Entrepreneurial Business Plan under Undiversifiable Idiosyncratic Risk[†]

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Abstract

This paper presents an entrepreneurial optimal business plan in which optimal consumption and portfolio rules, and optimal exit strategy for an entrepreneur are jointly determined in the presence of undiversifiable idiosyncratic risk. We find that the entrepreneur is more likely to exit from her risky business as investment opportunity worsens or as her risk aversion coefficient increases or as the idiosyncratic risk increases. When the entrepreneur decumulates wealth, she can achieve a partial hedging effect of a risky portfolio against the business risk by optimally increasing her risky portfolio as the idiosyncratic risk increases. Accordingly, stock market participation is of importance to the entrepreneur for the purpose of risk diversification and a smooth continuation of her risky business.

Keywords: Entrepreneurial Business, Optimal Exit, Undiversifiable Idiosyncratic Risk

JEL Classifications: C61, E21, G11

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1 Introduction

Modern economics and finance have greatly improved the understanding of household portfolio choice. Among important determinants are undiversifiable risks stemming from labor income and personal risky business. Portfolio theory contends that diversification and risk sharing are fundamental principles. Empirical evidence indicates that households typically invest in a single stock, and this strategy does not conform to the theoretical prediciton. Furthermore, entrepreneurs typically hold large equity shares in their risky business and hence bear substantial undiversifiable idiosyncratic risk. However, this behavior is counterintuitive because portfolio theory (Merton, 1969) implies that the underdiversification in entrepreneurs' portfolio is risky. Importantly, notwithstanding unattractive returns to entrepreneurial business,[1] households actively engage in entrepreneurship.

Entrepreneurs should be well compensated by the risk-return trade-off against their commitment in their own business to mitigate moral hazard and adverse selection. However, they are likely to be poorly compensated relative to investment in a public equity that guarantees a positive risk premium (Moskowitz and Vissing-Jørgensen, 2002; Hall and Woodward, 2007). The gap between (1) the theoretical prediction that entrepreneurs

[1]Following Cochrane (2005), the volatility of log returns to venture capital investments is up to 89%, whereas one of the log S&P return is only 14.9%. Although venture capital puts entrepreneurs at a large risk captured by the high volatility, the annual expected return of venture capital is only about 15% (Cochrane, 2005). The expected returns to private equity are lower than the returns to public equity (Moskowitz and Vissing-Jørgensen, 2002). Moreover, entrepreneurs' median life-time profits are lower than those of similarly skilled wage-earners (Hamilton, 2000).

enhance the demand for risky portfolio to diversify away their business risk, and (2) empirical observations for underdiversification in entrepreneurial portfolio choice is the so-called *private equity premium puzzle*.

Numerous authors have attempted to solve the private equity premium puzzle. To diversify the business risk entrepreneurs willingly choose a conservative portfolio composition (Heaton and Lucas, 2000). Entrepreneurs would have a significant safe portfolio of financial assets because of their liquidity constraints to finance their own business in its final steps (Faig and Shum, 2002). Puri and Robinson (2006) have concluded that entrepreneurs behave in a more optimistic and risk-loving manner than do wage earners. To hedge the business risk, entrepreneurs do appear to decrease their investment in other risky assets (Fang and Nofsinger, 2009). Social status concerns significantly affect entrepreneurial business plans in the sense that entrepreneurs have concentrated risky asset composition in equilibrium; these concerns provide a rational reason for the private equity premium puzzle (Roussanov, 2010). Credit constraints, heterogeneous risk attitudes, and nonpecuniary benefits from self-employment could also contribute to give a partial explanation for the private equity premium puzzle (Fossen, 2012).

Our paper generates important implications for an entrepreneurial business plan such as optimal consumption and portfolio selection, and optimal exit strategy from a risky business. We focus on how entrepreneurial undiversifiable idiosyncratic risk influences entrepreneurial investment behavior in their own business and their asset composition. For our model design, we follow Miao and Wang (2007b) who have studied the effects of uninsurable idiosyncratic risk on an entrepreneurial learning about the quality of risky business and the optimal exit strategy. However, they did not consider the role of a market portfolio in entrepreneurial business. Importantly, financial assets including a market portfolio can be used to hedge against bad outcomes of entrepreneurial business (Faig and Shum, 2002). Many researchers have tried to investigate the relationship between entrepreneurship and household portfolio composition (Heaton and Lucas, 2000, 2009; Faig and Shum, 2002; Miao and Wang 2007a; Chen *et al.*, 2010; Wang *et al.*, 2012; Leippold and Stromberg, 2014), by considering all realistic ramifications of an entrepreneurial business.[2]

In this paper, we present an entrepreneur's optimal business plan that jointly determines (1) optimal consumption and portfolio rules, and (2) optimal exit strategy, for an entrepreneur in the presence of undiversifiable idiosyncratic risk. As far as we know, this is the first study that clarifies the relationship between the idiosyncratic risk and optimal exit strategy from a risky business as well as the role of a market portfolio on a business plan under a constant relative risk aversion (CRRA) utility. Miao and Wang (2007a), and Chen have *et al.* (2010) successfully solved the problem of nondiversifiable investment risk in incomplete financial markets for a risk-averse entrepreneur. However, for tractability these authors adopted a constant absolute risk aversion (CARA) utility to solve the incomplete market problem, despite CARA's shortcoming that it does not capture wealth effects.

The objective of this paper is to study an entrepreneur's business plan to maximize her CRRA lifetime utility by controlling per-period consumption, risky portfolio, and the time

[2]Portfolio allocations of entrepreneurs in aggregate account for about 30% of the stock market (Heaton and Lucas, 2000). Further, the entrepreneur's investment in the stock market can be used as a measure of her risk tolerance (Fang and Nofsinger, 2009).

to quit a business and accept a safe job in the presence of undiversifiable idiosyncratic risk. The entrepreneur faces undiversifiable idiosyncratic risk from her risky business and thus, receives future income at random rates.[3] The entrepreneur is currently in the business and obtains low-quality income. Due to the undiversifiable idiosyncratic risk, she has a small probability of succeeding in the business (Miao and Wang, 2007b). If she succeeds in the business, then she obtains high-quality income afterward. We assume that the entrepreneur has an option to quit the business and accept a safe job, and that after she exercises the option, she obtains constant income infinitely.

Our paper is distinct from the existing optimal stopping problems in complete financial markets in that a market incompleteness is caused by undiversifiable idiosyncratic risk. In the presence of undiversifiable idiosyncratic risk, an entrepreneur should consider not only her consumption but also her wealth at the time of success in a business. Specifically, the entrepreneur is willing to maximize her utility value after the business success as well as her intermediate consumption before this success. The entrepreneurial business plan is characterized by two regions: a continuation region in which the entrepreneur's optimal choice is to retain an option liquidates her risky business; and a stopping region in which she should exercise this option, exit from the risky business, and accept a safe job. The continuation and stopping regions are determined by the so-called *critical wealth level* under which it is optimal for an entrepreneur to exit from her risky business and accept a safe job.

[3] Heaton and Lucas (2000) find that the uncertainty of business income is a large source of undiversifiable idiosyncratic risk.

The main contribution of this paper is to show that undiversifiable idiosyncratic risk significantly influences an entrepreneur's optimal strategies which depend crucially on the level of idiosyncratic risk, risk aversion, wealth, and investment opportunity. We establish two main results by numerical analysis:

- The entrepreneur is more likely to exit from her risky business (1) as investment opportunity worsens, or (2) as her risk aversion coefficient increases or (3) as the idiosyncratic risk increases.

- When the entrepreneur has significant wealth, the amount that she willingly invests in the stock market decreases as the idiosyncratic risk increases. However, when the entrepreneur decumulates wealth, she can achieve a partial hedging effect of the risky portfolio against the business risk by optimally increasing her risky portfolio as the idiosyncratic risk increases.

We measure the entrepreneurial value of running a risky business and the hedging effect of risky portfolio against undiversifiable idiosyncratic risk by using a concept of certainty equivalent wealth (CEW). The CEW induced by running a risky business is an increasing function of the entrepreneur's initial wealth. Further, the relationship between the CEW and wealth is highly nonlinear; this is consistent with the result of Hurst and Lusardi (2004). Importantly, we find that the CEW shows different patterns according to different degrees of idiosyncratic risk. The value of running the risky business significantly increases as idiosyncratic risk decreases, but the undiversifiable idiosyncratic risk might lead to non-trivial entrepreneurship; the probability that this happens depends crucially on the entrepreneur's wealth.

The hedging effect of risky portfolio also increases as the idiosyncratic risk increases. This result is consistent with results of Miao and Wang (2007a), Leippold and Stromberg (2014); a private equity premium can be generated by an increase of idiosyncratic volatility. In this paper, the positive and sizable hedging effect measured by the CEW can represent a large source of the private equity premium. Accordingly, stock market participation is of importance to the entrepreneur for the purpose of risk diversification and a smooth continuation of her risky business.

The rest of this paper is organized as follows. In Section 2 we describe a financial market in the presence of undiversifiable idiosyncratic risk and provide an entrepreneurial business plan. In Section 3 we show the implications of our model through numerical results. Specifically, we analyze the effects of idiosyncratic risk on the entrepreneurial optimal strategies, the value of running a risky business, and the hedging effect of a risky portfolio. In Section 4 we conclude the paper.

