Competitive Investment with Varying Risk Premia

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Abstract

We consider a model where the risk premium is varying. The risk premium is driven by a continuous time Markov chain, representing the state in the economy, and the stochastic process generating the cash flows is a Markov-modulated geometric Brownian motion. An existing firm is facing the possibility of competitors entering the market, and due to this, cash flows are limited at levels which are dependent on the state of the economy. This results in a regulated Markov-modulated geometric Brownian motion, and the resulting accumulated supply can have jumps, something that is not possible in a model with only one regime.

Keywords: Valuation, competition, Markov-modulated Brownian motion, regulated processes.

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1 Introduction

In many valuation problems it is assumed that the underlying value follows a geometric Brownian motion. In this paper we revisit a problem considered in Chapter 8 in Dixit & Pindyck [3] and in Grenadier [6] regarding the value of an investment in the presence of competitors, generalizing it from the geometric Brownian motion model used there, to a model where an observable Markov chain determines the state of the economy. These 'regime-switching' or 'Markov-modulated' models have been used to extend the irreversible investment problem of McDonald & Siegel [15]; see e.g. Driffill et al [4], Guo [8], Guo & Zhang [9] and Jobert & Rogers [13]. An early example of regime-switching models is given in Hamilton [10]. In our models, we consider a regime-dependent, i.e. time-varying, risk premium.

The fact that we introduce competitors in our model means that as the underlying cash flow increases, at a given level it will be profitable for competitors to enter the market. This results in two things: That the rent level is reflected and that there is added supply. Mathematically, the cumulative supply curve is a continuous increasing function. One simulated example of the reflected rent level and the cumulative supply is given in Figure 1

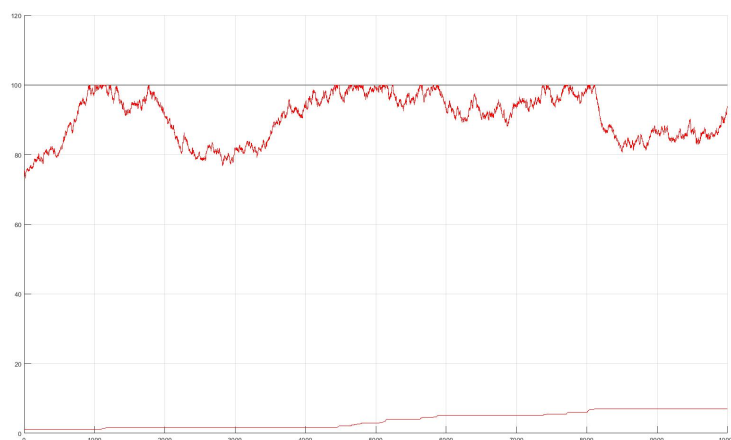

Figure 1: Cash flows reflected in the level 100 (top trajectory) and the cumulative supply created by firms entering the market when the cash flow level reaches 100 (lower trajectory). The unit on the y-axis refers to the cash flow value. This decribes the case as in Chaper 8 of Dixit & Pindyck [3] and in Grenadier [6].

When there are regime shifts present, there will, in general, be different levels at which it is profitable for firms to enter the market. There will still be reflection in the barriers, resulting in a continuously increasing cumulative supply, but as the regimes shift, there could also be a jump in the supply (in contrast to the one regime case, where there is only continuous increase in supply). Our regime-switching models can be seen as a simplified and tractable way of modelling the typically continous variation over time in risk premia. A common example of a real life application of our model is the boom-and-bust cycles observed in many commercial real estate markets. One example of simulated trajectories is given in Figure 2.

The geometric Brownian motion model with one upper barrier, i.e. where there is no switching, is studied in Bentolila & Bertola [1], and the mathematical problem of a reflected Markov-modulated Brownian motion with (in general state-dependent drift and diffusion) is studied in D'Auria & Kella [2].

Figure 2: The two-regime case (see Section 4 for details). In this simulation, the cash flow processes are started at the same value, but the red one is only reflected in the upper barrier with value 100, while the trajectory in blue shows a process reflected in state-dependent levels; values 50 and 100 respectively. The two lower trajectories represents the cumulative supply in the two cases (again, the unit on the y-axis refers to the cash flow value).

The cash flow process is not assumed to be the price of a traded asset, which means that we have two stochastic processes (the cash flow process and the process marking the state of the economy), none of which is traded. The type of models we consider are, in the laguage of mathematical finance, in general incomplete. This means that there exists more than one equivalent martingale, or pricing, measure. In order to choose which pricing measure to use, there are several principles available. In Elliott et al [5] Esscher transforms are used, and in Siu [16] a general martingale representation is the starting point. In both these approaches, the resulting measure is the minimal entropy martingale measure (MEMM). In Siu & Yang [17] an Esscher transformation technique which does not result in the MEMM is used. Our approach is to assume that the dynamics of the process marking the state of the world is not changed, and change the drift of the cash flow process using a state-dependent market price of risk which is not determined within the model.

2 The model

2.1 General theory

We consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$, where the filtration is assumed to satisfy the usual assumptions of right-continuity and $\mathcal{F}_0$ containing all null sets of $\mathcal{F}$. The pricing measure, or martingale measure, $\mathbf{Q}$ is the equivalent measure we use when valuing cash flows. The expected value under $\mathbf{Q}$ is denoted E^Q. We assume the existence of a bank account with constant interest rate $r > 0$, and we value cash flows by discounting them using r as discount rate and taking expectations under $\mathbf{Q}$. We use the notation $E_x^Q[\cdot] = E^Q[\cdot | X_0 = x]$.

The cash flows per time unit (e.g. the rent a building is generating) is given by P_t, and the inverse demand function is

$$P_t = Y_t D(Q_t),$$

where $D(\cdot)$ is a decreasing continuous function of accumulated supply Q and (Y_t) is a random shock. This modelling setup is used in e.g. Grenadier [6], [7] and Chapter 8 of Dixit & Pindyck, and we refer to these and to references therein for further aspects of this type of model. We assume that

$$Y_t = e^{X_t},$$

where X is a continuous strong Markov process to be defined below. It follows that

$$\ln P_t = \ln Y_t + \ln D(Q_t) = X_t + \ln D(Q_t).$$

Defining

$$Z_t = \ln P_t \quad \text{and} \quad U_t = -\ln D(Q_t)$$

we can write

$$Z_t = X_t - U_t.$$

In Grenadier [6] the stochastic process (X_t) is assumed to be a Brownian motion (so (Y_t) is a geometric Brownian motion), and $D(x) = x^{-1/\gamma}$ for some $\gamma > 0$.

