A model problem of stochastic optimal control subject to ambiguous jump intensity

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Abstract

A model problem on optimal control of stochastic jump-driven systems subject to an ambiguous jump intensity is considered. The problem is formulated on the basis of the multiplier-robust control whose resolution reduces to solving a Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation with a term that is nonlinear and nonlocal. Mathematical analysis focusing on this term is carried out in this paper. We show that the equation admits a unique continuous viscosity solution: the value function. Furthermore, we present a convergent finite difference scheme that can numerically handle the equation, generating numerical solutions consistent with the mathematical analysis results.

Keywords

Multiplier-robust control, HJBI equation, Viscosity solution

1. Introduction

In this paper, a series of mathematical analysis results on a simplified stochastic control problem of jump-driven system dynamics subject to an ambiguous jump intensity is presented. The analysis results can be applied not only to environmental and ecological problems, our main motivations, but also to problems in finance especially real options.

Stochastic system dynamics are ubiquitous in many research areas of finance, economics, resource, environment, and ecology (Capasso and Bakstein, 2005). In problems related to finance, dynamics of asset and commodity prices are widely described as stochastic processes (Kwok, 2008). In problems related to environment and ecology, population dynamics are considered in the context of stochastic processes because of inherently nonlinear and complicated background biological and physical processes (Lande et al. 2003). Modeling, analysis, and control of stochastic system dynamics have thus been major research topics in mathematical sciences. The stochastic optimal control based on stochastic differential equations (SDEs) is an effective mathematical approach to handle these issues from a unified viewpoint (Fleming and Soner, 2006).

Accurate identification of parameters and coefficients appearing in system dynamics is a common key in optimal control problems. In general, controlled system dynamics are nonlinear as implied in many examples (Ji and Shi, 2017; Liang and Liu, 2017; Yaegashi et al., 2018), meaning that a slight difference of parameter values may lead to qualitatively different system behavior like bifurcations (Grass et al., 2015). Unfortunately, even in simple mathematical models, accurate identification of parameters and coefficients are still difficult tasks (Collie et al., 2016; Zhou et al., 2014), and model misspecification remains as a critical issue. In practice, it is desirable to have an optimal control policy that works robustly against worst-case perturbations and misspecifications.

One of the most successful approaches to efficiently handle stochastic system dynamics subject to model ambiguity is the multiplier-robust control (Hansen and Sargent, 2001). This approach is based on the concept of stochastic control (Øksendal and Sulem, 2005), in which the ambiguity is represented by nature as an opposite player of the decision-maker. Therefore, one of them is the maximizer and the other is the minimizer of some performance index, and the problem can be mathematically formulated as a differential game problem. In the framework of multiplier-robust control, the ambiguity is handled by the decision-maker through a penalization of a performance index based on a relative entropy, which statistically measures the distance between the true and believed, possibly distorted models. Then, finding the optimal control policy ultimately reduces to solving a Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation, which is a degenerate nonlinear (integro-) differential equation. In this sense, analysis of the control problem is equivalent to analysis of the HJBI equation. Problems related to finance (Shen et al., 2018), insurance (Zeng et al., 2016), environment and ecology (Manoussi et al., 2018;

Yoshioka and Yaegashi, 2018a), and fisheries (Yoshioka and Yaegashi, 2018b) have been investigated so far.

A potential issue in modeling stochastic system dynamics is identification of noise properties, such as the volatility modulating the strength of a Brownian motion and the jump size and jump intensity of a compound Poisson process. Their identification results would critically affect behavior of the resulting system dynamics. In fact, several mathematical models of biological population dynamics subject to jump disturbances suggest that the noise intensity determines whether the population encounters an eventual extinction (Li et al., 2015; Schloman, 2018). The jump ambiguity has been an important research topic in financial management as well (Aït-Sahalia and Matthys, 2019). The HJBI equations in the framework of the multiplier-robust control with ambiguous jumps have terms that are nonlinear and nonlocal whose mathematical analysis seems to be difficult. Often, this difficulty has been overcome by assuming certain functional shapes of the coefficients to derive exactly-solvable models (Aït-Sahalia and Matthys, 2019; Zhu, 2017). However, such a strategy would not always successfully work in more realistic and complicated problems. This issue motivates us to carry out detailed mathematical analysis of the HJBI equations having nonlinear and nonlocal terms.

The objective of this paper is to formulate a simplified stochastic control problem of jump-driven systems subject to an ambiguous jump intensity. This problem is motivated by optimal controls of severe algae blooming in river environment (Yoshioka and Yaegashi, 2018a; Yoshioka, 2019), but can be considered as a model problem of other jump-driven stochastic control problems as well. The system is autonomous and is driven by a compound Poisson process. The model ambiguity is penalized by an entropic term (Zeng et al., 2016). The present problem, due to its simplicity, is not essentially a differential game problem but rather a control problem. Nevertheless, the resulting HJBI equation still has a term that is both nonlinear and nonlocal. The main mathematical tool in this paper is the concept of viscosity solutions (Azimzadeh et al., 2018; Crandall et al., 1992), with which we can show that the HJBI equation admits a unique continuous viscosity solution: the value function. In addition, we show that a finite difference scheme can potentially generate numerical solutions converging toward the viscosity solution. Despite our analysis results are for the simplified problem, they can be to some extent applied to advanced models arising in problems that are more realistic.

The rest of this paper is organized as follows. Section 2 describes the setup of the mathematical model. Section 3 is devoted to analyzing the model. The numerical analysis is presented in Section 4, and Section 5 concludes this paper.

2. Mathematical model

2.1 Problem setting

Consider the usual setting of optimal control of jump-diffusion processes (Øksendal and Sulem, 2005). The compound Poisson process defined on a complete probability space is denoted as P_t at time t. Its jump intensity, which is the inverse of the mean time interval between each successive jumps, is denoted as $\lambda > 0$. The jump size z at each jump follows the probability distribution $g = g(z) \geq 0$ having the compact support Z in $(0,1)$. Clearly, we should have $\int_Z g(z)\mathrm{d}z = 1$. The filtration generated by P_t is denoted as $\mathcal{F}_t$. Set $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$. The continuous-time state variable at time t is denoted as X_t, which is assumed to be a càdlàg process adapted to $\mathcal{F}_t$. In addition, we assume that the process X_t has the bounded range $\Omega = [0,1]$. Set $\breve{\Omega} = (0,1]$.

The boundedness assumption is satisfied in many resource and population management problems under certain normalization of the state variables (Yoshioka, 2019). Assuming the unit interval as the domain is for the sake of simplicity of descriptions. The boundedness assumption can be reasonable in real options as well if we consider problems of non-renewable (exhaustible) resources.

2.2 SDE without ambiguity

The system dynamics equation without ambiguity is introduced as

$$\mathrm{d}X_t = -X_{t-0}\mathrm{d}P_t, \quad t > 0 \tag{1}$$

subject to the initial condition $X_0 = x \in \Omega$. This SDE represents a jump-driven decreasing process, and is explicitly solved as

$$X_t = x\prod_{k=1}^{N_t}(1 - z_k), \quad t \geq 0. \tag{2}$$

The SDE (1) admits the path-wise solution (2) valued in Ω.

Remark 2.1

Practical problems should have drift and/or diffusion terms, but are not considered in this paper. This is because they in general do not affect the nonlinear and nonlocal part of the HJBI equation presented later. If we consider a system dynamics with an ambiguous diffusion term instead of (1), then we face with a problem with ambiguous volatility. The difference between our and such models is that the former considers a system having a discontinuous noise, while the latter considers a system having a continuous noise.

Remark 2.2

A generalized SDE like

$$\mathrm{d}X_t = -X_{t-0}^{\omega}\mathrm{d}P_t, \quad t > 0 \tag{3}$$

with $\omega > 0$ may also be considered. The mathematical analysis results in this paper would apply to this generalized SDE with minor modifications.

2.3 SDE with ambiguity

We focus on optimal control of an ambiguous counterpart of the SDE (1). It is assumed that the jump intensity λ is ambiguous and is difficult to identify for the observer, the decision-maker. Such a situation can be encountered in modeling financial crisis (Jin et al., 2017) and flood disturbances (Yoshioka, 2019).

The ambiguity is represented by a positive measurable process $\phi_t > 0$ $(t \geq 0)$ adapted to $\mathcal{F}_t$ at each $t \geq 0$. The expectation is denoted as E. Throughout this paper, we assume that each ϕ satisfies

$$\mathrm{E}\left[\int_0^{\infty} e^{-\delta_0 s}\left(\phi_s \ln \phi_s + 1 - \phi_s\right)\mathrm{d}s\right] < +\infty \quad \text{and} \quad \mathrm{E}\left[\exp\left(\int_0^{t}\left(\phi_s \ln \phi_s + 1 - \phi_s\right)\mathrm{d}s\right)\right] < +\infty \tag{4}$$

with some $\delta_0 > 0$ and all $t > 0$. They are technical conditions employed so that the performance index defined later is bounded. Notice that $f(\phi) = \phi \ln \phi + 1 - \phi$, $f(0) = 1$, $f : [0, +\infty) \to \mathbb{R}$ is non-negative and convex, having the global minimum 0 at $\phi = 0$.