2 The Model

2.1 The Financial Market

An entrepreneur has the following CRRA lifetime utility:

$$E\Big[\int_0^{\tau\wedge\tau_\delta} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-\beta(\tau\wedge\tau_\delta)}\int_{\tau\wedge\tau_\delta}^{\infty} e^{-\beta(t-\tau\wedge\tau_\delta)}\frac{c_t^{1-\gamma}}{1-\gamma}dt\Big],$$

where E is the expectation taken at time 0, $\beta > 0$ is the entrepreneur's subjective discount rate, $\gamma > 0$ is her coefficient of relative risk aversion, c_t is per-period consumption, τ is the first time (to be determined *endogenously*) when the entrepreneur goes into liquidation, τ_δ is the first time (occurs *exogenously*) when the quality of the business is determined to be sufficiently high.

The entrepreneur can trade securities in a financial market. Following the conventional model, the financial market consists of two assets: a bond (or a risk-free asset) and a stock (or a risky asset). The bond price B_t follows

$$dB_t = rB_t dt,$$

where $r > 0$ is the risk-free interest rate, and the stock price S_t is given by the following geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ ($\mu > r$) is the expected rate of the stock return, $\sigma > 0$ is the volatility of the return on the stock, and W_t is a standard one-dimensional Brownian motion defined on an appropriate probability space. The expected stock return μ and the stock volatility σ summarize the investment opportunity provided by the stock. We assume that the investment opportunity is constant, i.e., r, μ, and σ are constants.[4]

[4]The investigation of effects of stochastic investment opportunity on an entrepreneur's business plan would be an interesting extension. In this paper, we try to focus on how undiversifiable idiosyncratic risk optimally affects the entrepreneur's optimal strategies.

2.2 Undiversifiable Idiosyncratic Risk

We consider an entrepreneur who runs a risky business and is exposed to *undiversifiable idiosyncratic risk* from the business. The wealth process X_t of the entrepreneur with initial wealth $X_0 = x$ follows

$$dX_t = (rX_t - c_t + \epsilon_t)dt + \pi_t \sigma(dW_t + \theta dt), \quad t \geq 0, \tag{1}$$

where π is the dollar amount invested in the stock, θ is the Sharpe ratio, $(\mu - r)/\sigma$, and ϵ_t is the rate of future income obtained from her business. The entrepreneur accumulates wealth at the rates equal to $(rX_t - c_t + \epsilon_t)$. She consumes at the rate of c_t and receives interests at the proportional rate r to her current wealth by investing in the risk-free bond. Note that the entrepreneur faces undiversifiable idiosyncratic risk from her risky business and thus receives future income ϵ_t at random rates. The entrepreneur is currently in the business and obtains low-quality income at the rate equal to $\epsilon_L > 0$. Due to the undiversifiable idiosyncratic risk, she has a small probability of succeeding in the business (Miao and Wang, 2007b). If she succeeds in the business, then she obtains high-quality income at the rate equal to ϵ_H ($\epsilon_H > \epsilon_L$) afterward. We assume that the entrepreneur has an option to quit the business and accept a safe job, and that after she exercises the option, she obtains constant income at the rate of $y > 0$ ($\epsilon_L < y < \epsilon_H$) infinitely. Then the entrepreneur's income streams, ϵ_t, follow

$$\epsilon_t = \begin{cases} \epsilon_L, & \text{if } \ 0 \leq t < \tau \wedge \tau_\delta, \\ y, & \text{if } \ t \geq \tau, \\ \epsilon_H, & \text{if } \ \tau_\delta \leq t < \tau. \end{cases}$$

We also assume that the first time τ_δ when the quality of the business is determined to be high is distributed according to an exponential distribution with a positive intensity δ, i.e.,

$$\text{probability of } \{t \leq \tau_\delta\} = 1 - e^{-\delta t}.$$[5]

The reciprocal of the intensity is the expected time that the business succeeds.

The entrepreneur is allowed to participate in the stock market. She is exposed to the market risk from her stock holdings, as a result, bears random fluctuations of her wealth. The market risk is captured by the term involving the Brownian motion W in the wealth process (1). More specifically, the wealth process randomly changes at the rate equal to $\pi\sigma$, the product of the dollar amount π invested in the stock market and the stock volatility σ representing the standard deviation of the return on the stock. In this sense, the stock volatility represents the risk in the stock market. The entrepreneur is compensated by a risk premium from the stock investment and hence her wealth accumulation is increased by the rate equal to $\pi(\mu - r)$, the product of the stock investment π and the risk premium $\mu - r$, relative to the investment only in the risk-free bond.

We have two risk sources: the market risk (or the Brownian motion) and the failure in the business (or the Poisson arrival of high-quality income stream). The market risk can be diversified away by investing in the stock. We assume that there is no financial vehicle (securities, financial contracts, or insurance contracts) to hedge against the business fail. In this sense, the failure in the business is unhedgeable and subsequently, considered to

[5] In later numerical analysis, we set the value of the intensity δ to the annual probability that the business succeeds. This parameter set-up is due to the approximate relationship that $1 - e^{-\delta t} \approx \delta t$.

be undiversifiable idiosyncratic risk.[6] Hence, the financial market consisting of securities market and insurance market is essentially *incomplete*.

2.3 A Business Plan

We consider a business plan for an entrepreneur who is exposed to undiversifiable idiosyncratic risk and participates in the stock market. We assume that the entrepreneur can borrow money with her future income obtained from her business.[7] The present value of future income discounted at the risk-free interest rate r follows

$$E\Big[\int_0^{\tau_\delta} e^{-rt}\epsilon_t dt\Big] = \frac{1}{r+\delta}\Big(\epsilon_L + \epsilon_H\frac{\delta}{r}\Big).$$

For the limiting case of $\delta = 0$, the entrepreneur cannot succeed in the business, so that she continuously receives the low-quality income ϵ_L. In this case, she can borrow money up to the present value ϵ_L/r of the low-quality income discounted at the risk-free interest rate. The entrepreneur has a small probability of succeeding in the business, i.e., $\delta > 0$. In this case, the amount of income increases from the low-quality income ϵ_L to the sum of ϵ_L

[6]For the technical simplicity, the Brownian motion and Poisson arrival event are assumed to be independent. The independent assumption can be extended by imposing a correlation between the Brownian motion and Poisson arrival event. Specifically, we can introduce a time-varying probability of success in the business and allow an additional Brownian motion to have the correlation with the Brownian motion of the stock price. However, the correlation adds a technical difficulty such as solving a problem in multi-dimensions. We leave this as an extension for future research.

[7]Various wealth constraints such as a non-negative wealth constraint can be imposed (Farhi and Panageas, 2007; Dybvig and Liu, 2010). We abstract away the role of wealth constraints to focus on the effects of undiversifiable idiosyncratic risk on an entrepreneur's business plan.

and the idiosyncratic risk-adjusted high-quality income $\epsilon_H \times \delta/r$. Then the entrepreneur can borrow money up to the level $(\epsilon_L + \epsilon_H \frac{\delta}{r})/(r+\delta)$, which is the present value of the business income $(\epsilon_L + \epsilon_H \frac{\delta}{r})$ discounted at the sum of the risk-free interest rate r and the intensity δ of the timing for undiversifiable idiosyncratic risk.

For the other extreme case where $\delta = +\infty$, the entrepreneur always succeeds in the business, as a result, receives the high-quality income ϵ_H infinitely. Therefore, in that case the entrepreneur can borrow money up to the present value ϵ_H/r of the high-quality income ϵ_H discounted at the risk-free interest rate. To sum up, the wealth constraint of the entrepreneur while running the business is given by

$$X_t \geq -\frac{1}{r+\delta}\Big(\epsilon_L + \epsilon_H \frac{\delta}{r}\Big), \quad 0 \leq t \leq \tau. \tag{2}$$

After exiting the business and entering a safe job, the entrepreneur borrows money with insurable income y obtained from the safe job, i.e.,

$$X_t \geq -\frac{y}{r}, \quad t \geq \tau. \tag{3}$$

The entrepreneurial business plan is to maximize her CRRA lifetime utility by controlling per-period consumption c, risky portfolio π, and the time τ to quit the business and accept the safe job in the presence of undiversifiable idiosyncratic risk, or equivalently, to find the following value function:

$$\Phi(x) \equiv \max_{(c,\pi,\tau)} E\Big[\int_0^{\tau\wedge\tau_\delta} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-\beta(\tau\wedge\tau_\delta)}\int_{\tau\wedge\tau_\delta}^{\infty} e^{-\beta(t-\tau\wedge\tau_\delta)}\frac{c_t^{1-\gamma}}{1-\gamma}dt\Big]. \tag{4}$$

We reformulate the value function (4) by the following lemma.

Lemma 2.1 *The value function (4) can be rewritten by*

$$\Phi(x) = \max_{(c,\pi,\tau)} E\Big[\int_0^\tau e^{-(\beta+\delta)t}\Big\{\frac{c_t^{1-\gamma}}{1-\gamma} + \delta\widetilde{U}(X_t;\epsilon_H)\Big\}dt + e^{-(\beta+\delta)\tau}\widetilde{U}(X_\tau;y)\Big], \tag{5}$$

where

$$\widetilde{U}(x;a) = K\frac{(x+a/r)^{1-\gamma}}{1-\gamma},$$

and

$$K = \Big(\frac{1}{A}\Big)^\gamma, \quad A = \Big\{\frac{\gamma-1}{\gamma}\Big(r + \frac{\theta^2}{2\gamma}\Big) + \frac{\beta}{\gamma}\Big\}.$$

Proof. See Appendix. **Q.E.D.**

The term $\widetilde{U}(x;a)$ in Lemma 2.1 denotes the value function of an entrepreneur who has initial wealth x and receives income at the rate of a infinitely. Equation (5) shows that in the presence of undiversifiable idiosyncratic risk, an entrepreneur should consider not only her consumption but also wealth at the time of success in a business. Specifically, the second term $\delta\widetilde{U}(X_t;\epsilon_H)$ captures the utility value of an entrepreneur after the business success. The term is the product of the intensity δ of the timing for the idiosyncratic risk and the maximized value of the entrepreneur's utility after the business succeeds i.e., the income rate obtained from the business is increased by ϵ_H. For the limiting case where $\delta = 0$, the entrepreneur cannot succeed in the business, so that she maximizes an objective function which is a function of only intermediate consumption (Merton, 1969). For the other extreme case of $\delta = +\infty$, the entrepreneurial business plan is trivial because the business success is immediate and the income is provided by the rate equal to ϵ_H infinitely. In this case, the business plan reduces to the Merton's optimal consumption and portfolio selection problem.

