Finite project life and (in)finite options duration: Effect on timing and size of capacity investment

Anne G. Balter* Kuno J.M. Huisman‡ Peter M. Kort§

Tilburg University† Tilburg University Tilburg University

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VERY PRELIMINARY¶

Abstract

The literature on real investment in general departs from two assumptions. First, it is assumed that the project length is infinite, i.e. the firm produces forever after the investment has been undertaken. Second, the option to invest exists forever. This paper relaxes both assumptions. In a monopoly setting we find that, in case the inverse demand function linearly depends on a Geometric Brownian Motion process, a reduction of the project length delays the investment time whereas the investment size is not affected. Having a finite life of the investment option, caused for instance by technological progress making the product obsolete, accelerates investment whereas the investment size is reduced. We also investigate the effect of relaxing these two assumptions on the investment decisions of firms in a duopoly setting.

Keywords Investment; Uncertainty; Finite project length; Capacity; Real options model

*Email: a.g.balter@uvt.nl

†Department of Econometrics and Operations Research, Warandelaan 2, 5037AB, Tilburg, The Netherlands

‡Email: k.j.m.huisman@tilburguniversity.edu

§Email: kort@uvt.nl

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1 Introduction

In this paper we investigate the impact of finiteness of projects. We build upon the real options approach of Dixit and Pindyck (1994) which is used to optimize the capacity investment decision under uncertainty. In particular we combine the finite project life from Gryglewicz et al. (2008) in which the investment decision was a timing problem, with the capacity optimization idea that was carried out for the first time in Dangl (1999) and Bar-Ilan and Strange (1999). We extend Huisman and Kort (2015) to finite projects. Note that we embed the infinite horizon too. We consider a monopoly in which a probabilistic event causes the project to terminate, either exogenously or endogenously. We distinguish between the case in which this can only happen once the investor has entered the market and the case in which the option to invest can disappear before the firm actually invested. We can analyse these effects analytically. We also investigate this in a duopoly in which one event determines the end. In this respect, we derive the accommodation and deterrence strategies. And we derive the optimal decisions when the finiteness of both firms are triggered by independent events. This latter case can only be solved numerically.

In Section 2 we describe the setting of a stochastic finite project life of the projects. In Section 3 we derive the optimal capacity and investment trigger for the monopolist when having linear demand. We can distinguish two cases, either the project can terminate with a certain probability once it has started or the option can terminate with the same probability before the investment has actually started. We find that, if the option always remains, the capacity is not affected for the lifetime of the project compared to an infinite project horizon. We investigate the robustness of this and find that for all demand function that are linear in the risk source X this is true. We also see that more will be invested. We also derive and analyse the impact of a possible disappearance of the option to invest. In Section 4 we investigate a duopoly setting in which one event can cause the end of the project either the investment region or in both the investment and continuation region. In Section 5 we investigate via ordinary differential equations the duopoly setting with different probabilities for each firm. For the duopoly we derive the Stackelberg accommodation, the deterrence and preemption strategy. **Note that this is a very preliminary version. Among others, numerical results, the associated interpretations and comparisons still need to be added!**

2 Setting

Departing from Huisman and Kort (2015), we first consider a monopoly, thus a setting in which there is one firm who has to decide when to enter the market and how much to produce. The price at time t in this market is given by the inverse demand function

$$P(t) = X(t)\big(1-\eta Q(t)\big), \tag{1}$$

where $Q(t)$ is the total market output/production/capacity, $\eta > 0$ is a constant and $X(t)$ follows a geometric Brownian motion

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \tag{2}$$

where μ is the drift, $\sigma > 0$ is the volatility and $W(t)$ is the Wiener process.

The firm produces from the moment of investment onwards, so that $Q(t) = \begin{cases} 0 & \text{if } t < t_I \\ K & \text{if} t \geq t_I \end{cases}$. The investment costs are proportional to the capacity K and we assume that the firm produces up to capacity. Denoting I as investment, we thus have $I = \delta K$.

The investment problem that the firm is facing is to maximise the expected profit from the moment t_I that the investment is made until the finite horizon on the project T years ahead. The control variables are thus the time at which the investment is undertaken, and the capacity level that the firm acquires at t_I.

$$\max_{t_I \geq 0, Q(t_I)=K\geq 0} V(X,Q) = \max_{t_I \geq 0, Q(t_I)=K\geq 0} \mathbb{E}\left[\int_{t=t_I}^{t_I+T} e^{-rt}Q(t_I)\cdot P(t)dt - e^{-rt_I}\delta Q(t_I)|X(0)=X\right] \tag{3}$$

where r is the discount rate. We now transform the optimal t_I^* by the trigger point X^*. Let X^* be the value at which the firm is indifferent between investing and not. Thus for $X > X^*$, it is optimal to invest immediately, whereas for $X < X^*$ demand is still too low to undertake the investment and thus the firm waits. The optimal investment time t_I^* equals the first time that the stochastic process X reaches this level X^*.

We can solve the double maximisation in two steps, first for a given X we maximise V w.r.t. Q. Both the dynamic programming and contingent claims approach value the real option that is present in the discussed optimisation problem. The option of waiting is added to the net present valuation technique. As in Gryglewicz et al. (2008), this paper is based on the contingent claim approach.

By $V(X)$ we denote the pure NPV value and let $F(X)$ be the option value that solves the ODE in case of the dynamic programming approach

$$\frac{1}{2}\sigma^2 X^2 F''(X) + \mu X F'(X) - rF = 0 \tag{4}$$

The boundary conditions are

$$
\begin{aligned}
F(X^*) &= V(X^*,Q) - \delta Q && (5)\\
\frac{\partial F(X)}{\partial X}|_{X=X^*} &= \frac{\partial V(X,Q)}{\partial X}|_{X=X^*} && (6)
\end{aligned}
$$

Where (5) is the value-matching condition that states that when the firm invests at optimality the net payoff V equals the option value F. For $X < X^*$ the option value $F(X) > V(X) - \delta Q$ and thus it is better to wait until $F(X) = V(X) - \delta Q$ after which it is optimal to invest and receive the value function V. And (6) is the smooth pasting condition.

To find $F(X)$ the ODE has to be solved which can be done by introducing/guessing a functional form, which we call the Ansatz and check whether this solves the ODE. The Ansatz is

$$
F(X) = AX^{\beta} \tag{7}
$$

where A has to be determined and β is a known constant that depends on σ, r, μ.

The lifetime T is random and follows a Poisson process. At any time t the project terminates with probability λdt. The cumulative probability that the project terminates before t from the initial time zero onwards, equals $1 - e^{-\lambda t}$. And the density of the Poisson distribution is $\lambda e^{-\lambda t}$.

We first investigate the demand and capacity for a monopolist, whereafter we consider a duopoly.

3 Monopolist

The optimal strategies of the monopolist can be obtained by two different procedures; either by maximizing the expected payoff in (3) under the additional integration with respect to the termination probability to derive the optimal capacity and subsequently use the value matching and smooth pasting condition to derive the optimal trigger. Or by applying the same operations to the ordinary differential equation, i.e. the instantaneous objective. Throughout the paper we use the ODE method, especially suitable for the more complicated duopoly setting.

The value of the project at time t can be expressed as the sum of the operating profit over the interval $(t, t+dt)$ and the continuation value beyond $t+dt$.

$$
rV^M(X,Q) = Profit^M(X,Q) + \lim_{dt \downarrow 0} \frac{1}{dt}\mathbb{E}[dV^M] \tag{8}
$$

where by Ito's Lemma

$$\mathbb{E}[dV^M] = \frac{\partial V^M}{\partial X}\mu X dt + \frac{1}{2}\frac{\partial^2 V^M}{\partial X^2}\sigma^2 X^2 dt + \lambda dt(0 - V^M) \tag{9}$$

Together the stopping region is described by

$$(r+\lambda)V^M = KX(1-\eta K) + \frac{\partial V^M}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V^M}{\partial X^2}\sigma^2 X^2 \tag{10}$$

and thus the non-homogeneous equation is

$$V^M = \frac{1}{r+\lambda}\Big(KX(1-\eta K) + \frac{\partial V^M}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V^M}{\partial X^2}\sigma^2 X^2\Big) \tag{11}$$

The homogeneous equation (terms involving value function) is

$$0 = \frac{\partial V^M}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V^M}{\partial X^2}\sigma^2 X^2 - (r+\lambda)V^M \tag{12}$$

with solution

$$V^M(X) = B_1 X^{\beta_\lambda^+} + B_2 X^{\beta_\lambda^-} \tag{13}$$

where $\beta_\lambda^+ > 1, \beta_\lambda^- < 0$. For a particular solution of the total equation we propose

$$V^M(X) = aX + b \tag{14}$$

$$(r+\lambda)(aX+b) = KX(1-\eta K) + a\mu X \tag{15}$$

$$a = \frac{K(1-\eta K)}{r-\mu+\lambda} \tag{16}$$

$$b = 0. \tag{17}$$

The total solution is the sum of the homogeneous solution and particular solution

$$V^M(X) = \frac{XK(1-\eta K)}{r-\mu+\lambda} + B_1 X^{\beta_\lambda^+} + B_2 X^{\beta_\lambda^-} \tag{18}$$

with boundary conditions

$$V^M(0) = 0 \tag{19}$$

$$\lim_{X\to\infty} V^M(X) = wX \tag{20}$$

Since $\beta_\lambda^- < 0$, $X^{\beta_\lambda^-}$ will go to infinity when X goes to zero. Thus (19) leads to $B_2 = 0$. And (20) refers to the exclusion of speculative bubbles, i.e. in the limit the value function is linear in X where w is a constant implying $B_1 = 0$. Thus, if the solution is of the form

$$V^M(X) = aX + b \tag{21}$$

then we get

$$(r+\lambda)(aX+b) = KX(1-\eta K) + a\mu X \tag{22}$$

and thus

$$a = \frac{K(1-\eta K)}{r-\mu+\lambda} \tag{23}$$

$$b = 0 \tag{24}$$

$$V^M(X) = \frac{XK(1-\eta K)}{r-\mu+\lambda} \tag{25}$$

Derivative of $V^M - \delta K$ w.r.t. K yields

$$\frac{\partial V^M - cK}{\partial K} = 0 \tag{26}$$

$$K^*(X) = \frac{1}{2\eta}\left(1 - \frac{\delta(r-\mu+\lambda)}{X}\right) \tag{27}$$

The continuation region, under the assumption that the option to invest always exists, is defined by

$$rF^M = \frac{\partial F^M}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 F^M}{\partial X^2}\sigma^2 X^2 \tag{28}$$

which solves for a form of $F^M(X) = AX^\beta$. If the option to invest can stop before the project has begun then the continuation region is defined by

$$rF^M = \frac{\partial F^M}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 F^M}{\partial X^2}\sigma^2 X^2 + \lambda dt(0 - F^M) \tag{29}$$

which solves for a form of $F^M(X) = A_{\beta_\lambda^+} X^{\beta_\lambda^+}$. We continue the derivations with β, however β can simply be replaced by β_λ^+ to incorporate the probability of a final continuation time. The value matching condition is

$$V^M - \delta K = F^M \tag{30}$$

and the smooth pasting

$$\frac{\partial V^M(X)}{\partial X} = \frac{\partial F^M(X)}{\partial X} \tag{31}$$

together they imply

$$\frac{\partial V^M(X)}{\partial X} = (V^M(X) - \delta K)\beta X^{-1} \tag{32}$$

to get X.