Let us first consider a model where D is a constant (thus independent of accumulated supply). This can be seen as the model above with $\gamma = \infty$. Without loss of generality we set $D = 1$. In this case, the value of the incumbent firm is

$$v_0(x) = E_x^Q \left[\int_0^\infty e^{-rs} Y_s ds \right] = E_x^Q \left[\int_0^\infty e^{-rs} e^{X_s} ds \right].$$

In the general case with a non-constant function D, the value of a producing firm is

$$v(x) = E_x^Q \left[\int_0^\infty e^{-rs} P_s ds \right] = E_x^Q \left[\int_0^\infty e^{-rs} e^{Z_s} ds \right].$$

For a firm not in the market, the cost of entering the market is $I > 0$. Firms will enter the market if it is profitable, and since there are infinitely many potential entrants, the value of an incumbent firm will always satisfy $v(x) \leq I$.

For any $b \in \mathbb{R}$ we set

$$T_b = \inf\{t \geq 0 \,|\, X_t \geq b\},$$

and to shorten the notation we introduce

$$L(x; b) = E_x^Q \left[e^{-rT_b} \right].$$

In Grenadier [6], [7] these type of values are calculated by solving differential equations, but we will use probabilistic methods. The first example of this technique is in the proof of the following proposition. See also Harrison [11].

Proposition 2.1 *With notation as above, assume that there exists a unique level b_0 such that*

$$v_0(b_0) - \frac{v_0'(b_0)}{L'(b_0; b_0)} = I. \tag{1}$$

Then the value $v(x)$ when (X_t) is starting at $x \leq b_0$ and is reflected in the upper level b_0 satisfies is given by

$$v(x) = v_0(x) - (v_0(b_0) - I)L(x; b_0), \tag{2}$$

and satisfies

$$v(b_0) = I. \tag{3}$$

The level $\bar{P} = e^{b_0}$ is the level at which firms outside the market will enter the market and the effect will be that the rent will never rise above the level $\bar{P}$. Here is the proof of the proposition.

Proof. We recall the following version of Dynkin's formula: For a strong time-homogenous Markov process X such that

$$E_x \left[\int_0^\infty e^{-rs} |f(X_s)| ds \right] < \infty$$

define

$$u(x) = E_x \left[\int_0^\infty e^{-rs} f(X_s) ds \right].$$

For any stopping time τ it holds that

$$u(x) = E_x\left[\int_0^\tau e^{-rs} f(X_s)ds\right] + E_x\left[e^{-r\tau}u(X_\tau)\mathbf{1}(\tau < \infty)\right] \qquad (4)$$

(see e.g Karlin & Taylor [14] p. 297 ff.). Using this version of Dynkin's formula with the stopping time T_b together with the facts that $X = Z$ on $[0, T_b]$ and $X_{T_b} = Z_{T_b} = 0$ on $\{T_b < \infty\}$ yields

$$\begin{aligned} v(x) &= v_0(x) + (v(b) - v_0(b))E_x^Q\left[e^{-rT_b}\right] \\ &= v_0(x) + (v(b) - v_0(b))L(x; b). \end{aligned}$$

With $b = b_0$ we get

$$v(x) = v_0(x) + (v(b_0) - v_0(b_0))L(x; b_0).$$

Differenting this and setting $x = b_0$ yields

$$v'(b_0) = v_0'(b_0) + (v(b_0) - v_0(b_0))L'(b_0; b_0),$$

and this relation leads to

$$v(b_0) - \frac{v'(b_0)}{L'(b_0; b_0)} = v_0(b_0) - \frac{v_0'(b_0)}{L'(b_0; b_0)} = I$$

Since (Z_t) is reflected at the level b_0 we have

$$v'(b_0) = 0,$$

from which it follows that

$$v(b_0) = I,$$

and from this

$$v(x) = v_0(x) - (v_0(b_0) - I)L(x; b_0).$$

□

The strength with this approach is that we only need $v_0(x)$ and $L(x; b)$ in order to determine the value v of the firm facing competition: Find b_0 by solving Equation (1) and then insert this in Equation (2) to get v.

Remark 2.2 *It follows from general diffusion theory that when*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

then we can write

$$L(x; b) = \frac{\psi_r(x)}{\psi_r(b)},$$

where ψ_r solves

$$\frac{\sigma^2(x)}{2}\psi_r''(x) + \mu(x)\psi_r'(r) = r\psi_r(x).$$

From this we get

$$L'(b_0; b_0) = \frac{\psi_r'(b_0)}{\psi_r(b_0)}.$$

Example 2.3 Let us look at the case considered in Grenadier [6], where it is derived using PDE's (see also Chapter 8 in Dixit & Pindyck.). In Grenadier [6] the cost is assumed to vary accoding to a geometric Brownian motion, but here we only consider the solution when the cost is constant (it is possible to extend the approach used here to the case with stochastic cost). Let

$$dX_t = (\mu - \sigma^2/2)dt + \sigma dW_t.$$

Then $Y_t = e^{X_t}$ satisfies

$$dY_t = \mu Y_t dt + \sigma Y_t dW_t.$$

In this case

$$v_0(x) = \frac{e^x}{r-\mu}$$

and

$$L(x; b_0) = e^{a(x-b_0)},$$

where

$$a = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1.$$

It follows that

$$L'(x; b_0) = ae^{a(x-b_0)} \quad \Rightarrow \quad L'(b_0; b_0) = a.$$

We want to find the rent level b_0 that satisfies Equation (1), which in this case can be written

$$\frac{e^{b_0}}{r-\mu} - \frac{\frac{e^{b_0}}{r-\mu}}{a} = I \quad \Rightarrow \quad b_0 = \ln\left(\frac{a(r-\mu)}{a-1} I\right).$$

Finally, using Equation (2), we get

$$v(x) = \frac{e^x}{r-\mu} - \frac{I}{a-1}\left(\frac{a-1}{a(r-\mu)I}\right)^a e^{ax}.$$

□

2.2 Markov-modulated models

We now describe the Markov-modulated model we will use. Let (J_t) be a continuous-time Markov chain with state space $\mathcal{J} = \{1, 2, \ldots, n\}$ and constant intensity matrix Π. Further let (W_t) be a Brownian motion independent of (J_t). The dynamics of the underlying stochastic process is given by

$$dX_t = \mu(X_t, J_t)dt + \sigma(X_t, J_t)dW_t; \; X_0 = x \text{ and } J_0 = j.$$

Given that the functions $\mu(x, j)$ and $\sigma(x, j)$ satisfy some growth and continuity conditions, the two-dimensional process (X_t, J_t) is a strong time-homogeneous Markov process (see Chapter 2 in Yin & Zhu [18] for details). The generator A

of (X_t, J_t) acting on a function $f : \mathbb{R} \times \mathcal{J} \to \mathbb{R}$ such that $f(\cdot, j) \in C^2$ for every $j \in \mathcal{J}$ is given by

$$Af(x,j) = \mu(x,j)\frac{df(x,j)}{dx} + \frac{1}{2}\sigma^2(x,j)\frac{d^2 f(x,j)}{dx^2} + [\Pi f](x,j),$$

where

$$[\Pi f](x,j) = \sum_{i=1}^{n} \Pi_{ji} f(x,i)$$

(again, see Chapter 2 in Yin & Zhu [18]).

