Quasi-analytical solution of an investment problem with decreasing investment cost due to technological innovations

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Abstract

In this paper we address, in the context of real options, an investment problem with two sources of uncertainty: the price (reflected in the revenue of the firm) and the level of technology. The level of technology impacts in the investment cost, that decreases when there is a technology innovation. The price follows a geometric Brownian motion, whereas the technology innovations are driven by a Poisson process. As a consequence, the investment region may be attained in a continuous way (due to an increase of the price) or in a discontinuous way (due to a sudden decrease of the investment cost).

For this optimal stopping problem no analytical solution is known, and therefore we propose a quasi-analytical method to find an approximated solution that preserves the qualitative features of the exact

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solution. This method is based on a truncation procedure and we prove that the truncated solution converges to the solution of the original problem.

We provide results for the comparative statics for the investment thresholds. These results show interesting behaviors, particularly, the investment may be postponed or anticipated with the intensity of the technology innovations and with their impact on the investment cost.

1 Introduction

The optimal time to undertake an investment opportunity has been an important research question for both economists and mathematicians, mainly, since the pioneering works of Dixit and Pindyck [4] and Trigeorgis [21]. Over time, the models to solve these optimal stopping time problems have become more complex, since researchers and practitioners intend to represent the economic reality in a more realistic way. As a consequence, both the number of sequential decisions studied and the number of sources of uncertainty in these models have increased.

One particular aspect that has been under the spots of real options literature is the impact of technology innovations, that may lead to significant changes in the revenue and costs of the companies. In the past few years, given a large number of breakthroughs innovations, some industries have seen their investment costs decreasing over time. Therefore, nowadays, companies understand better the value of technology innovations (Guney et al. [9]), as they have been realizing that such innovations may create incentives for an early investment due to the lower costs. Due to the importance of these innovations to companies and governments, this has been broadly reported in the media (some examples are presented in the next paragraphs) and discussed in the economic literature (see for instance Flor and Hansen [7] and Murto [15]).

For example, in the renewable energy sector we see a large impact of falling costs. In December 2018, the International Renewable Energy Agency (IRENA), in its latest report, mentioned that the solar photovoltaic module prices have fallen by around 80% since 2009, and the wind turbine prices have fallen around 30-40% [1]. In the same line, Hanno Schoktlish, CEO of Kaiserwetter argues that " the decreases in the cost of renewable

[1] The report entitled *Renewable Power: Climate-Safe Energy Competes on Cost Alone* can be found at https://www.irena.org/-/media/Files/IRENA/Agency/Publication/2018/Dec/IRENA_COP24_costs_update_2018.pdf.

energy... has occurred for several reasons. These include technological improvements..."[2]. Also Hunt and Shuttleworth [12] reports about decreasing investment cost in energy investment. According to these authors, as a result of studies sponsored by space programs, it was possible to build turbines much more efficient and smaller than before, reducing in a drastic way the optimal power plant size, with enormous cost reduction. Aside from technological innovations, other factors may lead to a sudden decrease in the investment cost. For instance, as a result of govern interventions in key areas of the economy (as the renewable energies), as Deng et al. [3] analyse.

A recent global survey of IT and finance leaders by independent research firm Vanson Bourne showed that in manufacturing, 42% of the CEO's of 900 companies said that they have already reduced operational costs through innovation[3]. This impact is particularly important in high-tech companies, where the progress in technology takes advantage of other industries, as it is the case of pharmaceutical companies. For example, advances in technology related to biomarkers, as well as developments in the understanding of the human genome, have changed the cost structure for firms developing products targeting small patient populations[4]. Still in the health sector, research studies (see for instance Lee and Choi [13]) have demonstrated that investing in health IT in a hospital setting has potential benefits, that impact in the reduction of the cost, by increasing efficiency and productivity metrics.

In this paper we address, in the context of real options, an investment model with two sources of uncertainty: the price (reflected in the revenue of the firm) and the level of technology. The firm needs to optimize his investment decision by taking into account the random fluctuations of the revenue and the changing investment cost. Additionally, it is considered that the investment costs are decreasing with technology innovations. We borrow the probabilistic terminology to say that a significant drop in the investment cost is a jump. Problems with two sources of uncertainty have been studied in the real options literature, as one can see, for example, in Alghalith [1], Dixit and Pindyck [4], McDonald and Siegel [14], Murto [15], Pennings and Sereno [20], and Zambujal-Oliveira and Duque [22], among others.

[2] Available at https://es.kaiserwetter.energy/speakers-corner/single-view/production-cost-of-renewables-undercuts-fossil-fuel-energy-nuclear-power-for-the-first-time/

[3] Available at https://www.industryweek.com/leadership/why-it-so-hard-invest-technology

[4] Available at https://www.nber.org/reporter/2018number3/garthwaite.html

Along the paper, we consider that such jumps in the investment cost, which are due to an innovation process (assumed exogeneous to the company), are driven by a Poisson process. Furthermore, the size of the downward jump in the investment cost is known beforehand. These assumptions are common with most literature about real options models for technology innovations (see for instance, Farzin et al. [6], Doraszelski [5] and Hagspiel at al. [10]).

Our framework is very close to the one assumed by Murto [15], since we both study the timing of investment under effects of technological and revenue uncertainty. Additionally, we both assume that the revenue stream generated by the investment follows a geometric Brownian motion (GBM), and that the technology progress follows a Poisson process.

In his paper, Murto [15] states that, although the problem is well-posed, there is no analytical solution to the Hamilton-Jacobi-Bellman (HJB) equation that characterizes the value function. The reason of this difficulty lays in the fact that the stopping region may be attained continuously (due to an increase in the price) or discontinuously (as consequence of a downward jump in the investment cost). Then Murto [15] proposes an analytical solution only in the following some particular cases: either the price process is deterministic (meaning that he assumes that there is no volatility in the GBM); or the technological progress is deterministic (leading to an exponential decline in the investment cost). Therefore, in the cases he studied, instead of having a problem with two sources of uncertainty (that would lead to an exercise boundary and not to a point), he transforms it in a problem with just one source of uncertainty, where the classic tools (including verification theorems) may be used.

A similar model, where an analytical solution is obtained, can be found in Nunes and Pimentel [16]. In this paper, the authors consider that both the revenue and the costs are jump-diffusion processes, where the jumps in the revenue are downward jumps and the jumps in the investment cost are upwards. The direction of these jumps is such that, contrary to the case that we analise in the current paper, the stopping region is always attained through a continuous movement. This, combined with the fact that the value function is homogeneous (and therefore one may consider a change of variable, as proposed in Dixit and Pindyck [4]), leads to an optimal stopping time problem where an analytical solution can be found.

Our contribution to the state of the art is to provide a mathematical method to find an approximation for the value function and the prices that trigger the investment decision. The method is based in a truncation approach, and we prove that the truncated problem converges to the solution of the original problem. The truncated method was firstly addressed, in the field of real options, by Guerra et al. [8]. Using the results found for the truncated problem, we are also able to provide insights about the comparative statics.

The quasi-analytical method proposed in this paper can be useful for other problems with the same features. For instance, it may be used to analyze the problem addressed by Nunes and Pimentel [16], but assuming now that jumps may also lead directly to an investment decision. In the same line, it can be used in the context of the problem addressed by Couto et al. [2], where it is also assumed that the processes that model the uncertainty follow jump-diffusions. This method can also be applied to the problem presented in Hagspiel et al. [11], where, similar to our case, the stopping region may be attained by a jump, in a discontinuous way.

The paper is organized as follows: in Section 2 we define the investment model, in Section 3 we solve the truncated problem and in Section 4 we present a convergence result. In Section 5 we present the comparative statics for the investment threshold with respect to the relevant parameter and in Section 6 the main conclusions of the paper are presented. Finally, there are two appendixes: the first one where we present all the proofs and the second one where we present some numerical results.

2 Problem set-up

In this paper we consider a firm that has the option to make a singular and irreversible investment in a certain market producing a single good. We assume that the price of this product, P, evolves randomly in time according with a geometric Brownian motion, namely

$$dP_t = \mu P_t dt + \sigma P_t dW_t, \quad \text{with } P_0 = p > 0,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

The investment cost depends on the level of technology on the market: the higher the technology level, the lower the investment cost. The level of technology evolves according to a point process, due to technology innovations. The sequence of times between consecutive innovations is a sequence of independent and identically exponentially distributed random variables, with parameter $\lambda > 0$. Therefore, if we let $N = \{N_t : t \geq 0\}$, with $N_0 = n$ and N_t representing the number of technology innovations occurred until time t, then N is a Poisson process with rate λ. Moreover, we assume that the process N is independent of the process P. Finally, the investment process $I = \{I_t : t \geq 0\}$ is intrinsically related with the process N by:

$$I_t = I\phi^{N_t}, \quad \text{with} \quad I_0 = I\phi^n,$$

where I_t represents the investment cost at time t and $\phi \in]0, 1[$. Therefore, each time there is an innovation, the investment cost decreases by a factor of ϕ.