$$X^*(K) = \frac{\beta}{\beta - 1}\frac{\delta(r-\mu+\lambda)}{1-\eta K} \tag{33}$$

Theorem 3.1. *The optimal capacity and investment trigger for a project with a lifetime that terminates with probability λ equals*

$$(X^M)^* = \frac{(r-\mu)}{1-\frac{\lambda}{r-\mu+\lambda}}\delta\frac{\beta+1}{\beta-1} = (r-\mu+\lambda)\delta\frac{\beta+1}{\beta-1} \tag{34}$$

$$(K^M)^* = \frac{1}{\eta(\beta+1)} \tag{35}$$

We obtain A via plugging in the optimal K^* and X^* in $V^M(X) - \delta K = AX^\beta$

$$V^M(X) - \delta K = \frac{\delta}{\eta(\beta^2 - 1)} \tag{36}$$

$$A = V^M(X)(X^*)^{-\beta} \tag{37}$$

$$A = \frac{\delta}{\eta(\beta^2-1)}\left((r-\mu+\lambda)\delta\frac{\beta+1}{\beta-1}\right)^{-\beta} \tag{38}$$

$$A_{\beta_\lambda^+} = \frac{\delta}{\eta\left(\left(\beta_\lambda^+\right)^2-1\right)}\left((r-\mu+\lambda)\delta\frac{\beta_\lambda^+ + 1}{\beta_\lambda^+ - 1}\right)^{-\beta_\lambda^+} \tag{39}$$

and β by the continuation region, i.e. the second-order homogeneous differential equation for which we try the function AX^β, for which we see by substitution that it satisfies the equation provided β is a root of the quadratic equation

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0 \tag{40}$$

Two contrary effects on capacity size K^*: Given X, the firm invests less for larger λ. And simultaneously, since the trigger for the finite project is higher than for projects of infinite length, the firm invests more. These two effects exactly cancel out resulting in the robust capacity decision.

In case of a final continuation time, the β_λ is defined by the positive and negative root of

$$\frac{1}{2}\sigma^2\beta_\lambda(\beta_\lambda - 1) + \mu\beta_\lambda - (r+\lambda) = 0 \tag{41}$$

Since β_λ is increasing in λ, this implies that the capacity will be lower in case of the possibility that the option to invest can terminate before the monopolist has invested compared to this option to exist forever and compared to no ending at all. The moment of investment will be later if the project is finite compared to the classical situation where the project cannot end. Though when also the option of the project is finite, the moment to invest happens earlier than without a finite option time. However, compared to an infinite project length it is ambiguous what is dominating, the increase in $(X^M)^*$ due to λ or the decrease due to the indirect of λ via β_λ. In this case the capacity decision is affected by λ.

3.1 Isoelastic demand

We now replace the linear demand function by an isoelastic demand function in which the price at time t in the market is given by

$$P(t) = X(t)\,(Q(t))^{-\gamma} \tag{42}$$

where $\gamma \in (0,1)$ is the elasticity parameter. We assume that the investment costs are of the form $\delta_0 + \delta_1 K$. The value

$$V^M = \frac{1}{r+\lambda}\left(K^{1-\gamma}X + \frac{\partial V^M}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V^M}{\partial X^2}\sigma^2 X^2\right) \tag{43}$$

The homogeneous equation (terms involving value function)

$$0 = \frac{\partial V^M}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V^M}{\partial X^2}\sigma^2 X^2 - (r+\lambda)V^M \tag{44}$$

has as solution

$$V^M(X) = B_1 X^{\beta_\lambda^+} + B_2 X^{\beta_\lambda^-} \tag{45}$$

where $\beta_\lambda^+ > 1, \beta_\lambda^- < 0$. For a particular solution of the total equation we propose

$$V^M(X) = aX + b \tag{46}$$

$$(r+\lambda)(aX+b) = K^{1-\gamma}X + a\mu X \tag{47}$$

$$a = \frac{K^{1-\gamma}}{r-\mu+\lambda} \tag{48}$$

$$b = 0. \tag{49}$$

The total solution is the sum of the homogeneous solution and particular solution

$$V^M(X) = \frac{XK^{1-\gamma}}{r-\mu+\lambda} + B_1 X^{\beta_\lambda^+} + B_2 X^{\beta_\lambda^-} \tag{50}$$

with boundary conditions

$$V^M(0) = 0 \tag{51}$$

$$\lim_{X\to\infty} V^M(X) = wX \tag{52}$$

Since $\beta_\lambda^- < 0$, $X^{\beta_\lambda^-}$ will go to infinity when X goes to zero. Thus (19) leads to $B_2 = 0$. And (20) to $B_1 = 0$. Thus, if the solution is of the form

$$V^M(X) = aX + b \tag{53}$$

then we get

$$V^M(X) = \frac{XK^{1-\gamma}}{r-\mu+\lambda} \tag{54}$$

Derivative of $V^M - \delta_0 - \delta_1 K$ w.r.t. K yields

$$\frac{\partial V^M - \delta_0 - \delta_1 K}{\partial K} = 0 \tag{55}$$

$$K^*(X) = \left(\frac{\delta_1(r-\mu+\lambda)}{(1-\gamma)X}\right)^{-\frac{1}{\gamma}} \tag{56}$$

The continuation region is defined by

$$rF^M = \frac{\partial F^M}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 F^M}{\partial X^2}\sigma^2 X^2 \tag{57}$$

which solves for a form of $F^M(X) = AX^\beta$. The value matching condition is

$$V^M - \delta_0 - \delta_1 K = F^M \tag{58}$$

and the smooth pasting

$$\frac{\partial V^M(X)}{\partial X} = \frac{\partial F^M(X)}{\partial X} \tag{59}$$

together they imply

$$\frac{\partial V^M(X)}{\partial X} = (V^M(X) - \delta_0 - \delta_1 K)\beta X^{-1} \tag{60}$$

to get X.

$$X^*(K) = \frac{\beta}{\beta - 1} \frac{(\delta_0 + \delta_1 K)(r - \mu + \lambda)}{K^{1-\gamma}} \tag{61}$$

Theorem 3.2. *The optimal capacity and investment trigger for a project with a lifetime that terminates with probability λ equals*

$$K^* = \frac{\beta(\gamma - 1)\delta_0}{(1 - \beta\gamma)\delta_1} \tag{62}$$

$$X^* = \frac{\delta_1(r - \mu + \lambda)}{1 - \gamma} \left(\frac{\beta(\gamma - 1)\delta_0}{(1 - \beta\gamma)\delta_1} \right)^\gamma \tag{63}$$

3.2 Generality Multiplicative

In general, we can conclude that for all demand functions that are linear in $X(t)$ and the option to invest cannot end before the project has started, the optimal capacity is independent from the lifetime of the project.

Let

$$P(t) = X f(Q) \tag{64}$$

then the value function becomes

$$V^M(X) = \frac{XKf(K)}{r - \mu + \lambda} \tag{65}$$

Derivative of w.r.t. K yields

$$\begin{aligned} \frac{\partial V^M - \delta_0 - \delta_1 K}{\partial K} &= 0 && (66)\\ \frac{X}{r-\mu+\lambda}\left(K\frac{\partial f(K)}{\partial K} + f(K)\right) - \delta_1 &= 0 && (67) \end{aligned}$$

The optimal $K^*(X)$ is defined by the implicit function above. We use the Ansatz $F(X) = AX^\beta$ for the value matching and smooth pasting conditions

$$\begin{aligned} \frac{\partial V^M(X)}{\partial X} &= (V^M(X) - \delta_0 - \delta_1 K)\beta X^{-1} && (68)\\ \frac{Kf(K)}{r-\mu+\lambda} &= \frac{\beta K f(K)}{r-\mu+\lambda} - \frac{(\delta_0+\delta_1 K)\beta}{X} && (69) \end{aligned}$$

This results in

$$X^*(K) = \frac{\beta}{\beta-1}\frac{(\delta_0+\delta_1 K)(r-\mu+\lambda)}{Kf(K)} \quad (70)$$

If we plug $X^*(K)$ into the implicit function, we find that K^* does not depend on λ as

$$\frac{\beta}{\beta-1}\frac{(\delta_0+\delta_1 K)}{Kf(K)}\left(K\frac{\partial f(K)}{\partial K} + f(K)\right) - \delta_1 = 0 \quad (71)$$

The β is obtained from plugging the AX^β into the continuation region (57) which is independent from λ.

4 Projects end with one event

4.1 Follower

For firm 2, the follower, in a duopoly the value function equals

$$rV_F^D(X, Q_F, Q_L) = Profit_F^D(X, Q_F, Q_L) + \lim_{dt\downarrow 0}\frac{1}{dt}\mathbb{E}[dV_F^D] \quad (72)$$

where

$$\mathbb{E}[dV_F^D] = \frac{\partial V_F^D}{\partial X}\mu X dt + \frac{1}{2}\frac{\partial^2 V_F^D}{\partial X^2}\sigma^2 X^2 dt + \lambda dt(0 - V_F^D) \quad (73)$$

since if the project terminates, the value function of firm 2 becomes zero. The profit function is $P \cdot Q$ when both firm 1 and 2 are in the market,

$$Profit_F^D = K_F X(1-\eta(K_F+K_L)) \tag{74}$$

This leads to the ODE

$$V_F^D = \frac{1}{(r+\lambda)}\left(K_F X(1-\eta(K_F+K_L)) + \frac{\partial V_F^D}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V_F^D}{\partial X^2}\sigma^2 X^2\right) \tag{75}$$

The homogeneous equation (terms involving value function) is

$$0 = \frac{\partial V_F^D}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V_F^D}{\partial X^2}\sigma^2 X^2 - (r+\lambda)V_F^D \tag{76}$$

having as solution

$$V_F^D(X) = B_1 X^{\beta_\lambda^+} + B_2 X^{\beta_\lambda^-} \tag{77}$$

where $\beta_\lambda^+ > 1, \beta_\lambda^- < 0$. A particular solution is

$$\begin{aligned} V_F^D(X) &= aX + b && (78)\\ (r+\lambda)(aX+b) &= K_F X(1-\eta(K_F+K_L)) + a\mu X \\ a &= \frac{K_F(1-\eta(K_F+K_L))}{r-\mu+\lambda} && (79)\\ b &= 0. && (80) \end{aligned}$$

Such that the total solution is

$$V_F^D(X) = \frac{K_F(1-\eta(K_F+K_L))}{r-\mu+\lambda}X + B_1 X^{\beta_\lambda^+} + B_2 X^{\beta_\lambda^-} \tag{81}$$

with boundary conditions

$$\begin{aligned} V_F^D(0) &= 0 && (82)\\ \lim_{X\to\infty} V_F^D(X) &= wX && (83) \end{aligned}$$

This leads to $B_2 = 0$, and $B_1 = 0$. Therefore the solution is

$$V_F^D(X) = \frac{K_F X(1-\eta(K_F+K_L))}{r-\mu+\lambda} \tag{84}$$

Maximizing w.r.t. K_F.

$$\begin{aligned}\frac{\partial V_F^D - \delta_F K_F}{\partial K_F} &= 0 \quad (85)\\ \frac{X(1-\eta(2K_F+K_L))}{r-\mu+\lambda} - \delta_F &= 0 \quad (86)\end{aligned}$$

And thus

$$\begin{aligned}K_F^D(X,K_L) &= \frac{1}{2\eta}\left(1-\frac{\delta_F(r+\lambda-\mu)}{X}\right)-\frac{1}{2}K_L \quad (87)\\ &= K_F^M(X)-\frac{1}{2}K_L \quad (88)\end{aligned}$$