The ambiguity introduced here is mathematically the same with that of Zeng

(2016), which is based on the Girsanov's theorem. If

$$\int_0^t \ln \phi_s \mathrm{d}P_s + \int_0^t \int_Z (1-\phi_s)\lambda g(z)\mathrm{d}z\mathrm{d}s < +\infty, \quad t \geq 0, \tag{5}$$

then set the process

$$\Lambda_t^{(\phi)} = \exp\left(\int_0^t \ln \phi_s \mathrm{d}P_s + \int_0^t \int_Z (1-\phi_s)\lambda g(z)\mathrm{d}z\mathrm{d}t\right), \tag{6}$$

which is a positive martingale under the current probability measure. We assume (5) holds throughout this paper, and set the Radon-Nikodym derivative $\left.\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{F}_t} = \Lambda_t^{(\phi)}$, where $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ represent the current and distorted measures, respectively. All the expectations appearing below are defined in the sense of the distorted probability measure $\tilde{\mathbb{Q}}$.

The admissible set $\mathcal{A}$ of ϕ_t $(t \geq 0)$ is defined as follows.

Definition 2.1

The set $\mathcal{A}$, which is referred to as the admissible set, is defined as

$$\mathcal{A} = \left\{ \phi \,\middle|\, \begin{array}{l} \phi_t \text{ is positive, bounded, measurable, adapted to } \mathcal{F}_t, \\ \text{and satisfies (4) and (5) for } t \geq 0. \end{array} \right\}. \tag{7}$$

Under the distorted measure $\tilde{\mathbb{Q}}$, based on the Girsanov's theorem, P_t becomes a compound Poisson process $\tilde{P}_t$, which has the same jump size distribution g but has the modulated jump intensity $\lambda\phi_t$ (Zeng et al., 2016). Based on the non-ambiguous counterpart (1), the SDE having ambiguity in the jump intensity is formulated as

$$\mathrm{d}X_t = -X_{t-0}\mathrm{d}\tilde{P}_t, \quad t > 0 \tag{8}$$

subject to the same initial condition $X_0 = x \in \Omega$. The Poisson process with the jump intensity $\lambda\phi_t$ is denoted as $\tilde{N}_t$. As in (1), the SDE (8) is explicitly solved as

$$X_t = x\prod_{k=1}^{\tilde{N}_t}(1-z_k), \quad t \geq 0, \tag{9}$$

which is càdlàg, decreasing, and valued in Ω.

2.4 Performance index

The performance index to be optimized is formulated. For the sake of simplicity of analysis, assume that the state variable represents the amount of a quantity that would cause some disutility, like the population of harmful algae. In more realistic models than that considered in this paper, the performance index would have additional terms of the cost of human interventions to control the state variable. Then, the maximization problem should be replaced by a min-max or a max-min problem.

The performance index to be optimized is set as

$$p(x;\phi) = \mathrm{E}\left[\int_0^{\infty} e^{-\delta s}\left(X_s^m - \frac{\lambda}{\psi}\left(\phi_s \ln \phi_s + 1 - \phi_s\right)\right)\mathrm{d}s\right]. \quad (10)$$

Here, $m > 0$ is a constant that represents the sensitivity of disutility, and $\delta > 0$ is the discount rate representing how myopic the observer is: larger δ means that the observer is more myopic and puts larger weight on information near future. $\psi > 0$ is the ambiguity-aversion parameter of the observer: he/she is more ambiguity-averse with larger ψ. The first term of the performance index p is the disutility caused by the population, which simply represents the situation where a larger population causes a larger disutility. The second term represents the entropy penalization following the conventional models of multiplier-robust control (Zeng et al., 2016).

The value function is set as the worst-case performance index, which in the present model is

$$\Phi(x) = \sup_{\phi \in \mathcal{A}} p(x;\phi) \quad \text{in} \quad \Omega. \quad (11)$$

Notice that $\Phi(0) = 0$. The maximizing element ϕ, which gives the worst-case modulation of the jump intensity, is denoted as $\phi = \phi^*$. By the definition, we have

$$\Phi(x) \geq p(x;\phi) \quad \text{for all} \quad \phi \in \mathcal{A}. \quad (12)$$

2.5 HJBI equation

Set

$$\Delta\Phi = \Phi - \int_0^1 g(z)\Phi\left((1-z)x\right)\mathrm{d}z. \quad (13)$$

By the dynamic programming principle, the HJBI equation that governs Φ is formally derived as

$$\delta\Phi + \inf_{\phi>0}\left\{\lambda\phi\Delta\Phi - x^m + \frac{\lambda}{\psi}\left(\phi\ln\phi + 1 - \phi\right)\right\} = 0 \quad \text{in} \quad \Omega \tag{14}$$

subject to the boundary condition $\Phi(0) = 0$. The HJBI equation (14) is directly solved at the other boundary $x = 1$ because no information from the outside is required at this boundary. A straightforward calculation shows

$$\inf_{\phi>0}\left\{\lambda\phi\Delta\Phi + \frac{\lambda}{\psi}\left(\phi\ln\phi + 1 - \phi\right)\right\} = \frac{\lambda}{\psi}\left(1 - e^{-\psi\Delta\Phi}\right) \tag{15}$$

by

$$\arg\min_{\phi>0}\left\{\lambda\phi\Delta\Phi + \frac{\lambda}{\psi}\left(\phi\ln\phi + 1 - \phi\right)\right\} = e^{-\psi\Delta\Phi}. \tag{16}$$

Consequently, our HJBI equation (14) is compactly written as

$$\delta\Phi + \frac{\lambda}{\psi}\left(1 - e^{-\psi\Delta\Phi}\right) - x^m = 0 \quad \text{in} \quad \Omega, \quad \Phi(0) = 0. \tag{17}$$

In the context of Markov control, with an abuse of notation, the optimal control ϕ^* is considered as a function of the state x as

$$\phi^*(x) = e^{-\psi\Delta\Phi(x)} \quad \text{in} \quad \Omega. \tag{18}$$

The goal of the present optimal control problem is to find this ϕ^*.

Remark 2.3

Our HJBI equation is actually a HJB equation since it is related to a maximization problem. Nevertheless, we use the term ñHJBI equationò because we will analyze more realistic problem in future having game structures, based on the present simplified model.

Remark 2.4

The HJBI effectively reduces to

$$\delta\Phi + \lambda\Delta\Phi - x^m = 0 \quad \text{in} \quad \Omega \tag{19}$$

as $\psi \to +0$ (the observer is ambiguity-neutral), which is the equation with the no ambiguity ($\phi^* \equiv 0$). This is a reasonable result since the ambiguity-neutrality means ignorance of the ambiguity.

3. Mathematical analysis

The HJBI equation (17) is mathematically analyzed. Even in this simplified problem, existence, uniqueness, and regularity of solutions are not trivial because the equation is both non-linear and non-local.

Firstly, we show a basic theoretical result of the value function, with which an appropriate definition of viscosity solutions to the HJBI equation (17) is found.

Proposition 3.1

$$\left|\Phi(x_1)-\Phi(x_2)\right| \le \frac{1}{\delta}\left|x_1^m - x_2^m\right| \text{ for all } x_1, x_2 \in \Omega . \tag{20}$$

In addition, Φ *is increasing and non-negative in* Ω.

(Proof of Proposition 3.1)

The solution X_t of (9) with the initial condition $x = x_i$ is denoted as $X_{t,i}$ ($i = 1,2$). Fix one $\phi \in \mathcal{A}$. Then, by (9), we have

$$\begin{aligned}
\left|p(x_1;\phi) - p(x_2;\phi)\right| &= \mathrm{E}\left[\int_0^\infty e^{-\delta s}\left|X_{s,1}^m - X_{s,2}^m\right|\mathrm{d}s\right] \\
&= \mathrm{E}\left[\int_0^\infty e^{-\delta s}\left|x_1^m - x_2^m\right|\prod_{k=1}^{\tilde{N}_t}(1-z_k)^m \mathrm{d}s\right] \\
&= \mathrm{E}\left[\int_0^\infty e^{-\delta s}\prod_{k=1}^{\tilde{N}_t}(1-z_k)^m \mathrm{d}s\right]\left|x_1^m - x_2^m\right| . \\
&\le \mathrm{E}\left[\int_0^\infty e^{-\delta s}\mathrm{d}s\right]\left|x_1^m - x_2^m\right| \\
&= \frac{1}{\delta}\left|x_1^m - x_2^m\right|
\end{aligned} \tag{21}$$

The inequality (21) gives

$$p(x_1;\phi) \le \frac{1}{\delta}\left|x_1^m - x_2^m\right| + p(x_2;\phi), \tag{22}$$

and thus

$$p(x_1;\phi) \le \frac{1}{\delta}\left|x_1^m - x_2^m\right| + \Phi(x_2), \tag{23}$$

$$\Phi\left(x_1\right) \le \frac{1}{\delta}\left|x_1^m - x_2^m\right| + \Phi\left(x_2\right). \tag{24}$$

Similarly, we obtain

$$\Phi\left(x_2\right) \le \frac{1}{\delta}\left|x_1^m - x_2^m\right| + \Phi\left(x_1\right). \tag{25}$$

Combining (24) and (25) proves the continuity result (20).