Assuming that the current levels of price and technology are, respectively, $p > 0$ and $n \in \mathbb{N}_0$, the value of the firm that undertakes the investment opportunity at time τ is given by the functional

$$J(p, n, \tau) = E_{p,n}\left[\int_{\tau}^{+\infty} e^{-rs} P_s ds - e^{-r\tau} I_\tau\right],$$

where $r > 0$ represents the instantaneous interest rate. Here, $E_{p,n}[\cdot]$ represents the expected value conditional to the information $P_0 = p$ and $N_0 = n$. We will assume that $r > \mu$ in order to ensure that $E_{p,n}\left[\int_0^{+\infty} e^{-rs} P_s ds\right] < +\infty$. Additionally, using the strong Markov property of the GBM, and taking into account that $E_{p,n}\left[\int_0^{+\infty} e^{-rs} P_s ds\right] = \frac{p}{r-\mu}$, it follows that

$$J(p, n, \tau) = E_{p,n}\left[e^{-r\tau}\left(\frac{P_\tau}{r-\mu} - I_\tau\right)\right] \equiv E_{p,n}\left[e^{-r\tau} g_{N_\tau}(P_\tau)\right],$$

with $g_n(p) = \frac{p}{r-\mu} - I\phi^n$. Therefore, throughout this paper, we intend to find the optimal investment time τ^* that maximizes the functional J. Equivalently, we intend to find the value function

$$V(p, n) = \sup_{\tau \geq 0} J(p, n, \tau). \tag{1}$$

Trivial financial arguments lead us to guess that the optimal strategy is to invest for high levels of price and high levels of technology (and, consequently, lower investment cost). We also expect that, given a certain level of technology, n, the price that triggers the investment decision, p_n^*, should be larger than the threshold price, $p_{\tilde{n}}^*$, corresponding to a higher level of technology ($\tilde{n} > n$). Therefore, we expect a stopping region of the form $S = \{(p, n) \in \mathbb{R}^+ \times \mathbb{N}_0 : p \geq p_n^*\}$, where $n \to p_n^*$ is a decreasing function of the technology level, here denoted by n. This is in accordance with the classical results in real options (see for instance Dixit and Pindyck [4]).

Using standard arguments (see, for example, Oksendal and Sulem [18]), the value function V must satisfy the following HJB equation:

$$\min\{rV(p,n) - (\mathcal{L}V)(p,n), V(p,n) - g_n(p)\} = 0, \quad \text{for all } (p,n) \in \mathbb{R}^+ \times \mathbb{N}_0, \tag{2}$$

where $\mathcal{L}$ is the infinitesimal generator of the bi-dimensional process (P,N), given by:

$$(\mathcal{L}f)(p,n) = \mu p f'(p,n) + \frac{\sigma^2}{2} p^2 f''(p,n) + \lambda(f(p,n+1) - f(p,n))$$

where f' and f'' are respectively the first and second derivatives of f w.r.t. p. The solution of the differential-difference equation corresponding to the continuation decision in (2), $rV(p,n) - (\mathcal{L}V)(p,n) = 0$, is difficult to solve, since the solution in the state (p,n) depends on the solution of the same equation in the state $(p,n+1)$. Murto [15] also faces the same difficulty, even stating that there is no closed form solution for the differential-difference equation.

For a given p and n in the continuation region, after a jump in the technological process, one of the following situations hold: (1) $(p,n+1) \notin \mathcal{S}$, and, consequently, the decision is to continue postponing the investment or (2) $(p,n+1) \in \mathcal{S}$, which means that it is optimal to invest immediately after the jump (and thus $V(p,n+1) = g_{n+1}(p)$). This means that a jump may lead directly to the stopping region. As a consequence, the value of the firm at state (p,n) depends recursively on all the levels above $(n+1, n+2, \cdots)$.

In an attempt to solve such problem, Murto [15] reduces the dimension of the stochastic process, leading to a different HJB equation, as the new equations depend only on a single variable: instead of (p,n) as state variable he uses $\frac{p}{I\phi^n}$. This strategy was already developed by Dixit and Pindyck [4]; Nunes and Pimentel [16] also use the same idea to solve an investment problem with two sources of uncertainty and also Couto et al. [2]. But contrary to [4, 16] (as in both cases the stopping region may be reached only by the diffusion component and not by a jump), this technique is not successful to solve the current problem, as the new equation has exactly the same difficulty in the continuation region.

In the next section we present a different strategy, which allows us to approximate the solution of the problem. Additionally, this technique is efficient to study the qualitative behavior of the solution when there are some changes in the parameters that characterize the uncertainty of the market.

3 The truncated problem

In this section we introduce the truncated problem. By truncated problem we mean that we assume that the investment time is bounded by a random time, that we define next. For a given $\overline{n} \in \mathbb{N}$, we let $\tau_{\overline{n}} = \inf\{t \geq 0 \ : N_t \geq \overline{n}\}$ be the random time representing the first moment when the level of technology is, at least, $\overline{n}$. Then we define the following problem:

$$V^{\overline{n}}(p,n) = \sup_{0 \leq \tau \leq \tau_{\overline{n}}} J(p,n,\tau) \tag{3}$$

which is the *truncated* version of the one defined in (1) (in which the time horizon is infinite). We notice that, in our formulation, if no decision is taken until the moment $\tau_{\overline{n}}$, then this time will be the optimal investment time. When $\overline{n} \to +\infty$, then, using standard arguments, one can prove that $\tau_{\overline{n}} \nearrow +\infty$ and, as we will show later, the solution of the truncated problem also converges to the solution of the original problem.

In Figure 1 we illustrate, for $\overline{n} = 2, 3, 4$, the expected optimal strategy for the truncated investment problem. In this figure we use the following notation: $p_n^{\overline{n}^\star}$ is the trigger price for the investment when the maximum number of jumps of innovations is $\overline{n}$, and $n < \overline{n}$ jumps have already occurred. By construction, the firm immediately undertake the investment opportunity when the maximum number of innovations have occurred. Moreover, $p_n^{\overline{n}^\star}$ decreases with n: the smaller the technological level, the larger the investment cost and thus, the price needs to be larger to trigger the investment decision.

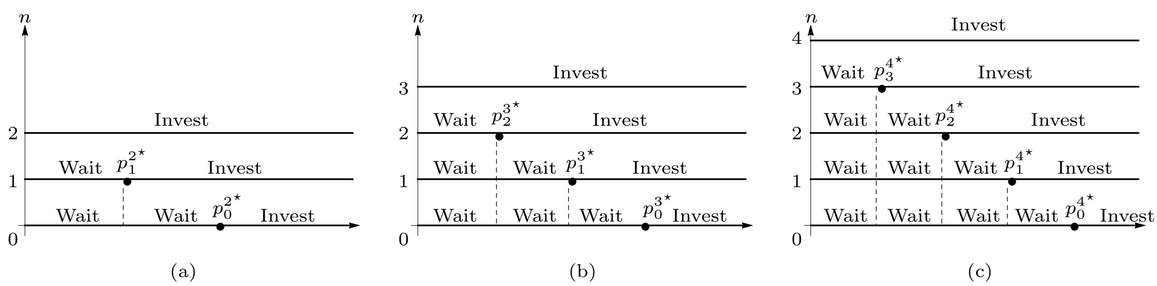


Figure 1: Continuation and stopping regions when (a) $\overline{n} = 2$, (b) $\overline{n} = 3$ and (c) $\overline{n} = 4$.

Following Oliveira and Perkowski [19], for a fixed level $\overline{n}$, the value function $V^{\overline{n}}(p,n)$ is a solution of the following HJB equations:

$$\min\{(r+\lambda)v_n^{\overline{n}}(p) - \mu p(v_n^{\overline{n}})'(p) - \frac{\sigma^2}{2}p^2(v_n^{\overline{n}})''(p) - \lambda v_{n+1}^{\overline{n}}(p), v_n^{\overline{n}}(p) - g_n(p)\} = 0, \tag{4}$$

with $n = 0, 1, 2, \cdots, \overline{n} - 1$. This accounts to solve a system of $\overline{n}$ HJB equations, with $v_{\overline{n}}^{\overline{n}}(p) = g_{\overline{n}}(p)$, and $v_n^{\overline{n}}(p)$ being C_1 (as a consequence of the usual smooth-fit conditions).

For each $\overline{n}$, we need to solve the following set of $\overline{n}$ ordinary differential equations (ODEs):

$$(r+\lambda)v_n^{\overline{n}}(p) - \mu p(v_n^{\overline{n}})'(p) - \frac{\sigma^2}{2}p^2(v_n^{\overline{n}})''(p) = \lambda v_{n+1}^{\overline{n}}(p). \tag{5}$$

The solution to the associated homogeneous ODE is a function of the type $Ap^{d_1} + Bp^{d_2}$, with $A, B \in \mathbb{R}$, and d_1 and d_2 being the roots of the characteristic polynomial $\frac{\sigma^2}{2}d(d-1) + \mu d - (r+\lambda)$:

$$d_1 = \frac{\left(\frac{\sigma^2}{2} - \mu\right) + \sqrt{\left(\frac{\sigma^2}{2} - \mu\right)^2 + 2\sigma^2(r+\lambda)}}{\sigma^2} > 0 \text{ and } d_2 = \frac{\left(\frac{\sigma^2}{2} - \mu\right) - \sqrt{\left(\frac{\sigma^2}{2} - \mu\right)^2 + 2\sigma^2(r+\lambda)}}{\sigma^2} < 0.$$

For future reference, we notice that d_1 and d_2 verify the following conditions

$$r + \lambda = -\frac{\sigma^2}{2}d_1 d_2 \quad \text{an} \quad \mu = \frac{\sigma^2}{2}(1 - d_1 - d_2). \tag{6}$$

The construction of the particular solution of the ODE relies on the method developed by Nunes et al. [17]. The derivations, as well the way to prove that the solution to the HJB equation is, indeed, the solution for the truncated problem, are very similar for every $\overline{n} \in \mathbb{N}$. Therefore, in the next section we present in detail the case $\overline{n} = 2$, and later we present the results for a general $\overline{n}$, without proofs.

3.1 The case $\overline{n} = 2$

When we fix $\overline{n}$, the value function at the level $\overline{n}$ is, by construction, given by $g_{\overline{n}}$, i.e. $v_{\overline{n}}^{\overline{n}}(p) = g_{\overline{n}}(p)$. Additionally, the ODE used to find $v_{\overline{n}-1}^{\overline{n}}(p)$ is very similar to the one considered in a standard investment problem. Then, we are left with the derivation of $v_i^{\overline{n}}(p)$ for $i = 0, 1, \ldots, \overline{n} - 2$.