2.4 Optimal Exit Time from Risky Business

In the previous subsection, we have formulated a problem for a business plan in which an entrepreneurial optimal consumption, optimal stock investment, and optimal exit time from the business are jointly determined. In fact, the problem is closely associated with an optimal stopping problem.[8] Specifically, we provide a lemma clarifying the relationship between the entrepreneurial business plan and the optimal stopping problem.

Lemma 2.2 *The value function formulated by (5) satisfies the following optimal stopping problem given by the variational inequality:*

$$\begin{cases} (\beta+\delta)\phi(x) - (rx+\epsilon_L)\phi'(x) + \dfrac{1}{2}\theta^2\dfrac{\phi'(x)^2}{\phi''(x)} - \dfrac{\gamma}{1-\gamma}\{\phi'(x)\}^{1-1/\gamma} \geq \delta\widetilde{U}(x;\epsilon_H), \\ \phi(x) \geq \widetilde{U}(x;y), \\ \Big[(\beta+\delta)\phi(x) - (rx+\epsilon_L)\phi'(x) + \dfrac{1}{2}\theta^2\dfrac{\phi'(x)^2}{\phi''(x)} \\ \qquad\qquad - \dfrac{\gamma}{1-\gamma}\{\phi'(x)\}^{1-1/\gamma} - \delta\widetilde{U}(x;\epsilon_H)\Big]\Big(\phi(x) - \widetilde{U}(x;y)\Big) = 0. \end{cases} \tag{6}$$

Proof. See Appendix. **Q.E.D.**

The entrepreneurial business plan is characterized by two regions: a continuation region in which the entrepreneur's optimal choice is to retain an option liquidates her risky

[8] When we deal with the standard optimal stopping problems (Farhi and Panageas, 2007; Dybvig and Liu, 2010), we can solve the differential equation derived by the conventional dual approach (Karatzas and Wang, 2000) up to explicit solutions. However, we cannot apply the dual approach to our problem because a state price density (or a stochastic discount factor) is not uniquely determined because of the market incompleteness induced by undiversifiable idiosyncratic risk. To address this issue, we can apply a modified convex-duality approach (Bensoussan *et al.*, 2016).

business; and a stopping region in which she should exercise this option, exit from the risky business, and accept a safe job. The first inequality in the variational inequality (6) shows that the strict inequality holds in the stopping region and the equality holds in the continuation region. In particular, the equality is the Hamilton-Jacobi-Bellman equation obtained when we solve an optimal consumption and portfolio selection problem (Merton, 1969). Importantly, the strict inequality in the second inequality is the case where the entrepreneurial value function with the liquidation option is strictly larger than the value function after exiting from the risky business and jumping into the safe job. Therefore, in that case the entrepreneur is in the continuation region and should hold the liquidation option. When the entrepreneur's value function is exactly same with the value function after exercising the option (the equality in the second inequality), the entrepreneur is in the stopping region and thus, optimally exits the business and accepts the safe job. The third equality in (6) is necessary because the strict inequalities (the first inequality represents the stopping region and the second one denotes the continuation region) cannot hold simultaneously.

The continuation and stopping regions are determined by the so-called *critical wealth level* under which it is optimal for an entrepreneur to exit from her risky business and accept a safe job.[9] We can construct a problem with an optimal stopping boundary (or a free

[9] Actually, this type business plan resembles the optimal strategy for an investor with an American put option in which the investor optimally exercises the put option whenever the underlying asset price approaches the optimal exercise boundary from above. The difference between the entrepreneur's liquidation option and the investor's American put option is attributable to the underlying asset on the option.

boundary) to solve the optimal stopping problem formulated by the variational inequality (6). That is, we would like to find a function $\phi(x)$ such that it is C^1 and piecewise C^2 and determine the free boundary $\underline{x}$. Specifically, the function $\phi(x)$ satisfies the following relationships:

$$\begin{cases} (\beta+\delta)\phi(x) - (rx+\epsilon_L)\phi'(x) + \dfrac{1}{2}\theta^2 \dfrac{\phi'(x)^2}{\phi''(x)} - \dfrac{\gamma}{1-\gamma}\{\phi'(x)\}^{1-1/\gamma} = \delta \widetilde{U}(x;\epsilon_H), \quad \underline{x} < x, \\ \phi(x) = \widetilde{U}(x;y), \quad -\dfrac{1}{r+\delta}\Big(\epsilon_L + \epsilon_H \dfrac{\delta}{r}\Big) < x \le \underline{x}, \\ \phi(\underline{x}) = \widetilde{U}(\underline{x};y), \\ \phi'(\underline{x}) = \widetilde{U}'(\underline{x};y), \end{cases} \tag{7}$$

where $\underline{x}$ is the critical wealth level.[10] In the free boundary problem (7), the continuation and stopping regions are explicitly characterized by $\{\underline{x} < x\}$ and $\{-\frac{1}{r+\delta}\Big(\epsilon_L + \epsilon_H \frac{\delta}{r}\Big) < x \le \underline{x}\}$, respectively.

3 Numerical Implications

Parameter Values

Specifically, the underlying asset on the liquidation option is the entrepreneurial wealth controlled by her optimal consumption and risky portfolio policies, and optimal exit strategy, whereas the underlying asset on the put option is typically the stock price.

[10]The solution to the variational inequality (6) clearly satisfies the free boundary problem (7). The converse statement that whether or not the solution to the free boundary problem (7) is the solution to the variational inequality (6) should be verified. The verification can be done by modifying the idea of Bensoussan *et al.* (2016) and is provided in Appendix. In Appendix, we provide the details of solving the free boundary problem (7).

We investigate various properties of optimal strategies for an entrepreneur in the presence of undiversifiable idiosyncratic risk by using numerical solutions. Default parameters are set as follows: $r = 3.71\%$, the annual rate of return from rolling-over of 1-month T-bills during the time period of 1926-2009,[11] $\mu = 11.23\%$ and $\sigma = 19.54\%$, the return and standard deviation of a portfolio consisting of the world's large stocks during the same time period.[12] We assume that $\beta = r$ and set $\gamma = 2$.

To set the parameter values for income streams, we follow Miao and Wang (2007b). More specifically, the income rate y from the safe job dominates the low-quality income rate ϵ_L. Moreover, the entrepreneur receives the higher income rate ϵ_H than y if she succeeds in a business. Due to the presence of undiversifiable idiosyncratic risk, she can fail in the business. However, a small possibility of succeeding in the business will be an incentive for the entrepreneur to stay in the business. To reflect this set-up, we set the income rates as follows: $\epsilon_L = 0.25$, $y = 1.5$, and $\epsilon_H = 2.5$. We also set $\delta = 0.10$, which means that the annual probability that the business succeeds is 10%.

Optimal Exit Strategy

In the previous subsection, we have shown that it is optimal for an entrepreneur to exit from her risky business and accept a safe job as soon as her initial wealth approaches the critical wealth level $\underline{x}$ from above. The entrepreneur's borrowing limits (see the wealth constraint (2)) for three values of $\delta \in \{0.20, 0.15, 0.10\}$ are computed as -57.8958, -55.3598,

[11]Source: Bureau of Labor Statistics

[12]Source: pp.170 of Bodie *et al.* (2011)

-50.9741, respectively. The results of sensitivity analysis of the critical wealth level $\underline{x}$ with respect to changes in μ, σ, and γ (Table 1) demonstrate that the values of $\underline{x}$ are negative and exceed the borrowing limits. The entrepreneur is more likely to exit from the business as investment opportunity worsens; i.e., as the expected rate of stock return μ decreases or the stock volatility σ increases, or as the risk aversion coefficient γ increases, or as undiversifiable idiosyncratic risk increases, or equivalently, as the intensity δ decreases. The value of the option to quit the business and accept a safe job is larger in a financial market with a bad investment opportunity than in one with a good investment opportunity. Moreover, a highly risk-averse entrepreneur is reluctant to take idiosyncratic risk from risky investment, and hence willingly exits from the business earlier than does an entrepreneur with low risk aversion. As the idiosyncratic risk to which an entrepreneur is exposed increases, the advantage of abandoning the business and accepting a safe job increases.

[Insert Table 1 here.]

Optimal Consumption and Risky Portfolio Strategies

Undiversifiable idiosyncratic risk has a wealth-dependent effect on optimal strategies. The entrepreneur tends to formulate aggressive consumption and risky portfolio strategies when she has small idiosyncratic risk (or a high δ) and has wealth that exceeds the critical wealth level (Figure 1).

[Insert Figure 1 here.]

When she has significantly larger amount of wealth than the critical wealth level, she willingly invests more in the stock as idiosyncratic risk decreases, i.e., as the intensity δ increases (Figure 1). However, this might not be a reasonable strategy while the entrepreneur decumulates wealth. The entrepreneur optimally increases risky portfolio as idiosyncratic risk increases (as the intensity δ decreases). Because a risky portfolio could be a good substitute for the risky business, the entrepreneur willingly increases her risky portfolio to absorb the idiosyncratic risk.