The non-negativity of Φ follows directly from (11). Its increasing property is a consequence of the fact that X_t is increasing with respect to x by (9) and that p is increasing with respect to x as well by (10).

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Remark 3.1

Similar continuity results are obtained if we replace X_s^m by an increasing Hölder continuous function on Ω.

Remark 3.2

As a byproduct of **Proposition 3.1**, we obtain the boundedness result

$$0 \le \Phi \le \frac{1}{\delta} x^m \quad \text{in} \quad \Omega. \tag{26}$$

Secondly, we show that the equation (17) admits a unique continuous viscosity solution and also show that the solution is the value function. The definition of viscosity solutions follows that of Definition 1 of Azimzadeh et al. (2018). We notice the monotonicity property

$$\int_0^1 g(z)\Phi_1\left(\left(1-z\right)x\right)\mathrm{d}z \le \int_0^1 g(z)\Phi_2\left(\left(1-z\right)x\right)\mathrm{d}z \quad \text{in} \quad \overline{\Omega} \tag{27}$$

for any bounded $\Phi_1, \Phi_2 \in C\left(\overline{\Omega}\right)$ such that $\Phi_1 \le \Phi_2$ in $\overline{\Omega}$ because of the non-negativity of g. Therefore, the relationship equivalent to the second equation in Section 2 of Azimzadeh et al. (2018) holds true. This result motivates us to define viscosity solutions to the HJBI equation (17) in the following manner.

Definition 3.1

(a) A function $\Psi \in C(\bar{\Omega}) \cap USC(\Omega)$ *with* $\Psi(0) \leq 0$ *is a viscosity sub-solution if for all* $x_0 \in \bar{\Omega}$ *and for all* $\varphi \in C^1(\Omega)$, $\varphi - \Psi$ *is globally minimized at* $x = x_0$, $\varphi(x_0) = \Psi(x_0)$, $\varphi \geq \Psi$ *on* Ω, *and*

$$\delta\varphi(x_0) + \frac{\lambda}{\psi}\left(1 - e^{-\psi\left(\varphi(x_0) - \int_0^1 g(z)\Psi((1-z)x_0)\mathrm{d}z\right)}\right) - x_0^m \leq 0 . \qquad (28)$$

(b) A function $\Psi \in C(\bar{\Omega}) \cap LSC(\Omega)$ *with* $\Psi(0) \geq 0$ *is a viscosity super-solution to if for all* $x_0 \in \bar{\Omega}$ *and for all* $\varphi \in C^1(\Omega)$, $\varphi - \Psi$ *is globally maximized at* $x = x_0$, $\varphi(x_0) = \Psi(x_0)$, $\varphi \leq \Psi$ *on* Ω, *and*

$$\delta\varphi(x_0) + \frac{\lambda}{\psi}\left(1 - e^{-\psi\left(\varphi(x_0) - \int_0^1 g(z)\Psi((1-z)x_0)\mathrm{d}z\right)}\right) - x_0^m \geq 0 . \qquad (29)$$

(c) A function $\Psi \in C(\Omega)$ *is a viscosity solution if it is a viscosity sub-solution in the sense of* ***Definition 3.1(a)*** *as well as a viscosity super-solution in the sense of* ***Definition 3.1(b)***.

Now, we prove that the value function is a viscosity solution.

Proposition 3.2

The value function Φ *is a viscosity solution.*

(Proof of Proposition 3.2)

The proof is based on that of Theorem 9.8 of Øksendal and Sulem (2005). Firstly, we show that the value function is a viscosity super-solution ($\varphi - \Phi$ is globally maximized at x, $\varphi(x) = \Phi(x)$, $\varphi \leq \Phi$ on Ω). Set a test function φ for viscosity super-solutions. By the dynamic programming principle, for any stopping time $\tau > 0$ adapted to the filtration $\mathcal{F}$, we have

$$\Phi(x) = \sup_{\phi \in \mathcal{A}} \mathrm{E}\left[\int_0^{\tau} e^{-\delta s}\left(X_s^m - \frac{\lambda}{\psi}(\phi_s \ln \phi_s + 1 - \phi_s)\right)\mathrm{d}s + e^{-\delta \tau}\Phi(X_{\tau})\right] \qquad (30)$$

with $\check{E} = \min(\tau, \rho)$ and a constant $\rho > 0$. Fix one $\phi \in \mathcal{A}$. By **Definition 3.1(b)**, we obtain

$$\begin{aligned}\Phi(x) &\geq \mathrm{E}\left[\int_0^{\check{E}} e^{-\delta s}\left(X_s^m - \frac{\lambda}{\psi}(\phi_s \ln\phi_s + 1 - \phi_s)\right)\mathrm{d}s + e^{-\delta\check{E}}\Phi(X_{\check{E}})\right] \\ &\geq \mathrm{E}\left[\int_0^{\check{E}} e^{-\delta s}\left(X_s^m - \frac{\lambda}{\psi}(\phi_s \ln\phi_s + 1 - \phi_s)\right)\mathrm{d}s + e^{-\delta\check{E}}\varphi(X_{\check{E}})\right] \\ &\geq \varphi(x) + \mathrm{E}\left[\int_0^{\check{E}} e^{-\delta s}\left(-\delta\varphi(X_s) - \lambda\phi_s\Delta\varphi(X_s) + X_s^m - \frac{\lambda}{\psi}(\phi_s \ln\phi_s + 1 - \phi_s)\right)\mathrm{d}s\right]\end{aligned}, \quad (31)$$

which leads to

$$0 \geq \mathrm{E}\left[\int_0^{\check{E}} e^{-\delta s}\left(-\delta\varphi(X_s) - \lambda\phi_s\Delta\varphi(X_s) + X_s^m - \frac{\lambda}{\psi}(\phi_s \ln\phi_s + 1 - \phi_s)\right)\mathrm{d}s\right] \quad (32)$$

by $\Phi(x) = \varphi(x)$ and an application of the classical Dynkinôs formula to $e^{-\delta s}\varphi(X_s)$. Divided by $\mathrm{E}[\check{E}]$ and taking the limit $\rho \to +0$ in (32) gives

$$0 \geq -\delta\varphi(x) - \phi\Delta\varphi(x) + x^m - \frac{\lambda}{\psi}(\phi\ln\phi + 1 - \phi) \quad (33)$$

and thus

$$\delta\varphi(x) + \phi\Delta\varphi(x) + \frac{\lambda}{\psi}(\phi\ln\phi + 1 - \phi) - x^m \geq 0, \quad (34)$$

where $\phi = \phi_0$ with an abuse of notation. Taking the minimum of the left hand-side with respect to $\phi > 0$ gives the desired inequality

$$\delta\varphi(x) + \frac{\lambda}{\psi}\left(1 - e^{-\psi\Delta\Phi(x)}\right) - x^m \geq 0 \quad (35)$$

by the monotonicity (27) and Proposition 3 of Azimzadeh et al. (2018).