For $\overline{n} = 2$, it follows that

$$v_2^2(p) = \frac{p}{r-\mu} - I\phi^2, \quad \text{for } p > 0 \tag{7}$$

$$v_1^2(p) = \begin{cases} A_{1,0,0}^2 p^{d_1} + \lambda\left[\frac{p}{(r-\mu)(r+\lambda-\mu)} - \frac{I\phi^2}{r+\lambda}\right], & 0 < p < p_1^{2^\star} \\ \frac{p}{r-\mu} - I\phi, & p \geq p_1^{2^\star} \end{cases} \tag{8}$$

where (7) is the value of the investment in perpetuity, and (8) is the value function of a standard investment problem. In particular, the usual value matching and smooth pasting conditions lead to

$$p_1^{2^\star} = \frac{d_2 - 1}{d_2} I\phi \left[r + \lambda(1-\phi) \right], \tag{9}$$

$$A_{1,0,0}^2 = \frac{(p_1^{2^\star})^{1-d_1}}{d_1(r+\lambda-\mu)}. \tag{10}$$

where, as previously introduced, $p_n^{\overline{n}^*}$ is the investment threshold where the maximum number of technology innovations is $\overline{n}$ (corresponding to the truncated problem) and n is the number of innovations already occurred.

Once we found v_1^2, we are now in position to derive v_0^1. Note that when $n = 0$ and $\overline{n} = 2$, we need to take into account three possible regions for p: for $0 < p < p_1^{2^\star}$, we know that we will not invest for sure even if the next jump occurs; when $p_1^{2^\star} \le p < p_0^{2^\star}$, then we will invest surely after the next jump; finally, for $p \ge p_0^{2^\star}$ we invest right away (see Figure 1). Using this reasoning and the results derived in Nunes et al. [17] to solve the differential-difference equations corresponding to the continuation region, we end up with the following result:

$$v_0^2(p) = \begin{cases} \left(A_{0,0,0}^2 + A_{0,0,1}^2 \ln p\right) p^{d_1} + \lambda^2 \left[\frac{p}{(r-\mu)(r+\lambda-\mu)^2} - \frac{I\phi^2}{(r+\lambda)^2}\right], & 0 < p < p_1^{2^\star} \\ A_{0,1,0}^2 p^{d_1} + B_{0,1,0}^2 p^{d_2} + \lambda \left[\frac{p}{(r-\mu)(r+\lambda-\mu)} - \frac{I\phi}{r+\lambda}\right], & p_1^{2^\star} \le p < p_0^{2^\star} \\ \frac{p}{r-\mu} - I, & p \ge p_0^{2^\star}. \end{cases} \tag{11}$$

The interpretation of (11) is as follows: first we note that the value function for the case $0 < p < p_1^{2^\star}$ can be re-written as

$$\left(A_{0,0,0}^2 + A_{0,0,1}^2 \ln p\right) p^{d_1} + \left(\frac{\lambda}{r+\lambda}\right)^2 \left[\frac{p(r+\lambda)^2}{(r-\mu)(r+\lambda-\mu)^2} - I\phi^2\right]$$

where the first part accounts for the fact that the investment may occur due to an increase of the price. The term involving $\ln p$ is related with the value of the option when a jump in the technology level happens but the price is not large enough to trigger the investment, and, therefore, we stay in the continuation region. The second part is the perpetual value of the investment undertaken right after the two technological innovations take place. The term $\frac{\lambda}{r+\lambda}$ is the stochastic discount factor under a Poisson process and the investment cost in this case is $I\phi^2$, as we need to wait for two jumps, meaning that the investment cost is reduced by a factor of ϕ^2.

The interpretation of the value function for the case $p_1^{2^\star} \leq p \leq p_0^{2^\star}$ is similar, but now the firm needs to wait for one jump. In this case the first part of the expression has both the positive and negative roots of the characteristic equation, where the term corresponding to the negative root, d_2, takes into consideration the fact that the price may decrease, and in that case the investment will only occur after two jumps.

In order to derive the expression for the constant term $A_{0,0,1}^2$, we use the method of undetermined coefficients to see that

$$A_{0,0,1}^2 = -2\frac{\lambda A_{1,0,0}^2}{\sigma^2(d_1 - d_2)}. \tag{12}$$

For the rest of the terms, we use value matching and smooth pasting conditions, which results in:

$$A_{0,0,0}^2 = A_{0,1,0}^2 + \frac{(p_1^{2^*})^{-d_1}}{d_1}\left[d_2 B_{0,1,0}^2 (p_1^{2^*})^{d_2} - A_{0,0,1}^2\left[1 + d_1 \ln p_1^{2^*}\right](p_1^{2^*})^{d_1} + \frac{\lambda p_1^{2^*}}{(r+\lambda-\mu)^2}\right] \tag{13}$$

$$A_{0,1,0}^2 = \frac{(p_0^{2^*})^{-d_1}}{d_1 - d_2}\left[(1-d_2)\frac{p_0^{2^*}}{r+\lambda-\mu} + d_2 I\frac{r+\lambda(1-\phi)}{r+\lambda}\right] \tag{14}$$

$$B_{0,1,0}^2 = \frac{(p_0^{2^*})^{-d_2}}{d_1 - d_2}\left[(d_1-1)\frac{p_0^{2^*}}{r+\lambda-\mu} - d_1 I\frac{r+\lambda(1-\phi)}{r+\lambda}\right]. \tag{15}$$

The threshold level $p_0^{2^*}$ is the unique solution of the following equation:

$$(d_1 - d_2)B_{0,1,0}^2(p_1^{2^*})^{d_2} + A_{0,0,1}^2(p_1^{2^*})^{d_1} + \lambda\left[\frac{(d_1-1)p_1^{2^*}}{(r+\lambda-\mu)^2} - \frac{d_1 I\phi(r+\lambda(1-\phi))}{(r+\lambda)^2}\right] = 0.^5 \tag{16}$$

In view of these results, we have the following proposition.

Proposition 1 *Consider the truncated optimal stopping problem defined by* (3)*, when* $\overline{n} = 2$*. Then, the value function* V^2 *is such that, for each* $n = 0, 1, 2$*,* $V^2(p, n) = v_n^2(p)$*, for* $p > 0$*, with* v_n^2 *defined by* (7)*,* (8) *and* (11)*, and the parameters* $A_{1,0,0}^2$*,* $A_{0,0,1}^2$*,* $A_{0,0,0}^2$*,* $A_{0,1,0}^2$ *and* $B_{0,1,0}^2$ *are given by* (10) *and* (12)-(15)*. Additionally, the threshold* $p_1^{2^\star}$ *is given by the expression* (9) *and* $p_0^{2^\star}$ *is the unique solution to the equation* (16) *satisfying* $p_0^{2^\star} > p_1^{2^\star}$*.*

3.2 The general case

In this section, we present the value function for the truncated optimal stopping problem (3) for $\overline{n} \in \mathbb{N}$. We use the same notation as before to denote the solution to the ODEs, $v_n^{\overline{n}}(p)$, and the thresholds price, $p_n^{\overline{n}^\star}$.

[5] Note that $B_{0,1,0}$ is indeed a function of $p_0^{2^\star}$.

We also borrow the arguments explained in the previous section to state that

$$v_{\overline{n}}^{\overline{n}}(p) = \frac{p}{r-\mu} - I\phi^{\overline{n}}, \quad \text{for all } p > 0 \equiv p_{\overline{n}}^{\overline{n}\star},$$

which is the value function in the stopping region. This accounts for the perpetual investment value when the maximum number of technologies innovations has occurred and therefore investment takes place right away.

In order to find $v_n^{\overline{n}}(p)$ in the continuation region and the corresponding threshold price, we need to follow a similar reasoning as the one presented for the case $\overline{n} = 2$. As before, this function is a solution of the ODE (5) and needs to be found backwards in n. But now the calculations are more cumbersome, as in order to find $v_n^{\overline{n}}(p)$ we have to study $\overline{n} - n$ regions in the continuation region (for instance, in the case $\overline{n} = 2$ and $n = 0$, we had already 2 different regions for p in the continuation region). The fact that we have the continuation region splitted in $\overline{n} - n + 1$ regions implies that we have also $\overline{n} - n$ different expressions for the value function. Additionally, since the function $v_n^{\overline{n}}$ is C^1 for each region, we have to check the usual smooth-fit conditions. We refer to Figure 1 for an illustration of this reasoning.

To take into account the different regions, we introduce further notation: we denote by $v_{n,k}^{\overline{n}}(p)$ the value function $v_n^{\overline{n}}(p)$ when $p_{\overline{n}-k}^{\overline{n}\star} \leq p < p_{\overline{n}-1-k}^{\overline{n}\star}$, with $k = 0, 1, 2, ..., \overline{n} - 1 - n$. Now we are in position to present the results for the value function:

$$v_n^{\overline{n}}(p) = \begin{cases} v_{n,k}^{\overline{n}}(p) & p_{\overline{n}-k}^{\overline{n}\star} \leq p < p_{\overline{n}-1-k}^{\overline{n}\star}, \ k = 0, 1, 2, ..., \overline{n} - 1 - n \\ \frac{p}{r-\mu} - I\phi^n & p > p_n^{\overline{n}\star} \end{cases},$$

with $v_{n,k}^{\overline{n}}(p)$ given by:

$$\begin{aligned} v_{n,k}^{\overline{n}}(p) = & \sum_{j=0}^{\overline{n}-1-n-k} \left[A_{n,k,j}^{\overline{n}} \ (\ln p)^j \ p^{d_1} + B_{n,k,j}^{\overline{n}} \ (\ln p)^j \ p^{d_2} \right] \\ & + \lambda^{\overline{n}-n-k} \left[\frac{p}{(r-\mu)\,(r+\lambda-\mu)^{\overline{n}-n-k}} - \frac{I\phi^{\overline{n}-k}}{(r+\lambda)^{\overline{n}-n-k}} \right]. \end{aligned}$$

As in the case $\overline{n} = 2$, this formula has a clear interpretation in economical terms: the first term (the sum) accounts for the the different ways that the investment may occur due to an increase of the price, whereas the second term accounts for the perpetual value of investment due to $\bar{n} - n - k$ jumps (meaning, in particular, that the investment region is crossed due to the jumps). We note that $\bar{n} - n - k$ is the number of jumps in the technological innovation process that are needed in order to achieve the stopping region, assuming that there are no movements in the price.

