

The Value of Investments when there is Time Varying Risk Premia

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February 1, 2019

Abstract

We consider a model where the risk premium is varying. The risk premium is driven by a continuous time Markov chain, representing the state in the economy, and the stochastic process generating the cash flows is a Markov-modulated geometric Brownian motion. An existing firm is facing the possibility of competitors entering the market, and due to this, cash flows are limited at levels which are dependent on the state of the economy. This results in a regulated Markov-modulated geometric Brownian motion, and the resulting accumulated supply can have jumps, something that is not possible in a model with only one regime.

Keywords: Valuation, competition, Markov-modulated Brownian motion, regulated processes.

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1 Introduction

In many valuation problems it is assumed that the underlying value follows a geometric Brownian motion. In this paper we revisit a problem considered in Chapter 8 in Dixit & Pindyck [3] and in Grenadier [6] regarding the value of an investment in the presence of competitors, generalizing it from the geometric Brownian motion model used in that paper, to a model where an observable Markov chain determines the state of the economy. These 'regime-switching' or 'Markov-modulated' models have been used to extend the irreversible investment problem of McDonald & Siegel [13]; see e.g. Driffill et al [4], Guo [7], Guo & Zhang [8] and Jobert & Rogers [11]. An early example of using regime-switching models is Hamilton [9].

The fact that we introduce competitors in our model means that the cash flows to the firm in the market will be a regulated Markov-modulated geometric Brownian motion. The geometric Brownian motion model with one upper barrier, i.e. where there is no switching, is solved in Bentolila & Bertola [1]. The mathematical problem of a regulated Markov-modulated Brownian motion with (in general state-dependent drift and diffusion) is studied in D'Auria & Kella [2].

These type of models are in general incomplete. This means that there exists more than one equivalent martingale, or pricing, measure. In order to choose which pricing measure to use, there are several principles we can use. In Elliott et al [5] Esscher transforms are used, and in Siu [14] a general martingale representation is the starting point. In both these approaches, the resulting measure is the minimal entropy martingale measure (MEMM). In Siu & Yang [15] an Esscher transformation technique which does not result in the MEMM is used. The cash flow process is not assumed to be the price of a traded asset, which means that we have two stochastic process (the cash flow process and the process marking the state of the economy), none of which is traded. We will assume that the dynamics of the process marking the state of the world is not changed, and change the drift of the cash flow process using a market price of risk which is not determined within the model.

2 The model

We consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$, where the filtration is assumed to satisfy the usual assumptions of right-continuity and \mathcal{F}_0 containing all null sets of \mathcal{F} .

A building is yielding a rent of P_t per time unit. The inverse demand function is given by

$$P_t = Y_t D(Q_t),$$

where $D(\cdot)$ is a decreasing function of accumulated supply Q and Y is a random shock. See Grenadier [6]. We use the model

$$Y_t = e^{X_t},$$

where X is a continuous strong Markov process to be defined below. It follows that

$$\ln P_t = \ln Y_t + \ln D(Q_t) = X_t + \ln D(Q_t).$$

Defining

$$Z_t = \ln P_t \quad \text{and} \quad U_t = -\ln D(Q_t)$$

we can write

$$Z_t = X_t - U_t.$$

In Grenadier [6] the stochastic process (X_t) is assumed to be a Brownian motion (so (Y_t) is a geometric Brownian motion), and $D(x) = x^{-1/\gamma}$ for some $\gamma > 0$.

Let us first consider a model where D is a constant; thus independent of accumulated supply. Without loss of generality we set $D = 1$. In this case the value of the firm is

$$v_0(x) = E_x^Q \left[\int_0^\infty e^{-rs} Y_s ds \right] = E_x^Q \left[\int_0^\infty e^{-rs} u(X_s) ds \right],$$

where $u(x) = e^x$. In the general case with a non-constant function D , the value of a producing firm is

$$v(x) = E_x^Q \left[\int_0^\infty e^{-rs} P_s ds \right] = E_x^Q \left[\int_0^\infty e^{-rs} u(Z_s) ds \right],$$

where again $u(x) = e^x$.

For any $b \in \mathbb{R}$ we set

$$T_b = \inf\{t \geq 0 \mid X_t = b\},$$

and to shorten the notation we introduce

$$L(x; b) = E_x [e^{-rT_b}].$$

Finally, for a firm not in the market, the cost of entering the market is $I > 0$.