The continuation region is defined by

$$rF_F^D = \frac{\partial F_F^D}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 F_F^D}{\partial X^2}\sigma^2 X^2 + \lambda\left(0 - F_F^{D2}\right) \quad (89)$$

The continuation value has the form

$$F_F^D(X) = AX^{\beta_\lambda^+} \quad (90)$$

where β_λ solves

$$\frac{1}{2}\beta_\lambda(1-\beta_\lambda)\sigma^2 + \beta_\lambda\mu - (r+\lambda) = 0, \quad (91)$$

The value matching condition is

$$V_F^D - \delta_F K_F = F_F^D \quad (92)$$

and the smooth pasting

$$\frac{\partial V_F^D(X)}{\partial X} = \frac{\partial F_F^D(X)}{\partial X} \quad (93)$$

together they imply

$$\frac{\partial V_F^D(X)}{\partial X} = (V_F^D(X) - \delta_F K_F)\beta_\lambda^+ X^{-1} \quad (94)$$

to get X.

$$X_F^*(K_F, K_L) = \frac{\beta_\lambda^+}{\beta_\lambda^+ - 1} \frac{\delta_F(r - \mu + \lambda)}{1 - \eta(K_F + K_L)} \tag{95}$$

Theorem 4.1. *The optimal capacity and investment trigger of the follower for a project with a lifetime that terminates with probability λ and*

(i) if the option exists forever then

$$(X_F^D)^*(K_L) = \frac{(r - \mu + \lambda)\delta_F}{1 - \eta K_L} \frac{\beta + 1}{\beta - 1} \tag{96}$$

$$(K_F^D)^*(K_L) = \frac{1 - \eta K_L}{(\beta + 1)\eta} \tag{97}$$

(ii) if the option can terminate before the project started with probability λ then

$$(X_F^D)^*(K_L) = \frac{(r - \mu + \lambda)\delta_F}{1 - \eta K_L} \frac{\beta_\lambda^+ + 1}{\beta_\lambda^+ - 1} \tag{98}$$

$$(K_F^D)^*(K_L) = \frac{1 - \eta K_L}{(\beta_\lambda^+ + 1)\eta} \tag{99}$$

4.2 Leader

4.2.1 Accommodation

The leader knows the strategies of the follower, and anticipates to these by the Stackelberg accommodation strategy as follows.

$$rV_L^D(X, K_F, K_L) = Profit_L^D(X, K_F, K_L) + \lim_{dt \downarrow 0} \frac{1}{dt} \mathbb{E}[dV_L^D] \tag{100}$$

where

$$\mathbb{E}[dV_L^D] = \frac{\partial V_L^D}{\partial X} \mu X dt + \frac{1}{2} \frac{\partial^2 V_L^D}{\partial X^2} \sigma^2 X^2 dt + \lambda dt(0 - V_L^D) \tag{101}$$

Now we know the strategy of firm 2, thus

$$Profit_L^D = K_L X(1 - \eta(K_F^*(X, K_L) + K_L)) \tag{102}$$

This implies that

$$V_L^D = \frac{1}{(r+\lambda)}\Big(K_L X\left(1-\eta(K_F^*+K_L)\right)+\frac{\partial V_L^D}{\partial X}\mu X+\frac{1}{2}\frac{\partial^2 V_L^D}{\partial X^2}\sigma^2 X^2\Big) \tag{103}$$

The solution of the homogeneous equation (terms involving value function) is

$$0 = \frac{\partial V_L^D}{\partial X}\mu X+\frac{1}{2}\frac{\partial^2 V_L^D}{\partial X^2}\sigma^2 X^2-(r+\lambda)V_L^D \tag{104}$$

$$V_L^D(X) = B_1 X^{\beta_\lambda^+}+B_2 X^{\beta_\lambda^-} \tag{105}$$

The particular solution (terms without value function) is

$$V_L^D(X) = aX+b \tag{106}$$

$$(r+\lambda)(aX+b) = K_L X\left(1-\eta(K_F^*+K_L)\right)+a\mu X$$

$$a = \frac{K_L(1-\eta(K_F^*+K_L))}{\left(r+\lambda-\mu\right)} \tag{107}$$

$$b = 0. \tag{108}$$

The total solution is

$$V_L^D(X) = \frac{K_L(1-\eta(K_F^*+K_L))}{\left(r+\lambda-\mu\right)}X+B_1 X^{\beta_\lambda^+}+B_2 X^{\beta_\lambda^-}$$

with boundary conditions

$$V_L^D(0) = 0 \tag{109}$$

$$\lim_{X\to\infty} V_L^D(X) = wX \tag{110}$$

Thus both $B_2=0$ and $B_1=0$. And therefore

$$V_L^D(X) = \frac{K_L(1-\eta(K_F^*+K_L))}{\left(r+\lambda-\mu\right)}X \tag{111}$$

Now we get K_L by

$$\frac{\partial V_L^D-\delta_L K_L}{\partial K_L} = 0 \tag{112}$$

where we have plugged in $K_F^*(X, K_L)$,

$$K_L^*(X) \;=\; \left(1 - \frac{(2\delta_L - \delta_F)(r+\lambda-\mu)}{X}\right)\frac{1}{2\eta} \tag{113}$$

First plug $K_L^*(X)$ in V_L^D to get

$$V_L^D(X) - \delta_L K_L \;=\; \frac{\left(X + (\delta_F - 2\delta_L)(r+\lambda-\mu)\right)^2}{8X\eta(r+\lambda-\mu)} \tag{114}$$

and then by the value and smooth pasting conditions we solve for $X_L(K_L)$. The stopping value is determined based on either the assumption that the option to invest always exists or on the assumption that the option to invest can disappear for both firms with different probabilities. These imply

(i) $F(X) = AX^{\beta}$ if the option always remains (β is without λ).

(ii) $F(X) = A_{\beta_\lambda^+}X^{\beta_\lambda^+}$ if the option to invest can vanish before the firm has invested

Case (i):

$$\frac{\partial V_L^D(X)}{\partial X} = (V_L^D(X) - \delta_L K_L)\beta X^{-1} \tag{115}$$

for $X_L(K_L)$.This gives two solutions.

$$X_L^* \;=\; \begin{cases} -(\delta_F - 2\delta_L)(r+\lambda_F-\mu) \\ -\frac{(1+\beta)(\delta_F-2\delta_L)(r+\lambda-\mu)}{(\beta-1)} \end{cases} \tag{116}$$

The one with β is the optimal decision, and

$$K_L^* = K_L^*(X_L^*) \;=\; \frac{1}{(1+\beta)\eta} \tag{117}$$

This coincides with the Huisman and Kort (2015) solution for $\lambda_F = \lambda_L = 0$, as

$$X_L^* \;=\; -\frac{(\delta_F - 2\delta_L)(1+\beta)(r-\mu)}{(\beta-1)} \tag{118}$$

and

$$K_L^* \;=\; \frac{1}{(1+\beta)\eta} \tag{119}$$

Case (ii): The same as case (i) but then replace β by β_λ^+.

Theorem 4.2. *The optimal capacity level and moment of entry for the follower and the leader in a Stackelberg accommodation equilibrium, based on a project that terminates with a probability λ for both the leader and the follower and*

(i) if the option exists forever then

$$X_L^{acc} = X_L^* = -\frac{(1+\beta)(\delta_F - 2\delta_L)(r+\lambda-\mu)}{(\beta-1)} \tag{120}$$

$$K_L^{acc} = K_L^* = \frac{1}{(1+\beta)\eta} \tag{121}$$

(ii) if the option terminates by one event with probability λ for both firms then

$$X_L^{acc} = X_L^* = -\frac{(1+\beta_\lambda^+)(\delta_F - 2\delta_L)(r+\lambda-\mu)}{(\beta_\lambda^+-1)} \tag{122}$$

$$K_L^{acc} = K_L^* = \frac{1}{(1+\beta_\lambda^+)\eta} \tag{123}$$

We define $X_1^{thres,acc}$ as

$$X_1^{thres,acc} = X_F^*(Q_L^{acc}(X_1^{thres,acc})) \tag{124}$$

where $X_F^* = (X_F^D)^*(K_L)$ is (96) or (98) and $Q_L^{acc} = K_L^*(X)$ is (113). Thus

(i)

$$X_1^{thres,acc} = (r-\mu+\lambda)\frac{(1+3\beta)\delta_F + 2\delta_L(1-\beta)}{\beta-1} \tag{125}$$

(ii)

$$X_1^{thres,acc} = (r-\mu+\lambda)\frac{(1+3\beta_\lambda^+)\delta_F + 2\delta_L(1-\beta_\lambda^+)}{\beta_\lambda^+-1} \tag{126}$$

4.2.2 Deterrence

For the deterrence strategy, the leader takes the strategy of the follower into account. Given the current level of X, the leader knows that the follower will invest later if it chooses its capacity $K_L = K_L$ such that $X_F^*(K_L) > X$. Thus the leader aims at letting the follower wait, this is called the deterrence strategy. As long as the demand level is low enough, the leader is a monopolist.

The value function of the leader at the moment of investment for the deterrence strategy is given by

$$rV_L^D(X,K_F,K_L) = \begin{cases} Profit_L^{DM}(X,K_F,K_L) + \lim_{dt\downarrow 0}\frac{1}{dt}\mathbb{E}[dV_L^D] & \text{if } X < X_F^*(K_L) \\ Profit_L^{DD}(X,K_F,K_L) + \lim_{dt\downarrow 0}\frac{1}{dt}\mathbb{E}[dV_L^D] & \text{if } X > X_F^*(K_L) \end{cases} \tag{127}$$

where

$$Profit_L^{DM} = K_L X(1-\eta K_L) \tag{128}$$

$$Profit_L^{DD} = K_L X(1-\eta(K_F^*(K_L)+K_L)) \tag{129}$$

$$\mathbb{E}[dV_L^D] = \frac{\partial V_L^D}{\partial X}\mu X dt + \frac{1}{2}\frac{\partial^2 V_L^D}{\partial X^2}\sigma^2 X^2 dt + \lambda dt(0-V_L^D). \tag{130}$$

For $X < X_F^*(K_L)$ the leader is still the only investor in the duopoly and thus acts temporarily as the monopolist which we denote by DM. The associated ODE is

$$(r+\lambda)V_L^{DM} = K_L X(1-\eta K_L) + \frac{\partial V_L^{DM}}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V_L^{DM}}{\partial X^2}\sigma^2 X^2 \tag{131}$$

the homogeneous solution is

$$0 = \frac{\partial V_L^{DM}}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V_L^{DM}}{\partial X^2}\sigma^2 X^2 - (r+\lambda)V_L^{DM} \tag{132}$$

$$V_L^{DM}(X) = B_F^{DM} X^{\beta_\lambda^+} + B_L^{DM} X^{\beta_\lambda^-} \tag{133}$$

where $\beta_\lambda^+ > 1, \beta_\lambda^- < 0$ solve (209). And the particular solution to the total differential equation with $V_L^{DM}(X) = a^{DM}X + b^{DM}$ leads to

$$(r+\lambda)(a^{DM}X + b^{DM}) = K_L X(1-\eta K_L) + a^{DM}\mu X \tag{134}$$

$$a^{DM} = \frac{K_L(1-\eta K_L)}{r+\lambda-\mu} \tag{135}$$

$$b^{DM} = 0 \tag{136}$$

Thus the total solution is

$$V_L^{DM}(X) = B_F^{DM} X^{\beta_\lambda^+} + B_L^{DM} X^{\beta_\lambda^-} + \frac{K_L(1-\eta K_L)}{r+\lambda-\mu}X \tag{137}$$

and the boundary condition is

$$V_L^{DM}(0) = 0 \tag{138}$$

which leads to $B_2^{DM} = 0$. But $B_1^{DM} \neq 0$ because if X goes to infinity we leave the monopoly state. Hence

$$V_L^{DM}(X) \quad = \quad B_F^{DM} X^{\beta_\lambda^+} + \frac{K_L(1-\eta K_L)}{r+\lambda-\mu} X \tag{139}$$

where β_λ^+ is the positive root of (209).