In order to completely define these functions, we use the following smooth-pasting conditions to calculate the constant terms and the thresholds,

$$v^{\overline{n}}_{n,k}(p^{\overline{n}\star}_{\overline{n}-1-k}) = v^{\overline{n}}_{n,k+1}(p^{\overline{n}\star}_{\overline{n}-1-k}) \quad \text{and} \quad (v^{\overline{n}}_{n,k})'(p^{\overline{n}\star}_{\overline{n}-1-k}) = (v^{\overline{n}}_{n,k+1})'(p^{\overline{n}\star}_{\overline{n}-1-k}),$$

for all $n = 0, 1, 2, ..., \overline{n} - 1$ and $k = 0, 1, 2, ..., \overline{n} - n - 1$. We start showing the expressions for the constants that are not multiplied by the logarithms.

$$\begin{aligned}
A^{\overline{n}}_{n,k,0} = A^{\overline{n}}_{n,(k+1),0} &+ \frac{\left(p^{\overline{n}\star}_{\overline{n}-1-k}\right)^{-d_1}}{d_1 - d_2} \left[\lambda^{\overline{n}-1-n-k} \left[(1 - d_2) \frac{p^{\overline{n}\star}_{\overline{n}-1-k}}{(r + \lambda - \mu)^{\overline{n}-n-k}} + d_2 I \phi^{\overline{n}-k-1} \frac{r + \lambda (1 - \phi)}{(r + \lambda)^{\overline{n}-n-k}}\right]\right. \\
&+ \left\{\left\{\sum_{j=1}^{\overline{n}-n-k-2} \left[\left(A^{\overline{n}}_{n,k,j} - A^{\overline{n}}_{n,k+1,j}\right) \left((d_2 - d_1) \ln p^{\overline{n}\star}_{\overline{n}-1-k} - j\right) (\ln p^{\overline{n}\star}_{\overline{n}-1-k})^{j-1} (p^{\overline{n}\star}_{\overline{n}-1-k})^{d_1}\right.\right.\right. \\
&- \left.\left.\left(B^{\overline{n}}_{n,k,j} - B^{\overline{n}}_{n,k+1,j}\right) j (\ln p^{\overline{n}\star}_{\overline{n}-1-k})^{j-1} (p^{\overline{n}\star}_{\overline{n}-1-k})^{d_2}\right]\right\} \chi_{\{k \neq \overline{n}-n-2\}} \\
&+ A^{\overline{n}}_{n,k,\overline{n}-n-k-1} \left((d_2 - d_1) \ln p^{\overline{n}\star}_{\overline{n}-1-k} - (\overline{n} - n - k - 1)\right) (\ln p^{\overline{n}\star}_{\overline{n}-1-k})^{\overline{n}-n-k-2} (p^{\overline{n}\star}_{\overline{n}-1-k})^{d_1} \\
&\left.\left. - (\overline{n} - n - k - 1) B^{\overline{n}}_{n,k,\overline{n}-n-k-1} (\ln p^{\overline{n}\star}_{\overline{n}-1-k})^{\overline{n}-n-k-2} (p^{\overline{n}\star}_{\overline{n}-1-k})^{d_2}\right\} \chi_{\{k \neq \overline{n}-n-1\}}\right]
\end{aligned}$$

and

$$\begin{aligned}
B^{\overline{n}}_{n,k,0} = B^{\overline{n}}_{n,(k+1),0} &+ \frac{\left(p^{\overline{n}\star}_{\overline{n}-1-k}\right)^{-d_2}}{d_1 - d_2} \left[\lambda^{\overline{n}-1-n-k} \left[(d_1 - 1) \frac{p^{\overline{n}\star}_{\overline{n}-1-k}}{(r + \lambda - \mu)^{\overline{n}-n-k}} - d_1 I \phi^{\overline{n}-k-1} \frac{r + \lambda (1 - \phi)}{(r + \lambda)^{\overline{n}-n-k}}\right]\right. \\
&+ \left\{\left\{\sum_{j=1}^{\overline{n}-n-k-2} \left[\left(B^{\overline{n}}_{n,k,j} - B^{\overline{n}}_{n,k+1,j}\right) \left((d_2 - d_1) \ln p^{\overline{n}\star}_{\overline{n}-1-k} + j\right) (\ln p^{\overline{n}\star}_{\overline{n}-1-k})^{j-1} (p^{\overline{n}\star}_{\overline{n}-1-k})^{d_2}\right.\right.\right. \\
&+ \left.\left.\left(A^{\overline{n}}_{n,k,j} - A^{\overline{n}}_{n,k+1,j}\right) j (\ln p^{\overline{n}\star}_{\overline{n}-1-k})^{j-1} (p^{\overline{n}\star}_{\overline{n}-1-k})^{d_1}\right]\right\} \chi_{\{k \neq \overline{n}-2-n\}} \\
&- B^{\overline{n}}_{n,k,\overline{n}-n-k-1} \left((d_1 - d_2)(\ln p^{\overline{n}\star}_{\overline{n}-1-k}) - (\overline{n} - n - k - 1)\right) (\ln p^{\overline{n}\star}_{\overline{n}-1-k})^{\overline{n}-n-k-2} (p^{\overline{n}\star}_{\overline{n}-1-k})^{d_2} \\
&\left.\left. + (\overline{n} - n - k - 1) A^{\overline{n}}_{n,k,\overline{n}-n-k-1} (\ln p^{\overline{n}\star}_{\overline{n}-1-k})^{\overline{n}-n-k-2} (p^{\overline{n}\star}_{\overline{n}-1-k})^{d_1}\right\} \chi_{\{k < \overline{n}-n-1\}}\right],
\end{aligned}$$

for $n = 0, 1, 2, ..., \overline{n} - 2$ and $k = 1, 2, ..., \overline{n} - 1 - n$, where we are assuming that $A^{\overline{n}}_{n,(\overline{n}-n),0} = 0$ and $B^{\overline{n}}_{n,(\overline{n}-n),0} = 0$. For $0 \equiv p^{\overline{n}\star}_{\overline{n}} < p \leq p^{\overline{n}\star}_{\overline{n}-1}$ (which implies that $k = 0$), the constants multiplied by p^{d_2} are all zero, i.e $B^{\bar{n}}_{n,0,j} = 0$, for all $n = 0, 1, 2, ..., \overline{n} - 1$ and $j = 0, 1, 2, \cdots, \overline{n} - 1 - n$. In particular, $B^{\overline{n}}_{n,0,0} = 0$, for

$n=0,1,2,...,\overline{n}-1$. Furthermore, the constants multiplied by p^{d_1} take the form

$$A^{\overline{n}}_{n,0,0} = A^{\overline{n}}_{n,1,0} + \frac{\left(p^{\overline{n}^\star}_{\overline{n}-1}\right)^{-d_1}}{d_1}\left[\left\{\left\{\sum_{l=1}^{\overline{n}-2-n}\left[\left(A^{\bar{n}}_{n,1,l} - A^{\bar{n}}_{n,0,l}\right)\left[l + d_1 \ln p^{\overline{n}^\star}_{\overline{n}-1}\right]\left(\ln p^{\overline{n}^\star}_{\overline{n}-1}\right)^{l-1}\left(p^{\overline{n}^\star}_{\overline{n}-1}\right)^{d_1}\right.\right.\right.\right.$$
$$\left.\left. + B^{\bar{n}}_{n,1,l}\left[l + d_2 \ln p^{\overline{n}^\star}_{\overline{n}-1}\right]\left(\ln p^{\overline{n}^\star}_{\overline{n}-1}\right)^{l-1}\left(p^{\overline{n}^\star}_{\overline{n}-1}\right)^{d_2}\right]\right\}\chi_{\{n\neq\overline{n}-2\}} + d_2 B^{\bar{n}}_{n,1,0}\left(p^{\overline{n}^\star}_{\overline{n}-1}\right)^{d_2}$$
$$\left.\left. - A^{\bar{n}}_{n,0,(\overline{n}-1-n)}\left[(\overline{n}-1-n) + d_1 \ln p^{\overline{n}^\star}_{\overline{n}-1}\right]\left(\ln p^{\overline{n}^\star}_{\overline{n}-1}\right)^{\overline{n}-2-n}\left(p^{\overline{n}^\star}_{\overline{n}-1}\right)^{d_1}\right\}\chi_{\{n\neq\overline{n}-1\}} + \frac{\lambda^{\overline{n}-1-n} p^{\overline{n}^\star}_{\overline{n}-1}}{(r+\lambda-\mu)^{\overline{n}-n}}\right],$$

for $n=0,1,2,...,\overline{n}-1$. As above, this representation is correct when we assume that $A^{\overline{n}}_{(\overline{n}-1),1,0}=0$. Indeed, in the case $n=\overline{n}-1$, we can get a simpler representation, which is

$$A^{\overline{n}}_{\overline{n}-1,0,0} = \frac{\left(p^{\overline{n}^\star}_{\overline{n}-1}\right)^{1-d_1}}{d_1\,(r+\lambda-\mu)}.$$

As in the case $\overline{n}=2$, the remaining constants can be found by using the method of undetermined coefficients (see the proof of this result in Nunes et al. [17]), resulting in the following

$$A^{\overline{n}}_{n,k,j} = -\frac{2\lambda}{\sigma^2}\sum_{l=j-1}^{\overline{n}-2-n-k}(-1)^{l+1-j}\,\frac{l!}{j!}\,\frac{A^{\overline{n}}_{(n+1),k,l}}{(d_1-d_2)^{l+2-j}},$$
$$B^{\overline{n}}_{n,k,j} = -\frac{2\lambda}{\sigma^2}\sum_{l=j-1}^{\overline{n}-2-n-k}(-1)^{l+1-j}\,\frac{l!}{j!}\,\frac{B^{\overline{n}}_{(n+1),k,l}}{(d_2-d_1)^{l+2-j}},$$

for $n=0,1,2,...,\overline{n}-2$, $k=0,1,2,...,\overline{n}-2-n$ and $j=1,2,\cdots,\overline{n}-1-n-k$. As before, this representation is correct when we assume that $B^{\bar{n}}_{\bar{n}-1,0,0}=0$.