Proposition 2.1 *With notation as above, assume that there exists a unique rent level \bar{P} such that*

$$v_0(\bar{P}) - \frac{v'_0(\bar{P})}{L'(\bar{P}; \bar{P})} = I. \quad (1)$$

Then the value $v(x)$ when (X_t) is starting at x and is reflected in the upper level \bar{P} satisfies

$$v(\bar{P}) = I. \quad (2)$$

The level \bar{P} in this proposition is the rent level at which firms outside the market will enter the market and the effect will be that the rent will never rise above the level \bar{P} . Here is the proof of the proposition.

Proof. We recall the following version of Dynkin's formula: For a strong time-homogenous Markov process X such that

$$E_x \left[\int_0^\infty e^{-rs} |f(X_s)| ds \right] < \infty$$

define

$$u(x) = E_x \left[\int_0^\infty e^{-rs} f(X_s) ds \right].$$

For any stopping time τ it holds that

$$u(x) = E_x \left[\int_0^\tau f(X_s) ds \right] + E_x [e^{-r\tau} u(X_\tau) \mathbf{1}(\tau < \infty)] \quad (3)$$

(see e.g Karlin & Taylor [12] p. 297 ff.). Using this version of Dynkin's formula with the stopping time T_b together with the facts that $X = Z$ on $[0, T_b]$ and $X_{T_b} = Z_{T_b} = 0$ on $\{T_b < \infty\}$ yields

$$v(x) = v_0(x) + (v(b) - v_0(b)) E_x [e^{-rT_b}] = v_0(x) + (v(b) - v_0(b)) L(x; b).$$

With $b = \bar{P}$ we get

$$v(x) = v_0(x) + (v(\bar{P}) - v_0(\bar{P})) L(x; \bar{P}).$$

Differentiating this and setting $x = \bar{P}$ yields

$$v'(\bar{P}) = v'_0(\bar{P}) + (v(\bar{P}) - v_0(\bar{P})) L'(\bar{P}; \bar{P}).$$

This relation leads to

$$v(\bar{P}) - \frac{v'(\bar{P})}{L'(\bar{P}; \bar{P})} = v_0(\bar{P}) - \frac{v'_0(\bar{P})}{L'(\bar{P}; \bar{P})} = I$$

and

$$v'(\bar{P}) = (v(\bar{P}) - I) L'(\bar{P}; \bar{P}).$$

Since (X_t) is reflected in \bar{P} we have

$$v'(\bar{P}) = 0,$$

from which it follows that

$$v(\bar{P}) = I.$$

□

Given that $v'(\bar{P}) = 0$ it holds that

$$v(\bar{P}) = I \Leftrightarrow v_0(\bar{P}) - \frac{v'_0(\bar{P})}{L'(\bar{P}; \bar{P})} = I,$$

where we have used that $L(\bar{P}, \bar{P}) = 1$.

The strength with this approach is that we only need $v_0(x)$ and $L(x; b)$ in order to determine the value v of the firm facing competition: Solve for \bar{P} and then insert this in the equation for v .

Remark 2.2 *It follows from general diffusion theory that when*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

then we can write

$$L(x; b) = \frac{\psi_r(x)}{\psi_r(b)},$$

where ψ_r solves

$$\sigma^2(x)\psi_r''(x) + \mu(x)\psi_r'(x) = r\psi_r(x).$$

From this we get

$$\frac{1}{L'(\bar{P}; \bar{P})} = \frac{\psi_r(\bar{P})}{\psi_r'(\bar{P})}.$$

As an example of the technique presented, let us look at the case considered in Grenadier [6], where it is derived using PDE's. In Grenadier [6] the cost is assumed to vary according to a geometric Brownian motion, but here we only consider the solution when the cost is constant. Let

$$dX_t = (\mu - \sigma^2/2)dt + \sigma dW_t.$$

Then $Y_t = e^{X_t}$ satisfies

$$dY_t = \mu Y_t dt + \sigma Y_t dW_t.$$

In this case

$$v_0(x) = \frac{e^x}{r - \mu}$$

and

$$L(x; \bar{P}) = \left(\frac{x}{\bar{P}}\right)^a,$$

where

$$a = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1.$$

It follows that

$$L'(x; \bar{P}) = \frac{a}{x} \left(\frac{x}{\bar{P}}\right)^a \Rightarrow L'(\bar{P}; \bar{P}) = \frac{a}{\bar{P}}.$$