[**Insert Figure 2 here.**]

Given the undiversifiable idiosyncratic risk, $\delta = 0.10$, the effects of changes in the coefficient of relative risk aversion on optimal consumption are trivial if an entrepreneur has significantly large wealth, which is near the zero wealth level in Figure 2; consumption decreases as risk aversion increases. However, as the entrepreneur decumulates wealth, she willingly takes more aggressive consumption strategy with respect to an increase of risk aversion. A constant income stream obtained from quitting a risky business and jumping into a safe job would induce the aggressive consumption behavior of the highly risk-averse entrepreneur as her wealth approaches the critical wealth level $\underline{x}$. Relative to an optimal risky portfolio, a more risk-averse entrepreneur invests less in the stock than does a less risk-averse one. This response follows the traditional investment rule that an investor decreases her stockholdings as her risk aversion increases.

[**Insert Table 2 here.**]

The entrepreneurial optimal strategies are affected by changes in investment opportunity (Table 2). Given the undiversifiable idiosyncratic risk, $\delta = 0.10$, the entrepreneur optimally reduces her consumption and risky portfolio as the expected rate of stock return μ decreases or as the stock volatility σ increases.

Value of Running Risky Business

Until now, we have investigated how undiversifiable idiosyncratic risk optimally influences components of an entrepreneurial business plan such as optimal exit strategy, optimal consumption and risky portfolio policies. In this section, we try to determine how much benefit an entrepreneur obtains from her own risky business by bearing the undiversifiable idiosyncratic risk. To address this question, we will compute the value of running the risky business by introducing a concept of certainty equivalent wealth (CEW). We define the CEW induced by running a risky business as follows.

Definition 3.1 *The certainty equivalent wealth $\Delta(x)$ induced by running a risky business is defined as*

$$U(x + \Delta(x); y) = \Phi(x),$$

where $U(\cdot; y)$ is the value function after exiting business and accepting a safe job in which an entrepreneur receives income y infinitely, and $\Phi(x)$ is the value function given by (5) while staying in risky business.

[Insert Figure 3 here.]

[Insert Figure 4 here.]

The CEW induced by running a risky business is an increasing function of an entrepreneurial initial wealth x (Figure 3). Further, the relationship between the CEW and wealth is highly nonlinear; this observation is consistent with the result of Hurst and Lusardi (2004). The value of running a risky business (or the CEW) sharply decreases as the entrepreneurial wealth decumulates from above zero, the value also steadily decreases as the wealth approaches the critical wealth level (Figure 3). When the value becomes zero, the entrepreneur has no incentive to run the risky business, so she decides to exit from it and accept a safe job. We also find that a highly risk-averse entrepreneur values a risky business less than does the entrepreneur with low risk aversion; therefore the former would exit the business sooner than would the latter (Figure 4).

Importantly, we find that the CEW patterns differ according to values of the intensity δ (i.e., the annual probability that the business succeeds). Specifically, when the entrepreneur is exposed to high idiosyncratic risk or has a low value of δ ($\delta = 0.10$), the CEW and wealth have a convex relation, which means that entrepreneurs appear to have more benefits from their business as their wealth increases. The rationale behind this trend is that the effects of idiosyncratic risk decrease as an entrepreneur's wealth increases, so that her willingness to stay in the risky business also increases. This result is compatible that of Puri and Robinson (2006): that entrepreneurs are more optimistic and risk tolerant than normal wage earners, especially in the upper percentile of the wealth distribution.

In contrast, when the entrepreneur is exposed to low idiosyncratic risk or has a high value of δ ($\delta = 0.20$), the value of running the risky business significantly increases as idiosyncratic risk decreases, but the undiversifiable idiosyncratic risk might lead to non-trivial entrepreneurship depending crucially on the entrepreneurial wealth. More specifically, the relation between the CEW and wealth shows both convex and concave patterns. Entrepreneurs with low wealth show the convex trend, whereas those with large wealth follow the concave one. The former trend implies that entrepreneurs with relatively little wealth seem to have more incentive to run their own business with respect to an increase of wealth; the latter trend shows that wealthy entrepreneurs are not necessarily more beneficial as wealth increases up to a point.

The results of the analysis of the effects of CEW can offer an intuition to resolve the private equity premium puzzle. In the standard option pricing theory proposed by Black-Scholes-Merton shows the positive convexity effect that the option value increases as market volatility increases. According to the standard real options theory (Dixit and Pindyck, 1994), entrepreneurial option value for investment also increases as project volatility increases. Contrary to these predictions under complete financial markets, we confirm that the value for an option to quit a risky business and accept a safe job decreases when idiosyncratic risk is undiversifiable: this fact induces a negative relationship between option value and idiosyncratic volatility and is consistent with the result of Chen *et al.* (2010). Whereas Moskowitz and Vissing-Jørgensen (2002) have found that private equity premium against idiosyncratic risk is low in the U.S., Mueller (2011) have showed the opposite. In support of Mueller (2011), we predict that if entrepreneurs demand a substantial idiosyncratic risk premium that mitigates the effect of a significant decrease in CEW (or the value for running risky business) due to idiosyncratic risk, then they might obtain a high private equity premium by committing in their own risky business.[13]

[13] Wang *et al.* (2012) obtain a quantitative result for a significant idiosyncratic risk premium, especially for entrepreneurs with small wealth.

Hedging Effect of Risky Portfolio against Undiversifiable Idiosyncratic Risk

An entrepreneur cannot fully eliminate idiosyncratic risk by diversifying her portfolio, but she can achieve a partial hedging effect from her stock holdings against the riskiness of a business. To quantify the hedging effect, we compute the certainty equivalent wealth by comparing two value functions: one that is allowed to participate in the stock market and one that is not.[14]

Definition 3.2 *The hedging effect $HE(x)$ of risky portfolio against undiversifiable idiosyncratic risk is quantified as the following:*

$$\Psi(x;\delta) = \Phi(x - HE(x);\delta),$$

where $\Phi(x;\delta)$ and $\Psi(x;\delta)$ are the value functions with and without the access to the stock market, respectively. The average time to succeed in a risky business is given by $1/\delta$.

[**Insert Figure 5 here.**]

The hedging effect $HE(x)$ of risky portfolio against undiversifiable idiosyncratic risk that arises from running a risky business increases increases as the initial wealth of an entrepreneur increases (Figure 5). An entrepreneur with large initial wealth may be more willing to absorb idiosyncratic risk than does an entrepreneur with small wealth. Simultaneously, as wealth increases, the stock investment becomes increasingly attractive to the

[14]The value function of an entrepreneur who has limited access to the stock market is derived by considering a consumption-saving model for the entrepreneur. For the details, see Appendix.

wealthy entrepreneur due to the positive risk premium from the investment. Then the wealthy entrepreneur can increase her hedging effect by trading the market portfolio.

The hedging effect also increases as the idiosyncratic risk increases, i.e., as δ decreases (Figure 5). This result is consistent with results of Miao and Wang (2007a) and Leippold and Stromberg (2014); a private equity premium can be generated by an increase of idiosyncratic volatility. In this paper, the positive and sizable hedging effect measured by the CEW can represent a large source of the private equity premium. Accordingly, stock market participation is of importance to the entrepreneur for the purpose of risk diversification and a smooth continuation of her risky business.

Our results for the hedging effect complement the existing literature regarding portfolio allocations of entrepreneurs. Heaton and Lucas (2009) have investigated the relation between capital structure and portfolio selection in financial markets that consist of a stock market and a bond market, and show that entrepreneurs optimally hold a sizable stock investment relative to the investment in a risk-free bond. Although entrepreneurs willingly hold a safe asset composition to diversify away their own business risk, they in aggregate account for about 30% of the stock market (Heaton and Lucas, 2000). Faig and Shum (2002) also stress that entrepreneurs' portfolios of financial assets can be used as a hedging tool against bad outcomes of their business.

[Insert Figure 6 here.]

[Insert Table 3 here.]

A more risk-averse entrepreneur is likely to obtain a larger hedging effect from her stock investment against the idiosyncratic risk than is a risk-tolerant entrepreneur (Figure 6). According to the results of the sensitivity analysis of the hedging effect $HE(x)$ to the changes in investment opportunity, an entrepreneur's hedging effect increases as investment opportunities increase in the market in which she participates (Table 3).

4 Conclusion

We have provided an entrepreneur's optimal business plan in the presence of undiversifiable idiosyncratic risk, and have investigated the relationship between the idiosyncratic risk and optimal exit strategy for an entrepreneur as well as the role of a market portfolio on a business plan under a CRRA utility. The entrepreneurial business plan is characterized by two regions: a continuation region in which the entrepreneur's optimal choice is to retain an option liquidates her risky business; and a stopping region in which she should exercise this option, exit from the business, and accept a safe job. The continuation and stopping regions are determined by the so-called critical wealth level under which it is optimal for an entrepreneur to exit from her risky business and accept a safe job.

By numerical analysis, we find that the entrepreneur is more likely to exit from her risky business as (1) investment opportunity worsens, or (2) as her risk aversion coefficient increases, or (3) as the idiosyncratic risk increases. When the entrepreneur has significant wealth, the amount that she willingly invests in the stock market decreases as the idiosyncratic risk increases. However, when the entrepreneur decumulates wealth, she can achieve a partial hedging effect of the risky portfolio against the business risk by optimally increasing her risky portfolio as the idiosyncratic risk increases. We measure an entrepreneurial value of running a risky business and hedging effect of risky portfolio against undiversifiable idiosyncratic risk by using a concept of certainty equivalent wealth (CEW). The CEW induced by running a risky business is an increasing function of an entrepreneurial initial wealth. The hedging effect of risky portfolio also increases as the idiosyncratic risk increases.