For $X > X_F^*(K_L)$ the ODE is

$$(r+\lambda) V_L^{DD} \quad = \quad K_L X(1-\eta(K_F^*(K_L)+K_L)) + \frac{\partial V_L^{DD}}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V_L^{DD}}{\partial X^2}\sigma^2 X^2 \tag{140}$$

the homogeneous solution is

$$0 \quad = \quad \frac{\partial V_L^{DD}}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V_L^{DD}}{\partial X^2}\sigma^2 X^2 - (r+\lambda) V_L^{DD} \tag{141}$$

$$V_L^{DD}(X) \quad = \quad B_F^{DD} X^{\beta_\lambda^+} + B_L^{DD} X^{\beta_\lambda^-} \tag{142}$$

where $\beta_\lambda^+ > 1, \beta_\lambda^- < 0$. And the particular solution of the differential equation with $V_L^D(X) = a^{DD}X + b^{DD}$ leads to

$$(r+\lambda)(a^{DD}X + b^{DD}) \quad = \quad K_L X(1-\eta(K_F^*(K_L)+K_L)) + a^{DD}\mu X \tag{143}$$

$$a^{DD} \quad = \quad \frac{K_L(1-\eta(K_F^*(K_L)+K_L))}{r+\lambda-\mu} \tag{144}$$

$$b^{DD} \quad = \quad 0 \tag{145}$$

Thus the total solution is

$$V_L^{DD}(X) \quad = \quad B_F^{DD} X^{\beta_\lambda^+} + B_L^{DD} X^{\beta_\lambda^-} + \frac{K_L(1-\eta(K_F^*(K_L)+K_L))}{r+\lambda-\mu} X \tag{146}$$

and the boundary condition is

$$V_L^{DD}(0) \quad = \quad 0 \tag{147}$$

$$\lim_{X\to\infty} V_L^{DD}(X) \quad = \quad wX \tag{148}$$

which leads to $B_2^{DD} = 0$ and $B_1^{DD} = 0$. Thus

Proposition 4.3.

$$V_L^{DM}(X) \quad = \quad B_F^{DM} X^{\beta_\lambda^+} + \frac{K_L(1-\eta K_L)}{r+\lambda-\mu} X \tag{149}$$

and

$$V_L^{DD}(X) \quad = \quad \frac{K_L(1-\eta(K_F^*(K_L)+K_L))}{r+\lambda-\mu}X \tag{150}$$

However, V_L^{DD} occurs when $X > X_F^*(K_L)$ and thus coincides with the Stackelberg accommodation strategy. Therefore for deriving the deterrence strategy we only need to concentrate on

$$rV_L^{DM}(X,K_F,K_L) = Profit_L^{DM}(X,K_F,K_L) + \lim_{dt\downarrow 0}\frac{1}{dt}\mathbb{E}[dV_L^D] \tag{151}$$

Though we also need the DD part to obtain B_1^{DM}.

At $X = X_F^*(K_L) = (X_F^D)^*(K_L)$ which is either equation (96) or (98) and $(K_F^D)^*(K_L)$ which is given by either equation (97) or (99), it should hold that

$$V_L^{DM}(X_F^*(K_L)) \quad = \quad V_L^{DD}(X_F^*(K_L)) \tag{152}$$

$$B_F^{DM}X^{\beta_\lambda^+} + \frac{K_L(1-\eta K_L)}{r+\lambda-\mu}X \quad = \quad \frac{K_L(1-\eta(K_F^*(K_L)+K_L))}{r+\lambda-\mu}X \tag{153}$$

which we can solve for

$$B_1^{DM} \quad = \quad \frac{-K_L\eta K_F^*(K_L)}{r+\lambda-\mu}X_F^*(K_L)^{1-\beta_\lambda^+} \tag{154}$$

Depending on whether the option of the *follower* is finite or not this implies

(i)

$$B_1^{DM} \quad = \quad \frac{-K_L(1-\eta K_L)}{(r+\lambda-\mu)(\beta+1)}\left(\frac{(r-\mu+\lambda)c}{1-\eta K_L}\frac{\beta+1}{\beta-1}\right)^{1-\beta_\lambda^+} \tag{155}$$

(ii)

$$B_1^{DM} \quad = \quad \frac{-K_L(1-\eta K_L)}{(r+\lambda-\mu)(\beta_\lambda^++1)}\left(\frac{(r-\mu+\lambda)c}{1-\eta K_L}\frac{\beta_\lambda^++1}{\beta_\lambda^+-1}\right)^{1-\beta_\lambda^+} \tag{156}$$

After this, solve for K_L by

$$\frac{\partial V_L^D - \delta_L K_L}{\partial K_L} = 0 \tag{157}$$

and use the smooth pasting and value matching condition to obtain X_L^*.

Since V_L^{DD} is exactly the same as the Stackelberg objective, we know already what the strategy in that domain will be.

For V_L^{DM} in case (i) we have

$$V_L^{DM}(X) = \frac{-K_L(1-\eta K_L)}{(r+\lambda-\mu)(\beta+1)}\left(\frac{(r-\mu+\lambda)c}{1-\eta K_L}\frac{\beta+1}{\beta-1}\right)^{1-\beta_\lambda^+} X^{\beta_\lambda^+} + \frac{K_L(1-\eta K_L)}{r+\lambda-\mu}X \tag{158}$$

$$= \frac{-K_L\delta_F}{\beta-1}\left(\frac{X(1-\eta K_L)}{(r-\mu+\lambda)\delta_F}\frac{\beta-1}{\beta+1}\right)^{\beta_\lambda^+} + \frac{K_L(1-\eta K_L)}{r+\lambda-\mu}X \tag{159}$$

while in case (ii) all β's are replaced by β_λ^+. Maximizing with respect to K_L gives the following first order condition.

$$\frac{\partial V_L^{DM} - \delta_L K_L}{\partial K_L} = 0 \tag{160}$$

$$\phi(X,K_L) = 0 \tag{161}$$

$$\begin{aligned}\phi(X,K_L) = {} & \frac{-\delta_F}{\beta-1}\left(\frac{X(1-\eta K_L)}{(r-\mu+\lambda)\delta_F}\frac{\beta-1}{\beta+1}\right)^{\beta_\lambda^+} \\ & + \frac{-K_L\delta_F\beta_\lambda^+}{\beta-1}\left(\frac{X(1-\eta K_L)}{(r-\mu+\lambda)\delta_F}\frac{\beta-1}{\beta+1}\right)^{\beta_\lambda^+-1}\left(\frac{-X\eta}{(r-\mu+\lambda)\delta_F}\frac{\beta-1}{\beta+1}\right) \\ & + \frac{(1-2\eta K_L)}{r+\lambda-\mu}X - \delta_L\end{aligned} \tag{162}$$

$$\begin{aligned} = {} & \frac{-\delta_F(1-(\beta_\lambda^+ +1)\eta K_L)}{(\beta-1)(1-\eta K_L)}\left(\frac{X(1-\eta K_L)}{(r-\mu+\lambda)\delta_F}\frac{\beta-1}{\beta+1}\right)^{\beta_\lambda^+} \\ & + \frac{(1-2\eta K_L)}{r+\lambda-\mu}X - \delta_L\end{aligned} \tag{163}$$

From this equation, it can be derived that the optimal entry deterrence capacity level is increasing in X. It follows that by putting K_L equal to zero a value for X is found, denoted by $X_1^{thres,det}$, below which an entry deterrence strategy will not occur. Then, the demand level is simply too low for an investment to be profitable. Hence, $X_1^{thres,det}$ is implicitly determined from putting K_L equal to zero in $\phi(X,0)$. Also an upper bound exists, which we denote by $X_2^{thres,det}$, above which entry deterrence cannot occur. The rationale for this is that for large enough X the output price is so high that it is always optimal for both firms to invest. To determine $X_2^{thres,det}$, which is by definition the lowest value of X for which the follower invests at the same time as the leader, we should recognize that it should hold that the entry deterrence threshold is such that it equals the follower threshold. We define $X_2^{thres,det}$ as

$$X_F^*(K_L^{det}(X_2^{thres,det})) = X_2^{thres,det} \tag{164}$$

For this reason we substitute equation (96) for X into (161) leads to

$$\frac{(1-2\eta K_L)}{1-\eta K_L}\frac{(\beta+1)\delta_F}{\beta-1}-\frac{\delta_F(1-(\beta_\lambda^+ +1)\eta K_L)}{(\beta-1)(1-\eta K_L)}-\delta_L=0 \tag{165}$$

so that

$$K_L=\frac{\beta\delta_F-(\beta-1)\delta_L}{\eta(\beta(\delta_F-\delta_L)+\delta_F+\delta_L)+(\beta-\beta_\lambda^+)\delta_F} \tag{166}$$

Substituting this into (96) or (98) gives

(i) if the option exists forever for the follower

$$X_2^{thres,det} \quad = \quad (r-\mu+\lambda)\frac{(\beta(\delta_F-\delta_L)+\delta_F+\delta_L+(\beta-\beta_\lambda^+)\delta_F)}{1+(\beta-\beta_\lambda^+)}\frac{\beta+1}{\beta-1} \tag{167}$$

(ii) if the option can end for the follower

$$X_2^{thres,det} \quad = \quad (r-\mu+\lambda)(\beta_\lambda^+(\delta_F-\delta_L)+\delta_F+\delta_L)\frac{\beta_\lambda^+ +1}{\beta_\lambda^+ -1} \tag{168}$$

Before the leader has invested, thus when $X<X_L^{det}$, the firm holds an option to invest. The option value is The stopping value is determined based either on the assumption that the option to invest always exists or on the assumption that the option to invest can disappear for both firms simultaneously. These imply

(i) $F(X)=AX^\beta$ if the option always remains (β is without λ).

(ii) $F(X)=AX^{\beta_\lambda^+}$ if the option to invest can vanish before the firm has invested

Case (i): The value matching and smooth pasting condition together lead to

$$\begin{aligned}\frac{\partial V_L^{DM}(X)}{\partial X} &= (V_L^{DM}(X)-\delta_L K_L)\beta X^{-1} &(169)\\ \frac{XK_L(1-\eta K_L)}{r+\lambda-\mu}(\beta-1)-\delta_L K_L\beta+B_1^{DM}X^{\beta_\lambda^+}(\beta-\beta_\lambda^+) &= 0 &(170)\end{aligned}$$

TO BE ADDED: Numerical results.