We want to find the rent level \bar{P} that satisfies

$$\frac{\bar{P}}{r - \mu} - \frac{1}{\frac{r - \mu}{\bar{P}}} = I \Rightarrow \bar{P} = \frac{a(r - \mu)}{a - 1} I.$$

Finally we get

$$v(x) = \frac{e^x}{r - \mu} - \frac{I}{a - 1} \left(\frac{e^x(a - 1)}{a(r - \mu)I}\right)^a.$$

2.1 A Markov-modulated model

We now describe the Markov-modulated model we will use. Let (J_t) be a continuous-time Markov chain with state space $\mathcal{J} = \{1, 2, \dots, n\}$ and constant intensity matrix Π . Further let (W_t) be a Brownian motion independent of (J_t) . The dynamics of the underlying stochastic process is modelled as

$$dX_t = \mu(X_t, J_t)dt + \sigma(X_t, J_t)dW_t; \quad X_0 = x.$$

We note that the two-dimensional process (X_t, J_t) is a time-homogeneous Markov process (Yin & Zhu [16]). The generator A of (X_t, J_t) acting on a function $f : \mathbb{R} \times \mathcal{J} \rightarrow \mathbb{R}$ such that $f(\cdot, j) \in C^2$ for every $j \in \mathcal{J}$ is given by

$$Af(x, j) = \mu(x, j)\frac{df(x, j)}{dx} + \frac{1}{2}\sigma^2(x, j)\frac{d^2f(x, j)}{dx^2} + [\Pi f](x, j),$$

where

$$[\Pi f](x, j) = \sum_{i=1}^n \Pi_{ji}f(x, i)$$

(Yin & Zhu [16]).

The pricing measure, or martingale measure, \mathbf{Q} is the equivalent measure we use when valuing cash flows. The expected value under \mathbf{Q} is denoted $E^{\mathbf{Q}}$. We assume the existence of a bank account with constant interest rate $r > 0$, and we value cash flows by discounting them using r as discount rate and taking expectations under \mathbf{Q} . We assume that the dynamics of the Markov chain (J_t) is the same under \mathbf{P} and \mathbf{Q} , i.e. the intensity matrix is the same under \mathbf{P} and \mathbf{Q} . The measure change will change the dynamics of (X_t) according to

$$dX_t = (\mu(X_t, J_t) - \lambda(X_t, J_t)\sigma(X_t, J_t))dt + \sigma(X_t, J_t)dW_t^{\mathbf{Q}},$$

where $W_t^{\mathbf{Q}}$ is a \mathbf{Q} -Brownian motion.

The specific model we use is based on the Markov switching model

$$dX_t = \mu^P dt + \sigma dW_t,$$

where $\mu^P \in \mathbb{R}$ and $\sigma > 0$ are two constants; i.e. (X_t) is a Brownian motion with drift under \mathbf{P} . We use a market price of risk λ that only depends on J_t :

$$dX_t = (\mu^P - \lambda(J_t)\sigma)dt + \sigma dW_t^{\mathbf{Q}} =: \mu(J_t)dt + \sigma dW_t^{\mathbf{Q}}. \quad (4)$$

The ‘geometric’ version of (X_t) is $Y_t = e^{X_t}$, with explicit expression

$$Y_t = Y_0 e^{\int_0^t (\mu(J_s) - \frac{\sigma^2}{2}) ds + \sigma W_t}.$$

We define the stochastic process

$$V_0(t) = E_{x,j}^{\mathbf{Q}} \left[\int_t^\infty e^{-r(s-t)} Y_s ds \middle| \mathcal{F}_t \right]$$

and the function

$$v_0(x, j) = E_{x, j}^Q \left[\int_0^\infty e^{-rs} Y_s ds \right].$$

Time-homogeneity and the Markov property implies that

$$V_0(t) = v_0(X_t, J_t).$$

The function v_0 is in this case given by

$$\begin{aligned} v_0(x, j) &= \int_0^\infty e^{-rs} E_{x, j}^Q [Y_s] ds \\ &= \int_0^\infty e^{-rs} E_{x, j}^Q \left[e^{x + \int_0^s (\mu(J_u) - \frac{\sigma^2}{2}) du + \sigma W_s} \right] ds \\ &= e^x \int_0^\infty e^{-rs} E_{x, j}^Q \left[e^{\int_0^s \mu(J_u) du} \right] ds \\ &= e^x E_{x, j}^Q \left[\int_0^\infty e^{-rs} e^{\int_0^s \mu(J_u) du} ds \right] \\ &= e^x \left[(rI - \Pi - D(\mu))^{-1} \mathbf{1} \right]_j \\ &= e^x h(j), \end{aligned}$$