5 Appendix

5.1 Details of Deriving Optimal Strategies

Critical Wealth Level

We have shown that an entrepreneurial optimal business plan is characterized by her critical wealth level. Specifically, the entrepreneur should hold a liquidation option to exit from her risky business and accept a safe job as far as initial wealth of the entrepreneur is larger than the critical wealth level. Such optimal policy is in a continuation region. If the entrepreneurial initial wealth approaches the critical wealth level from above, then it is optimal for the entrepreneur to exercise the liquidation option. In this case, the optimal strategy is in a stopping region.

To determine the critical wealth level which takes a key role in an entrepreneurial optimal business plan, we have to solve the highly non-linear differential equation described as in (7). It seems to be hardly possible to obtain analytical results for the solution to the problem (7). Indeed, we cannot derive a closed form solution for the critical wealth level. Instead, we can obtain lower and upper bounds for the critical wealth level.

We suggest a modified convex-duality approach to solve our problem. Actually, the approach is developed by Bensoussan *et al.* (2016) for solving the retirement problem with unemployment risks. We modify the idea of Bensoussan *et al.* (2016) and apply it to our problem. We provide the following lemma reformulating the free boundary problem (7) by using the convex-duality approach.

Lemma 5.1 *The first relationship of the free boundary problem (7) is reformulated by*

$$-\frac{1}{2}\theta^2\lambda^2 G''(\lambda) - \lambda G'(\lambda)(\theta^2+\beta+\delta-r) + rG(\lambda) + \delta K\left(G(\lambda) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\right)^{-\gamma} G'(\lambda) = \lambda^{-1/\gamma}, \;\; 0 < \lambda < \overline{\lambda}, \tag{8}$$

where G is a convex-dual function of the value function ϕ,[15] *λ is the marginal value of the value function ϕ, and $\overline{\lambda}$ is a free boundary to be determined according to the value-matching and smooth-pasting conditions.*

Proof. See Appendix. **Q.E.D.**

We call the function G the convex-dual function. In the later section, we will verify that the function G is monotonically-decreasing with respect to an increase in initial wealth x. Furthermore, the function G has the implicit relationship with the marginal value of the value function ϕ as follows: $G(\phi'(x)) = x + \epsilon_L/r$. In this sense, G is the dual function of the value function ϕ satisfying increasing and concave properties. Note that the convexity

[15] The existence of such convex-dual function G satisfying the differential equation (8) is verified in Appendix.

of the dual function G can be verified numerically under the reasonable parameter values.

In Lemma 5.1, the free boundary $\overline{\lambda}$ takes an important role in determining the critical wealth level $\underline{x}$ under which it is optimal for an entrepreneur to exit from her risky business and accept a safe job. In fact, the free boundary $\overline{\lambda}$ has an inverse relationship with the critical wealth level $\underline{x}$ as follows: $\overline{\lambda} = K(\underline{x} + y/r)^{-\gamma}$. To determine the free boundary $\overline{\lambda}$ we use the value-matching and smooth-pasting conditions. Specifically, we use the boundary conditions of ϕ and ϕ' at $\underline{x}$: $\phi(\underline{x}) = \widetilde{U}(\underline{x}; y)$ and $\phi'(\underline{x}) = \widetilde{U}'(\underline{x}; y)$.

For the next, we present an important lemma that gives an analytic solution to the non-linear differential equation (8).

Theorem 5.1 *An analytic solution to the non-linear differential equation (8) follows*

$$\begin{aligned} G(\lambda) = {} & \frac{\gamma\lambda^{-1/\gamma}}{\gamma A + \delta} + B^*(\overline{\lambda})\lambda^{-\alpha_\delta^*} \\ & + \frac{2\delta K}{\theta^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta - 1)\lambda^{-\alpha_\delta}\int_0^\lambda \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} d\mu \\ & + (\alpha_\delta^* - 1)\lambda^{-\alpha_\delta^*}\int_\lambda^{\overline{\lambda}} \mu^{\alpha_\delta^* - 2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} d\mu\Big], \end{aligned} \tag{9}$$

where $\alpha_\delta > 0$ and $\alpha_\delta^* < 0$ are the two roots of the following characteristic equation:

$$I(\alpha; \delta) \equiv -\frac{1}{2}\theta^2\alpha(\alpha - 1) + \alpha(\beta + \delta - r) + r = 0 \tag{10}$$

and $B^*(\overline{\lambda})$ is a constant to be determined according to the smooth-pasting conditions.

Proof. See Appendix. **Q.E.D.**

For the next, we determine the free boundary $\overline{\lambda}$ and the constant $B^*(\overline{\lambda})$. We use the value-matching and smooth-pasting conditions (or equivalently, the boundary conditions) of the value function ϕ at the free boundary $\overline{\lambda}$. Note that when we reformulated the first relationship of the free boundary problem (7) as the non-linear differential equation (8), we defined λ as the marginal value of the value function ϕ, introduced a dual variable $\overline{\lambda}$ of the free boundary $\underline{x}$, and employed a function G that is the so-called convex-dual function (see Proof of Lemma 5.1). Recall such variables

$$\lambda(x) = \phi'(x), \quad \overline{\lambda} = K(\underline{x} + y/r)^{-\gamma}, \quad \text{and} \quad G(\lambda(x)) = x + \frac{\epsilon_L}{r}.$$

Then the boundary condition $\phi'(\underline{x}) = \widetilde{U}'(\underline{x}; y)$ in (8) is easily rewritten by the convex-dual function G through its definition. More specifically,

$$G(\overline{\lambda}) = K^{1/\gamma}\overline{\lambda}^{-1/\gamma} - \frac{y}{r} + \frac{\epsilon_L}{r} \tag{11}$$

and subsequently,

$$\begin{aligned} K^{1/\gamma}\overline{\lambda}^{-1/\gamma} - \frac{y}{r} + \frac{\epsilon_L}{r} &= \frac{\gamma\overline{\lambda}^{-1/\gamma}}{\gamma A + \delta} + B^*(\overline{\lambda})\overline{\lambda}^{-\alpha_\delta^*} \\ &+ \frac{2\delta K(\alpha_\delta - 1)\overline{\lambda}^{-\alpha_\delta}}{\theta^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} \int_0^{\overline{\lambda}} \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} d\mu. \end{aligned} \tag{12}$$

We give a lemma to rewrite the boundary condition $\phi(\underline{x}) = \widetilde{U}(\underline{x}; y)$ in (7) in terms of the convex-dual function G.

Lemma 5.2 *The boundary condition $\phi(\underline{x}) = \widetilde{U}(\underline{x}; y)$ given in (7) is rewritten by the convex-dual function G as the following:*

$$\begin{aligned} (\beta + \delta)\frac{K^{1/\gamma}\overline{\lambda}^{1-1/\gamma}}{1-\gamma} &= r\overline{\lambda}\Big(K^{1/\gamma}\overline{\lambda}^{-1/\gamma} - \frac{y}{r} + \frac{\epsilon_L}{r}\Big) - \frac{1}{2}\theta^2\overline{\lambda}^2 G'(\overline{\lambda}) \\ &+ \frac{\gamma}{1-\gamma}\overline{\lambda}^{1-1/\gamma} + \frac{\delta K}{1-\gamma}\Big(K^{1/\gamma}\overline{\lambda}^{-1/\gamma} - \frac{y}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}. \end{aligned} \tag{13}$$

Proof. See Appendix. **Q.E.D.**

We rearrange the relationship (12) as the following:

$$B^*(\overline{\lambda}) = \Big[K^{1/\gamma}\overline{\lambda}^{-1/\gamma} - \frac{y}{r} + \frac{\epsilon_L}{r} - \frac{\gamma\overline{\lambda}^{-1/\gamma}}{\gamma A + \delta} - \frac{2\delta K(\alpha_\delta - 1)\overline{\lambda}^{-\alpha_\delta}}{\theta^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}\int_0^{\overline{\lambda}} \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} d\mu\Big]\overline{\lambda}^{\alpha_\delta^*}. \tag{14}$$

If we determine the free boundary $\overline{\lambda}$, then the constant $B^*(\overline{\lambda})$ is also determined by the relationship (14). We suggest a lemma in which $\overline{\lambda}$ can be determined numerically.

Lemma 5.3 *The free boundary $\overline{\lambda}$ can be determined numerically by solving the following equation:*

$$\Big[\Big\{\frac{\beta+\delta}{1-\gamma} - r - \frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma} - \frac{\gamma}{1-\gamma} - \frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta}\Big]\overline{\lambda}^{-1/\gamma} = \Big(1 + \frac{\theta^2}{2r}\alpha_\delta^*\Big)(-y+\epsilon_L) + \frac{\delta K(\alpha_\delta - 1)\overline{\lambda}^{-\alpha_\delta}}{1-\gamma}\int_0^{\overline{\lambda}} \mu^{\alpha_\delta-2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} d\mu. \tag{15}$$

Proof. See Appendix. **Q.E.D.**

A little rearrangement of (15) shows that

$$\Big(1 + \frac{\theta^2}{2r}\alpha_\delta^*\Big)(-y+\epsilon_L) = \Big[\Big\{\frac{\beta+\delta}{1-\gamma} - r - \frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma} - \frac{\gamma}{1-\gamma} - \frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta}\Big]\overline{\lambda}^{-1/\gamma} + \frac{\delta K(\alpha_\delta - 1)\overline{\lambda}^{-\alpha_\delta}}{\gamma - 1}\int_0^{\overline{\lambda}} \mu^{\alpha_\delta-2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} d\mu. \tag{16}$$

Let $M_\delta(\overline{\lambda})$ and N_δ be the right-hand and left-hand side of (16), respectively. We also set

$$\underline{M}_\delta(\overline{\lambda}) \equiv \Big[\Big\{\frac{\beta+\delta}{1-\gamma} - r - \frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma} - \frac{\gamma}{1-\gamma} - \frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta}\Big]\overline{\lambda}^{-1/\gamma},$$

and

$$\overline{M}_\delta(\overline{\lambda}) \equiv \underline{M}_\delta(\overline{\lambda}) + \frac{\delta K}{(\gamma-1)\overline{\lambda}}\Big(K^{1/\gamma}\overline{\lambda}^{-1/\gamma} - \frac{y}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma},$$

which are lower and upper bounds of $M_\delta(\overline{\lambda})$ respectively. Then we can obtain lower and upper bounds for the free boundary $\overline{\lambda}$.