Secondly, we show that the value function is a viscosity sub-solution. With a $\varepsilon > 0$, set constant an ε-optimal policy $\phi^{(\varepsilon)} \in \mathcal{A}$ such that

$$\Phi(x) \leq \mathrm{E}\left[\int_0^{\check{E}} e^{-\delta s}\left(X_s^m - \frac{\lambda}{\psi}\left(\phi_s^{(\varepsilon)} \ln\phi_s^{(\varepsilon)} + 1 - \phi_s^{(\varepsilon)}\right)\right)\mathrm{d}s + e^{-\delta\check{E}}\Phi(X_{\check{E}})\right] - \varepsilon\rho. \quad (36)$$

Set a test function φ for viscosity sub-solutions ($\varphi - \Phi$ is globally minimized at x, $\varphi(x) = \Phi(x)$, $\varphi \geq \Phi$ on Ω). By **Definition 3.1(a)**, we have $\varphi \geq \Phi$ and

$\varphi(x) = \Phi(x)$ and obtain

$$\varphi(x) \le \mathrm{E}\left[\int_0^{\check{\tau}} e^{-\delta s}\left(X_s^m - \frac{\lambda}{\psi}\left(\phi_s^{(\varepsilon)} \ln \phi_s^{(\varepsilon)} + 1 - \phi_s^{(\varepsilon)}\right)\right)\mathrm{d}s + e^{-\delta\check{\tau}}\varphi\left(X_{\check{\tau}}\right)\right] - \varepsilon\rho . \tag{37}$$

Again by the Dynkin's formula, we have

$$0 \le \mathrm{E}\left[\int_0^{\check{\tau}} e^{-\delta s}\left(-\delta\varphi(X_s) - \lambda\phi_s^{(\varepsilon)}\Delta\varphi(X_s) + X_s^m - \frac{\lambda}{\psi}\left(\phi_s^{(\varepsilon)} \ln \phi_s^{(\varepsilon)} + 1 - \phi_s^{(\varepsilon)}\right)\right)\mathrm{d}s\right] - \varepsilon\rho . \tag{38}$$

Then, as in the proof for viscosity super-solutions, we have

$$\varepsilon \le -\delta\varphi(x) - \phi\Delta\varphi(x) - \frac{\lambda}{\psi}(\phi \ln \phi + 1 - \phi) + x^m . \tag{39}$$

Since ε is arbitrary, we have

$$0 \le -\delta\varphi(x) - \phi\Delta\varphi(x) - \frac{\lambda}{\psi}(\phi \ln \phi + 1 - \phi) + x^m \tag{40}$$

and consequently

$$\delta\varphi(x) + \phi\Delta\varphi(x) + \frac{\lambda}{\psi}(\phi \ln \phi + 1 - \phi) - x^m \le 0 . \tag{41}$$

Taking the minimum of the left hand-side with respect to $\phi > 0$ gives the desired result

$$\delta\varphi(x) + \frac{\lambda}{\psi}\left(1 - e^{-\psi\Delta\Phi(x)}\right) - x^m \le 0 \tag{42}$$

by the monotonicity (27) and Proposition 3 of Azimzadeh et al. (2018).

□

Remark 3.3

Proposition 3 of Azimzadeh et al. (2018), which is a non-trivial result on a linkage between different notions of viscosity solutions, is an essential element in the proof. In fact, we can take $\omega(\varepsilon) = \varepsilon$ in the assumption of the proposition in the literature.

The comparison (unique solvability) result below applies to our HJBI equation.

Proposition 3.3

The HJBI equation (17) admits at most one viscosity solution.

(Proof of Proposition 3.3)

As in the standard argument for comparison theorems (Crandall et al., 1992), it is sufficient to show that for any couple of a viscosity sub-solution $\underline{\Phi}$ and a viscosity super-solution $\overline{\Phi}$, $\overline{\Phi} \geq \underline{\Phi}$ in Ω. We already have $\overline{\Phi}(0) \geq \underline{\Phi}(0)$ by the definition. Therefore, what we have to prove here is $\overline{\Phi} \geq \underline{\Phi}$ in $\breve{\Omega}$. This statement is proven with a contradiction argument.

Assume $\sup_{\breve{\Omega}} \{\underline{\Phi} - \overline{\Phi}\} > 0$. Set $\varphi_\varepsilon : \Omega \times \Omega \to \mathbb{R}$ as $\varphi_\varepsilon(x, y) = \frac{1}{\varepsilon}(x - y)^2$ and $f_\varepsilon : \Omega \times \Omega \to \mathbb{R}$ as $f_\varepsilon(x, y) = \underline{\Phi}(x) - \overline{\Phi}(y) - \varphi_\varepsilon(x, y)$. Then, f_ε attains a maximum at some point in $\Omega \times \Omega$ because it is upper semi-continuous. A point at which f_ε is maximized is denoted as $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$. Then, we have

$$f_\varepsilon(x_\varepsilon, y_\varepsilon) \geq f_\varepsilon(x, x) = \underline{\Phi}(x) - \overline{\Phi}(x) \tag{43}$$

for all $x \in \Omega$.

Following the standard argument of comparison theorems (Crandall et al., 1992), we can choose a sequence $\varepsilon = \varepsilon_k$ with $\lim_{k \to +\infty} \varepsilon_k = 0$ such that

$$\lim_{k \to +\infty} x_{\varepsilon_k} = \lim_{k \to +\infty} y_{\varepsilon_k} = x_0 \tag{44}$$

and

$$\lim_{k \to +\infty} \frac{1}{\varepsilon_k}\left(x_{\varepsilon_k} - y_{\varepsilon_k}\right) = \lim_{k \to +\infty} \frac{1}{\varepsilon_k}\left(x_{\varepsilon_k} - y_{\varepsilon_k}\right)^2 = 0 \tag{45}$$

with some $x_0 \in \breve{\Omega}$ such that $\underline{\Phi}(x_0) = \overline{\Phi}(x_0) = \sup_{\breve{\Omega}} \{\underline{\Phi} - \overline{\Phi}\} > 0$. Hereafter, we only consider such a sub-sequence. Taking this sequence with (43) gives

$$\underline{\Phi}(x_0) - \overline{\Phi}(x_0) \geq \underline{\Phi}(x) - \overline{\Phi}(x) \tag{46}$$

or equivalently

$$\overline{\Phi}(x) - \underline{\Phi}(x) \geq \overline{\Phi}(x_0) - \underline{\Phi}(x_0) \tag{47}$$

for all $x \in \breve{\Omega}$.

We see that $\underline{\Phi}(x) - \left(\overline{\Phi}(y_\varepsilon) + \varphi_\varepsilon(x, y_\varepsilon)\right)$ is maximized at x_ε and $\overline{\Phi}(y) - \left(\underline{\Phi}(x_\varepsilon) - \varphi_\varepsilon(x_\varepsilon, y)\right)$ is minimized at y_ε. Therefore, we can use $\underline{\Phi}(x_\varepsilon) + \varphi_\varepsilon(x, y_\varepsilon) - \varphi_\varepsilon(x_\varepsilon, y_\varepsilon)$ as a test function for the viscosity sub-solution and

$\overline{\Phi}(y_\varepsilon)-\varphi_\varepsilon(x_\varepsilon,y)+\varphi_\varepsilon(x_\varepsilon,y_\varepsilon)$ as that for the viscosity super-solution. Now, we have

$$\delta\underline{\Phi}(x_\varepsilon)+\frac{\lambda}{\psi}\left(1-e^{-\psi\left(\underline{\Phi}(x_\varepsilon)-\int_0^1 g(z)\underline{\Phi}((1-z)x_\varepsilon)\mathrm{d}z\right)}\right)-x_\varepsilon^m\le 0 \tag{48}$$

and

$$\delta\overline{\Phi}(y_\varepsilon)+\frac{\lambda}{\psi}\left(1-e^{-\psi\left(\overline{\Phi}(y_\varepsilon)-\int_0^1 g(z)\overline{\Phi}((1-z)y_\varepsilon)\mathrm{d}z\right)}\right)-y_\varepsilon^m\ge 0\,. \tag{49}$$

Letting $\varepsilon\to+0$ in (48) and (49) yields

$$\delta\underline{\Phi}(x_0)+\frac{\lambda}{\psi}\left(1-e^{-\psi\left(\underline{\Phi}(x_0)-\int_0^1 g(z)\underline{\Phi}((1-z)x_0)\mathrm{d}z\right)}\right)-x_0^m\le 0 \tag{50}$$

and

$$\delta\overline{\Phi}(x_0)+\frac{\lambda}{\psi}\left(1-e^{-\psi\left(\overline{\Phi}(x_0)-\int_0^1 g(z)\overline{\Phi}((1-z)x_0)\mathrm{d}z\right)}\right)-x_0^m\ge 0\,, \tag{51}$$

respectively. Combining (50) and (51) yields

$$\delta\left(\overline{\Phi}(x_0)-\underline{\Phi}(x_0)\right)-e^{-\psi\left(\overline{\Phi}(x_0)-\int_0^1 g(z)\overline{\Phi}((1-z)x_0)\mathrm{d}z\right)}+e^{-\psi\left(\underline{\Phi}(x_0)-\int_0^1 g(z)\underline{\Phi}((1-z)x_0)\mathrm{d}z\right)}\ge 0\,. \tag{52}$$