We assume that the dynamics of the Markov chain (J_t) is the same under **P** and **Q**, i.e. the intensity matrix is the same under **P** and **Q**, and that the measure change will change the dynamics of (X_t) according to

$$dX_t = (\mu(X_t, J_t) - \lambda(X_t, J_t)\sigma(X_t, J_t))dt + \sigma(X_t, J_t)dW_t^Q,$$

where W^Q is a **Q**-Brownian motion and the Girsanov kernel $\lambda : \mathbb{R} \times \mathcal{J} \to \mathbb{R}$ represents the market price of risk with respect to the risk in the Wiener process.

2.3 A Markov-modulated Brownian motion model

The specific model we use is

$$dX_t = \mu^P dt + \sigma dW_t,$$

where $\mu^P \in \mathbb{R}$ and $\sigma > 0$ are two constants; i.e. (X_t) is a Brownian motion with drift under **P**. We further use a market price of risk λ that only depends on J_t:

$$dX_t = (\mu^P - \lambda(J_t)\sigma)dt + \sigma dW_t^Q =: \mu(J_t)dt + \sigma dW_t^Q. \tag{5}$$

This means that the market price of risk is constant in each state j, and does not depend on any other quantity than the state. The 'geometric' version of (X_t) is $Y_t = e^{X_t}$, with explicit expression

$$Y_t = Y_0 e^{\int_0^t \left(\mu(J_s) - \frac{\sigma^2}{2}\right)ds + \sigma W_t^Q}.$$

This represents the cash flows generated by an investment. We define the stochastic process

$$V_0(t) = E_{x,j}^Q \left[\int_t^\infty e^{-r(s-t)} Y_s ds \middle| \mathcal{F}_t \right],$$

representing the value at time $t \geq 0$ of the stream of cash flows (Y_t), and the function

$$v_0(x,j) = E_{x,j}^Q \left[\int_0^\infty e^{-rs} Y_s ds \right].$$

Here

$$E_{x,j}^Q [\cdot] = E_{x,j}^Q [\cdot | X_0 = x,\ J_0 = j].$$

Time-homogeneity and the Markov property implies that

$$V_0(t) = v_0(X_t, J_t).$$

The function v_0 is in this case given by

$$\begin{aligned}
v_0(x,j) &= \int_0^\infty e^{-rs} E^Q_{x,j}\left[Y_s\right] ds \\
&= \int_0^\infty e^{-rs} E^Q_{x,j}\left[e^{x+\int_0^s \left(\mu(J_u)-\frac{\sigma^2}{2}\right)du+\sigma W^Q_s}\right] ds \\
&= e^x \int_0^\infty e^{-rs} E^Q_{x,j}\left[e^{\int_0^s \mu(J_u)du}\right] ds \\
&= e^x E^Q_{x,j}\left[\int_0^\infty e^{-rs} e^{\int_0^s \mu(J_u)du} ds\right] \\
&= e^x \left[\left(rI - \Pi - D(\mu)\right)^{-1}\mathbf{1}\right]_j \\
&= e^x h(j),
\end{aligned}$$

where

$$D(\mu) = \mathrm{diag}(\mu(1), \ldots, \mu(n))$$

and

$$h(j) = \left[\left(rI - \Pi - D(\mu)\right)^{-1}\mathbf{1}\right]_j.$$

Remark 2.4 *The same formula will hold if we replace the constant σ with a function $\sigma(t, J_t)$ if the function $\sigma(\cdot,\cdot)$ is nice enough and under the assumption that (J_t) and (X_t) are independent.*

To calculate $(rI - \Pi - D(\mu))^{-1}$ we can use the fact that for a matrix A such that $(sI - A)^{-1}$ is well defined we have

$$(sI - A)^{-1} = \frac{N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_n}{s^n + a_1 s^{n-1} + \cdots + a_n} \tag{6}$$

(see Hou [12] for a discussion and a simple proof of this result). The denominator is the characteristic polynomial of A evaluated at s, and the matrices as well as the constants can be determined by the recursions

$$\begin{array}{ll}
N_1 = I & a_1 = -\mathrm{tr}A \\
N_2 = A + a_1 I & a_2 = -\frac{1}{2}\mathrm{tr}AN_2 \\
\vdots & \vdots \\
N_n = AN_{n-1} + a_{n-1}I & a_n = -\frac{1}{n}\mathrm{tr}AN_n.
\end{array}$$

Example 2.5 Let us consider the function $v_0(x,j) = e^x h(j)$ when $n = 2$. With

$$\Pi = \left[\begin{array}{cc} -\nu_1 & \nu_1 \\ \nu_2 & -\nu_2 \end{array}\right]$$

we let

$$A = \Pi + D(\mu) = \begin{bmatrix} \mu_1 - \nu_1 & \nu_1 \\ \nu_2 & \mu_2 - \nu_2 \end{bmatrix}.$$

Introducing

$$\begin{aligned} N_1 &= I & a_1 &= \nu_1 - \mu_1 + \nu_2 - \mu_2 \\ N_2 &= \begin{bmatrix} \nu_2 - \mu_2 & \nu_1 \\ \nu_2 & \nu_1 - \mu_1 \end{bmatrix} & a_2 &= \mu_1\mu_2 - \mu_1\nu_2 - \mu_2\nu_1, \end{aligned}$$

$(rI - A)^{-1}$ can be calculated using Equation (6), and this yields

$$\begin{bmatrix} h(1) \\ h(2) \end{bmatrix} = (rI - \Pi - D(\mu))^{-1}\,\mathbf{1} = \frac{1}{r^2 + a_1 r + a_2} \begin{bmatrix} r + \nu_1 + \nu_2 - \mu_2 \\ r + \nu_1 + \nu_2 - \mu_1 \end{bmatrix} =$$

$$\frac{1}{r^2 + (\nu_1 - \mu_1 + \nu_2 - \mu_2)r + \mu_1\mu_2 - \mu_1\nu_2 - \mu_2\nu_1} \begin{bmatrix} r + \nu_1 + \nu_2 - \mu_2 \\ r + \nu_1 + \nu_2 - \mu_1 \end{bmatrix}.$$