where

$$D(\mu) = \text{diag}(\mu(1), \dots, \mu(n))$$

and

$$h(j) = \left[(rI - \Pi - D(\mu))^{-1} \mathbf{1} \right]_j.$$

Remark 2.3 *The same formula will hold if we replace the constant σ with a function $\sigma(t, J_t)$ if the function $\sigma(\cdot, \cdot)$ is nice enough and under the assumption that (J_t) and (X_t) are independent.*

To calculate $(rI - \Pi - D(\mu))^{-1}$ we use the fact that for a matrix A such that $(sI - A)^{-1}$ is well defined we have

$$(sI - A)^{-1} = \frac{N_1 s^{n-1} + N_2 s^{n-2} + \dots + N_n}{s^n + a_1 s^{n-1} + \dots + a_n} \quad (5)$$

(see Hou [10] for a discussion and a simple proof of this result). The denominator is the characteristic polynomial of A evaluated at s , and the matrices as well as the constants can be determined by the recursions

$$\begin{array}{ll} N_1 = I & a_1 = -\text{tr}A \\ N_2 = A + a_1 I & a_2 = -\frac{1}{2} \text{tr}AN_2 \\ \vdots & \vdots \\ N_n = AN_{n-1} + a_{n-1} I & a_n = -\frac{1}{n} \text{tr}AN_n. \end{array}$$

Now consider the case of a firm which operates in an environment where there is a possibility of other firms to enter the market. The level at which

entry happens is dependent of the underlying state. For each $j = 1, \dots, n$ we let $b(j)$ denote the price level at which entry occurs if the state is j .¹ The states are ordered in the way so that

$$b(1) \leq b(2) \leq \dots \leq b(n).$$

The stochastic process (Z_t) regulated at the state-dependent barrier $b(J_t)$ represents the cash flows to a firm acting in a market where there is entry of competing firms when the price level reaches $b(J_t)$.

The value of an existing firm is given by

$$V(t) = E_{x,j}^Q \left[\int_t^\infty e^{-r(s-t)} P_s ds \middle| \mathcal{F}_t \right].$$

Introducing the function

$$v(x, j) = E_{x,j}^Q \left[\int_0^\infty e^{-rs} P_s ds \right]$$

we have (again using the strong Markov property and time-homogeneity)

$$V(t) = u(Z_t, J_t).$$

3 A Markov-modulated Brownian motion model

Now let (X_t) be the Markov-modulated process defined in Equation (4), and define the cash flows generated by a firm when there are no potential competitors by

$$Y_t = e^{X_t}.$$

Futhermore let Z denote the regulated version of X , and let P denote the cash flows for an incumbent firm when it faces the possibility of market entry from competitors:

$$P_t = e^{Z_t}.$$

Generalizing the version of Dynkin's formula given in Equation (3) yields that for any stopping time τ it holds that

$$v_0(x, j) = E_{x,j}^Q \left[\int_0^\tau e^{-rs} Y_s ds \right] + E_{x,j}^Q [e^{-r\tau} v_0(X_\tau, J_\tau) \mathbf{1}(\tau < \infty)]$$

and

$$v(x, j) = E_{x,j}^Q \left[\int_0^\tau e^{-rs} P_s ds \right] + E_{x,j}^Q [e^{-r\tau} v(Z_\tau, J_\tau) \mathbf{1}(\tau < \infty)].$$

Now let

$$\tau = \inf\{t \geq 0 | X_t \geq b(J_t)\} = \inf\{t \geq 0 | Z_t = b(J_t)\}.$$

¹The case $n = 1$ was considered above, with $\bar{P} = b(1)$.