By $\overline{\Phi}(x_0)-\underline{\Phi}(x_0)<0$ and $\delta>0$, (52) gives

$$0>e^{-\psi\left(\overline{\Phi}(x_0)-\int_0^1 g(z)\overline{\Phi}((1-z)x_0)\mathrm{d}z\right)}-e^{-\psi\left(\underline{\Phi}(x_0)-\int_0^1 g(z)\underline{\Phi}((1-z)x_0)\mathrm{d}z\right)} \tag{53}$$

and thus

$$-\psi\left(\overline{\Phi}(x_0)-\int_0^1 g(z)\overline{\Phi}\left((1-z)x_0\right)\mathrm{d}z\right)<-\psi\left(\underline{\Phi}(x_0)-\int_0^1 g(z)\underline{\Phi}\left((1-z)x_0\right)\mathrm{d}z\right), \tag{54}$$

which can be rearranged as

$$\overline{\Phi}(x_0)-\int_0^1 g(z)\overline{\Phi}\left((1-z)x_0\right)\mathrm{d}z>\underline{\Phi}(x_0)-\int_0^1 g(z)\underline{\Phi}\left((1-z)x_0\right)\mathrm{d}z\,. \tag{55}$$

Consequently, we have

$$\overline{\Phi}(x_0)-\underline{\Phi}(x_0)>\int_0^1 g(z)\overline{\Phi}\left((1-z)x_0\right)\mathrm{d}z-\int_0^1 g(z)\underline{\Phi}\left((1-z)x_0\right)\mathrm{d}z\,. \tag{56}$$

By (47), we obtain

$$
\begin{aligned}
&\int_0^1 g(z)\overline{\Phi}\left((1-z)x_0\right)\mathrm{d}z-\int_0^1 g(z)\underline{\Phi}\left((1-z)x_0\right)\mathrm{d}z\\
&=\int_0^1 g(z)\left\{\overline{\Phi}\left((1-z)x_0\right)-\underline{\Phi}\left((1-z)x_0\right)\right\}\mathrm{d}z\\
&\geq\int_0^1 g(z)\left\{\overline{\Phi}(x_0)-\underline{\Phi}(x_0)\right\}\mathrm{d}z\\
&=\left(\int_0^1 g(z)\mathrm{d}z\right)\left(\overline{\Phi}(x_0)-\underline{\Phi}(x_0)\right)\\
&=\overline{\Phi}(x_0)-\underline{\Phi}(x_0)
\end{aligned}
\quad . \tag{57}
$$

Combining (56) and (57) leads to the contradiction

$$
\overline{\Phi}(x_0)-\underline{\Phi}(x_0)>\overline{\Phi}(x_0)-\underline{\Phi}(x_0). \tag{58}
$$

Therefore, we must have $\overline{\Phi}(x_0)-\underline{\Phi}(x_0)\geq 0$. Since x_0 is arbitrary in $\overline{\Omega}$, $\overline{\Phi}\geq\underline{\Phi}$ in $\overline{\Omega}$. The proof is thus completed.

□

Remark 3.4

The comparison result still holds true when we replace x^m in the source term by $f\in C(\Omega)$.

Proposition 3.3 shows that the HJBI equation (17) admits at most one viscosity solution. Consequently, by **Proposition 3.2**, in the present case it turns out that the unique viscosity solution is the value function.

Theorem 3.1

The value function Φ is the unique viscosity solution to the HJBI equation (17).

Finally, we show that the solution has the following asymptotic properties. The proof is omitted here since it is by a direct substitution of $\Phi(x)=C_1x^{C_2}$ with unknown constants $C_1,C_2\in\mathbb{R}$.

Proposition 3.5

The solution admits the following formal asymptotic expansion for small $x>0$:

$$\Phi(x) \sim \frac{1}{\delta + \lambda\left(1-(1-z)^m\right)} x^m . \tag{59}$$

4. Numerical analysis

Asymptotic behavior of the unique viscosity solution to the HJBI equation (17) was analyzed in **Proposition 3.5**, but its full explicit expression has not been found. Therefore, we present a finite difference scheme for discretization of the equation. The scheme is simple, but its stability, monotonicity, and consistency, which are key elements in analyzing convergence of numerical solutions to the viscosity solution (Azimzadeh et al., 2018), are nontrivial issues. These issues are thus analyzed here. The present scheme can also be applied to problems having possibly nonlinear drift and diffusion terms with their appropriate discretization (Koleva and Vulkov, 2018; Yoshioka and Yaegashi, 2019).

4.1 Discretization scheme

The domain $\Omega = [0,1]$ is divided into I cells and $I+1$ vertices $x = x_i$ as

$$0 = x_0 < x_1 < ... < x_{I-1} < x_I = 1 . \tag{60}$$

The ith cell is denoted as $\Omega_i = [I_i, I_{i+1}]$ $(0 \le i \le I-1)$. For the sake of simplicity, we use the uniform discretization $x_i = i / I$. The value function Φ approximated at $x = x_i$ is denoted as Φ_i. We directly specify the boundary condition $\Phi_0 = 0$. The discretized (17) at $x = x_i$ $(i \ge 1)$ is set as

$$\delta\Phi_i + \frac{\lambda}{\psi}\left(1 - e^{-\psi[\Delta\Phi]_i}\right) - x_i^m = 0 , \tag{61}$$

where $[\Delta\Phi]_i$ is the discretization of $\Delta\Phi$ specified below.

Set a natural number J. The possible range $[0,1]$ of the jump intensity is discretized as

$$0 = z_0 < z_1 < ... < z_J = 1 . \tag{62}$$

Set $z_{j+1/2} = \left(z_j + z_{j+1}\right)/2$ $(0 \le j \le J-1)$. The jump density g is approximated at each $z_{j+1/2}$, and the approximation value of g is denoted as g_j. Set the uniform discretization $z_j = j/J$ with $\Delta z = 1/J$. We assume the condition of probability normalization in a discrete sense:

$$\sum_{j=0}^{J-1} g_j \Delta z = 1. \tag{63}$$

For each i ($0 \le i \le I$) and j ($0 \le j \le J$), there is exactly one $l(\le I-1)$ such that

$$x_l \le x_i\left(1 - z_{j+1/2}\right) < x_{l+1}. \tag{64}$$

This l is denoted as $l_{i,j}$. Set

$$w_{i,j} = \frac{x_{l_{i,j}+1} - x_{l_{i,j}}\left(1 - z_{j+1/2}\right)}{x_{l_{i,j}+1} - x_{l_{i,j}}}, \tag{65}$$

which satisfies $0 \le w_{i,j} \le 1$. In addition, set the interpolated value

$$\Phi_{i,j} = w_{i,j}\Phi_{l_{i,j}} + \left(1 - w_{i,j}\right)\Phi_{l_{i,j}+1}. \tag{66}$$

Then, we propose the discretization

$$[\Delta\Phi]_i \to \Phi_i - \sum_{j=0}^{J-1} g_j \Phi_{i,j} \Delta z. \tag{67}$$

Consequently, the fully-discretized equation is derived as

$$\delta\Phi_i + \frac{\lambda}{\psi}\left(1 - e^{-\psi\left(\Phi_i - \sum_{j=0}^{J-1} g_j \Phi_{i,j}\Delta z\right)}\right) - x_i^m = 0. \tag{68}$$

Collecting the equation (68) for $1 \le i \le I$ with the boundary condition $\Phi_0 = 0$ leads to I non-linear equations and I unknowns, meaning that the total number of the equations and that of the unknowns are the same.

4.2 Analysis of the scheme

Monotonicity, stability, consistency, and convergence of the scheme are analyzed (Azimzadeh et al. 2018; Barles and Souganidis, 1991). For their definitions, see Section 3.1 of Neilan et al. (2017). We note

$$
\begin{aligned}
\Phi_i - \sum_{j=0}^{J-1} g_j \Phi_{i,j} \Delta z &= \sum_{j=0}^{J-1} g_j \Phi_i \Delta z - \sum_{j=0}^{J-1} g_j \Phi_{i,j} \Delta z \\
&= \Delta z \sum_{j=0}^{J-1} g_j \left(\Phi_i - \Phi_{i,j} \right) \\
&= \Delta z \sum_{j=0}^{J-1} g_j \left\{ \Phi_i - \left(w_{i,j} \Phi_{l_{i,j}} + \left(1 - w_{i,j}\right) \Phi_{l_{i,j}+1} \right) \right\} \\
&= \Delta z \sum_{j=0}^{J-1} g_j \left\{ w_{i,j} \left(\Phi_i - \Phi_{l_{i,j}} \right) + \left(1 - w_{i,j}\right) \left(\Phi_i - \Phi_{l_{i,j}+1} \right) \right\}
\end{aligned}
\quad . \qquad (69)
$$

Introduce the notation $u_{i,j} = \Phi_i - \Phi_j$ when $i > j$. Then, the scheme is expressed as

$$
S_i\left(\Phi_i, \left\{u_{i,j}\right\}_{0 \le j \le I}\right) \equiv \delta \Phi_i + \frac{\lambda}{\psi}\left(1 - e^{-\psi \Delta z \left(\sum_{j=0}^{J-1} g_j \left\{ w_{i,j} u_{i,l_{i,j}} + \left(1 - w_{i,j}\right) u_{i,l_{i,j}+1} \right\} \right)}\right) - x_i^m = 0 \quad (1 \le i \le I) \qquad (70)
$$

and $S_0 \equiv \Phi_0 = 0$ with the abbreviation $S_i\left(\Phi_i, \left\{u_{i,j}\right\}_{0 \le j \le I}\right) = S_i\left(\Phi_i, u_{i,0}, u_{i,1}, \ldots u_{i,I},\right)$. By this notation, the scheme is monotone if each S_i is increasing with respect to the first argument and decreasing with respect to the other arguments (Oberman, 2006). This is directly checked from (70).