Case (ii): The value matching and smooth pasting condition together lead to

$$\frac{\partial V_L^{DM}(X)}{\partial X} = (V_L^{DM}(X) - \delta_L K_L)\beta_\lambda^+ X^{-1} \tag{171}$$

$$\frac{XK_L(1-\eta K_L)}{r+\lambda-\mu}(\beta_\lambda^+ - 1) - \delta_L K_L \beta_\lambda^+ = 0 \tag{172}$$

so that the leader threshold is given by

$$X_L^{det} = \frac{\beta_\lambda^+ + 1}{\beta_\lambda^+ - 1}\delta_L(r+\lambda-\mu) \tag{173}$$

and

$$K_L^{det} = \frac{1}{(\beta_\lambda^+ + 1)\eta} \tag{174}$$

All results are equal to the infinite horizon case in which r is replaced for $r+\lambda$. And λ is included in the β if the event that the project can terminate both before it has started is incorporated.

Proposition 4.4. *In case (ii): The optimal capacity level of the leader satisfies*

$$Q_{L,\lambda}^*(X) = \begin{cases} Q_L^{det}(X_L^{det}) & if\, X < X_L^{det} \\ Q_L^{det}(X) & if\, X_L^{det} < X < \hat{X} \\ Q_L^{acc}(X) & if\, X > \hat{X} \end{cases} \tag{175}$$

where $\hat{X} = \min\left\{X \in \left(X_1^{thres,acc}, X_2^{thres,det}\right) \middle| V_L^{acc}(X) = V_L^{det}(X)\right\}$.

The value of the leader is given by

$$V_{L,\lambda_L}^*(X) = \begin{cases} \left(\frac{X}{X_L^{det}}\right)^\beta V_L^{det}(X_L^{det}) & if\, X < X_L^{det} \\ V_L^{det}(X) & if\, X_L^{det} < X < \hat{X} \\ V_L^{acc}(X) & if\, X > \hat{X} \end{cases} \tag{176}$$

$$X_L^{det} = \frac{\beta_\lambda^+ + 1}{\beta_\lambda^+ - 1}\delta_L(r+\lambda-\mu) \tag{177}$$

$$\phi(X, K_L^{det}) = 0 \tag{178}$$

$$K_L^{acc}(X) = \left(1 - \frac{(2\delta_L - \delta_F)(r+\lambda-\mu)}{X}\right)\frac{1}{2\eta} \tag{179}$$

$$X_L^{thres,det} = (r-\mu+\lambda)(\beta_\lambda^+(\delta_F - \delta_L) + \delta_F + \delta_L)\frac{\beta_\lambda^+ + 1}{\beta_\lambda^+ - 1} \tag{180}$$

$$X_F^{thres,acc} = (r-\mu+\lambda)\frac{(1+3\beta_\lambda^+)\delta_F + 2\delta_L(1-\beta_\lambda^+)}{\beta_F - 1} \tag{181}$$

$$V_L^{DM}(X) = V_L^{det}(X) = \frac{-K_L\delta_F}{\beta_\lambda^+ - 1}\left(\frac{X(1-\eta K_L)}{(r-\mu+\lambda)\delta_F}\frac{\beta_\lambda^+ - 1}{\beta_\lambda^+ + 1}\right)^{\beta_\lambda^+} + \frac{K_L(1-\eta K_L)}{r+\lambda-\mu}X \tag{182}$$

$$V_L^D(X) = V_L^{acc}(X) = \frac{K_L(1-\eta(K_F+K_L))}{r+\lambda-\mu}X \tag{183}$$

$$V_L^{acc}(X) = V_L^D(X) - \delta_L K_L = \frac{\big(X + (\delta_F - 2\delta_L)(r+\lambda-\mu)\big)^2}{8X\eta(r+\lambda-\mu)} \tag{184}$$

5 Duopoly

In this section we consider competition in the terminating project life's investment problem. Let $Q_L = Q_1$ be the capacity of the leader and $Q_F = Q_2$ of the follower. The total production in the market is $Q = Q_L + Q_F$. Given the investment of the leader, the follower cannot influence the leader's decision and the follower has to determine the optimal trigger and capacity given the leader's strategies. Thus we look for $Q_F^*(Q_L)$ and $X_F^*(Q_L)$, that is why we first solve for the follower.

The value function of the follower is denoted by V_F^* and depends on X, Q_L, Q_F. As long as the leader is in the market, the demand is determined by both Q_L and Q_F but after the leader's project terminates, the total demand implied by Q_F is for the follower. Therefore, we first derive the value function for the follower based on the assumption that the leader leaves the market with probability $\lambda_L dt$.

5.1 Follower

We refer to the follower as firm 2. Recall the demand function

$$P(t) = X(t)\big(1 - \eta Q(t)\big), \tag{185}$$

where

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \tag{186}$$

and discount rate r. For firm 2, the follower, in a duopoly the value function equals

$$rV_F^D(X, Q_F, Q_L) = Profit_F^D(X, Q_F, Q_L) + \lim_{dt\downarrow 0} \frac{1}{dt}\mathbb{E}[dV_F^D] \tag{187}$$

where

$$\mathbb{E}[dV_F^D] = \frac{\partial V_F^D}{\partial X}\mu X dt + \frac{1}{2}\frac{\partial^2 V_F^D}{\partial X^2}\sigma^2 X^2 dt + \lambda_F dt(0 - V_F^D) + \lambda_L dt(V_F^M - V_F^D) \tag{188}$$

since if the project terminates, the value function of the follower becomes zero and if the project of the leader terminates, the follower becomes a monopoly with value function V_F^M. From the previous section we know

$$V_F^M - \delta_F K_F = XK_F(1 - \eta K_F)\frac{\lambda_F}{r-\mu}\left(\frac{1}{\lambda_F} - \frac{1}{r-\mu+\lambda_F}\right) - \delta_F K_F \tag{189}$$

Which is the same as (25), without the costs $\delta_F K_F$

$$V_F^M(X) = \frac{XK_F(1-\eta K_F)}{r-\mu+\lambda_F} \tag{190}$$

we plug in the above equation for general K and X. And the profit function is $P \cdot Q$ when both firms are in the market,

$$Profit_F^D = K_F X(1 - \eta(K_F + K_L)) \tag{191}$$

This leads to the ODE

$$V_F^D = \frac{1}{(r+\lambda_F+\lambda_L)}\Big(K_F X(1-\eta(K_F+K_L)) + \frac{\partial V_F^D}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V_F^D}{\partial X^2}\sigma^2 X^2 + \lambda_L V_F^M\Big) \tag{192}$$

The homogeneous equation (terms involving value function) is

$$0 = \frac{\partial V_F^D}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V_F^D}{\partial X^2}\sigma^2 X^2 - (r+\lambda_F+\lambda_L)V_F^D \tag{193}$$

having as solution

$$V_F^D(X) = B_1 X^{\beta^+_{\lambda_{F,L}}} + B_2 X^{\beta^-_{\lambda_{F,L}}} \tag{194}$$

where $\beta^+_{\lambda_{F,L}} > 1, \beta^-_{\lambda_{F,L}} < 0$.

A particular solution is

$$
\begin{aligned}
V_F^D(X) &= aX + b && (195)\\
V_F^M(X) &= \frac{XK_F(1-\eta K_F)}{r-\mu+\lambda_F} && (196)\\
(r+\lambda_F+\lambda_L)(aX+b) &= K_F X(1-\eta(K_F+K_L)) + a\mu X + V_F^M(X) &&\\
a &= \frac{K_F(1-\eta(K_F+K_L)) + \frac{\lambda_L}{r-\mu+\lambda_F}K_F(1-\eta K_F)}{r-\mu+\lambda_F+\lambda_L} && (197)\\
b &= 0. && (198)
\end{aligned}
$$

Such that the total solution is

$$
V_F^D(X) = \frac{K_F(1-\eta(K_F+K_L)) + \frac{\lambda_L}{r-\mu+\lambda_F}K_F(1-\eta K_F)}{r-\mu+\lambda_F+\lambda_L} X + B_1 X^{\beta^+_{\lambda_{F,L}}} + B_2 X^{\beta^-_{\lambda_{F,L}}} \qquad (199)
$$

with boundary conditions

$$
\begin{aligned}
V_F^D(0) &= 0 && (200)\\
\lim_{X\to\infty} V_F^D(X) &= wX && (201)
\end{aligned}
$$

This leads to $B_2 = 0$, and $B_1 = 0$. Therefore the solution is

$$
V_F^D(X) = \frac{K_F X(1-\eta(K_F+K_L)) + \frac{\lambda_L}{r-\mu+\lambda_F}K_F X(1-\eta K_F)}{r-\mu+\lambda_F+\lambda_L} \qquad (202)
$$

Maximizing w.r.t. K_F.

$$
\begin{aligned}
\frac{\partial V_F^D - \delta_F K_F}{\partial K_F} &= 0 && (203)\\
\frac{X(1-\eta(2K_F+K_L)) + \frac{\lambda_L}{r-\mu+\lambda_F}X(1-2\eta K_F)}{r-\mu+\lambda_F+\lambda_L} - \delta_F &= 0 && (204)
\end{aligned}
$$

And thus

$$
\begin{aligned}
K_F^D(X, K_L) &= \frac{(r+\lambda_F-\mu)(1-\eta K_L)+\lambda_L}{2\eta(r+\lambda_F+\lambda_L-\mu)} - \frac{\delta_F(r+\lambda_F-\mu)}{2\eta X} && (205)\\
&= K_F^M(X) - \frac{K_L(r+\lambda_F-\mu)}{2(r+\lambda_F+\lambda_L-\mu)} && (206)
\end{aligned}
$$

The continuation region is defined by

$$rF_F^D : \begin{cases} rF_F^{D1} = \frac{\partial F_F^{D1}}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 F_F^{D1}}{\partial X^2}\sigma^2 X^2 + \lambda_L\left(F_F^M - F_F^{D1}\right) & \text{if } 0 < X < (X^M)^* \\ rF_F^{D2} = \frac{\partial F_F^{D2}}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 F_F^{D2}}{\partial X^2}\sigma^2 X^2 + \lambda_L\left(V_F^M - \delta_F K_F - F_F^{D2}\right) & \text{if } (X^M)^* < X < (X^F)^* \end{cases} \tag{207}$$

where $(X^M)^*$ is derived in (34)

$$(X^M)^* = (r - \mu + \lambda_F)\delta_F \frac{\beta + 1}{\beta - 1}$$

Or in case we incorporate the probability that the project ends before it has started, then

$$(X^M)^* = (r - \mu + \lambda_F)\delta_F \frac{\beta_{\lambda_F}^+ + 1}{\beta_{\lambda_F}^2 - 1}$$

The continuation values have the form

$$F_F^D(X) = \begin{cases} A_1 X^{\beta_{\lambda_L}^+} + A_2 X^{\beta_{\lambda_L}^-} & \text{if } 0 < X < (X^M)^* \\ A_3 X^{\beta_{\lambda_L}^+} + A_4 X^{\beta_{\lambda_L}^-} & \text{if } (X^M)^* < X < (X^F)^* \end{cases} \tag{208}$$

in the homogeneous part. Since $F_F^{D1}(0) = 0$ it follows that $A_2 = 0$. By plugging these forms into the ODEs above we obtain that

$$\frac{1}{2}\beta_{\lambda_L}(1 - \beta_{\lambda_L})\sigma^2 + \beta_{\lambda_L}\mu - (r + \lambda_L) = 0, \tag{209}$$

where $\beta_{\lambda_L}^+$ is the positive root of this equation, and $\beta_{\lambda_L}^-$ the negative root.