Lemma 5.4 *Assume that*

$$\Big\{\frac{\beta+\delta}{1-\gamma} - r - \frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma} - \frac{\gamma}{1-\gamma} - \frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta} > 0.$$

The free boundary $\overline{\lambda}$ to be determined in (16) satisfies

$$\lambda_\delta^l \le \overline{\lambda} \le \lambda_\delta^u,$$

where λ_δ^l and λ_δ^u are obtained from

$$\underline{M}_\delta(\lambda_\delta^l) = N_\delta, \quad \text{and} \quad \overline{M}_\delta(\lambda_\delta^u) = N_\delta.$$

Proof. See Appendix. **Q.E.D.**

Now it remains to get lower and upper bounds for the critical wealth level $\underline{x}$. Due to the definition of the convex-dual function G given by $G(\lambda(x)) = x + \epsilon_L/r$, the lower bound λ^l and the upper bound λ^u given in Lemma 5.4, the following theorem is easily followed.

Theorem 5.2 *The lower and upper bounds for the critical wealth level $\underline{x}$ are given as the following:*

$$G(\lambda_\delta^u) - \frac{\epsilon_L}{r} \le \underline{x} \le G(\lambda_\delta^l) - \frac{\epsilon_L}{r}. \tag{17}$$

An entrepreneur is exposed to undiversifiable idiosyncratic risk and hence, she should manage the idiosyncratic risk by controlling optimal exit time from her risky business. It is optimal for the entrepreneur to liquidate the risky business as soon as her initial wealth approaches the critical wealth level from above. Moreover, Theorem 5.2 suggests that the lower and upper bounds given by (17) for the critical wealth level might give a hint for another business planning. Even though utilizing the exit strategy at wealth levels between the lower and upper bounds is a suboptimal policy for the business plan, the entrepreneur could run her risky business a little bit longer or shorter than the optimal exit time, in exchange for giving up the optimality.

Optimal Consumption and Risky Portfolio Strategies.

We state a theorem concerning the optimal consumption and risky portfolio strategies in the presence of undiversifiable idiosyncratic risk.

Theorem 5.3 *The entrepreneur's optimal consumption c^* and optimal portfolio π^* prior to exit from risky business follow*

$$
\begin{aligned}
c_t^* =& \Big(A+\frac{\delta}{\gamma}\Big)\Big(x+\frac{\epsilon_L}{r}\Big) - \Big(A+\frac{\delta}{\gamma}\Big)B^*(\overline{\lambda})\lambda^*(x)^{-\alpha_\delta^*} \\
&- \frac{2\delta K(A+\delta/\gamma)}{\theta^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta-1)\lambda^*(x)^{-\alpha_\delta}\int_0^{\lambda^*(x)}\mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{\epsilon_L}{r}+\frac{\epsilon_H}{r}\Big)^{-\gamma+1}d\mu \\
&+ (\alpha_\delta^*-1)\lambda^*(x)^{-\alpha_\delta^*}\int_{\lambda^*(x)}^{\overline{\lambda}}\mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{\epsilon_L}{r}+\frac{\epsilon_H}{r}\Big)^{-\gamma+1}d\mu\Big],
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\pi_t^* =& \frac{\theta}{\gamma\sigma}\Big(x+\frac{\epsilon_L}{r}\Big) + \Big(\alpha_\delta^*-\frac{1}{\gamma}\Big)\frac{\theta}{\sigma}B^*(\overline{\lambda})\lambda^*(x)^{-\alpha_\delta^*} - \frac{2\delta K}{\sigma\theta(1-\gamma)}\frac{1}{\lambda^*(x)}\Big(x+\frac{\epsilon_H}{r}\Big)^{-\gamma+1} \\
&+ \frac{2\delta K}{\sigma\theta(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big[\Big(\alpha_\delta-\frac{1}{\gamma}\Big)(\alpha_\delta-1)\lambda^*(x)^{-\alpha_\delta}\int_0^{\lambda^*(x)}\mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{\epsilon_L}{r}+\frac{\epsilon_H}{r}\Big)^{-\gamma+1}d\mu \\
&+ \Big(\alpha_\delta^*-\frac{1}{\gamma}\Big)(\alpha_\delta^*-1)\lambda^*(x)^{-\alpha_\delta^*}\int_{\lambda^*(x)}^{\overline{\lambda}}\mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{\epsilon_L}{r}+\frac{\epsilon_H}{r}\Big)^{-\gamma+1}d\mu\Big],
\end{aligned}
\tag{19}
$$

where $\lambda^*(x)$ is a decreasing function of initial wealth x, satisfying

$$
\begin{aligned}
x + \frac{\epsilon_L}{r} =& \frac{\gamma \lambda^*(x)^{-1/\gamma}}{\gamma A + \delta} + B^*(\overline{\lambda})\lambda^*(x)^{-\alpha_\delta^*} \\
&+ \frac{2\delta K}{\theta^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} \Big[(\alpha_\delta - 1)\lambda^*(x)^{-\alpha_\delta} \int_0^{\lambda^*(x)} \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{-\gamma+1} d\mu \\
&+ (\alpha_\delta^* - 1)\lambda^*(x)^{-\alpha_\delta^*} \int_{\lambda^*(x)}^{\overline{\lambda}} \mu^{\alpha_\delta^* - 2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{-\gamma+1} d\mu\Big],
\end{aligned}
$$

and

$$
B^*(\overline{\lambda}) = \Big[K^{1/\gamma}\overline{\lambda}^{-1/\gamma} - \frac{y}{r} + \frac{\epsilon_L}{r} - \frac{\gamma\overline{\lambda}^{-1/\gamma}}{\gamma A + \delta} - \frac{2\delta K(\alpha_\delta - 1)\overline{\lambda}^{-\alpha_\delta}}{\theta^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} \int_0^{\overline{\lambda}} \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} d\mu\Big]\overline{\lambda}^{\alpha_\delta^*}.
$$

Proof. See Appendix. **Q.E.D.**

An entrepreneurial optimal consumption and risky portfolio strategies are largely affected by the option to quit the business and accept the safe job to avoid undiversifiable idiosyncratic risk. The effects of the idiosyncratic risk are reflected in the last terms consisting of two integral parts in (18) and (19).

5.2 Iterative Algorithm and Convergence

In this section, we propose a simple iterative algorithm to solve the implicit equation suggested by (9) and determine the free boundary $\overline{\lambda}$.

A Simple Iterative Algorithm

Step 0. Set $\delta = 0$ in (9). Then we obtain $G(\lambda) = \frac{1}{A}\lambda^{-1/\gamma}$. We use it as the initial guess for $G(\lambda)$ satisfying (9).

Step 1. Given the initial $G(\lambda)$, we determine $B^*(\overline{\lambda})$ and $\overline{\lambda}$ by (12) and (15), respectively.

Step 2. Use the relationship (9) to update $G(\lambda)$. Then we set the updated $G(\lambda)$ as the new initial value.

Step 3. Repeat steps 1-2 until $\overline{\lambda}$ converges.

We successfully solve the equation (9) and determine $\overline{\lambda}$ by using the above iterative procedure. Now we show that the function $G(\lambda)$ obtained from the iterative procedure converges by using the Banach fixed-point theorem.

Consider the domain of $\lambda(\cdot)$ as

$$X = [\lambda_0, \underline{\lambda}],$$

where λ_0 is a value corresponding to a sufficiently large wealth $\hat{x}$ by the relationship of $\lambda_0 \equiv \lambda(\hat{x}) = \phi'(\hat{x})$. Denote $\mathcal{R}$ by the set of real numbers. We also consider the set of all bounded functions $y : X \to \mathcal{R}$ as

$$B(X, \mathcal{R}).$$

Then $B(X, \mathcal{R})$ is a complete metric space with the supremum norm

$$d(y, z) \equiv \sup\{|y(x) - z(x)| : x \in X\},$$

due to the fact that $\mathcal{R}$ is complete. We let $C(X, \mathcal{R})$ be the set of all continuous bounded functions $y : X \to \mathcal{R}$. Then $C(X, \mathcal{R})$ is a closed subspace of $B(X, \mathcal{R})$. Therefore, $C(X, \mathcal{R})$ is also a complete metric space. Because we have shown that $G(\lambda)$ has a monotonic decreasing property, we obtain

$$G(\overline{\lambda}) \leq G(\lambda) \leq G(\lambda_0),$$

accordingly $G(\lambda)$ is in $C(X,\mathcal{R})$.