Proposition 4.1

The present scheme is monotone.

Stability of the scheme follows if we can find a global positive constant M such that $\left|\Phi_i\right| \le M$ ($0 \le i \le I$). This is proven as follows.

Proposition 4.2

The present scheme is stable. Moreover, $0 \le \Phi_i \le \frac{1}{\delta}$ *for* $0 \le i \le I$.

(Proof of Proposition 4.2)

Set $i_0 = \arg\max_i \Phi_i$. Then, we must show $\Phi_{i_0} \le M$ with a global constant $M > 0$. By the definition of i_0, we have

$$\sum_{j=0}^{J-1} g_j \left\{ w_{i_0,j} \left(\Phi_{i_0} - \Phi_{l_{i_0,j}} \right) + \left(1 - w_{i_0,j}\right)\left(\Phi_{i_0} - \Phi_{l_{i_0,j}+1} \right) \right\} \geq 0 , \tag{71}$$

leading to

$$\begin{aligned} \delta\Phi_{i_0} &= x_{i_0}^m - \frac{\lambda}{\psi}\left(1 - e^{-\psi\Delta z \sum_{j=0}^{J-1} g_j \left\{ w_{i_0,j}\left(\Phi_{i_0} - \Phi_{l_{i_0,j}}\right) + \left(1-w_{i_0,j}\right)\left(\Phi_{i_0} - \Phi_{l_{i_0,j}+1}\right)\right\}} \right) \\ &\leq x_{i_0}^m \\ &\leq 1 \end{aligned} , \tag{72}$$

by (63) and (69). Therefore, we obtain the stability result by choosing $M = \dfrac{1}{\delta}$.

Next, we show $\Phi_i \geq 0$ ($0 \leq i \leq I$). If this is true, then $0 \leq \Phi_i \leq M = \dfrac{1}{\delta}$ and the proof is completed. We use an induction argument. Assume $\Phi_1 < 0$. Then, we have

$$\delta\Phi_1 = x_1^m - \frac{\lambda}{\psi}\left(1 - e^{-\psi[\Delta\Phi]_1}\right) < 0 \tag{73}$$

and thus

$$x_1^m < \frac{\lambda}{\psi}\left(1 - e^{-\psi[\Delta\Phi]_1}\right). \tag{74}$$

Therefore, we must have

$$\left[\Delta\Phi\right]_1 > 0 . \tag{75}$$

On the other hand, combining (66) and (67) gives

$$\left[\Delta\Phi\right]_1 = \Phi_1 - \sum_{j=0}^{J-1} g_j \Phi_{1,j} \Delta z = \left(\sum_{j=0}^{J-1} g_j w_{1,j} \Delta z \right) \Phi_1 < 0 \tag{76}$$

since $\Phi_0 = 0$. This contradict with $\Phi_1 < 0$, showing $\Phi_1 \geq 0$.

Assume that we have $\Phi_i \geq 0$ ($0 \leq i \leq i_0$) for some $1 \leq i_0 \leq I$. If $I = 1$, then we have nothing to prove and thus assume $I \geq 2$. Assume $\Phi_{i_0+1} < 0$. Then, as in the case for $I = 1$, we have

$$x_{i_0}^m < \frac{\lambda}{\psi}\left(1 - e^{-\psi[\Delta\Phi]_{i_0+1}}\right), \tag{77}$$

leading to

$$\left[\Delta\Phi\right]_{i_0+1} > 0 . \tag{78}$$

On the other hand, we have

$$
\begin{aligned}
\left[\Delta\Phi\right]_{i_0+1} &= \Phi_{i_0+1} - \Delta z \sum_{j=0,}^{J-1} g_j \left\{ w_{i_0+1,j} \Phi_{l_{i_0+1,j}} + \left(1 - w_{i_0+1,j}\right) \Phi_{l_{i_0+1,j}+1} \right\} \\
&= \Phi_{i_0+1} - \Delta z \sum_{j=0, l_{i_0+1,j}=i_0}^{J-1} g_j \left\{ w_{i_0+1,j} \Phi_{l_{i_0+1,j}} + \left(1 - w_{i_0+1,j}\right) \Phi_{l_{i_0+1,j}+1} \right\} \\
&\quad - \Delta z \sum_{j=0, l_{i_0+1,j}\neq i_0}^{J-1} g_j \left\{ w_{i_0+1,j} \Phi_{l_{i_0+1,j}} + \left(1 - w_{i_0+1,j}\right) \Phi_{l_{i_0+1,j}+1} \right\} \\
&= \left(1 - \alpha_{i_0+1}\right) \Phi_{i_0+1} - \sum_{i=0}^{i_0} \alpha_i \Phi_i
\end{aligned}
\tag{79}
$$

with some non-negative constants $0 \le \alpha_i \le 1$ ($0 \le i \le i_0$), $0 \le \alpha_{i_0+1} < 1$. By the assumption, (79) gives

$$
\begin{aligned}
0 &< \left[\Delta\Phi\right]_{i_0+1} \\
&= \left(1 - \alpha_{i_0+1}\right) \Phi_{i_0+1} - \sum_{i=0}^{i_0} \alpha_i \Phi_i \\
&\le \left(1 - \alpha_{i_0+1}\right) \Phi_{i_0+1}
\end{aligned}
\tag{80}
$$

and thus $\Phi_{i_0+1} > 0$. The contradicts with $\Phi_{i_0+1} < 0$. Therefore, we obtain $\Phi_{i_0+1} \ge 0$ and $\Phi_i \ge 0$ ($0 \le i \le i_0 + 1$), which completes the proof by the induction argument.

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Remark 4.1

In an essentially the same way, it follows $0 \le \Phi_i \le \frac{1}{\delta} x_i^m$ ($0 \le i \le I$), which is consistent with **Remark 3.2.** In addition, it leads to $\left[\Delta\Phi\right]_i \ge 0$ ($0 \le i \le I$) by (68). Furthermore, we obtain the lower-bound $\Phi_i \ge \frac{1}{\delta} x_i^m - \frac{\lambda}{\psi}$, with which **Proposition 4.2** gives the sharper stability result

$$
\max\left\{0, \frac{1}{\delta} x_i^m - \frac{\lambda}{\psi}\right\} \le \Phi_i \le \frac{1}{\delta} x_i^m \quad (0 \le i \le I). \tag{81}
$$

Finally, we show that the present scheme is consistent in the sense of Section 2 of Azimzadeh et al. (2018), but this follows from continuity of viscosity (sub-, super-)

solutions in $\breve{\Omega}$ at least if g is a delta distribution concentrated at $z_0 \in (0,1)$ because the non-local term reduces to the point-wise term

$$\int_0^1 g(z)\Phi((1-z)x)\mathrm{d}z = \Phi((1-z_0)x). \tag{82}$$

For generic g, verifying the consistency property is a more complicated issue, but is possible under certain regularity conditions. A sufficient regularity condition for the probability density g is as follows:

$$\lim_{I\to+\infty}\int_0^1 |g_I(z) - g(z)|\mathrm{d}z = 0, \tag{83}$$

where $g \in C^1(0,1)$ and g has a compact support in $(0,1)$. Then, we can discretize this g as

$$g_I(z) = \sum_{j=0}^{J-1} g_j \chi_j(z), \tag{84}$$

$$g_j = \int_{z_j}^{z_{j+1}} g(z)\mathrm{d}z, \tag{85}$$

$$\chi_j(z) = \begin{cases} 1 & (z_j \le z \le z_{j+1}) \\ 0 & (\text{Otherwise}) \end{cases}, \tag{86}$$

and $J = O(I)$. In this case, we can check that a condition similar to that obtained in Lemma 12 of Azimzadeh et al. (2018), and thus the consistency.

Proposition 4.3

The present scheme is consistent with an appropriate g.

Propositions 4.1 through **4.3** lead to convergence of numerical solutions to the present scheme toward the viscosity solution to the equation (17).