Since $X = Z$ on $[0, \tau)$ we get

$$v(x, j) = v_0(x, j) + E_{x,j}^Q [e^{-r\tau} v(Z_\tau, J_\tau) \mathbf{1}(\tau < \infty)] - E_{x,j}^Q [v_0(X_\tau, J_\tau) \mathbf{1}(\tau < \infty)].$$

From $P_t = e^{Zt}$ we get

$$0 \leq P_t \leq e^{\max_j b(j)},$$

so

$$e^{-r\tau} v(Z_\tau, J_\tau) = 0 \text{ on } \{\tau = \infty\}.$$

We further assume that X is such that

$$e^{-rt} v_0(X_t, J_t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

(From Equation (5) it follows that a sufficient condition for this is that $\mu(j) < r$ for every $j = 1, \dots, n$.) Hence, we can write

$$v(x, j) = v_0(x, j) + E_{x,j}^Q [e^{-r\tau} v(Z_\tau, J_\tau)] - E_{x,j}^Q [e^{-r\tau} v_0(X_\tau, J_\tau)].$$

The expected values can be written

$$\begin{aligned} E_{x,j}^Q [e^{-r\tau} v(Z_\tau, J_\tau)] - E_{x,j}^Q [e^{-r\tau} v_0(X_\tau, J_\tau)] &= \sum_{i=1}^n E_{x,j}^Q [e^{-r\tau} v(Z_\tau, J_\tau) \mathbf{1}(J_\tau = i)] \\ &\quad - \sum_{i=1}^n E_{x,j}^Q [e^{-r\tau} v_0(X_\tau, J_\tau) \mathbf{1}(J_\tau = i)] \\ &= \sum_{i=1}^n v(b(i), i) E_{x,j}^Q [e^{-r\tau} \mathbf{1}(J_\tau = i)] \\ &\quad - \sum_{i=1}^n E_{x,j}^Q [e^{-r\tau} v_0(X_\tau, i) \mathbf{1}(J_\tau = i)]. \end{aligned}$$

We know that when X is modelled as a Markov-modulated process

$$v_0(x, j) = e^x h(j),$$

so the last term on the right-hand side is

$$\sum_{i=1}^n h(i) E_{x,j}^Q [e^{-r\tau} e^{X_\tau} \mathbf{1}(J_\tau = i)].$$

Letting

$$\begin{aligned} L_i(x, j) &= E_{x,j}^Q [e^{-r\tau} \mathbf{1}(J_\tau = i)] \\ H_i(x, j) &= E_{x,j}^Q [e^{-r\tau} e^{X_\tau} \mathbf{1}(J_\tau = i)] \end{aligned}$$

we can write

$$v(x, j) = e^x h(j) + \sum_{i=1}^n v(b(i), i) L_i(x, j) - \sum_{i=1}^n h(i) H_i(x, j).$$

We have the boundary conditions

$$v(b(j), j) = I_j \text{ and } v'(b(j), j) = 0 \text{ for } j = 1, \dots, n.$$

It follows from the first set of boundary conditions that

$$v(x, j) = xh(j) + \sum_{i=1}^n I_i L_i(x, j) - \sum_{i=1}^n h(i) H_i(x, j) \text{ for } j = 1, \dots, n.$$

This, in turn, leads to, using the second set of boundary conditions,

$$0 = h(j) + \sum_{i=1}^n I_i L'_i(b(j), j) - \sum_{i=1}^n h(i) H'_i(b(j), j) \text{ for } j = 1, \dots, n.$$

We now consider the case $n = 2$. On $\{\tau < \infty\}$

$$Z_\tau \in \{b(1), b(2)\} \text{ and } X_\tau \in [b(1), b(2)].$$

In fact:

$$\text{If } Z_\tau = b(2), \text{ then } X_\tau = b(2).$$

In the case $n = 2$ we have

$$\begin{aligned} E_{x,j}^Q [e^{-r\tau}(v(Z_\tau, J_\tau) - v_0(X_\tau, J_\tau))] &= (I_1 - v_0(b(1), 1))E_{x,j}^Q [e^{-r\tau}\mathbf{1}(J_\tau = 1)] \\ &\quad + I_2 E_{x,j}^Q [e^{-r\tau}\mathbf{1}(J_\tau = 2)] \\ &\quad - h(2)E_{x,j}^Q [e^{-r\tau}X_\tau\mathbf{1}(J_\tau = 2)] \\ &= (I_1 - v_0(b(1), 1))E_{x,j}^Q [e^{-r\tau}\mathbf{1}(J_\tau = 1)] \\ &\quad + E_{x,j}^Q [e^{-r\tau}(I_2 - h(2)X_\tau)\mathbf{1}(J_\tau = 2)]. \end{aligned}$$

4 Solving some hitting problems

4.1 General theory

The following result will be used to find the function v introduced above. The proof is a straightforward generalization of the proof of Proposition 2 in Jobert & Rogers [11].