To finalize this section, we note that the thresholds $p^{\overline{n}^\star}_n$ are generally not possible to find explicitly. However, in light of the smooth-pasting conditions, we can define $p^{\overline{n}^\star}_n$, for each $n=0,1,2,\cdots,\overline{n}-1$, as the unique solution of the equation

$$0 = \left\{\left\{\sum_{j=1}^{\overline{n}-2-n}\left[(A^{\overline{n}}_{n,0,j} - A^{\overline{n}}_{n,1,j})j(\ln p^{\overline{n}^\star}_{\overline{n}-1})^{j-1}(p^{\overline{n}^\star}_{\overline{n}-1})^{d_1}\right.\right.\right.$$
$$\left.\left. + B^{\overline{n}}_{n,1,j}\left[(d_1-d_2)(\ln p^{\overline{n}^\star}_{\overline{n}-1})^{j} - j(\ln p^{\overline{n}^\star}_{\overline{n}-1})^{j-1}\right](p^{\overline{n}^\star}_{\overline{n}-1})^{d_2}\right]\right\}\chi_{\{n\neq\overline{n}-2\}}$$
$$\left. + (d_1-d_2)B^{\overline{n}}_{n,1,0}\left(p^{\overline{n}^\star}_{\overline{n}-1}\right)^{d_2} + A^{\overline{n}}_{n,0,\overline{n}-1-n}(\overline{n}-1-n)(\ln p^{\overline{n}^\star}_{\overline{n}-1})^{\overline{n}-2-n}(p^{\overline{n}^\star}_{\overline{n}-1})^{d_1}\right\}\chi_{\{n\neq\overline{n}-1\}}$$
$$+ \lambda^{\overline{n}-n-1}\left[\frac{(d_1-1)p^{\overline{n}^\star}_{\overline{n}-1}}{(r+\lambda-\mu)^{\overline{n}-n}} - \frac{d_1 I\phi^{\overline{n}-1}(r+\lambda(1-\phi))}{(r+\lambda)^{\overline{n}-n}}\right].$$

As in the case $\overline{n}=2$, we note that $B^{\overline{n}}_{n,1,0}$ is indeed a function of the threshold $p^{\overline{n}^\star}_n$. When $n=\bar{n}-1$, an explicit solution to the previous equation is possible to obtain,

$$p^{\overline{n}^\star}_{\overline{n}-1} = \frac{d_2-1}{d_2}I\phi^{\overline{n}-1}[r+\lambda(1-\phi)].$$

4 The convergence result

In this section we show that the solution of the truncated optimal stopping problem (3) converges to the solution of the optimal stopping problem (1). Additionally, we illustrate this result numerically. In the next proposition, we state a convergence result that will be proved along the same lines of Proposition 4.1 in Guerra et al. [8].

Proposition 2 *The solution of the truncated problem* (3) *converges to the solution of the original one* (1), *i.e.,*

$$\lim_{\overline{n}\to\infty} V^{\overline{n}}(p,n) = V(p,n) \qquad \forall (p,n) \in]0,+\infty[\times\mathbb{N}.$$

In view of this result, we know that for sufficient large values of $\overline{n}$, the solution that we get for the truncated problem is arbitrarily close to the solution of the original problem. In particular, for sufficient large values of $\overline{n}$, the price thresholds $p_n^{\overline{n}^*}$ should be quite close to the price threshold $p_n^{(\overline{n}+1)^*}$. We notice that, by "sufficient large values of $\overline{n}$" we mean that $\overline{n}$ should be sufficiently away from the initial level technology n.

4.1 Numerical illustration

The quasi-analytical method proposed in this paper to solve the investment problem (1) is good enough if the approximated solution converges to the solution of the original optimal investment problem (which is true in light of Proposition 2, at least for $\overline{n}$ large enough). In practical terms, we are also interested in knowing what is the value of $\overline{n}$, for each initial condition n that allows us to get a good approximation. Throughout this section we will illustrate numerically that we can get a desired accuracy with acceptable values of $\overline{n}$, in the sense that it does not need to be very far from the initial value n.

Due to the Markovian nature of the technology innovation process and the fact that the investment cost decreases at a common ratio, ϕ, with the increase of the innovations, we may show that

$$E_{p,n}\left[e^{-r\tau}\left(\frac{P_\tau}{r-\mu} - I\phi^{N_\tau}\right)\right] = E_{p,0}\left[e^{-r\tau}\left(\frac{P_\tau}{r-\mu} - \tilde{I}\phi^{N_\tau}\right)\right],$$

where $\tilde{I} = I\phi^n$. Therefore, $V(p,n;I) = V(p,0;\tilde{I})$ and, consequently, the numerical results that we will show for the case $n = 0$ (in particular for $v_0^{\overline{n}}(p)$ and $p_0^{\overline{n}^*}$) play a fundamental role in our discussion about the convergence results of the truncated problem.

Henceforward, we will consider the set of parameters $r = 0.05, \sigma = 0.1, \mu = 0.03, \lambda = 0.1, \phi = 0.9$ and $I = 1$ in order to proceed with our numerical illustration[6].

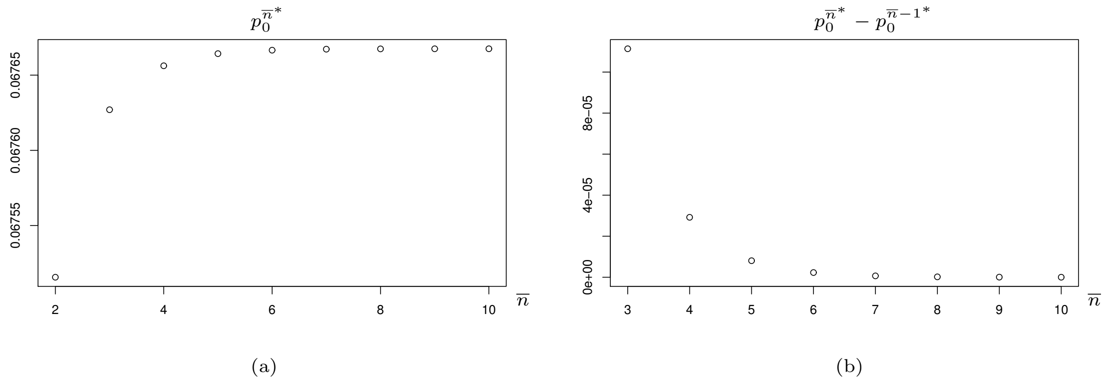


Figure 2: (a) Plot of the function $p_0^{\overline{n}^*}$, for $\overline{n} = 2, \cdots, 10$. (b) Plot of $p_0^{\overline{n}^*} - p_0^{\overline{n}-1^*}$, for $\overline{n} = 3, \cdots, 10$.

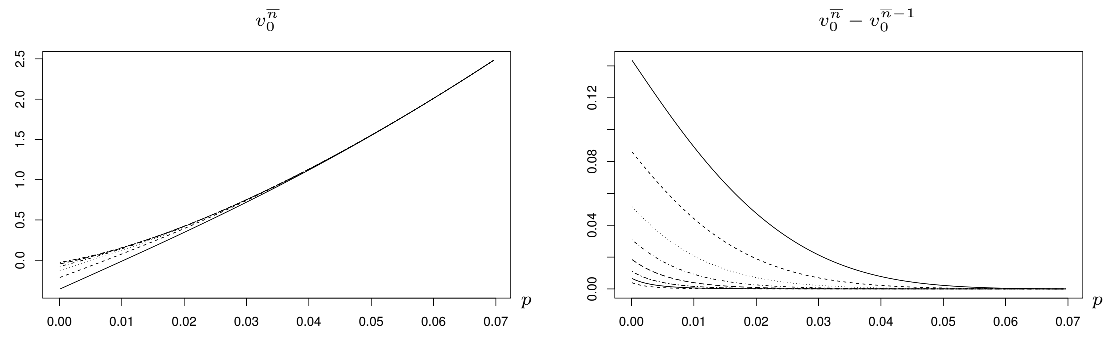
In Figure 2 (a) we show that $p_0^{\overline{n}^*}$ is converging to a particular value (see plot (a)) that is approximately 0.06766755 according to Table 1. Additionally, from Proposition 2, we know that $\lim_{\overline{n} \to +\infty} p_0^{\overline{n}^*} = p_0^*$, where p_0^* is the threshold of the original model. Combining the arguments above, we may conclude that $p_0^* \simeq 0.06766755$. Furthermore, Figure 2 (b) depicts the speed of this convergence. Indeed, the function $p_0^{\overline{n}^*} - p_0^{\overline{n}-1^*}$ decreases with increasing $\overline{n}$, being almost 0 for $\overline{n} \geq 6$.