Straightforward calculations yields

$$h(1) = \frac{1}{r - \mu_1 + \frac{\nu_1(\mu_1 - \mu_2)}{r + \nu_1 + \nu_2 - \mu_2}}$$

and

$$h(2) = \frac{1}{r - \mu_2 + \frac{\nu_2(\mu_2 - \mu_1)}{r + \nu_1 + \nu_2 - \mu_1}}$$

respectively. With

$$\tilde{\mu}_1 = \mu_1 + \frac{\nu_1(\mu_2 - \mu_1)}{r + \nu_1 + \nu_2 - \mu_2} = \mu_1 \cdot \alpha_1 + \mu_2(1 - \alpha_1),$$

where

$$\alpha_1 = \frac{r + \nu_2 - \mu_2}{r + \nu_1 + \nu_2 - \mu_2} \in (0, 1],$$

we can write

$$h(1) = \frac{1}{r - \tilde{\mu}_1} = \frac{1}{(r - \mu_1)\alpha_1 + (r - \mu_2) \cdot (1 - \alpha_1)},$$

and equivalently for $h(2)$. Hence, the state dependent discount factors $h(i) = 1/(r - \tilde{\mu}_i)$ are weighted harmonic means of the discount factors $1/(r - \mu_1)$ and $1/(r - \mu_2)$. □

Now consider the case of a firm which operates in an environment where there is a possibility of other firms to enter the market. The level at which entry happens is dependent of the underlying state $j = 1, \ldots, n$. For each $j = 1, \ldots, n$ we let $b(j)$ denote the level at which entry occurs if the state is j.[1] The states are ordered in the way so that

$$b(1) \le b(2) \le \ldots \le b(n).$$

[1] The case $n = 1$ was considered above; there $b_0 = b(1)$.

The stochastic process (Z_t) regulated at the state-dependent barrier $b(J_t)$ represents the cash flows to a firm acting in a market where there is entry of competing firms when the price level reaches $b(J_t)$.

The value of a firm in the market is given by

$$V(t) = E^Q_{x,j}\left[\int_t^\infty e^{-r(s-t)} P_s ds \middle| \mathcal{F}_t\right].$$

Introducing the function

$$v(x,j) = E^Q_{x,j}\left[\int_0^\infty e^{-rs} P_s ds\right]$$

we have,(again using the strong Markov property and time-homogeneity – see Harrison [11] for details)

$$V(t) = u(Z_t, J_t).$$

Now let (X_t) be the Markov-modulated process defined in Equation (5), and define the cash flows generated by a firm when there are no potential competitors by

$$Y_t = e^{X_t}.$$

Futhermore let Z denote the regulated version of X, and let P denote the cash flows for an incumbent firm when it faces the possibility of market entry from competitors:

$$P_t = e^{Z_t}.$$

Generalizing the version of Dynkin's formula given in Equation (4) yields that for any stopping time τ it holds that

$$v_0(x,j) = E^Q_{x,j}\left[\int_0^\tau e^{-rs} e^{X_s} ds\right] + E^Q_{x,j}\left[e^{-r\tau} v_0(X_\tau, J_\tau)\mathbf{1}(\tau < \infty)\right]$$

and

$$v(x,j) = E^Q_{x,j}\left[\int_0^\tau e^{-rs} e^{Z_s} ds\right] + E^Q_{x,j}\left[e^{-r\tau} v(Z_\tau, J_\tau)\mathbf{1}(\tau < \infty)\right].$$

Now let, with a slight abuse of previous notation,

$$T_b = \inf\{t \geq 0 | X_t \geq b(J_t)\} = \inf\{t \geq 0 | Z_t = b(J_t)\}.$$

Since $X = Z$ on $[0, T_b)$ we get

$$v(x,j) = v_0(x,j) + E^Q_{x,j}\left[e^{-rT_b} v(Z_{T_b}, J_{T_b})\mathbf{1}(T_b < \infty)\right] - E^Q_{x,j}\left[e^{-rT_b} v_0(X_{T_b}, J_{T_b})\mathbf{1}(T_b < \infty)\right].$$

From $P_t = e^{Z_t}$ we get

$$0 \leq P_t \leq e^{\max_j b(j)},$$

so

$$0 \leq v(x,j) \leq \frac{e^{\max_j b(j)}}{r},$$

from which it follows that

$$e^{-rT_b} v(Z_{T_b}, J_{T_b}) = 0 \text{ on } \{T_b = \infty\}.$$

We further assume that X is such that

$$e^{-rt} v_0(X_t, J_t) \to 0 \text{ as } t \to \infty.$$

(It follows from Equation (6) that a sufficient condition for this is that $\mu(j) < r$ for every $j = 1, \dots, n$.) Hence, we can write

$$v(x, j) = v_0(x, j) + E^Q_{x,j}\left[e^{-rT_b} v(Z_{T_b}, J_{T_b})\right] - E^Q_{x,j}\left[e^{-rT_b} v_0(X_{T_b}, J_{T_b})\right].$$

The expected values can be written

$$\begin{aligned} E^Q_{x,j}\left[e^{-rT_b} v(Z_{T_b}, J_{T_b})\right] &= \sum_{i=1}^{n} E^Q_{x,j}\left[e^{-rT_b} v(Z_{T_b}, J_{T_b}) \mathbf{1}(J_{T_b} = i)\right] \\ &= \sum_{i=1}^{n} v(b(i), i) E^Q_{x,j}\left[e^{-rT_b} \mathbf{1}(J_{T_b} = i)\right] \end{aligned}$$

and

$$\begin{aligned} E^Q_{x,j}\left[e^{-rT_b} v_0(X_{T_b}, J_{T_b})\right] &= \sum_{i=1}^{n} E^Q_{x,j}\left[e^{-rT_b} v_0(X_{T_b}, J_{T_b}) \mathbf{1}(J_{T_b} = i)\right] \\ &= \sum_{i=1}^{n} E^Q_{x,j}\left[e^{-rT_b} v_0(X_{T_b}, i) \mathbf{1}(J_{T_b} = i)\right] \end{aligned}$$

respectively. We know that when X is modelled according to Equation (5), then

$$v_0(x, j) = e^x h(j),$$

so

$$E^Q_{x,j}\left[e^{-rT_b} v_0(X_{T_b}, J_{T_b})\right] = \sum_{i=1}^{n} h(i) E^Q_{x,j}\left[e^{-rT_b} e^{X_{T_b}} \mathbf{1}(J_{T_b} = i)\right]$$

in this case. Letting

$$\begin{aligned} L_i(x, j) &= E^Q_{x,j}\left[e^{-rT_b} \mathbf{1}(J_{T_b} = i)\right] \\ H_i(x, j) &= E^Q_{x,j}\left[e^{-rT_b} e^{X_{T_b}} \mathbf{1}(J_{T_b} = i)\right] \end{aligned}$$

we can write

$$v(x, j) = e^x h(j) + \sum_{i=1}^{n} v(b(i), i) L_i(x, j) - \sum_{i=1}^{n} h(i) H_i(x, j).$$