Proposition 4.1 *Let $f = (f(\cdot, 1), \dots, f(\cdot, n))$ be a bounded solution to the system of ODE's*

$$\begin{aligned} \frac{\sigma^2(j)}{2} \frac{d^2 f(x, j)}{dx^2} + \mu(j) \frac{df(x, j)}{dx} - r(j)f(x, j) + \sum_{k=1}^n q_{jk} f(x, k) &= 0 \text{ when } x \leq b_j \\ f(x, j) &= \psi_j(x) \text{ when } x \geq b_j. \end{aligned}$$

Then

$$f(x, j) = E_{x,j} \left[e^{-\int_0^\tau r(J_u) du} \sum_{k=1}^n \psi_k(X_\tau) \mathbf{1}(J_\tau = k) \right], \quad (6)$$

where

$$\tau = \inf\{t \geq 0 | X(t) \geq b(J(t))\}.$$

Proof. Let $n \in \mathbb{Z}_+$, An application of Ito's formula yields

$$\begin{aligned} e^{-\int_0^{n \wedge \tau} r(J_u) du} f(X_{n \wedge \tau}, J_{n \wedge \tau}) &= f(x, j) \\ &+ \int_0^{n \wedge \tau} (Af(X_u, J_u) - r(J_u)f(X_u, J_u)) du \\ &+ M_{n \wedge \tau}. \end{aligned}$$

Since f solves the systems of ODE's above, $Af(X_u, J_u) = r(J_u)f(X_u, J_u)$ on $[0, n \wedge \tau]$, so

$$e^{-\int_0^{n \wedge \tau} r(J_u) du} f(X_{n \wedge \tau}, J_{n \wedge \tau}) = f(x, j) + M_{n \wedge \tau}.$$

Taking $E_{x,j}[\cdot \cdot \cdot]$ of this equation, letting $n \rightarrow \infty$ and using bounded convergence results in Equation (6). \square

A special case of this proposition is when for a fixed $i = 1, \dots, n$

$$\psi_j(x) = \varphi(x)\delta_{ij}$$

for $j = 1, \dots, n$ and a given function φ . In this case Equation (6) takes the form

$$f(x, j) = E_{x,j} \left[e^{-\int_0^\tau r(J_u) du} \varphi(X_\tau) \mathbf{1}(J_\tau = i) \right].$$

4.2 The two-state case

Let us now consider the case $n = 2$. The same technique we use below has been used in e.g. Guo [7]. We have to consider the three intervals $(-\infty, b(1))$, $[b(1), b(2)]$ and $[b(2), \infty)$.

4.2.1 When $x \in [b(2), \infty)$

On this interval

$$f(x, j) = \psi_j(x)$$

for $j = 1, 2$.

4.2.2 When $x \in [b(1), b(2)]$

Now

$$f(x, 1) = \psi_1(x)$$

and

$$\frac{1}{2}\sigma^2(2)f''(x, 2) + \mu(2)f'(x, 2) - r(2)f(x, 2) + q_{21}\psi_1(x) + q_{22}f(x, 2) = 0.$$

The solution to this ODE is

$$f(x, 2) = A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x} + g(x),$$

where g is the particular solution, $\lambda_1 < 0 < \lambda_2$ are solutions to the quadratic equation

$$\frac{1}{2}\sigma^2(2)x^2 + \mu(2)x - r(2) + q_{22} = 0$$

and $A_1, A_2 \in \mathbb{R}$.

4.2.3 When $x \in (-\infty, b(1)]$

In this case

$$\begin{aligned} \frac{1}{2}\sigma^2(1)f''(x, 1) + \mu(1)f'(x, 1) - r(1)f(x, 1) + q_{11}f(x, 1) + q_{12}f(x, 2) &= 0 \\ \frac{1}{2}\sigma^2(2)f''(x, 2) + \mu(2)f'(x, 2) - r(2)f(x, 2) + q_{21}f(x, 2) + q_{22}f(x, 2) &= 0. \end{aligned}$$