Proposition 4.4

With an appropriate g*, numerical solutions to the present numerical scheme converge to the unique viscosity solution to the HJBI equation (17), the value function, locally uniformly in* $\breve{\Omega}$.

Remark 4.2

Uniqueness of numerical solutions follows from Theorems 5 of Oberman (2006) because the scheme is degenerate elliptic and proper in the sense of the literature. Existence of numerical solutions, however, is not a trivial issue in this framework because the scheme is not Lipschitz in the sense of Oberman (2006). We can overcome this issue through replacing (70) by

$$S_i\left(\Phi_i,\{u_{i,j}\}_{0\le j\le I}\right)\equiv\delta\Phi_i+\frac{\lambda}{\psi}\left(1-e^{-\psi\Delta z\left(\sum_{j=0}^{J-1}g_j\left\{w_{i,j}\breve{u}_{l,l_{i,j}}+(1-w_{i,j})\breve{u}_{l,l_{i,j}+1}\right\}\right)}\right)-x_i^m\quad(1\le i\le I)\quad(87)$$

with $\breve{u}_{i,j}=\min\left\{2M,\max\left\{u_{i,j},-2M\right\}\right\}$ ($0\le i,j\le I$). Then, the modified scheme admits a unique solution since it is Lipchitz, and thus admits a solution. The modified scheme is degenerate elliptic and proper, and thus admits a unique numerical solution. By choosing M sufficiently large, by **Proposition 4.2**, the unique solution to the modified scheme is a solution to the present scheme, showing existence of numerical solutions. Consequently, the present scheme admits a unique numerical solution.

Several computational results are attached to this material (**Figures 1** through **6**). The computational conditions are $I=1{,}000$, $J=500$, $m=1.5$, $\delta=3$, $g(z)=1.25\chi_{\{0.1\le z\le 0.9\}}$, $\lambda=0.1$ or 1, and $\psi=1,\ 10,$ or 100. The numerical solutions and asymptotic results (59) are plotted, suggesting their good agreement especially for the smaller λ. Notice that the value function is normalized as $\delta\Phi$ in these figures. Here, the numerical solutions are computed with the recursion

$$\Phi_i^{(n+1)}=\frac{1}{\delta}x_i^m-\frac{\lambda}{\delta\psi}\left(1-e^{-\psi\left(\Phi_i-\sum_{j=0}^{J-1}g_j\Phi_{i,j}\Delta z\right)}\right)^{(n)}\quad n=0,1,2,\ldots\quad(88)$$

with the initial guess $\Phi_i^{(0)}$ until $\max_i\left|\Phi_i^{(n+1)}-\Phi_i^{(n)}\right|\le 10^{-14}$. Typically, $n=O\left(10^0-10^1\right)$.

Our computational results on the optimal control ϕ^* are consistent with the fundamental assumptions (4) and (6) of the model and in accordance with the asymptotic result (**Proposition 3.5**), especially for small λ.

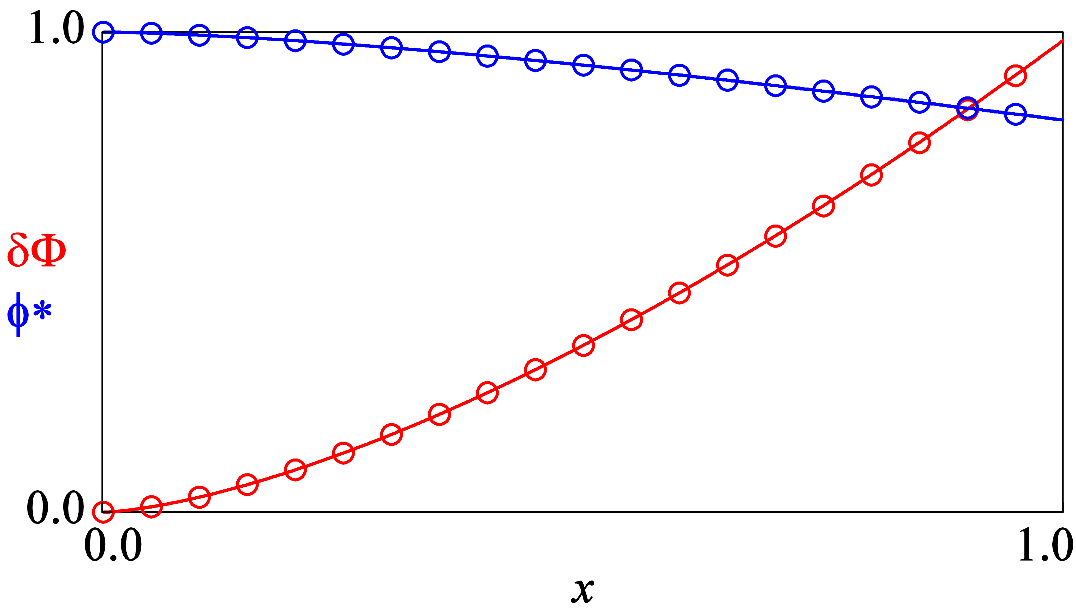

Figure 1: The computed and asymptotic $\delta\Phi(x)$ and $\phi^*(x)$ with $\lambda = 0.1$ and $\psi = 1$.

Line: computed result, Circle: asymptotic result.

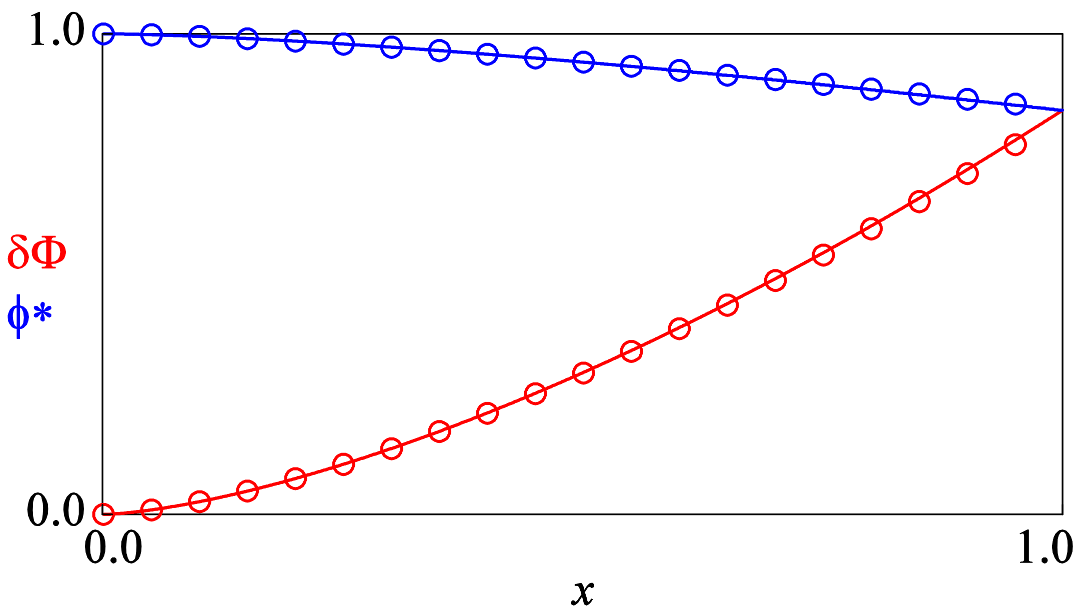

Figure 2: The computed and asymptotic $\delta\Phi(x)$ and $\phi^*(x)$ with $\lambda = 1$ and $\psi = 1$.

Line: computed result, Circle: asymptotic result.

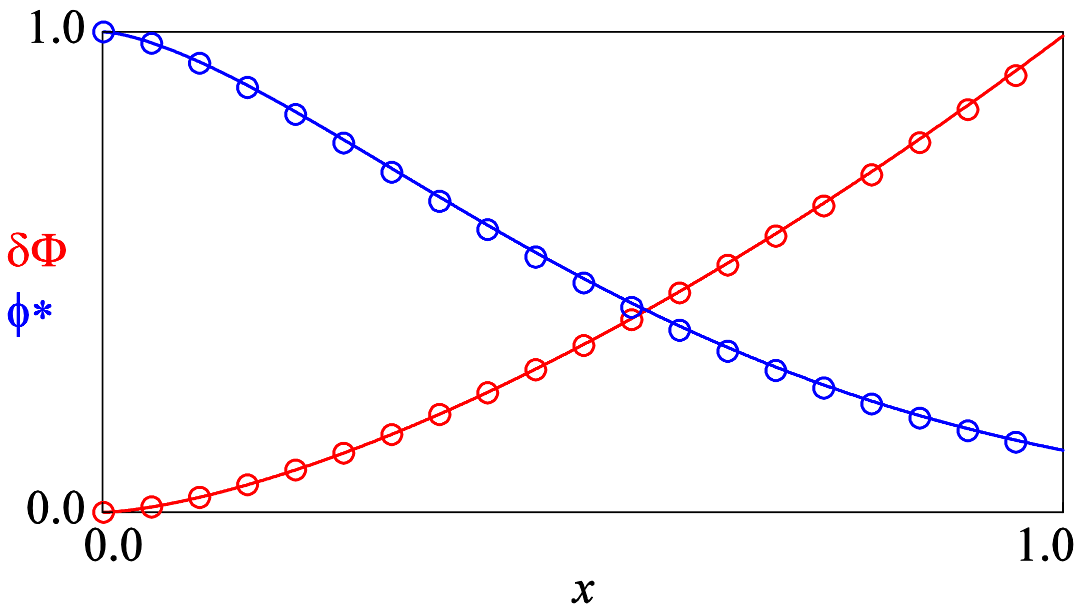

Figure 3: The computed and asymptotic $\delta\Phi(x)$ and $\phi^*(x)$ with $\lambda = 0.1$ and $\psi = 10$. Line: computed result, Circle: asymptotic result.