In Table 1 we present the investment thresholds $p_n^{\overline{n}^*}$ for $\overline{n} = 2, \ldots, 10$ and $n = 0, \ldots, \overline{n} - 1$. From this numerical illustration we conclude that for a fixed value n, the investment threshold $p_n^{\overline{n}^*}$ converges very fast with increasing $\overline{n}$. This suggests that one may get a good approximation of the solution of the original problem without the need to consider very large values of $\overline{n}$, which would induce a costly computation. We also present in Table 2 the differences $p_n^{\overline{n}^*} - p_n^{\overline{n}-1^*}$. From this table, one can easily conclude that the consecutive differences converge to zero in a very fast way. Moreover, the behavior is similar for all the initial values, n, considered in the table.

Although the thresholds, $p_n^{\overline{n}^*}$, converge quite fast for a particular value $p_n^{\overline{n}^*}$, that is the threshold of the

[6]This set of values will also be used as a baseline in the comparative statistics section. Moreover, a sketch of the numerical implementation is available from the authors, in case one intend to use a different baseline.

original model, the speed of the convergence of the function should also be verified. In Figure 3 we depict the behavior of $v_0^{\overline{n}}$ when $\overline{n}$ increases. In Figure 3 (a) we can see that the value functions $v_0^{\overline{n}}$ with $\overline{n} = 5, 6, 7$



(a) (b)

Figure 3: (a) Plot of the functions $v_0^{\overline{n}}$, for $\overline{n} = 2, \cdots, 7$ where the functions appear in an increasing way in $\overline{n}$. (b). Plot of the functions $v_0^{\overline{n}} - v_0^{\overline{n}-1}$, for $\overline{n} = 3, \cdots, 10$, where the functions appear in a decreasing way in $\overline{n}$.

are close to each others. From Figure 3 (b) we conclude that the difference between the value functions for two consecutive values of $\overline{n}$ decreases significantly with increasing $\overline{n}$.

Finally, just for illustration purposes, we present in Figure 4 the value function considering $\overline{n} = 30$, mostly when p belongs to the continuation region. We note that, for this level, the value function is, as expected, always positive, which means that, for this parameters we can consider that $V(p, 0) \approx v_0^{30}(p)$.

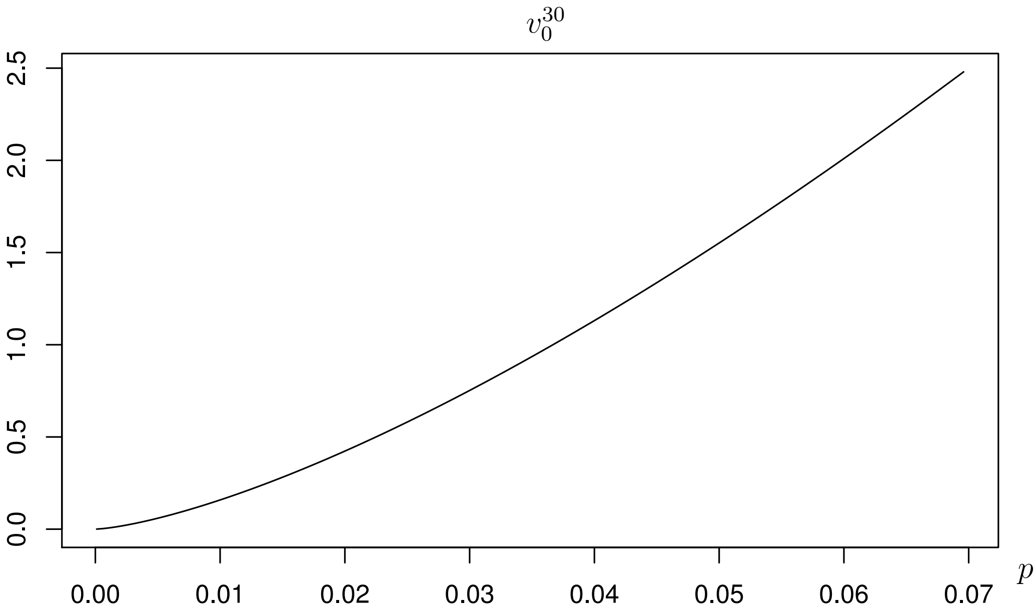


Figure 4: Plot of the function v_0^{30}.

5 Comparative Statics

In this section we provide some insights about the behavior of the investment thresholds with the parameters that influence each one of the uncertainties. It follows from the convergence of the truncated problem to the original one that the behavior of the thresholds is the same, independently of the particular $\overline{n}$ and n considered. As we can obtain a closed expression for $p_1^{2^\star}$ (defined in Equation (9)), we study analytically the influence of σ, μ, ϕ and λ in the investment decision for $\overline{n} = 2$ and $n = 1$. The results are presented in the next proposition.

Proposition 3 *The investment threshold $p_1^{2^\star}$ is increasing with σ and decreasing with μ. For ϕ, the behavior depends on the relationship between other parameters, as follows: if $r \geq \lambda$, then it increases with ϕ; in case $r < \lambda$, it increases with ϕ if $\phi < \frac{1}{2}\frac{r+\lambda}{\lambda}$ and decreases afterwards. Finally, $p_1^{2^\star}$ has monotonically behavior* *with λ, for "small values" of ϕ, and a non-monotonically behavior, for "large values" of ϕ.*

We note that the results that we obtain for σ and μ agree with the standard case: increasing the volatility usually postpones the investment decision, whereas increasing the drift anticipates it, as we expect larger profits in the future. But the results for the investment parameters, ϕ and λ, are somehow unexpected.

Intuitively, we would expect that increasing ϕ (and thus, the investment cost decreases less with a technology innovation) would always postpone the investment decision. In particular, this would imply waiting more for a decrease in the investment cost. The result presented in Proposition 3 shows that, indeed, this is the case in almost all situations. There is an exception when the intensity of new arrivals is larger than the interest rate and, simultaneously, ϕ is larger than $\frac{1}{2}\frac{r+\lambda}{\lambda}$. This indicates that waiting for a new innovation that will decrease the investment cost is worthwhile depending on: (i) how much it decreases; (ii) the relationship between the interest rate and the expected time until the next innovation. In fact, if the decrease in the investment cost is large (low ϕ) and the expected time until the next innovation is small (large λ), then the decision to invest should be postponed. This happens because as the expected time until the next arrival is small, the expected return of the investment during this period, if we would invest earlier, would be smaller than the decrease in the investment cost. But if the impact of the innovation is not so large, then it happens the other way around, i.e, it is more profitable for the company to invest earlier than to wait for the next innovation, which would lead to a small impact in the cost. Finally, we note that the function $\lambda \to \frac{1}{2}\frac{r+\lambda}{\lambda}$ is decreasing. For $\lambda > r$, we have that

$$\lim_{\lambda \to r} \frac{1}{2}\frac{r+\lambda}{\lambda} = 1 \quad \text{and} \quad \lim_{\lambda \to +\infty} \frac{1}{2}\frac{r+\lambda}{\lambda} = \frac{1}{2},$$

which means that the investment decision is anticipated with increasing ϕ in an interval. In fact, the greater the λ, the larger the interval.

Regarding the influence of changing λ in the investment decision, it is possible to find two distinct behaviors, for different sets of parameters: (i) $p_1^{2^\star}$ is non-monotonic, in particular for large values of ϕ, which means low impact in the decrease of the investments costs due to technological innovations; (ii) $p_1^{2^\star}$ is monotonic for small values of ϕ. In Figure 5 we draw $p_1^{2^\star}$ as a function of λ for small and large values of ϕ, using as base-case parameters the ones mentioned in the previous section.

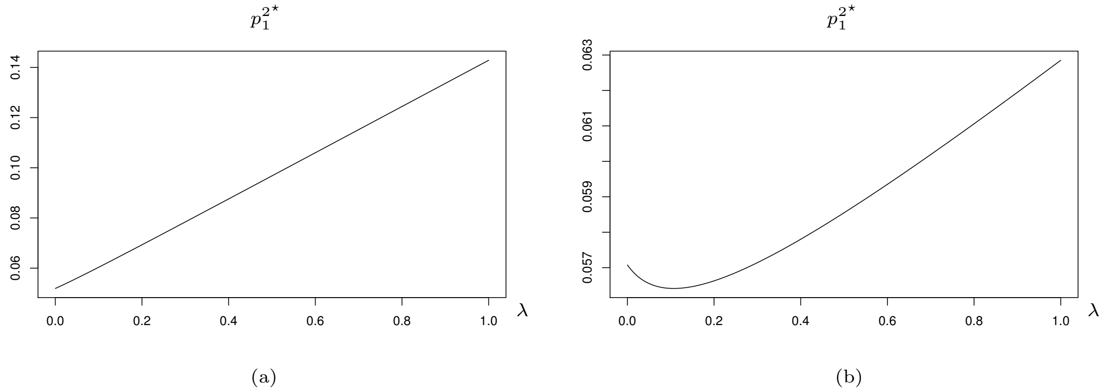


Figure 5: Investment threshold $p_1^{2^\star}$ as a function of λ, for (a) $\phi = 0.9$ and (b) $\phi = 0.99$.

These illustrations show that when the impact of technology innovations is significant in the reduction of the investment costs (corresponding to the case illustrated in the left hand panel of Figure 5), then the decision to invest should be postponed. This holds as increasing λ means that the expected time until next innovation decreases. Thus it is worthwhile to wait for a cheaper investment, because the value that could be accumulated during this period (if the investment would take place earlier) does not compensate the reduction in the investment cost. On the other hand, when the impact of the innovations in the investment cost is small (right hand panel of Figure 5)and the expected time until the next innovation is large (meaning small values of λ), there is no reason to wait for such innovation. Indeed, the firm expects to gain more starting producing than waiting for the cost reduction, therefore the decision to invest is anticipated. However, if the expected time until the next innovation is small (meaning large values of λ), then it pays back to invest latter, for similar reasons as the ones previously explained.