It is known, see e.g. Remark 2.1 in Guo [7], that if the interest rates, the intensities and the volatility are all strictly positive, then there exists constants $\beta_1 < \beta_2 < 0 < \beta_3 < \beta_4$ such that the solution can be written

$$f(x, j) = \sum_{k=1}^4 B_{jk} e^{\beta_k x}$$

for $B_{jk} \in \mathbb{R}$ to be determined. First of all we always have the relation

$$B_{2k} = \ell_k B_{1k}$$

for $k = 1, 2$ where

$$\ell_1 = -\frac{\sigma(1)^2 \beta_3^2 / 2 + \mu(1) \beta_3 - (r(1) - q_{11})}{q_{12}}$$

and

$$\ell_2 = -\frac{\sigma(1)^2 \beta_4^2 / 2 + \mu(1) \beta_4 - (r(1) - q_{11})}{q_{12}}.$$

In our cases, for every $j = 1, 2$ the function $f(\cdot, j)$ must be bounded as $x \rightarrow -\infty$, so

$$B_{j1} = B_{j2} = 0$$

for every $j = 1, 2$:

$$f(x, j) = B_{j3} e^{\beta_3 x} + B_{j4} e^{\beta_4 x}.$$

We can write this as

$$\begin{aligned} f(x, 1) &= B_{13} e^{\beta_3 x} + B_{14} e^{\beta_4 x} \\ f(x, 2) &= \ell_1 B_{13} e^{\beta_3 x} + \ell_2 B_{14} e^{\beta_4 x}. \end{aligned}$$

4.2.4 The complete solution

To determine the remaining constants we use continuity of $f(\cdot, 2)$ at $b(2)$:

$$f(b(2), 2) = \psi_2(b(2)),$$

continuity of $f(\cdot, j)$, $j = 1, 2$, at $b(1)$:

$$f(b(1), 1) = \psi_1(b(1)) \quad \text{and} \quad f(b(1), 2) = h(b(1)) + A_1 e^{\lambda_1 b(1)} + A_2 e^{\lambda_2 b(1)},$$

and finally smoothness at $b(1)$ for $f(\cdot, 2)$:

$$f'(b(1), 2) = g'(b(1)) + A_1 \lambda_1 e^{\lambda_1 b(1)} + A_2 \lambda_2 e^{\lambda_2 b(1)}.$$

Summarizing these relations we get the following system of equations:

$$\begin{aligned} A_1 e^{\lambda_1 b(2)} + A_2 e^{\lambda_2 b(2)} + g(b(2)) &= \psi_2(b(2)) \\ B_{13} e^{\beta_3 b(1)} + B_{14} e^{\beta_4 b(1)} &= \psi_1(b(1)) \\ B_{13} \ell_1 e^{\beta_3 b(1)} + B_{14} \ell_2 e^{\beta_4 b(1)} &= A_1 e^{\lambda_1 b(1)} + A_2 e^{\lambda_2 b(1)} + g(b(1)) \\ B_{13} \ell_1 \beta_3 e^{\beta_3 b(1)} + B_{14} \ell_2 \beta_4 e^{\beta_4 b(1)} &= A_1 \lambda_1 e^{\lambda_1 b(1)} + A_2 \lambda_2 e^{\lambda_2 b(1)} + g'(b(1)) \end{aligned}$$

5 The value of the investment

We now want to find the value of an existing firm that faces the possibility of other firms entering the market. We have

$$r(2) = r(1) = r, \quad \mu(j) = \mu^P - \lambda(j)\sigma \quad \text{and} \quad \sigma(1) = \sigma(2) = \sigma.$$

We need to find the four functions L_1 , L_2 , H_1 and H_2 , i.e. we need to solve the problem from Section 4.2 when

$$\psi_j(x) = \varphi(x) \delta_{ij}, \quad i, j = 1, 2,$$

with

$$\varphi(x) = 1 \quad \text{and} \quad \varphi(x) = e^x.$$

We use the following parameter names:

Function $\varphi(x)$	Parameters for $x \in [b(1), b(2)]$	Parameters for $x \in [b(2), \infty)$
1	A_1, A_2, B_{13}, B_{14}	C_1, C_2, D_{13}, D_{14}
e^x	$\hat{A}_1, \hat{A}_2, \hat{B}_{13}, \hat{B}_{14}$	$\hat{C}_1, \hat{C}_2, \hat{D}_{13}, \hat{D}_{14}$

To solve for the unknown parameters we go through the following four steps:

- 1) The particular solutions when $\varphi(x) = 1$ are

$$g_1(x) = \frac{q_{21}}{r - q_{22}} \quad \text{and} \quad g_2(x) = 0$$

respectively. Hence, when $x \in [b(1), b(2)]$ we have

$$f_1(x, 2) = A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x}$$

and

$$f_2(x, 2) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \frac{q_{21}}{r - q_{22}}.$$