We have the boundary conditions

$$v(b(j), j) = I_j \text{ and } v'(b(j), j) = 0 \text{ for } j = 1, \dots, n.$$

It follows from the first set of boundary conditions that

$$v(x,j) = e^x h(j) + \sum_{i=1}^{n} I_i L_i(x,j) - \sum_{i=1}^{n} h(i) H_i(x,j) \text{ for } j = 1, \dots, n.$$

This, in turn, leads to, using the second set of boundary conditions,

$$0 = e^{b(j)} h(j) + \sum_{i=1}^{n} I_i L_i'(b(j),j) - \sum_{i=1}^{n} h(i) H_i'(b(j),j) \text{ for } j = 1, \dots, n. \quad (7)$$

In order to be able to find the value function $v(x,j)$, we need to find the levels $b(1), \dots, b(n)$, and the functions $L_1(x,j), \dots L_n(x,j)$ and $H_1(x,j), \dots, H_n(x,j)$. The idea is to find general expressions for L_i and H_i as functions of the levels $b(1), \dots, b(n)$, and then use the n boundary conditions (7) to find the levels.

Later on, we will consider the model under the following assumptions.

Assumption 2.6

- The number of states is two: $n = 2$.
- The cost of the investment is the same in both states: $I_1 = I_2 = I$.

Under these assumptions,

$$\begin{aligned} v(x,1) &= e^x h(1) + IL(x,1) - h(1)H_1(x,1) - h(2)H_2(x,1) \\ v(x,2) &= e^x h(2) + IL(x,2) - h(1)H_1(x,2) - h(2)H_2(x,2), \end{aligned}$$

where for $j = 1, 2$

$$L(x,j) = L_1(x,j) + L_2(x,j) = E^Q_{x,j}\left[e^{-rT_b}\right].$$

The boundary conditions in Equation (7) simplifies to

$$\begin{aligned} 0 &= e^{b(1)} h(1) + IL'(b(1),1) - h(1)H_1'(b(1),1) - h(2)H_2'(b(1),1) \\ 0 &= e^{b(2)} h(2) + IL'(b(2),2) - h(1)H_1'(b(2),2) - h(2)H_2'(b(2),2), \end{aligned}$$

3 Solving some hitting problems

3.1 General theory

The following result will be used to find the functions L_i and H_i introduced above. The proof is a straightforward generalization of the proof of Proposition 2 in Jobert & Rogers [13].

Proposition 3.1 *Let $f = (f(\cdot,1),\ldots,f(\cdot,n))$ be a bounded solution to the system of ODE's*

$$\begin{aligned}\frac{\sigma^2(x,j)}{2}\frac{d^2f(x,j)}{dx^2}+\mu(x,j)\frac{df(x,j)}{dx}-r(j)f(x,j)+\sum_{k=1}^{n}\Pi_{jk}f(x,k) &= 0 \text{ when } x\le b(j)\\ f(x,j) &= \psi_j(x) \text{ when } x\ge b(j).\end{aligned}$$

Then

$$f(x,j)=E_{x,j}\left[e^{-\int_0^\tau r(J_u)du}\sum_{k=1}^{n}\psi_k(X_\tau)\mathbf{1}(J_\tau=k)\right], \tag{8}$$

where

$$dX_t=\mu(X_t,J_t)dt+\sigma(X_t,J_t)dW_t$$

with (W_t) being a Brownian motion, (J_t) is a continuous time Markov chain with generator $\Pi=(\Pi_{ij})$, $i,j=1,\ldots,n$ independent of (W_t) and

$$\tau=\inf\{t\ge 0|X(t)\ge b(J(t))\}.$$

Proof. Let $n\in\mathbb{Z}_+$, An application of Ito's formula yields

$$\begin{aligned}e^{-\int_0^{n\wedge\tau}r(J_u)du}f(X_{n\wedge\tau},J_{n\wedge\tau}) &= f(x,j)\\ &\quad+\int_0^{n\wedge\tau}\big(Af(X_u,J_u)-r(J_u)f(X_u,J_u)\big)du\\ &\quad+M_{n\wedge\tau}.\end{aligned}$$

Since f solves the systems of ODE's above, $Af(X_u,J_u)=r(J_u)f(X_u,J_u)$ on $[0,n\wedge\tau]$, so

$$e^{-\int_0^{n\wedge\tau}r(J_u)du}f(X_{n\wedge\tau},J_{n\wedge\tau})=f(x,j)+M_{n\wedge\tau}.$$

Taking $E_{x,j}[\cdots]$ of this equation, letting $n\to\infty$ and using bounded convergence results in Equation (8). □

3.2 Two states with a Brownian motion with drift

We now consider Proposition 3.1 when $n=2$ and

$$r(1)=r(2)=r>0.$$

We also let

$$\Pi=\begin{bmatrix}-\nu_1 & \nu_1\\ \nu_2 & -\nu_2\end{bmatrix},$$

and assume that X is a Brownian motion with drift, i.e.

$$\mu(x,j)=\mu(j) \text{ and } \sigma(x,j)=\sigma(j) \text{ for } j=1,2.$$

The same technique we use below has been used in e.g. Guo [8]. We have to consider the three intervals $(-\infty,b(1)]$, $[b(1),b(2)]$ and $[b(2),\infty)$.

3.2.1 When $x \in [b(2), \infty)$

On this interval

$$f(x,j) = \psi_j(x)$$

for $j = 1, 2$.

3.2.2 When $x \in [b(1), b(2)]$

Now

$$f(x,1) = \psi_1(x)$$

and

$$\frac{1}{2}\sigma^2(2)f''(x,2) + \mu(2)f'(x,2) - rf(x,2) + \nu_2\psi_1(x) - \nu_2 f(x,2) = 0.$$

The solution to this ODE is

$$f(x,2) = A_1 e^{\gamma_1 x} + A_2 e^{\gamma_2 x} + g(x),$$

where g is the particular solution, $\gamma_1 < 0 < \gamma_2$ are solutions to the quadratic equation

$$\frac{1}{2}\sigma^2(2)\gamma^2 + \mu(2)\gamma - r - \nu_2 = 0$$

and $A_1, A_2 \in \mathbb{R}$.