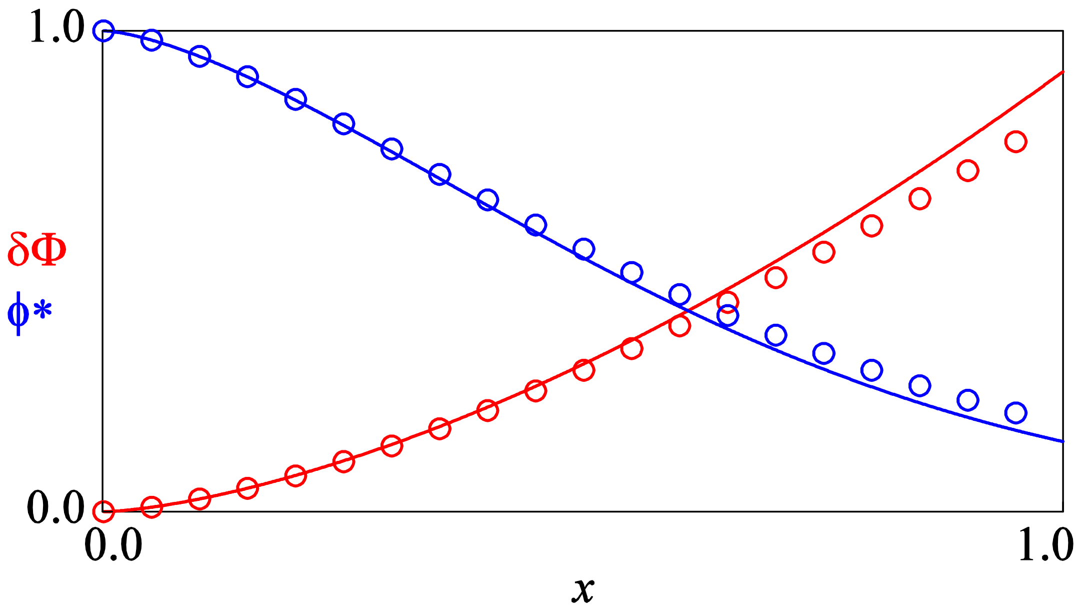

Figure 4: The computed and asymptotic $\delta\Phi(x)$ and $\phi^*(x)$ with $\lambda = 1$ and $\psi = 10$. Line: computed result, Circle: asymptotic result.

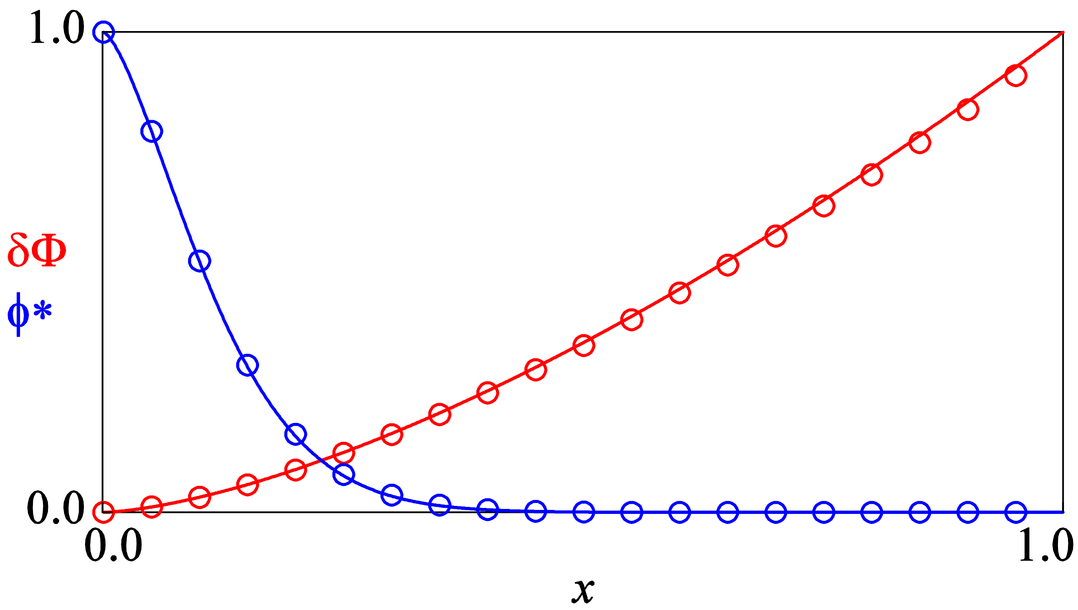

Figure 5: The computed and asymptotic $\delta\Phi(x)$ and $\phi^*(x)$ with $\lambda = 0.1$ and $\psi = 100$. Line: computed result, Circle: asymptotic result.

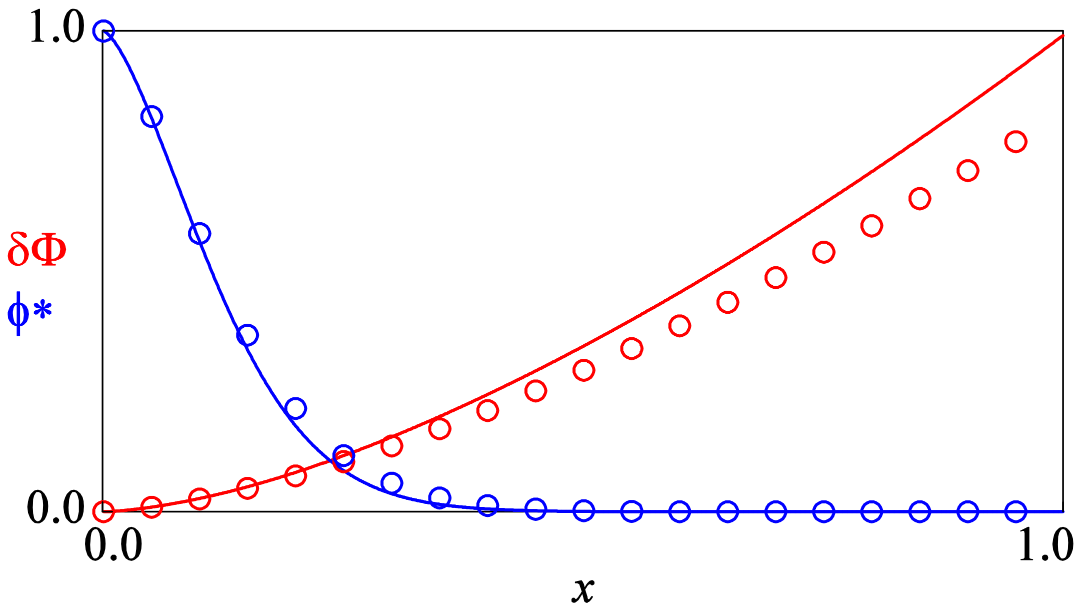

Figure 6: The computed and asymptotic $\delta\Phi(x)$ and $\phi^*(x)$ with $\lambda = 1$ and $\psi = 100$ Line: computed result, Circle: asymptotic result.

5. Conclusions

Mathematical analysis on a simplified stochastic control problem of jump-driven systems subject to an ambiguous jump intensity was carried out. The main result was unique solvability of the HJBI equation in the viscosity sense. Numerical computation of the HJBI equation was performed as well.

In future research, we will investigate to what extent our analysis results apply to problems with drift and diffusion terms. Analyzing advanced problems, such as ergodic control, impulse control, and singular control, is an interesting research topic as well. In addition, the model will be extended and applied to a management problem of biological population subject to a jump disturbance such that the observer has a difficulty in accurately identifying the jump intensity. This kind of problems frequently arise in environmental and ecological management in and around rivers. An example is management of algae population subject to floods.

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