A particular solution with Ansatz $F_F^{D1}(X) = a\left(A_5 X^{\beta_5}\right) + c$ solves the non-homogeneous equation for $A_5 = A$ and $\beta_5 = \beta$ coming from $F_F^M(X) = AX^\beta$ where A is derived in (38) or $\beta_5 = \beta_{\lambda_F}^+$ and A is adjusted accordingly.

$$\begin{aligned} (r + \lambda_L)F_F^{D1} &= \frac{\partial F_F^{D1}}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 F_F^{D1}}{\partial X^2}\sigma^2 X^2 + \lambda_L A X^\beta & (210) \\ (r + \lambda_L)\left(aAX^\beta + c\right) &= aAX^\beta \beta\mu + \frac{1}{2}aAX^\beta \beta(\beta - 1)\sigma^2 + \lambda_L A X^\beta & (211) \\ a &= \frac{\lambda_L}{r + \lambda_L - \beta\mu - \beta\frac{1}{2}(\beta - 1)\sigma^2} = 1 & (212) \\ c &= 0 & (213) \end{aligned}$$

Hence the total solution is, in case (i)

$$F_F^{D1}(X) = A_1 X^{\beta^+_{\lambda_L}} + AX^{\beta} \tag{214}$$

or, in case (ii)

$$F_F^{D1}(X) = A_1 X^{\beta^+_{\lambda_L}} + A_{\beta^+_{\lambda_F}} X^{\beta^+_{\lambda_F}} \tag{215}$$

where only A_1 is unknown. A particular solution for $F_F^{D2}(X)$ has to solve

$$(r+\lambda_L)F_F^{D2} = \frac{\partial F_F^{D2}}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 F_F^{D2}}{\partial X^2}\sigma^2 X^2 + \lambda_L\left(V_F^M(X, K_F^M(X)) - \delta_F K_F^M(X)\right) \tag{216}$$

where $K_F^M(X)$ is a function of X. Plugging (27) into (25) minus the costs yields

$$V_F^M(X, K_F^M(X)) - \delta_F K_F^M(X) = \frac{\left(X - \delta_F(r-\mu+\lambda_F)\right)^2}{4\eta(r-\mu+\lambda_F)X}. \tag{217}$$

Therefore we propose a functional form of $F_F^{D2}(X) = aX + b + cX^{-1}$ which solves for

$$a = \frac{\lambda_L}{4\eta\left(r-\mu+\lambda_L\right)\left(r-\mu+\lambda_F\right)} \tag{218}$$

$$b = -\frac{\lambda_L\delta_F}{2\eta(r+\lambda_L)} \tag{219}$$

$$c = \frac{\lambda_L\delta_F^2\left(r-\mu+\lambda_F\right)^2}{\left(r+\mu+\lambda_L-\sigma^2\right)4\eta} \tag{220}$$

Hence the total solution is

$$F_F^{D2}(X) = A_3 X^{\beta^+_{\lambda_L}} + A_4 X^{\beta^-_{\lambda_L}} + aX + b + cX^{-1} \tag{221}$$

where A_3 and A_4 are unknown.

Both $A_3(A_1)$ and $A_4(A_1)$ can be obtained by the equalities that

$$F_F^{D1}(X^M) = F_F^{D2}(X^M) \tag{222}$$

$$\left.\frac{\partial F_F^{D1}(X)}{\partial X}\right|_{X=X^M} = \left.\frac{\partial F_F^{D2}(X)}{\partial X}\right|_{X=X^M} \tag{223}$$

Leading to, in case (i)

$$
\begin{aligned}
A_3 &= \frac{c(1+\beta^-_{\lambda_L}) + X_M\left(AX_M^{\beta}(\beta-\beta^-_{\lambda_L}) + aX_M(\beta^-_{\lambda_L}-1) + b\beta^-_{\lambda_L} - A_1 X_M^{\beta^+_{\lambda_L}}(\beta^-_{\lambda_L}-\beta^+_{\lambda_L})\right)}{X_M^{1-\beta^+_{\lambda_L}}(\beta^+_{\lambda_L}-\beta^-_{\lambda_L})} \\
A_4 &= \frac{c(1+\beta^+_{\lambda_L}) + X_M\left(AX_M^{\beta}(\beta-\beta^+_{\lambda_L}) + aX_M(\beta^+_{\lambda_L}-1) + b\beta^+_{\lambda_L}\right)}{X_M^{1-\beta^-_{\lambda_L}}(\beta^-_{\lambda_L}-\beta^+_{\lambda_L})} \qquad (224)
\end{aligned}
$$

and to, in case (ii)

$$
A_3 = \frac{c(1+\beta^-_{\lambda_L}) + X_M\left(AX_M^{\beta^+_{\lambda_F}}(\beta^+_{\lambda_F}-\beta^-_{\lambda_L}) + aX_M(\beta^-_{\lambda_L}-1) + b\beta^-_{\lambda_L} - A_1 X_M^{\beta^+_{\lambda_L}}(\beta^-_{\lambda_L}-\beta^+_{\lambda_L})\right)}{X_M^{1-\beta^+_{\lambda_L}}(\beta^+_{\lambda_L}-\beta^-_{\lambda_L})} \qquad (225)
$$

$$
A_4 = \frac{c(1+\beta^+_{\lambda_L}) + X_M\left(AX_M^{\beta^+_{\lambda_F}}(\beta^+_{\lambda_F}-\beta^+_{\lambda_L}) + aX_M(\beta^+_{\lambda_L}-1) + b\beta^+_{\lambda_L}\right)}{X_M^{1-\beta^-_{\lambda_L}}(\beta^-_{\lambda_L}-\beta^+_{\lambda_L})} \qquad (226)
$$

The value matching condition is

$$
\begin{aligned}
V_F^D(X) - \delta_F K_F &= F_F^{D2}(X) \qquad (227) \\
\frac{K_F X(1-\eta(K_F+K_L)) + \frac{\lambda_L}{r-\mu+\lambda_F}K_F X(1-\eta K_F)}{r-\mu+\lambda_F+\lambda_L} - \delta_F K_F &= A_3 X^{\beta^+_{\lambda_L}} + A_4 X^{\beta^-_{\lambda_L}} + aX + b + cX^{-1} \qquad (228)
\end{aligned}
$$

at $X = X_F$ and the smooth pasting

$$
\begin{aligned}
\frac{\partial V_F^D(X)}{\partial X}\Big|_{X=X_F} &= \frac{\partial F_F^{D2}(X)}{\partial X}\Big|_{X=X_F} \qquad (229) \\
\frac{K_F(1-\eta(K_F+K_L)) + \frac{\lambda_L}{r-\mu+\lambda_F}K_F(1-\eta K_F)}{r-\mu+\lambda_F+\lambda_L} &= \beta^+_{\lambda_L} A_3 X^{\beta^+_{\lambda_L}-1} + \beta^-_{\lambda_L} A_4 X^{\beta^-_{\lambda_L}-1} + a - cX^{-2} \qquad (230)
\end{aligned}
$$

together they imply at $X = X_F$

$$
\begin{aligned}
\frac{\partial F_F^{D2}(X)}{\partial X} X - \delta_F K_F &= F_F^{D2}(X) \qquad (231) \\
A_3(A_1)X^{\beta^+_{\lambda_L}}(\beta^+_{\lambda_L}-1) + A_4(A_1)X^{\beta^-_{\lambda_L}}(\beta^-_{\lambda_L}-1) &= \delta_F K_F^D(X,K_L) + b \qquad (232) \\
&= \delta_F\left(K_F^D(X,K_L) - \frac{\lambda_L}{2\eta(r+\lambda_L)}\right) \qquad (233) \\
&= \frac{\delta_F}{2\eta}\left(\frac{(r+\lambda_F-\mu)(1-\eta K_L)+\lambda_L}{(r+\lambda_F+\lambda_L-\mu)} - \frac{\delta_F(r+\lambda_F-\mu)}{X} - \frac{\lambda_L}{(r+\lambda_L)}\right)
\end{aligned}
$$

to get $X_F(A_1)$ and we re-use the value matching condition to get A_1 at $X = X_F$

$$V_F^D(X) - \delta_F K_F = F_F^{D2}(X) \tag{234}$$

$$\frac{K_F^D(X,K_L)X(1-\eta(K_F^D(X,K_L)+K_L)) + \frac{\lambda_L}{r-\mu+\lambda_F}K_F^D(X,K_L)X(1-\eta K_F^D(X,K_L))}{r-\mu+\lambda_F+\lambda_L} - \delta_F K_F^D(X,K_L)$$
$$= A_3(A_1)X^{\beta^+_{\lambda_L}} + A_4(A_1)X^{\beta^-_{\lambda_L}} + aX + b + cX^{-1} \tag{235}$$

Since $X_F(A_1)$ has to be solved numerically, we need to solve for X_F and A_1 simultaneously. For fixed K_L we can find the roots of X_F and A_1 (FindRoot, but it matters a lot where we let the root search start from!!!)

Theorem 5.1. *The optimal capacity level and moment of entry for the follower given the leader's strategy, based on a project that terminates with a probability λ_L for the leader and λ_F for the follower are*

$$\begin{aligned} K_F^D(X_F^*, K_L) &= \frac{(r+\lambda_F-\mu)(1-\eta K_L)+\lambda_L}{2\eta(r+\lambda_F+\lambda_L-\mu)} - \frac{\delta_F(r+\lambda_F-\mu)}{2\eta X_F^*} && (236)\\ X_F^*(K_L) &= numerically && (237)\end{aligned}$$

The value of the follower is

$$V_{F,\lambda_F}(X,K_L) = \begin{cases} F_F^{D1}(X) & \text{if } X \le X_M^* \\ F_F^{D2}(X) & \text{if } X_M^* \le X \le X_F^*(K_L) \\ V_F^D(X) - \delta_F K_F^D(X,K_L) & \text{if } X \ge X_F^*(K_L) \end{cases} \tag{238}$$

(i) If the option exists forever then A_3 and A_4 are defined by (225) and $F_F^{D1}(X)$ is defined by (214) with β and A given by (38) and (40) with the parameters set for the follower.

(ii) If the option can disappear before the project has started with probability λ_F then A_3 and A_4 are defined by (225) and $F_F^{D1}(X)$ is defined by (215) with $\beta^+_{\lambda_F}$ and $A^+_{\lambda_F}$ given by (39) and (41) with the parameters set for the follower.

As a sanity check; This is equivalent to the equations (34) and (35) for the monopolist if we assume that $K_L = 0$.

5.2 Leader

We refer to the leader as firm 1. We can differentiate two cases for duopolies with probabilistic termination of projects

- Stackelberg accommodation
- Deterrence

If both firms optimize their strategies simultaneously, we obtain the Cournot accommodation equilibrium. While adjusting one's strategy to the known and anticipated strategy of the other firms yields the Stackelberg equilibrium. Given the derived strategies of the follower given the leader, we have obtained firm 2's decision $K_F^D(X_F^*, K_L)$ and thus firm 1 knows how the capacity of firm 2 depends on his own choice. Moreover, the accommodation strategy implies that both firms invest at the same time so X_F^* and X_L^* are redundant. It implies also that we need to consider all possible roles a firm can have, i.e. firm 1 can be both the leader or the follower, and vice versa. The deterrence strategy is obtained by constructing the ODE for the case in which the follower is not in the market yet, based on the previous section this occurs when $X < X_F^*(K_L)$, and the ODE for the duopoly when $X > X_F^*(K_L)$.