3.2.3 When $x \in (-\infty, b(1)]$

In this case

$$\begin{aligned}
\frac{1}{2}\sigma^2(1)f''(x,1) + \mu(1)f'(x,1) - rf(x,1) - \nu_1 f(x,1) + \nu_1 f(x,2) &= 0 \\
\frac{1}{2}\sigma^2(2)f''(x,2) + \mu(2)f'(x,2) - rf(x,2) + \nu_2 f(x,1) - \nu_2 f(x,2) &= 0.
\end{aligned}$$

It is known, see e.g. Remark 2.1 in Guo [8], that if the interest rate, the intensities and the volatility are all strictly positive, then there exists constants $\beta_1 < \beta_2 < 0 < \beta_3 < \beta_4$ solving the quadratic equation

$$\left(\frac{1}{2}\sigma(1)^2\beta^2 + \mu(1)\beta - (r + \nu_1)\right)\left(\frac{1}{2}\sigma(2)^2\beta^2 + \mu(2)\beta - (r + \nu_2)\right) = \nu_1\nu_2,$$

and such that the general solution to the system of ODE's can be written

$$f(x,j) = \sum_{k=1}^{4} B_{jk} e^{\beta_k x}$$

for $B_{jk} \in \mathbb{R}$. In our cases, for $j = 1, 2$ the functions $f(\cdot, j)$ must be bounded as $x \to -\infty$, so

$$B_{j1} = B_{j2} = 0$$

for every $j = 1, 2$, which leads to

$$f(x, j) = B_{j3}e^{\beta_3 x} + B_{j4}e^{\beta_4 x}.$$

Furthermore, we always have the relation

$$B_{2k} = \ell_k B_{1k}$$

for known constants ℓ_k, $k = 1, \ldots, 4$. We are only interested in the values ℓ_3 and ℓ_4:

$$\ell_3 = -\frac{\sigma(1)^2\beta_3^2/2 + \mu(1)\beta_3 - (r + \nu_1)}{\nu_1}$$

and

$$\ell_4 = -\frac{\sigma(1)^2\beta_4^2/2 + \mu(1)\beta_4 - (r + \nu_1)}{\nu_1}.$$

(Hence, we can write

$$\begin{aligned} f(x, 1) &= B_{13}e^{\beta_3 x} + B_{14}e^{\beta_4 x} \\ f(x, 2) &= \ell_3 B_{13}e^{\beta_3 x} + \ell_4 B_{14}e^{\beta_4 x}. \end{aligned}$$

3.2.4 The complete solution

To determine the constants A_1, A_2, B_{13} and B_{14} we use continuity of $f(\cdot, 2)$ at $b(2)$:

$$A_1 e^{\gamma_1 b(2)} + A_2 e^{\gamma_2 b(2)} + g(b(2)) = \psi_2(b(2)),$$

continuity of $f(\cdot, j)$, $j = 1, 2$, at $b(1)$:

$$B_{13}e^{\beta_3 b(1)} + B_{14}e^{\beta_4 b(1)} = \psi_1(b(1))$$

and

$$\ell_3 B_{13}e^{\beta_3 b(1)} + \ell_4 B_{14}e^{\beta_4 b(1)} = A_1 e^{\gamma_1 b(1)} + A_2 e^{\gamma_2 b(1)} + g(b(1)),$$

and finally smoothness at $b(1)$ for $f(\cdot, 2)$:

$$\ell_3 B_{13}\beta_3 e^{\beta_3 b(2)} + \ell_4 B_{14}\beta_4 e^{\beta_4 b(2)} = A_1\gamma_1 e^{\gamma_1 b(1)} + A_2\gamma_2 e^{\gamma_2 b(1)} + g'(b(1)).$$

Summarizing these relations we get the following system of equations:

$$\begin{aligned} A_1 e^{\gamma_1 b(2)} + A_2 e^{\gamma_2 b(2)} + g(b(2)) &= \psi_2(b(2)) \\ B_{13}e^{\beta_3 b(1)} + B_{14}e^{\beta_4 b(1)} &= \psi_1(b(1)) \\ B_{13}\ell_3 e^{\beta_3 b(1)} + B_{14}\ell_4 e^{\beta_4 b(1)} &= A_1 e^{\gamma_1 b(1)} + A_2 e^{\gamma_2 b(1)} + g(b(1)) \\ B_{13}\ell_3\beta_3 e^{\beta_3 b(1)} + B_{14}\ell_4\beta_4 e^{\beta_4 b(1)} &= A_1\gamma_1 e^{\gamma_1 b(1)} + A_2\gamma_2 e^{\gamma_2 b(1)} + g'(b(1)) \end{aligned}$$

For given $b(1)$ and $b(2)$, this is a linear system of equations in A_1, A_2, B_{13} and B_{14}:

$$\begin{bmatrix} e^{\gamma_1 b(2)} & e^{\gamma_2 b(2)} & 0 & 0 \\ 0 & 0 & e^{\beta_3 b(1)} & e^{\beta_4 b(1)} \\ -e^{\gamma_1 b(1)} & -e^{\gamma_2 b(1)} & \ell_3 e^{\beta_3 b(1)} & \ell_4 e^{\beta_4 b(1)} \\ -\lambda_3 e^{\gamma_1 b(1)} & -\gamma_2 e^{\gamma_2 b(1)} & \ell_3\beta_3 e^{\beta_3 b(1)} & \ell_4\beta_4 e^{\beta_4 b(1)} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ B_{13} \\ B_{14} \end{bmatrix} = \begin{bmatrix} \psi_2(b(2)) - g(b(2)) \\ \psi_1(b(1)) \\ g(b(1)) \\ g'(b(1)) \end{bmatrix}$$

Remark 3.2 *An important special case is when*

$$\psi_j(x) = \varphi(x)\delta_{ij} \text{ for } i,j = 1,2.$$

With

$$f_i(x,j) = E_{x,j}\left[e^{-r\tau}\varphi(X_\tau)\mathbf{1}(J_\tau = i)\right],$$

we get the following. When $i = 1$*:*

$$\begin{array}{lll} I & f_1(x,1) = \varphi(x) & f_1(x,2) = 0 \\ II & f_1(x,1) = \varphi(x) & f_1(x,2) = A_1 e^{\gamma_1 x} + A_2 e^{\gamma_2 x} + g_1(x) \\ III & f_1(x,1) = B_{13}e^{\beta_3 x} + B_{14}e^{\beta_4 x} & f_1(x,2) = B_{13}\ell_1 e^{\beta_3 x} + B_{14}\ell_2 e^{\beta_4 x}. \end{array}$$