Define

$$Y(G(\lambda)) \equiv \frac{\gamma\lambda^{-1/\gamma}}{\gamma A+\delta} + B^*(\overline{\lambda})\lambda^{-\alpha_\delta^*}$$
$$+ \frac{2\delta K}{\theta^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta-1)\lambda^{-\alpha_\delta}\int_0^\lambda \mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{\epsilon_L}{r}+\frac{\epsilon_H}{r}\Big)^{1-\gamma}d\mu$$
$$+ (\alpha_\delta^*-1)\lambda^{-\alpha_\delta^*}\int_\lambda^{\overline{\lambda}} \mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{\epsilon_L}{r}+\frac{\epsilon_H}{r}\Big)^{1-\gamma}d\mu\Big],$$

for any $G(\lambda)\in C(X,\mathcal{R})$. Then $Y(\cdot)$ is continuous and is in $C(X,\mathcal{R})$. This is because

$$|Y(G(\lambda))| \le \frac{2\delta K}{\theta^2(\alpha_\delta-\alpha_\delta^*)(\gamma-1)\overline{\lambda}}\sup_\mu |G(\mu)|.$$

Assume that

$$\frac{2\delta K}{\theta^2(\alpha_\delta-\alpha_\delta^*)(\gamma-1)\overline{\lambda}} < 1.$$

Then the map $Y: C(X,\mathcal{R})\to C(X,\mathcal{R})$ is a contraction mapping. Certainly, for any $G_1(\lambda)$, $G_2(\lambda)\in C(X,\mathcal{R})$, Y satisfies

$$\sup_\lambda |Y(G_1(\lambda))-Y(G_2(\lambda))| = \frac{2\delta K}{\theta^2(\alpha_\delta-\alpha_\delta^*)(\gamma-1)\overline{\lambda}}\sup_\lambda |G_1(\lambda)-G_2(\lambda)|.$$

Let $G^i(\lambda)$, $B^*(\overline{\lambda})^i$, $\overline{\lambda}^i$ be the values from the i-th iteration. If we apply the Banach fixed-point theorem, then $G^i(\lambda)$ converges uniformly to $G(\lambda)$ on $[\lambda_0,\overline{\lambda}]$. Moreover, $B^*(\overline{\lambda})^i\to B^*(\overline{\lambda})$ and $\overline{\lambda}^i\to\overline{\lambda}$ as $i\to\infty$.

5.3 Various Properties of Convex-Dual Function G

5.3.1 Uniqueness of G

We show the uniqueness of $G(\lambda)$ proposed by the implicit equation (9) under suitable parameter conditions.

Theorem 5.4 *Suppose $\gamma > 1$. If we assume that*

$$\frac{2\delta K}{\theta^2(\alpha_\delta - \alpha_\delta^*)(\gamma - 1)\overline{\lambda}} < 1,$$

then $G(\lambda)$ given by (9) is unique.

Proof. See Appendix. **Q.E.D.**

5.3.2 Monotonic decreasing property of G

The function $G(\cdot)$ satisfying the implicit equation (9) is monotonically decreasing.

Theorem 5.5 *Suppose $\gamma > 1$. If we assume that*

$$\overline{\lambda}^{-1/\gamma}\frac{\delta}{A}\left\{\frac{1}{\gamma A + \delta} - \frac{2}{\theta^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}\right\} - \frac{y}{r} + \frac{\epsilon_L}{r} < 0,$$

then any solution to (8) satisfies $G'(\lambda) < 0$.

Proof. See Appendix. **Q.E.D.**

5.3.3 Uniqueness of free boundary $\overline{\lambda}$

Theorem 5.4 states that there exists a unique solution $G(\lambda)$ to the implicit equation (9) under appropriate parameter conditions. However, the conditions contain a free boundary $\overline{\lambda}$ to be determined with $G(\lambda)$ by two conditions (12), (15). Because it is of importance to check whether or not the conditions hold, we provide more detail parameter conditions in which not a free boundary $\overline{\lambda}$ is uniquely determined, just but corresponding $G(\lambda)$ is unique.

Theorem 5.6 *Suppose $\gamma > 1$. Assume that*

$$\Big[\Big\{\frac{\beta+\delta}{1-\gamma} - r - \frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma} - \frac{\gamma}{1-\gamma} - \frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta}\Big] \times \Big(\frac{2\delta K}{\theta^2(\alpha_\delta - \alpha_\delta^*)(\gamma-1)}\Big)^{-1/\gamma} > \Big(1+\frac{\theta^2}{2r}\alpha_\delta^*\Big)(-y+\epsilon_L),$$

$$\Big[\Big\{\frac{\beta+\delta}{1-\gamma} - r - \frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma} - \frac{\gamma}{1-\gamma} - \frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta}\Big] \times \frac{A}{\delta}\Big(\frac{y-\epsilon_L}{r}\Big)\Big/\Big\{\frac{1}{rA+\delta} - \frac{2}{\theta^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}\Big\} > \Big(1+\frac{\theta^2}{2r}\alpha_\delta^*\Big)(-y+\epsilon_L).$$

Then there exists a unique free boundary $\overline{\lambda}$ and a unique $G(\lambda)$ satisfying (12), (15).

Proof. See Appendix. **Q.E.D.**

5.3.4 The equivalence between optimal stopping problem (5) and free boundary problem (7)

We verify that the solution to the free boundary problem (7) is a solution to the variational inequality (6).

Theorem 5.7 *Suppose $\gamma > 1$. If we assume that*

$$\Big[\Big\{\frac{\beta+\delta}{1-\gamma} - r - \frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma} - \frac{\gamma}{1-\gamma} - \frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta}\Big] \times \frac{A}{\delta}\Big(\frac{y-\epsilon_L}{r}\Big)\Big/\Big\{\frac{1}{rA+\delta} - \frac{2}{\theta^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}\Big\} > \Big(1+\frac{\theta^2}{2r}\alpha_\delta^*\Big)(-y+\epsilon_L),$$

and

$$\Big[\Big\{\frac{\beta+\delta}{1-\gamma} - r - \frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma} - \frac{\gamma}{1-\gamma} - \frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta}\Big] K^{-1/\gamma}(y-\epsilon_L)\Big/\Big(-\frac{\beta+\delta}{1-\gamma} + \frac{\theta^2}{2\gamma} + \frac{\gamma}{1-\gamma}K^{-1/\gamma} + r\Big) \geq \Big(1+\frac{\theta^2}{2r}\alpha_\delta^*\Big)(-y+\epsilon_L),$$

then the solution to the free boundary problem (7) is a solution to the variational inequality (6).

Proof. See Appendix. **Q.E.D.**

5.4 Proofs of Lemmas and Theorems

5.4.1 Proof of Lemma 2.1

We consider the following wealth process X_t with initial wealth $X_0 = x$:

$$dX_t = (rX_t - c_t + a) + \pi_t \sigma (dW_t + \theta dt), \quad t \geq 0, \tag{20}$$

$a > 0$ is a constant. If we define the value function $\widetilde{U}(x; a)$ as

$$\widetilde{U}(x; a) \equiv \max_{(c,\pi)} E\Big[\int_0^\infty e^{-\beta t} \frac{c_t^{1-\gamma}}{1-\gamma}\Big],$$

which is subject to the wealth process (20), then the value function $\widetilde{U}(x; a)$ follows

$$\widetilde{U}(x; a) = K \frac{(x + a/r)^{1-\gamma}}{1-\gamma},$$

where

$$K = \Big(\frac{1}{A}\Big)^\gamma, \quad A = \Big\{\frac{\gamma - 1}{\gamma}\Big(r + \frac{\theta^2}{2\gamma}\Big) + \frac{\beta}{\gamma}\Big\}.$$

By the principle of dynamic programming, the value function $\Phi(x)$ formulated by (4) becomes

$$\Phi(x) = \max_{(c,\pi,\tau)} E\Big[\int_0^{\tau \wedge \tau_\delta} e^{-\beta t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + e^{-\beta(\tau \wedge \tau_\delta)} \widetilde{U}(X_{\tau \wedge \tau_\delta}; a)\Big],$$

where $a \in \{y, \epsilon_H\}$. A straightforward calculation by using the conditional expectation of τ_δ yields that

$$
\begin{aligned}
& E\Big[\int_0^{\tau\wedge\tau_\delta} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-\beta(\tau\wedge\tau_\delta)}\widetilde{U}(X_{\tau\wedge\tau_\delta}; a)\Big] \\
&= E\Big[E\Big[\int_0^{\tau\wedge\tau_\delta} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-\beta(\tau\wedge\tau_\delta)}\widetilde{U}(X_{\tau\wedge\tau_\delta}; a)\Big]\Big|\tau_\delta\Big] \\
&= E\Big[\int_0^{\infty} \delta e^{-\delta s}\int_0^{\tau\wedge s} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dtds + \int_0^{\infty}\delta e^{-\delta s}e^{-\beta(\tau\wedge s)}\widetilde{U}(X_{\tau\wedge s}; a)ds\Big] \\
&= E\Big[\int_0^{\tau} \delta e^{-\delta s}\int_0^{s} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dtds + \int_\tau^{\infty}\delta e^{-\delta s}\int_0^{\tau} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dtds \\
&\qquad + \int_0^{\tau}\delta e^{-\delta s}e^{-\beta s}\widetilde{U}(X_s; \epsilon_H)ds + \int_\tau^{\infty}\delta e^{-\delta s}e^{-\beta\tau}\widetilde{U}(X_\tau; y)ds\Big] \\
&= E\Big[\int_0^{\tau} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}\int_t^{\tau}\delta e^{-\delta s}dsdt + \int_0^{\tau} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}\int_\tau^{\infty}\delta e^{-\delta s}dsdt \\
&\qquad + \int_0^{\tau} e^{-(\beta+\delta)s}\delta\widetilde{U}(X_s; \epsilon_H)ds + e^{-(\beta+\delta)\tau}\widetilde{U}(X_\tau; y)\Big] \\
&= E\Big[\int_0^{\tau} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}\int_t^{\infty}\delta e^{-\delta s}dsdt + \int_0^{\tau} e^{-(\beta+\delta)s}\delta\widetilde{U}(X_s; \epsilon_H)ds + e^{-(\beta+\delta)\tau}\widetilde{U}(X_\tau; y)\Big] \\
&= E\Big[\int_0^{\tau} e^{-(\beta+\delta)t}\Big\{\frac{c_t^{1-\gamma}}{1-\gamma} + \delta\widetilde{U}(X_t; \epsilon_H)\Big\}dt + e^{-(\beta+\delta)\tau}\widetilde{U}(X_\tau; y)\Big].
\end{aligned}
$$

Therefore, we complete the proof of the Lemma 2.1.