6 Conclusion

In this paper we develop a quasi-analytical method to solve an investment problem with two sources of uncertainty: the price that follows a GBM and the number of technology innovations that is driven by a Poisson process.

The difficulty in the resolution of this investment problem comes from the fact that the investment region may be attained by an increase of either the level of technology available in the market or the price of the product. An interesting feature of the method develop in this paper is its flexibility and suitability to other problems with the same features.

The quasi-analytical method is based on the truncation of the number of technology innovations allowed. We prove that the approximated solution converges to the exact solution. Additionally, we illustrate that, in a numerical point of view, the convergence is quickly attained, which means that one can consider a reasonable level $\overline{n}$, in order to get a good approximation.

As a consequence of the convergence result, we know that the qualitative behavior of the threshold prices, $p_n^{\overline{n}^*}$, should be preserved for all $\overline{n} \in \mathbb{N}$ and $n < \overline{n}$, which allows us to provide an extensive sensitivity analysis. We prove that the standard results of real options still hold, notably the investment is postponed for increasing volatility and anticipated for increasing drift. However, we can theoretically prove non-monotonic behaviors for the price threshold when one increases the impact of the innovation in the investment cost and the intensity of the technology innovations.

A Proofs

In order to facilitate the proof of Proposition 1 we state an initial auxiliary result.

Lemma 1 *Consider the function*

$$\begin{aligned} f(p) =& A_{0,0,1}^2 (p_1^{2^\star})^{d_1} + \left[(d_1 - 1) \frac{p}{r + \lambda - \mu} - d_1 I \frac{r + \lambda(1-\phi)}{r + \lambda} \right] \left(\frac{p}{p_1^{2^\star}} \right)^{-d_2} \\ &+ \lambda \left[\frac{(d_1 - 1) p_1^{2^\star}}{(r + \lambda - \mu)^2} - \frac{d_1 I \phi (r + \lambda(1-\phi))}{(r+\lambda)^2} \right], \end{aligned}$$

where $A_{0,0,1}^2$ and $p_1^{2^\star}$ are defined, respectively, by (12) *and* (9)*, respectively. Then, the equation $f(p) = 0$ has a unique root, $p_0^{2^\star}$, verifying $p_0^{2^\star} > p_1^{2^\star} > 0$.*

Proof of Lemma 1 Firstly, we note that

$$A_{0,0,1}^2 (p_1^{2^\star})^{d_1} = -\frac{2\lambda}{\sigma^2 (d_1 - d_2)} \times \frac{p_1^{2^*}}{d_1 (r + \lambda - \mu)},$$

which combined with (6) allow us to make the calculations

$$f(0) = A^2_{0,0,1}(p_1^{2^*})^{d_1} + \lambda\left[\frac{(d_1-1)p_1^{2^*}}{(r+\lambda-\mu)^2} - \frac{d_1 I\phi(r+\lambda(1-\phi))}{(r+\lambda)^2}\right]$$
$$= -\frac{2\lambda p_1^{2^*}}{\sigma^2(r+\lambda-\mu)} \times \frac{1-d_1}{d_2(d_1-d_2)(1-d_2)} < 0.$$

Taking into account that the derivative of the function f is given by

$$f'(p) = \left(\frac{p}{p_1^{2^*}}\right)^{-d_2}\left[\frac{(1-d_2)(d_1-1)}{r+\lambda-\mu} + d_1 d_2 I\frac{r+\lambda(1-\phi)}{r+\lambda}p^{-1}\right],$$

we conclude that the function f is decreasing for $p < p*$ and increasing $p > p*$, where

$$p^* = d_1 d_2 I \times \frac{r+\lambda(1-\phi)}{r+\lambda} \times \frac{r+\lambda-\mu}{(d_2-1)(d_1-1)}.$$

Combining the previous information with the fact that $\lim_{p\to+\infty} f(p) = +\infty$, we can conclude that there is a unique $p = p_0^{2^*}$ that satisfies the equation $f(p) = 0$.

To complete the proof, we note that $f(p_1^{2^*}) = f(0) + \left[(d_1-1)\frac{p_1^{2^*}}{r+\lambda-\mu} - d_1 I\frac{r+\lambda(1-\phi)}{r+\lambda}\right] < 0$, which follows in light of the facts that $f(0) < 0$ and

$$\left[(d_1-1)\frac{p_1^{2^*}}{r+\lambda-\mu} - d_1 I\frac{r+\lambda(1-\phi)}{r+\lambda}\right] = -\frac{2(\phi-1)}{d_2\sigma^2}I\left[r+\lambda(1-\phi)\right] < 0.$$

Proof of Proposition 1 We start noticing that v_2^2 satisfies the boundary condition of Equation (4). To complete this proof we need to verify that:

$$v_1^2 \quad \text{is such that} \quad \min\{(r+\lambda)v_1^2(p) - \mu p(v_1^2)' - \frac{\sigma^2}{2}p^2(v_1^2)''(p) - \lambda v_2^2(p), v_1^2(p) - g_1(p)\} = 0, \tag{17}$$
$$v_0^2 \quad \text{is such that} \quad \min\{(r+\lambda)v_0^2(p) - \mu p(v_0^2)' - \frac{\sigma^2}{2}p^2(v_0^2)''(p) - \lambda v_1^2(p), v_0^2(p) - g_0(p)\} = 0. \tag{18}$$

Proof of (17): By construction, the function $p \to A^2_{1,0,0}p^{d_1} + \lambda\left[\frac{p}{(r-\mu)(r+\lambda-\mu)} - \frac{I\phi^2}{r+\lambda}\right]$ is a solution to the ODE $(r+\lambda)v_1^2(p) - \mu p(v_1^2)' - \frac{\sigma^2}{2}p^2(v_1^2)''(p) - \lambda v_2^2(p) = 0$. Additionally, trivial calculations show that the smooth-pasting conditions

$$v_1^2(p_1^{2^*}) = g_1(p_1^{2^*}) \quad \text{and} \quad (v_1^2)'(p_1^{2^*}) = g_1'(p_1^{2^*})$$

are satisfied if and only if $p_1^{2^*}$ and $A^2_{1,0,0}$ are given by (9) and (10). To finish this part of the proof, we still need to verify that the function v_1^2 satisfies the inequalities

$$(r+\lambda)v_1^2(p) - \mu p(v_1^2)' - \frac{\sigma^2}{2}p^2(v_1^2)''(p) - \lambda v_2^2(p) \geq 0, \quad \text{for all } p \geq p_1^{2^*} \tag{19}$$
$$v_1^2(p) - g_1(p) \geq 0, \quad \text{for all } 0 < p \leq p_1^{2^*}. \tag{20}$$

For $p \geq p_1^{2^*}$, we have $v_1^2(p) = \frac{p}{r-\mu} - I\phi$, which allow us to obtain

$$(r+\lambda)v_1^2(p) - \mu p(v_1^2)' - \frac{\sigma^2}{2}p^2(v_1^2)''(p) - \lambda v_2^2(p) = p - I\phi(r + \lambda(1-\phi)).$$

Consequently, condition (19) is equivalent to $p \geq I\phi(r+\lambda(1-\phi))$, and therefore, the first inequality holds true because

$$p_1^{2^*} = \frac{d_2 - 1}{d_2} I\phi(r+\lambda(1-\phi)) \geq I\phi(r+\lambda(1-\phi)).$$

To prove the inequality (20), we use (10) to see that

$$v_1^2(p) - g_1(p) = \frac{1-d_1}{d_1} \times \frac{p}{r+\lambda-\mu} + I\phi\frac{r}{r+\lambda}.$$

It is now trivial to observe that the function $p \to v_1^2(p) - g_1(p)$ is decreasing, which combined with the fact that $v_1^2(p_1^{2^*}) - g_1(p_1^{2^*}) = 0$ proves the inequality (20).

Proof of (18): It is a matter of calculations to see that the ODE $(r+\lambda)v_0^2(p) - \mu p(v_0^2)' - \frac{\sigma^2}{2}p^2(v_0^2)''(p) - \lambda v_1^2(p) = 0$ is satisfied by the functions $p \to \left(A_{0,0,0}^2 + A_{0,0,1}^2 \ln p\right) p^{d_1} + \lambda^2 \left[\frac{p}{(r-\mu)(r+\lambda-\mu)^2} - \frac{I\phi^2}{(r+\lambda)^2}\right]$, with $A_{0,0,1}^2$ as in (12), when $p \leq p_1^{2^*}$ and $p \to A_{0,1,0}^2 p^{d_1} + B_{0,1,0}^2 p^{d_2} + \lambda\left[\frac{p}{(r-\mu)(r+\lambda-\mu)} - \frac{I\phi}{r+\lambda}\right]$, when $p \geq p_1^{2^*}$. By using the C^1 continuity of the function v_0^2 it follows that the parameters $A_{0,0,1}^2$, $A_{0,0,0}^2$, $A_{0,1,0}^2$ and $B_{0,1,0}^2$ are given by (12)-(15) while $p_0^{2^*}$ is such that (16) holds true and $p_1^{2^*}$ is given by (9). Additionally, in light of Lemma 1, $p_0^{2^*}$ is the unique solution to the equation $f(p) = 0$ and verify $p_0^{2^*} > p_1^{2^*} > 0$. To finish this part of the proof, we have to verify that the function v_1^2 satisfies the inequalities

$$(r+\lambda)v_1^2(p) - \mu(r+\lambda)v_0^2(p) - \mu p(v_0^2)' - \frac{\sigma^2}{2}p^2(v_0^2)''(p) - \lambda v_1^2(p) \geq 0, \quad \text{for all } p \geq p_0^{2^*} \tag{21}$$

$$v_0^2(p) - g_0(p) \geq 0, \quad \text{for all } 0 < p \leq p_0^{2^*}. \tag{22}$$

The proof of conditions (21)-(22) follows in light of a similar argument to the one used to prove the conditions in (19)-(20).