When $i = 2$*:*

$$\begin{array}{lll} I & f_2(x,1) = 0 & f_2(x,2) = \varphi(x) \\ II & f_2(x,1) = 0 & f_2(x,2) = C_1 e^{\gamma_1 x} + C_2 e^{\gamma_2 x} + g_2(x) \\ III & f_2(x,1) = D_{13}e^{\beta_3 x} + D_{14}e^{\beta_4 x} & f_2(x,2) = D_{13}\ell_1 e^{\beta_3 x} + D_{14}\ell_2 e^{\beta_4 x}. \end{array}$$

In order to calculate the optimal boundary values $b(1)$ *and* $b(2)$ *we need the following derivatives.*

When $i = 1$*:*

$$\begin{array}{lll} I & & \\ II & & f_1'(x,2) = A_1\gamma_1 e^{\gamma_1 x} + A_2\gamma_2 e^{\gamma_2 x} + g_1'(x) \\ III & f_1'(x,1) = B_{13}\beta_3 e^{\beta_3 x} + B_{14}\beta_4 e^{\beta_4 x} & \end{array}$$

When $i = 2$*:*

$$\begin{array}{lll} I & & \\ II & & f_2'(x,2) = C_1\gamma_1 e^{\gamma_1 x} + C_2\gamma_2 e^{\gamma_2 x} + g_2'(x) \\ III & f_2'(x,1) = D_{13}\beta_3 e^{\beta_3 x} + D_{14}\beta_4 e^{\beta_4 x} & \end{array}$$

4 The value of the investment

4.1 The solution to the investment problem

We now want to find the value of an existing firm that faces the possibility of other firms entering the market. We have

$$r(1) = r(2) = r,\ \sigma(x,1) = \sigma(x,2) = \sigma > 0 \text{ and } \mu(x,j) = \mu^P - \lambda(j)\sigma,$$

and need to find the three functions L, H_1 and H_2 and the two constants $b(1)$ and $b(2)$. The two cases we have to consider are

- $\psi_j(x) = 1$ for $j = 1,2$.
- $\psi_j(x) = e^x\delta_{ij}$ for $i,j = 1,2$.

We use the following parameter names:

Function $\varphi(x)$	Parameters for $i=1$ when $x \in [b(1), b(2)]$	Parameters for $i=2$ when $x \in [b(2), \infty)$
1	A_1, A_2, B_{13}, B_{14}	A_1, A_2, B_{13}, B_{14}
e^x	$\hat{A}_1$, $\hat{A}_2$, $\hat{B}_{13}$, $\hat{B}_{14}$	$\hat{C}_1$, $\hat{C}_2$, $\hat{D}_{13}$, $\hat{D}_{14}$

We introduce the three parts I to III of $\mathbb{R}$ according to

$$\begin{array}{cl} \text{I} & x \in [b(2), \infty) \\ \text{II} & x \in [b(1), b(2)] \\ \text{III} & x \in (-\infty, b(1)] \end{array}$$

To solve for the unknown parameters we go through the following four steps:

1) The particular solution when $\psi_1(x) = \psi_2(x) = 1$ is

$$g(x) = \frac{\nu_2}{r + \nu_2}.$$

Hence, when $x \in [b(1), b(2)]$ we have

$$L(x, 2) = A_1 e^{\gamma_1 x} + A_2 e^{\gamma_2 x} + \frac{\nu_2}{r + \nu_2}.$$

This gives

$$\begin{array}{lll} \text{I} & L(x,1) = 1 & L(x,2) = 1 \\ \text{II} & L(x,1) = 1 & L(x,2) = A_1 e^{\gamma_1 x} + A_2 e^{\gamma_2 x} + \frac{\nu_2}{r+\nu_2} \\ \text{III} & L(x,1) = B_{13} e^{\beta_3 x} + B_{14} e^{\beta_4 x} & L(x,2) = B_{13} \ell_3 e^{\beta_3 x} + B_{14} \ell_4 e^{\beta_4 x}. \end{array}$$

2) The particular solutions when $\varphi(x) = e^x$ are

$$g_1(x) = \frac{\nu_2 e^x}{\frac{\sigma^2}{2} + \mu(2) - r - \nu_2} \quad \text{and} \quad g_2(x) = 0.$$

respectively. This gives

$$H_1(x, 2) = \hat{A}_1 e^{\gamma_1 x} + \hat{A}_2 e^{\gamma_2 x} + \frac{\nu_2 e^x}{\frac{\sigma^2}{2} + \mu(2) - r - \nu_2}$$

and

$$H_2(x, 2) = \hat{C}_1 e^{\gamma_1 x} + \hat{C}_2 e^{\gamma_2 x}.$$

When $i = 1$, the solution is

$$\begin{array}{lll}
\text{I} & H_1(x,1) = e^x & H_1(x,2) = 0 \\
\text{II} & H_1(x,1) = e^x & H_1(x,2) = \hat{A}_1 e^{\gamma_1 x} + \hat{A}_2 e^{\gamma_2 x} + \frac{\nu_2 e^x}{\frac{\sigma^2}{2} + \mu(2) - r - \nu_2} \\
\text{III} & H_1(x,1) = \hat{B}_{13} e^{\beta_3 x} + \hat{B}_{14} e^{\beta_4 x} & H_1(x,2) = \hat{B}_{13} \ell_3 e^{\beta_3 x} + \hat{B}_{14} \ell_4 e^{\beta_4 x},
\end{array}$$

and when $i = 2$, we get the solution

$$\begin{array}{lll}
\text{I} & H_2(x,1) = 0 & H_2(x,2) = e^x \\
\text{II} & H_2(x,1) = 0 & H_2(x,2) = \hat{C}_1 e^{\gamma_1 x} + \hat{C}_2 e^{\gamma_2 x} \\
\text{III} & H_2(x,1) = \hat{D}_{13} e^{\beta_3 x} + \hat{D}_{14} e^{\beta_4 x} & H_2(x,2) = \hat{D}_{13} \ell_3 e^{\beta_3 x} + \hat{D}_{14} \ell_4 e^{\beta_4 x}.
\end{array}$$