- 1) The particular solutions when $\varphi(x) = e^x$ are

$$g_1(x) = \frac{q_{21}}{\frac{\sigma^2}{2} + \mu(2) - r + q_{22}} \cdot e^x \quad \text{and} \quad g_2(x) = 0.$$

respectively. This gives

$$f_1(x, 2) = \hat{A}_1 e^{\lambda_1 x} + \hat{A}_2 e^{\lambda_2 x}$$

and

$$f_2(x, 2) = \hat{C}_1 e^{\lambda_1 x} + \hat{C}_2 e^{\lambda_2 x} + \frac{q_{21}}{\frac{\sigma^2}{2} + \mu(2) - r + q_{22}} \cdot e^x.$$

- 3) We have the following two equations from the property of zero derivative at the hitting levels:

$$\begin{aligned} e^{b(1)} h(1) + I_1 L'_1(b(1), 1) + I_2 L'_2(b(1), 1) - h(1) H'_1(b(1), 1) - h(2) H'_2(b(1), 1) &= 0 \\ e^{b(2)} h(2) + I_1 L'_1(b(2), 2) + I_2 L'_2(b(2), 2) - h(1) H'_1(b(2), 2) - h(2) H'_2(b(2), 2) &= 0. \end{aligned}$$

- 4) We now have four system of equations (four version of the set of equations given at the end of Section 4.2.4) together with the two equations from 3) above. Hence, we have 18 equations and 18 unknowns.

References

- [1] Bentolila, S. & Bertola, G. (1990), 'Firing Costs and Labour Demand: How Bad Is Euroclerosis?', *Review of Economic Studies* 57, pp. 381-402.
- [2] D'Auria, B. & Kella O. (2012), 'Markov modulation of a two-sided reflected Brownian motion with application to fluid queues', *Stochastic Processes and their Applications* 122, pp. 1566-1581.
- [3] Dixit A. K. & Pindyck R. S. (1994), 'Investment under Uncertainty', *Princeton University Press*.

- [4] Driffill J., Kenc, T. & Sola, M. (2013), 'Real Options with Priced Regime-Switching Risk', *International Journal of Theoretical and Applied Finance*, Vol. 16, No. 5.
- [5] Elliott, R. J., Chan, L. & Siu, T. K. (2005), 'Option pricing and Esscher transform under regime switching', *Annals of Finance*, 1, pp. 423-432.
- [6] Grenadier, S. R. (1995), 'Valuing lease contracts: A real-options approach', *Journal of Financial Economics* 38, pp. 297-331.
- [7] Guo, X. (2001), 'An explicit solution to an optimal stopping problem with regime shifting', *J. Appl. Prob.* 38, pp. 464-481.
- [8] Guo, X. & Zhang, Q. (2004), 'Closed form solutions for perpetual American put options with regime switching', *SIAM J. Appl. Math.*, Vol 64, No. 6, pp. 2034-2049.
- [9] Hamilton, J. D. (1989), 'A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle', *Econometrica*, Vol. 57, No. 2, pp. 357-384.
- [10] Hou, S.-H. (1998), 'Classroom note: A Simple Proof of the Leverrier-Faddeev Characteristic Polynomial Algorithm', *SIAM Review*, 40(3), pp. 706-709.
- [11] Jobert, A. & Rogers L. C. G. (2006), 'Option Pricing With Markov-Modulated Dynamics', *SIAM Journal on Control and Optimization*, 44(6), pp. 2063-2078.
- [12] Karlin, S. & Taylor H. M. (1981), 'A Second Course in Stochastic Processes', *Academic Press*.
- [13] McDonald, R. & Siegel D. (1986), 'The Value of Waiting to Invest', *The Quarterly Journal of Economics*, Vol. 101, No. 4 (Nov). pp. 707-728.
- [14] Siu, T. K. (2011), 'Regime-Switching Risk: To Price or not to Price?', *International Journal of Stochastic Analysis*, Vol. 2011, Article ID 843246, 14 pages.
- [15] Siu, T. K. & Yang H. (2009), 'Option Pricing when the Regime-Switching Risk is Priced', *Acta Mathematicae Applicatae Sinica, English Series*, Vol. 25, No. 3, pp. 369-388.
- [16] Yin, G. G. & Zhu, C. (2010), 'Hybrid Switching Diffusions: Properties and Applications', *Springer*.