Proof of Proposition 2 Notice that by definition $V(p,n) = \sup_{\tau \geq 0} J(p,n,\tau) \geq \sup_{\tau \geq 0} J(p,n,\tau \wedge \tau_{\overline{n}}) = V^{\overline{n}}(p,n)$, for any $\overline{n} \in \mathbb{N}$, which implies that $\limsup_{\overline{n} \to +\infty} V^{\overline{n}}(p,n) \leq V(p,n)$. In order to prove that $\liminf_{\overline{n} \to +\infty} V^{\overline{n}}(p,n) \geq V(p,n)$, we will show that, for all stopping times $\tau \geq 0$, $\liminf_{\overline{n} \to +\infty} J(p,n,\tau \wedge \tau_{\overline{n}}) \geq J(p,n,\tau)$. Fix $\tau \geq 0$ and notice that

$$J(p,n,\tau) - J(p,n,\tau\wedge\tau_{\overline{n}}) = E_{p,n}\left[-\int_{\tau\wedge\tau_{\overline{n}}}^{\tau} e^{-rs}P_s ds - e^{-r\tau}I_\tau + e^{-r(\tau\wedge\tau_{\overline{n}})}I_{\tau\wedge\tau_{\overline{n}}}\right]$$
$$\leq E_{p,n}\left[-e^{-r\tau}I_\tau + e^{-r(\tau\wedge\tau_{\overline{n}})}I_{\tau\wedge\tau_{\overline{n}}}\right].$$

For any $\tau \geq 0$, $\tau\wedge\tau_{\overline{n}} \leq \tau$, $N_{\tau\wedge\tau_{\overline{n}}} \leq N_\tau$ and, consequently, $I \geq e^{-r(\tau\wedge\tau_{\overline{n}})}I_{\tau\wedge\tau_{\overline{n}}} \geq e^{-r\tau}I_\tau$ almost surely. By construction, $e^{-r(\tau\wedge\tau_{\overline{n}})}I_{\tau\wedge\tau_{\overline{n}}} \to e^{-r\tau}I_\tau$ almost surely, when $\overline{n}\to+\infty$. Thus, it follows from the dominated convergence theorem that

$$\lim_{\overline{n}\to+\infty} E_{p,n}\left[-e^{-r\tau}I_\tau + e^{-r(\tau\wedge\tau_{\overline{n}})}I_{\tau\wedge\tau_{\overline{n}}}\right] = 0,$$

which is sufficient to conclude the proof.

Proof of Proposition 3 We will split the proof in three parts.

- *Monotony of $p_1^{2^*}$ regarding μ and σ:*

To prove this part, we notice that

$$\frac{\partial p_1^{2^*}}{\partial i} = d_2^{-2}\frac{\partial d_2}{\partial i} I\phi(r+\lambda(1-\phi))$$

with $i=\mu,\sigma^2$. The intended result follows taking into account that

$$\frac{\partial p_1^{2^*}}{\partial \mu} = \frac{-1}{\sigma^2}\left(1+\frac{\frac{\sigma^2}{2}-\mu}{\sqrt{\left(\frac{\sigma^2}{2}-\mu\right)^2+2\sigma^2(r+\lambda)}}\right) < 0$$

and

$$\frac{\partial p_1^{2^*}}{\partial \sigma^2} = \frac{2\mu^2+2\sigma^2(r+\lambda)+\mu\left(-\sigma^2+\sqrt{\left(\sigma^2-2\mu\right)^2+8\sigma^2(r+\lambda)}\right)}{\sigma^4\sqrt{\left(\sigma^2-2\mu\right)^2+8\sigma^2(r+\lambda)}} > 0.$$

The second inequality follows in light of the condition $r+\lambda>\mu$.

- *Monotony of $p_1^{2^*}$ regarding ϕ:*

It is a matter of calculations to see that

$$\frac{\partial p_1^{2^*}}{\partial \phi} = \frac{d_2-1}{d_2} I(r+\lambda(1-2\phi)).$$

Therefore, $\frac{\partial p_1^{2^*}}{\partial\phi} > 0$ if and only if $\phi < \frac{r+\lambda}{2\lambda}$. Additionally, we notice that $\frac{r+\lambda}{2\lambda} \Leftrightarrow r \geq \lambda \geq 1$, which concludes this part of the prof.

- *Monotony of $p_1^{2^*}$ regarding λ:*

After some calculations, one can obtain that

$$\frac{\partial p_1^{2\star}}{\partial \lambda} = I\phi d_2^{-2}\left[\frac{\partial d_2}{\partial \lambda}\left(r+\lambda(1-\phi)\right)+(d_2^2-d_2)(1-\phi)\right].$$

Additionally, one can easily see that, for every $0 \leq \phi < 1$,

$$\lim_{\lambda\to\infty}\frac{\partial d_2}{\partial \lambda}\left(r+\lambda(1-\phi)\right)+(d_2^2-d_2)(1-\phi)=\infty,$$

and, for $\phi = 1$, $\frac{\partial p_1^{2\star}}{\partial \lambda} = Id_2^{-2}\frac{\partial d_2}{\partial \lambda}r < 0$. Therefore, taking into account that $p_1^{2^\star}$ is a continuous function in ϕ, we conclude that for large values of ϕ (at least in a neighborhood of 1), the sign of the derivative $\frac{\partial p_1^{2\star}}{\partial \lambda}$ must change from negative to positive. Then, for such values of ϕ, the function $p_1^{2\star}$ is non-monotonic with respect to λ.

Moreover, for $\phi = 0$, some trivial calculations lead us to

$$\begin{aligned}\frac{\partial d_2}{\partial \lambda}\left(r+\lambda(1-\phi)\right)+(d_2^2-d_2)(1-\phi)|_{\phi=0} &= \frac{\partial d_2}{\partial \lambda}(r+\lambda)+(d_2^2-d_2)\\ &= d_2^2\left[\frac{\sigma^2}{2}+\sqrt{\left(\frac{\sigma^2}{2}-\mu\right)^2+2\sigma^2(r+\lambda)}\right],\end{aligned}$$

which is always positive. Thus, for small values of ϕ (at least in a neighborhood of 0), the sign of the derivative $\frac{\partial p_1^{2\star}}{\partial \lambda}$ must be always positive. Then, for such values of ϕ, the function $p_1^{2\star}$ increases with respect to λ.

B Tables

	$\bar{n}=2$	$\bar{n}=3$	$\bar{n}=4$	$\bar{n}=5$	$\bar{n}=6$	$\bar{n}=7$	$\bar{n}=8$	$\bar{n}=9$	$\bar{n}=10$
$p_0^{\bar{n}\star}$	0.06751563	0.06762701	0.06765623	0.06766430	0.06766660	0.06766727	0.06766747	0.06766753	0.06766755
$p_1^{\bar{n}\star}$	0.06033744	0.06076407	0.06086431	0.06089061	0.06089787	0.06089994	0.06090054	0.06090072	0.06090078
$p_2^{\bar{n}\star}$		0.05430369	0.05468766	0.05477788	0.05480155	0.05480808	0.05480994	0.05481049	0.05481065
$p_3^{\bar{n}\star}$			0.04887332	0.04921890	0.04930009	0.04932139	0.04932727	0.04932895	0.04932944
$p_4^{\bar{n}\star}$				0.04398599	0.04429701	0.04437008	0.04438925	0.04439454	0.04439605
$p_5^{\bar{n}\star}$					0.03958739	0.03986731	0.03993307	0.03995033	0.03995509
$p_6^{\bar{n}\star}$						0.03562865	0.03588058	0.03593977	0.03595529
$p_7^{\bar{n}\star}$							0.03206579	0.03229252	0.03234579
$p_8^{\bar{n}\star}$								0.02885921	0.02906327
$p_9^{\bar{n}\star}$									0.02597329

Table 1: $p_n^{\overline{n}^*}$ for $\overline{n} = 2 \cdots 10$ and $n = 0, \cdots, \overline{n} - 1$.

	$\bar{n}=3$	$\bar{n}=4$	$\bar{n}=5$	$\bar{n}=6$	$\bar{n}=7$	$\bar{n}=8$	$\bar{n}=9$	$\bar{n}=10$
$p_0^{\bar{n}\star} - p_0^{\bar{n}-1\star}$	0.00011137	0.00002922	0.00000807	0.00000230	0.00000067	0.00000020	0.00000006	0.00000002
$p_1^{\bar{n}\star} - p_1^{\bar{n}-1\star}$	0.00042664	0.00010024	0.00002630	0.00000726	0.00000207	0.00000060	0.00000018	0.00000005
$p_2^{\bar{n}\star} - p_2^{\bar{n}-1\star}$		0.00038397	0.00009021	0.00002367	0.00000653	0.00000186	0.00000054	0.00000016
$p_3^{\bar{n}\star} - p_3^{\bar{n}-1\star}$			0.00034558	0.00008119	0.00002130	0.00000588	0.00000168	0.00000049
$p_4^{\bar{n}\star} - p_4^{\bar{n}-1\star}$				0.00031102	0.00007307	0.00001917	0.00000529	0.00000151
$p_5^{\bar{n}\star} - p_5^{\bar{n}-1\star}$					0.00027992	0.00006577	0.00001725	0.00000476
$p_6^{\bar{n}\star} - p_6^{\bar{n}-1\star}$						0.00025192	0.00005919	0.00001553
$p_7^{\bar{n}\star} - p_7^{\bar{n}-1\star}$							0.00022673	0.00005327
$p_8^{\bar{n}\star} - p_8^{\bar{n}-1\star}$								0.00020406

Table 2: $p_n^{\overline{n}^*} - p_n^{\overline{n}-1^*}$ for $\overline{n} = 3 \cdots 10$ and $n = 0, \cdots, \overline{n} - 2$.

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