=\hat{q}_{(II)011}} - K_1 \, , \; \frac{\partial V_{(II)0}\left(q\right)}{\partial q}\Bigg|_{q=\hat{q}_{(II)011}} = \frac{\partial V_{(II)11}\left(q\right)}{\partial q}\Bigg|_{q=\hat{q}_{(II)011}} . \tag{B.32}$$

The respective value matching relationship and smooth pasting condition are:

$$A_{(II)011}\hat{q}_{(II)011}^{\beta_1} = A_{(II)111}\hat{q}_{(II)011}^{\beta_1} + A_{(II)112}\hat{q}_{(II)011}^{\beta_2} + \frac{pq_u}{r} - \frac{f_1}{r} - K_1 , \tag{B.33}$$

$$\beta_1 A_{(II)011}\hat{q}_{(II)011}^{\beta_1 - 1} = \beta_1 A_{(II)111}\hat{q}_{(II)011}^{\beta_1 - 1} + \beta_2 A_{(II)112}\hat{q}_{(II)011}^{\beta_2 - 1} . \tag{B.34}$$

From (B.34), $A_{(II)011} = A_{(II)111} + \beta_2 A_{(II)112} \, \hat{q}_{(II)011}^{\beta_2 - \beta_1} \big/ \beta_1$, which when substituted in (B.33) yields:

$$\hat{q}_{(II)011}^{\beta_2} = \frac{-\beta_1\left(pq_u - f_1 - rK_1\right)}{\left(\beta_1 - \beta_2\right) r A_{(II)112}} , \tag{B.35}$$

where $A_{(II)112}$ is determined from (B.15) if the stage-2 cap is not breached, or from (B.27) if otherwise. Also, from (B.34):

$$A_{(II)011} = A_{(II)111} + \frac{-\beta_2\left(pq_u - f_1 - rK_1\right)}{\left(\beta_1 - \beta_2\right) r} \hat{q}_{(II)011}^{-\beta_1} \tag{B.36}$$

where $A_{(II)111}$ is determined from (B.10) if the stage-2 cap is not breached, or from (B.26) if otherwise.

Figure 1a

Model I: Unbreached Cap

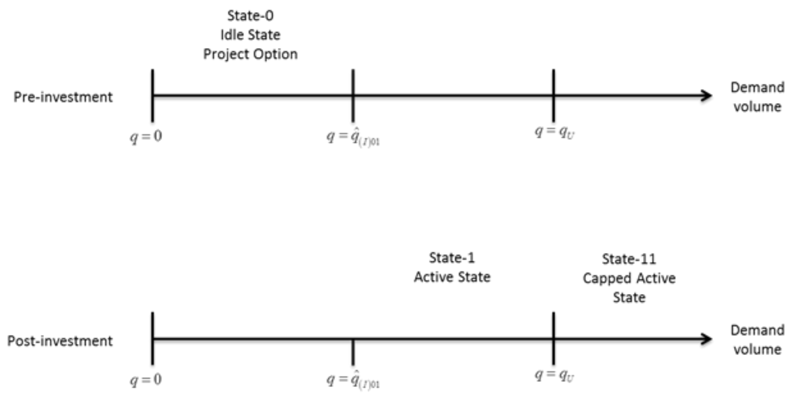


Figure 1b

Model I: Breached Cap

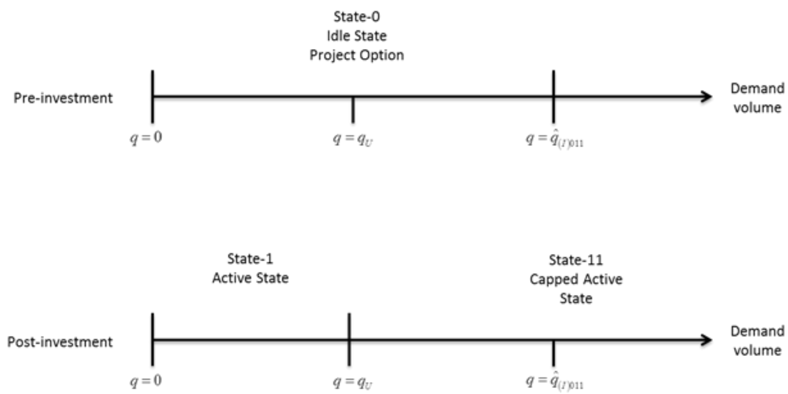


Figure 2

Model II