3) The two previous steps leads to the following system of equations:

$$\left\{\begin{array}{rcl}
A_1 e^{\gamma_1 b(2)} + A_2 e^{\gamma_2 b(2)} + \frac{\nu_2}{r+\nu_2} & = & 1 \\
B_{13} e^{\beta_3 b(1)} + B_{14} e^{\beta_4 b(1)} & = & 1 \\
B_{13} \ell_3 e^{\beta_3 b(1)} + B_{14} \ell_4 e^{\beta_4 b(1)} & = & A_1 e^{\gamma_1 b(1)} + A_2 e^{\gamma_2 b(1)} + \frac{\nu_2}{r+\nu_2} \\
B_{13} \ell_3 \beta_3 e^{\beta_3 b(1)} + B_{14} \ell_4 \beta_4 e^{\beta_4 b(1)} & = & A_1 \gamma_1 e^{\gamma_1 b(1)} + A_2 \gamma_2 e^{\gamma_2 b(1)} \\
\hat{A}_1 e^{\gamma_1 b(2)} + \hat{A}_2 e^{\gamma_2 b(2)} + \frac{\nu_2 e^{b(2)}}{\sigma^2/2 + \mu(2) - (r+\nu_2)} & = & 0 \\
\hat{B}_{13} e^{\beta_3 b(1)} + \hat{B}_{14} e^{\beta_4 b(1)} & = & e^{b(1)} \\
\hat{B}_{13} \ell_3 e^{\beta_3 b(1)} + \hat{B}_{14} \ell_4 e^{\beta_4 b(1)} & = & \hat{A}_1 e^{\gamma_1 b(1)} + \hat{A}_2 e^{\gamma_2 b(1)} \\
 & & + \frac{\nu_2 e^{b(1)}}{\sigma^2/2 + \mu(2) - (r+\nu_2)} \\
\hat{B}_{13} \ell_3 \beta_3 e^{\beta_3 b(1)} + \hat{B}_{14} \ell_4 \beta_4 e^{\beta_4 b(1)} & = & \hat{A}_1 \gamma_1 e^{\gamma_1 b(1)} + \hat{A}_2 \gamma_2 e^{\gamma_2 b(1)} \\
 & & + \frac{\nu_2 e^{b(1)}}{\sigma^2/2 + \mu(2) - (r+\nu_2)} \\
\hat{C}_1 e^{\gamma_1 b(2)} + \hat{C}_2 e^{\gamma_2 b(2)} & = & e^{b(2)} \\
\hat{D}_{13} e^{\beta_3 b(1)} + \hat{D}_{14} e^{\beta_4 b(1)} & = & 0 \\
\hat{D}_{13} \ell_3 e^{\beta_3 b(1)} + \hat{D}_{14} \ell_4 e^{\beta_4 b(1)} & = & \hat{C}_1 e^{\gamma_1 b(1)} + \hat{C}_2 e^{\gamma_2 b(1)} \\
\hat{D}_{13} \ell_3 \beta_3 e^{\beta_3 b(1)} + \hat{D}_{14} \ell_4 \beta_4 e^{\beta_4 b(1)} & = & \hat{C}_1 \gamma_1 e^{\gamma_1 b(1)} + \hat{C}_2 \gamma_2 e^{\gamma_2 b(1)}
\end{array}\right.$$

4) We have the following two equations from the property of zero derivative at the hitting levels:

$$\begin{array}{rcl}
e^{b(1)} h(1) + IL'(b(1),1) - h(1) H_1'(b(1),1) - h(2) H_2'(b(1),1) & = & 0 \\
e^{b(2)} h(2) + IL'(b(2),2) - h(1) H_1'(b(2),2) - h(2) H_2'(b(2),2) & = & 0.
\end{array}$$

5) We now have a system of 14 equations and 14 unknowns, which, at least numerically, are possible to solve.

4.2 A numerical example

We now consider the model from the previous section with the following parameter values:

$$\begin{aligned}\mu^P &= 0.10\\ \sigma &= 0.30\\ r &= 0.050\\ \lambda(1) &= 1.0\\ \lambda(2) &= 1.2\\ \nu_1 &= 1.0\\ \nu_2 &= 0.50\\ I &= 100\end{aligned}$$

The levels $b(1)$ and $b(2)$ are in this case given by

$$\begin{aligned}b(1) &= 3.5889\\ b(2) &= 3.5969.\end{aligned}$$

In levels, we have

$$\begin{aligned}e^{b(1)} &= 36.1960\\ e^{b(2)} &= 36.4855.\end{aligned}$$

The values for the two states are given in Figures 4.2 and 4.2.

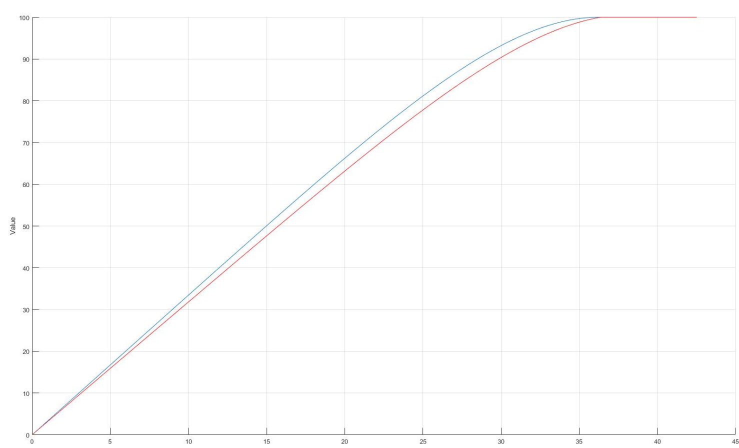

Figure 3: The value in the two regimes, with parameter values as given in this section. The blue curve is when $j = 1$, and the red curve when $j = 2$.

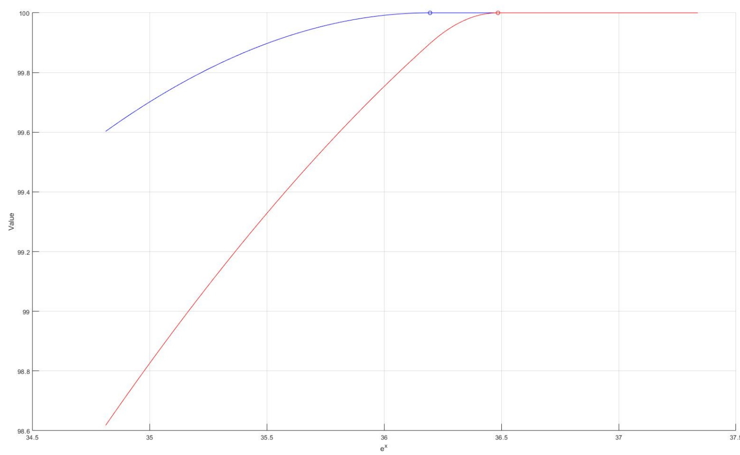

Figure 4: The values of the two regimes zoomed in on the values close to $e^{b(1)} = 36.1960$ and $e^{b(2)} = 36.4855$. Again, blue is for $j = 1$ and red for $j = 2$. The two circles show the points where the value functions reaches the cost $I = 100$ of investing.

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