5.2.1 Accommodation

The leader knows the strategies of the follower, and anticipates to these by the Stackelberg accommodation strategy as follows.

$$rV_L^D(X, K_F, K_L) = Profit_L^D(X, K_F, K_L) + \lim_{dt \downarrow 0} \frac{1}{dt} \mathbb{E}[dV_L^D] \tag{239}$$

where

$$\mathbb{E}[dV_L^D] = \frac{\partial V_L^D}{\partial X} \mu X dt + \frac{1}{2} \frac{\partial^2 V_L^D}{\partial X^2} \sigma^2 X^2 dt + \lambda_L dt(0 - V_L^D) + \lambda_F dt(V_L^M - V_L^D). \tag{240}$$

From (25), we know

$$V_L^M = \frac{XK_L(1-\eta K_L)}{r - \mu + \lambda_L} \tag{241}$$

And now we know the strategy of firm 1, thus

$$Profit_L^D = K_L X(1 - \eta(K_F^*(K_L) + K_L)) \tag{242}$$

In the profit function, $K_F(K_L)$ which depends on the X_F^* which is not analytically solvable. The objective is

$$V_L^D = \frac{1}{(r + \lambda_F + \lambda_L)} \Big(K_L X(1 - \eta(K_F^*(K_L) + K_L)) + \frac{\partial V_L^D}{\partial X} \mu X + \frac{1}{2} \frac{\partial^2 V_L^D}{\partial X^2} \sigma^2 X^2 + \lambda_F V_L^M \Big) \tag{243}$$

The solution of the homogeneous equation (terms involving value function) is

$$
\begin{aligned}
0 &= \frac{\partial V_L^D}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V_L^D}{\partial X^2}\sigma^2 X^2 - (r+\lambda_F+\lambda_L)V_L^D && (244)\\
V_L^D(X) &= B_1 X^{\beta_{F,L}^+} + B_2 X^{\beta_{F,L}^-} && (245)
\end{aligned}
$$

where $\beta_{F,L}^+ > 1, \beta_{F,L}^- < 0$. The particular solution (terms without value function) is

$$
\begin{aligned}
V_L^D(X) &= aX + b && (246)\\
(r+\lambda_F+\lambda_L)(aX+b) &= K_L X(1-\eta(K_F^*(K_L)+K_L)) + a\mu X + \lambda_F \frac{XK_L(1-\eta K_L)}{r-\mu+\lambda_L} \\
a &= \frac{K_L(1-\eta(K_F^*(K_L)+K_L)) + \lambda_F \frac{K_L(1-\eta K_L)}{r-\mu+\lambda_L}}{r-\mu+\lambda_F+\lambda_L} && (247)\\
b &= 0. && (248)
\end{aligned}
$$

The total solution is

$$
V_L^D(X) = X\frac{K_L(1-\eta(K_F^*(K_L)+K_L)) + \lambda_F \frac{K_L(1-\eta K_L)}{r-\mu+\lambda_L}}{r-\mu+\lambda_F+\lambda_L} + B_1 X^{\beta_{F,L}^+} + B_2 X^{\beta_{F,L}^-}
$$

with boundary conditions

$$
\begin{aligned}
V_L^D(0) &= 0 && (249)\\
\lim_{X\to\infty} V_L^D(X) &= wX && (250)
\end{aligned}
$$

Thus both $B_2 = 0$ and $B_1 = 0$. And therefore

$$
V_L^D(X) = X\frac{K_L(1-\eta(K_F^*(K_L)+K_L)) + \lambda_F \frac{K_L(1-\eta K_L)}{r-\mu+\lambda_L}}{r-\mu+\lambda_F+\lambda_L} \quad (251)
$$

Now we get K_L by

$$
\frac{\partial V_L^D - \delta_L K_L}{\partial K_L} = 0 \quad (252)
$$

If we plug in $K_F^*(X, K_L)$ as given by (206) then we get

$$\begin{aligned} K_F^D(X, K_L) &= \frac{(r+\lambda_F-\mu)(1-\eta K_L)+\lambda_L}{2\eta(r+\lambda_F+\lambda_L-\mu)} - \frac{\delta_F(r+\lambda_F-\mu)}{2\eta X} && (253)\\ &= K_F^M(X) - \frac{K_L(r+\lambda_F-\mu)}{2(r+\lambda_F+\lambda_L-\mu)} && (254)\\ K_F^M(X) &= \frac{1}{2\eta}\left(1-\frac{\delta_F(r-\mu+\lambda_F)}{X}\right) && (255)\\ \frac{\partial K_F^D(X,K_L)}{\partial K_L} &= -\frac{(r+\lambda_F-\mu)}{2(r+\lambda_F+\lambda_L-\mu)} && (256)\\ \frac{\partial V_L^D-\delta_L K_L}{\partial K_L} &= \frac{X}{r-\mu+\lambda_F+\lambda_L}\left((1-\eta(K_F^D(X,K_L)+K_L))-\eta K_L\left(\frac{\partial K_F^D(X,K_L)}{\partial K_L}+1\right)+\lambda_F\frac{(1-2\eta K_L)}{r-\mu+\lambda_L}\right)-\delta_L \end{aligned}$$

and thus

$$\begin{aligned} K_L^*(X) &= \frac{1}{2X\eta(r^2+2\lambda_F^2+2\lambda_L^2+r(3\lambda_F+3\lambda_L-2\mu)+3\lambda_F(\lambda_L-\mu)-3\lambda_L\mu+\mu^2)} \\ &\quad \left[\delta_F(r+\lambda_F-\mu)(r+\lambda_L-\mu)-2\delta_L(r+\lambda_L-\mu)(r+\lambda_F+\lambda_L-\mu)+X(r+2\lambda_F+\lambda_L-\mu)\right] \\ &\quad (r+\lambda_F+\lambda_L-\mu) \end{aligned} \tag{257}$$

First plug $K_L^*(X)$ in V_L^D to get

$$V_L^D(X) = \frac{\left[\delta_F(r+\lambda_F-\mu)(r+\lambda_L-\mu)-2\delta_L(r+\lambda_L-\mu)(r+\lambda_F+\lambda_L-\mu)+X(r+2\lambda_F+\lambda_L-\mu)\right]^2}{8X\eta(r+\lambda_L-\mu)(r^2+2\lambda_F^2+2\lambda_L^2+r(3\lambda_F+3\lambda_L-2\mu)+3\lambda_F(\lambda_L-\mu)-3\lambda_L\mu+\mu^2)}$$

and then by the value and smooth pasting conditions we solve for $X_L(K_L)$. The stopping value is determined based on the assumption that the option to invest always exists or that the option to invest can disappear for both firms with different probabilities. The case in which both options disappear at the same event with equal chances throughout is treated in Section 4. These imply

(i) $F(X) = AX^\beta$ if the option always remains

(ii) $F(X) = A_{\beta_L^+} X^{\beta_L^+}$ if the option to invest can vanish before the firm has invested

The value matching and smooth pasting conditions for case (i) yield

$$\frac{\partial V_L^D(X)}{\partial X} = (V_L^D(X) - \delta_L K_L)\beta X^{-1} \tag{258}$$

for $X_L(K_L)$.This gives two solutions.

$$X_L^* = \begin{cases} -\frac{(\delta_F(r+\lambda_F-\mu)-2\delta_L(r+\lambda_F+\lambda_L-\mu))(r+\lambda_L-\mu)}{r+2\lambda_F+\lambda_L-\mu} \\ -\frac{(1+\beta)(\delta_F(r+\lambda_F-\mu)-2\delta_L(r+\lambda_F+\lambda_L-\mu))(r+\lambda_L-\mu)}{(\beta-1)(r+2\lambda_F+\lambda_L-\mu)} \end{cases} \tag{259}$$

for both β and β_L^+ is obtained simply by replacing β for β_F. The one with β is the optimal decision, and

$$K_L^* = K_L^*(X_L^*) = \frac{(r+\lambda_F+\lambda_L-\mu)(r+2\lambda_F+\lambda_L-\mu)}{(1+\beta)\eta(r^2+2\lambda_F^2+2\lambda_L^2+r(3\lambda_F+3\lambda_L-2\mu)+3\lambda_F(\lambda_L-\mu)-3\lambda_L\mu+\mu^2)}$$

This coincides with the infinite horizon results from Huisman and Kort (2015) for $\lambda_F = \lambda_L = 0$, as

$$X_L^* = -\frac{(\delta_F-2\delta_L)(1+\beta)(r-\mu)}{(\beta-1)} \tag{260}$$

and

$$K_L^* = \frac{1}{(1+\beta)\eta} \tag{261}$$

Theorem 5.2. *The optimal capacity level and moment of entry for the follower and the leader in a Stackelberg accommodation equilibrium, based on a project that terminates with a probability λ_L for the leader and λ_F for the follower are,*

(i) If the option exists forever

$$\begin{aligned} X_L^{acc} = X_L^* &= -\frac{(1+\beta)(\delta_F(r+\lambda_F-\mu)-2\delta_L(r+\lambda_F+\lambda_L-\mu))(r+\lambda_L-\mu)}{(\beta-1)(r+2\lambda_F+\lambda_L-\mu)} \\ K_L^{acc} = K_L^* &= \frac{(r+\lambda_F+\lambda_L-\mu)(r+2\lambda_F+\lambda_L-\mu)}{(1+\beta)\eta(r^2+2\lambda_F^2+2\lambda_L^2+r(3\lambda_F+3\lambda_L-2\mu)+3\lambda_F(\lambda_L-\mu)-3\lambda_L\mu+\mu^2)} \end{aligned} \tag{262}$$

(ii) If the option of the leader can disappear before the project has started with probability λ_L

$$\begin{aligned} X_L^{acc} = X_L^* &= -\frac{(1+\beta_{\lambda_L}^+)(\delta_F(r+\lambda_F-\mu)-2\delta_L(r+\lambda_F+\lambda_L-\mu))(r+\lambda_L-\mu)}{(\beta_{\lambda_L}^+-1)(r+2\lambda_F+\lambda_L-\mu)} \\ K_L^{acc} = K_L^* &= \frac{(r+\lambda_F+\lambda_L-\mu)(r+2\lambda_F+\lambda_L-\mu)}{(1+\beta_{\lambda_L}^+)\eta(r^2+2\lambda_F^2+2\lambda_L^2+r(3\lambda_F+3\lambda_L-2\mu)+3\lambda_F(\lambda_L-\mu)-3\lambda_L\mu+\mu^2)} \end{aligned} \tag{263}$$

5.2.2 Deterrence

For the deterrence strategy, the leader takes the strategy of the follower into account. Given the current level of X, the leader knows that the follower will invest later if it chooses its capacity

$K_L = K_L$ such that $X_F^*(K_L) > X$. Thus the leader aims at letting the follower wait, this is called the deterrence strategy. As long as the demand level is low enough, the leader is a monopolist.

The value function of the leader at the moment of investment for the deterrence strategy is given by

$$rV_L^D(X, K_F, K_L) = \begin{cases} Profit_L^{DM}(X, K_F, K_L) + \lim_{dt\downarrow 0} \frac{1}{dt}\mathbb{E}[dV_L^{DM}] & \text{if } X < X_F^*(K_L) \\ Profit_L^{DD}(X, K_F, K_L) + \lim_{dt\downarrow 0} \frac{1}{dt}\mathbb{E}[dV_L^{DD}] & \text{if } X > X_F^*(K_L) \end{cases} \tag{264}$$

where

$$Profit_L^{DM} = K_L X(1 - \eta K_L) \tag{265}$$
$$Profit_L^{DD} = K_L X(1 - \eta(K_F^*(K_L) + K_L)) \tag{266}$$
$$\mathbb{E}[dV_L^{DM}] = \frac{\partial V_L^D}{\partial X}\mu X dt + \frac{1}{2}\frac{\partial^2 V_L^D}{\partial X^2}\sigma^2 X^2 dt + \lambda_L dt(0 - V_L^D) \tag{267}$$
$$\mathbb{E}[dV_L^{DD}] = \frac{\partial V_L^D}{\partial X}\mu X dt + \frac{1}{2}\frac{\partial^2 V_L^D}{\partial X^2}\sigma^2 X^2 dt + \lambda_L dt(0 - V_L^D) + \lambda_F dt(V_L^M - V_L^D). \tag{268}$$

For $X < X_F^*(K_L)$ the leader is still the only investor in the duopoly and thus acts temporarily as the monopolist which we denote by DM. The associated ODE is

$$(r + \lambda_L)V_L^{DM} = K_L X(1 - \eta K_L) + \frac{\partial V_L^{DM}}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V_L^{DM}}{\partial X^2}\sigma^2 X^2 \tag{269}$$

the homogeneous solution is

$$0 = \frac{\partial V_L^{DM}}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V_L^{DM}}{\partial X^2}\sigma^2 X^2 - (r + \lambda_L)V_L^{DM} \tag{270}$$
$$V_L^{DM}(X) = B_1^{DM} X^{\beta^+_{\lambda_L}} + B_2^{DM} X^{\beta^-_{\lambda_L}} \tag{271}$$

where $\beta^+_{\lambda_L} > 1, \beta^-_{\lambda_L} < 0$ solve (209). And the particular solution to the total differential equation with $V_L^{DM}(X) = a^{DM}X + b^{DM}$ leads to

$$(r + \lambda_L)(a^{DM}X + b^{DM}) = K_L X(1 - \eta K_L) + a^{DM}\mu X \tag{272}$$
$$a^{DM} = \frac{K_L(1 - \eta K_L)}{r + \lambda_L - \mu} \tag{273}$$
$$b^{DM} = 0 \tag{274}$$

Thus the total solution is

$$V_L^{DM}(X) = B_1^{DM} X^{\beta^+_{\lambda_L}} + B_2^{DM} X^{\beta^-_{\lambda_L}} + \frac{K_L(1-\eta K_L)}{r+\lambda_L-\mu} X \tag{275}$$

and the boundary condition is

$$V_L^{DM}(0) = 0 \tag{276}$$

which leads to $B_2^{DM} = 0$. But $B_1^{DM} \neq 0$ because if X goes to infinity we leave the monopoly state. Hence

$$V_L^{DM}(X) = B_1^{DM} X^{\beta^+_{\lambda_L}} + \frac{K_L(1-\eta K_L)}{r+\lambda_L-\mu} X \tag{277}$$

where β_F is the positive root of (209).

For $X > X_F^*(K_L)$ the ODE is

$$(r+\lambda_F+\lambda_L)V_L^{DD} = K_L X(1-\eta(K_F^*(K_L)+K_L)) + \frac{\partial V_L^{DD}}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V_L^{DD}}{\partial X^2}\sigma^2 X^2 + \lambda_F V_L^M \tag{278}$$

the homogeneous solution is

$$0 = \frac{\partial V_L^{DD}}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 V_L^{DD}}{\partial X^2}\sigma^2 X^2 - (r+\lambda_F+\lambda_L)V_L^{DD} \tag{279}$$

$$V_L^{DD}(X) = B_F^{DD} X^{\beta^+_{\lambda_{F,L}}} + B_L^{DD} X^{\beta^-_{\lambda_{F,L}}} \tag{280}$$

where $\beta^+_{\lambda_{F,L}} > 1, \beta^-_{\lambda_{F,L}} < 0$. And the particular solution of the differential equation with $V_L^D(X) = a^{DD}X + b^{DD}$ leads to

$$(r+\lambda_F+\lambda_L)(a^{DD}X+b^{DD}) = K_L X(1-\eta(K_F^*(K_L)+K_L)) + a^{DD}\mu X + \lambda_F \frac{XK_L(1-\eta K_L)}{r-\mu+\lambda_L} \tag{281}$$

$$a^{DD} = \frac{K_L(1-\eta(K_F^*(K_L)+K_L)) + \lambda_F \frac{K_L(1-\eta K_L)}{r-\mu+\lambda_L}}{r+\lambda_F+\lambda_L-\mu} \tag{282}$$

$$b^{DD} = 0 \tag{283}$$

where we used

$$V_L^M(X) = \frac{XK_L(1-\eta K_L)}{r-\mu+\lambda_L} \tag{284}$$

Thus the total solution is

$$V_L^{DD}(X) = B_F^{DD} X^{\beta^+_{\lambda_{F,L}}} + B_L^{DD} X^{\beta^+_{\lambda_{F,L}}} + \frac{K_L(1-\eta(K_F^*(K_L)+K_L)) + \lambda_F \frac{K_L(1-\eta K_L)}{r-\mu+\lambda_L}}{r+\lambda_F+\lambda_L-\mu} X \tag{285}$$

and the boundary condition is

$$V_L^{DD}(0) = 0 \tag{286}$$

$$\lim_{X\to\infty} V_L^{DD}(X) = wX \tag{287}$$

which leads to $B_2^{DD}=0$ and $B_1^{DD}=0$. Thus

Proposition 5.3.

$$V_L^{DM}(X) = B_1^{DM} X^{\beta^+_{\lambda_{F,L}}} + \frac{K_L(1-\eta K_L)}{r+\lambda_L-\mu} X \tag{288}$$

and

$$V_L^{DD}(X) = \frac{K_L(1-\eta(K_F^*(K_L)+K_L)) + \lambda_F \frac{K_L(1-\eta K_L)}{r-\mu+\lambda_L}}{r+\lambda_F+\lambda_L-\mu} X \tag{289}$$

However, V_L^{DD} occurs when $X > X_F^*(K_L)$ and thus coincides with the Stackelberg accommodation strategy. Therefore for deriving the deterrence strategy we only need to concentrate on

$$rV_L^{DM}(X,K_F,K_L) = Profit_L^{DM}(X,K_F,K_L) + \lim_{dt\downarrow 0} \frac{1}{dt}\mathbb{E}[dV_L^{DM}] \tag{290}$$

Though we also need the DD part to obtain B_1^{DM}.

At $X = X_F^*(K_L)$, the numerical value we obtained at (237), it should hold that

$$V_L^{DM}(X_F^*(K_L)) = V_L^{DD}(X_F^*(K_L)) \tag{291}$$

which we can solve for B_1^{DM}. After this, solve for K_L by

$$\frac{\partial V_L^D - \delta_L K_L}{\partial K_L} = 0 \tag{292}$$

and use the smooth pasting and value matching condition to obtain X_L^*.

Since V_L^{DD} is exactly the same as Stackelberg, we know already what the strategy in that domain will be.

For V_L^{DM} we have

$$V_L^{DM}(X) \quad = \quad \frac{\theta X_F^*(K_L)}{X_F^*(K_L)^{\beta_F}} X^{\beta^+_{\lambda_{F,L}}} + \frac{K_L(1-\eta K_L)}{r+\lambda_L-\mu} X \tag{293}$$

where $\theta = \left(\frac{K_L(1-\eta K_L - \eta K_F^*(K_L)) + \lambda_F \frac{K_L(1-\eta K_L)}{r-\mu+\lambda_L}}{r+\lambda_F+\lambda_L-\mu} - \frac{K_L(1-\eta K_L)}{r+\lambda_L-\mu} \right)$ and for fixed K_L we have found the numerical value of $X_F^*(K_L)$ and thus also the implied $K_F^*(K_L)$. However, note that this depends on a fixed K_L. Maximizing with respect to K_L gives the following first order condition.

$$\frac{\partial V_L^{DM} - \delta_L K_L}{\partial K_L} = 0 \tag{294}$$

$$\phi(X, K_L) = 0 \tag{295}$$

Again, the stopping value is determined based either on the assumption that the option to invest always exists or on the assumption that the option to invest can disappear for both firms with different probabilities. These imply

(i) $F(X) = AX^{\beta}$ if the option always remains

(ii) $F(X) = A_{\beta^+_{\lambda_L}} X^{\beta^+_{\lambda_L}}$ if the option to invest can vanish before the firm has invested

For case (i): Before the leader has invested, thus when $X < X_L$, the firm holds an option to invest. Therefore by the value matching and smooth pasting conditions with $F(X) = AX^{\beta}$ we solve

$$V_L^{DM}(X) - \delta_L K_L \quad = \quad AX^{\beta} \tag{296}$$

$$\frac{\partial V_L^{DM}(X)}{\partial X} \quad = \quad \beta A X^{\beta-1} \tag{297}$$

$$\frac{\partial V_L^{DM}(X)}{\partial X} - (V_L^{DM}(X) - \delta_L K_L)\beta X^{-1} \quad = \quad 0 \tag{298}$$

$$B_1^{DM}(\beta - \beta^+_{\lambda_{F,L}}) X^{\beta^+_{F,L}} + \frac{XK_L(1-\eta K_L)(\beta-1)}{r+\lambda_L-\mu} - \delta_L K_L \beta \quad = \quad 0 \tag{299}$$

Note that, B_1^{DM} is a function of K_L. **TO BE ADDED: Numerical results.**

For case (ii): Before the leader has invested, thus when $X < X_L$, the firm holds an option to invest. Therefore by the value and smooth pasting conditions with $F(X) = A_{\beta^+_{\lambda_L}} X^{\beta^+_{\lambda_L}}$ we solve

$$V_L^{DM}(X) - \delta_L K_L = A_{\beta^+_{\lambda_L}} X^{\beta^+_{\lambda_L}} \tag{300}$$

$$\frac{\partial V_L^{DM}(X)}{\partial X} = \beta^+_{\lambda_L} A_{\beta^+_{\lambda_L}} X^{\beta^+_{\lambda_L}-1} \tag{301}$$

$$\frac{\partial V_L^{DM}(X)}{\partial X} - (V_L^{DM}(X) - \delta_L K_L)\beta X^{-1} = 0 \tag{302}$$

$$B_1^{DM}(\beta^+_{\lambda_L} - \beta^+_{\lambda_{F,L}})X^{\beta^+_{\lambda_{F,L}}} + \frac{XK_L(1-\eta K_L)(\beta^+_{\lambda_L}-1)}{r+\lambda_L-\mu} - \delta_L K_L \beta^+_{\lambda_L} = 0 \tag{303}$$

TO BE ADDED: Numerical results.

References

Bar-Ilan, A. and Strange, W. C. (1999). The timing and intensity of investment. *Journal of Macroeconomics*, 21(1):57–77. 2

Dangl, T. (1999). Investment and capacity choice under uncertain demand. *European Journal of Operational Research*, 117(3):415–428. 2

Dixit, A. K. and Pindyck, R. S. (1994). *Investment under uncertainty*. Princeton university press. 2

Gryglewicz, S., Huisman, K. J., and Kort, P. M. (2008). Finite project life and uncertainty effects on investment. *Journal of Economic Dynamics and Control*, 32(7):2191–2213. 2, 3

Huisman, K. J. and Kort, P. M. (2015). Strategic capacity investment under uncertainty. *The RAND Journal of Economics*, 46(2):376–408. 2, 16, 34