5.4.2 Proof of Lemma 2.2

For a fixed stopping time τ, we define

$$
J_\tau(x) \equiv \max_{(c,\pi)} E\Big[\int_0^{\tau} e^{-(\beta+\delta)t}\Big\{\frac{c_t^{1-\gamma}}{1-\gamma} + \delta\widetilde{U}(X_t; \epsilon_H)\Big\}dt + e^{-(\beta+\delta)\tau}\widetilde{U}(X_\tau; y)\Big].
$$

Then an entrepreneur's decision problem (5) is to solve the following optimal stopping problem:

$$
\Phi(x) = \max_\tau J_\tau(x).
$$

We denote c_t^* and π_t^* by optimal consumption and optimal risky portfolio, respectively. We introduce the following partial differential operator L:

$$L \equiv \frac{\partial}{\partial t} + \Big(rx - c_t^* + \epsilon_L + \pi_t^* \sigma\theta\Big)\frac{\partial}{\partial x} + \frac{1}{2}(\pi_t^*)^2\sigma^2\frac{\partial^2}{\partial x^2}.$$

We define domains G and D as follows

$$G = \Big\{(x,t) \in \mathbf{R}\times\mathbf{R}; x \geq -\frac{1}{r+\delta}\Big(\epsilon_L + \epsilon_H\frac{\delta}{r}\Big), t \geq 0\Big\}$$

and

$$D = \{(x,t) \in G; \tilde{\phi}(x,t) > e^{-(\beta+\delta)t}\widetilde{U}(x;y)\}$$

for a function $\tilde{\phi} : \bar{G} \to \mathbf{R}$. Then we obtain

$$\begin{aligned} L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{\frac{(c_t^*)^{1-\gamma}}{1-\gamma} + \delta\widetilde{U}(x;\epsilon_H)\Big\} &= \frac{\partial\tilde{\phi}}{\partial t} + \Big(rx - c_t^* + \epsilon_L + \pi_t^*\sigma\theta\Big)\frac{\partial\tilde{\phi}}{\partial x} \\ &\quad + \frac{1}{2}(\pi_t^*)^2\sigma^2\frac{\partial^2\tilde{\phi}}{\partial x^2} + e^{-(\beta+\delta)t}\Big\{\frac{(c_t^*)^{1-\gamma}}{1-\gamma} + \delta\widetilde{U}(x;\epsilon_H)\Big\}. \end{aligned}$$

The optimal stopping problem (5) is equivalent to the following variational inequality (Bensoussan and Lions, 1982; Øksendal, 2007):

$$\begin{aligned} L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{\frac{(c_t^*)^{1-\gamma}}{1-\gamma} + \delta\widetilde{U}(x;\epsilon_H)\Big\} &= 0 \quad \text{on} \quad D, \\ L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{\frac{(c_t^*)^{1-\gamma}}{1-\gamma} + \delta\widetilde{U}(x;\epsilon_H)\Big\} &\leq 0 \quad \text{on} \quad G\backslash D. \end{aligned}$$

As a result, we get the following variational inequality:

$$\begin{aligned} L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{\frac{(c_t^*)^{1-\gamma}}{1-\gamma} + \delta\widetilde{U}(x;\epsilon_H)\Big\} &\leq 0, \\ \tilde{\phi}(x,t) &\geq e^{-(\beta+\delta)t}\widetilde{U}(x;y), \quad (21) \\ \Big[L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{\frac{(c_t^*)^{1-\gamma}}{1-\gamma} + \delta\widetilde{U}(x;\epsilon_H)\Big\}\Big]\Big(\tilde{\phi}(x,t) - e^{-(\beta+\delta)t}\widetilde{U}(x;y)\Big) &= 0. \end{aligned}$$

We conjecture the form of $\tilde{\phi}$ as the following:

$$\tilde{\phi}(x,t) = e^{-(\beta+\delta)t}\phi(x).$$

By substituting the conjectured $\tilde{\phi}$ into the variational inequality (21), we obtain

$$\Big[-(\beta+\delta)\phi(x) + \Big(rx - c_t^* + \epsilon_L + \pi_t^*\sigma\theta\Big)\phi'(x) + \frac{1}{2}(\pi_t^*)^2\sigma^2\phi''(x) + \frac{(c_t^*)^{1-\gamma}}{1-\gamma} + \delta\widetilde{U}(x;\epsilon_H)\Big] \leq 0,$$

$$\phi(x) \geq \widetilde{U}(x;y),$$

$$\Big[-(\beta+\delta)\phi(x) + \Big(rx - c_t^* + \epsilon_L + \pi_t^*\sigma\theta\Big)\phi'(x) + \frac{1}{2}(\pi_t^*)^2\sigma^2\phi''(x) + \frac{(c_t^*)^{1-\gamma}}{1-\gamma} + \delta\widetilde{U}(x;\epsilon_H)\Big]\Big(\phi(x) - \widetilde{U}(x;y)\Big) = 0.$$

Note that optimality conditions for optimal consumption and risky portfolio are given by

$$c_t^* = \phi'(x)^{-1/\gamma} \quad \text{and} \quad \pi_t^* = -\frac{\theta}{\sigma}\frac{\phi'(x)}{\phi''(x)}.$$

Hence, we derive the variational inequality (6). Finally, if we apply the verification theorem for an optimal stopping problem given by Øksendal (2007), then the solution to the variational inequality (6) is the solution to our optimal stopping problem (5).

5.4.3 Proof of Lemma 5.1

We define λ as the marginal value of the value function ϕ and introduce a dual variable $\overline{\lambda}$ of the free boundary $\underline{x}$. Specifically,

$$\lambda(x) \equiv \phi'(x), \quad \text{and} \quad \overline{\lambda} \equiv K(\underline{x} + y/r)^{-\gamma}.$$

Differentiating the first relationship in (7) with respect to x yields

$$(\beta+\delta)\lambda(x) - r\lambda(x) - (rx+\epsilon_L)\lambda'(x) + \frac{1}{2}\theta^2\frac{2\lambda(x)\lambda'(x)^2 - \lambda(x)^2\lambda''(x)}{\lambda'(x)^2} + \lambda(x)^{-1/\gamma}\lambda'(x) = \delta K(x+\epsilon_H/r)^{-\gamma}. \tag{22}$$

We employ a function G satisfying

$$G(\lambda(x)) \equiv x + \frac{\epsilon_L}{r}.$$

Then the differential equation (22) is rewritten by

$$-\frac{1}{2}\theta^2\lambda^2 G''(\lambda) - \lambda G'(\lambda)(\theta^2+\beta+\delta-r) + rG(\lambda) + \delta K\Big(G(\lambda) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{-\gamma} G'(\lambda) = \lambda^{-1/\gamma},\ \ 0 < \lambda < \overline{\lambda},$$

where $\overline{\lambda}$ is a free boundary to be determined according to the smooth-pasting conditions.

5.4.4 Proof of Theorem 5.1

We can always write the general solution of (8) as the following:

$$G(\lambda) = \frac{\gamma\lambda^{-1/\gamma}}{\gamma A + \delta} + \eta(\lambda)\lambda^{-\alpha_\delta} + \eta^*(\lambda)\lambda^{-\alpha^*_\delta}, \tag{23}$$

subject to

$$\eta'(\lambda)\lambda^{-\alpha_\delta} + (\eta^*(\lambda))'\lambda^{-\alpha^*_\delta} = 0,$$

where $\alpha_\delta > 0$ and $\alpha^*_\delta < 0$ are the two roots of the characteristic equation (10). The first and second derivatives of G follow

$$G'(\lambda) = -\frac{\lambda^{-1/\gamma-1}}{\gamma A + \delta} - \alpha_\delta\eta(\lambda)\lambda^{-\alpha_\delta-1} - \alpha^*_\delta\eta^*(\lambda)\lambda^{-\alpha^*_\delta-1}$$

and

$$G''(\lambda) = \Big(\frac{1}{\gamma} + 1\Big)\frac{\lambda^{-1/\gamma-2}}{\gamma A + \delta} - \alpha_\delta\eta'(\lambda)\lambda^{-\alpha_\delta-1} + \alpha_\delta(\alpha_\delta+1)\eta(\lambda)\lambda^{-\alpha_\delta-2}$$
$$- \alpha^*_\delta(\eta^*(\lambda))'\lambda^{-\alpha^*_\delta-1} + \alpha^*_\delta(\alpha^*_\delta+1)\eta^*(\lambda)\lambda^{-\alpha^*_\delta-2},$$

respectively. Using the general solution (23) of G and its first and second derivatives, we obtain

$$
\begin{aligned}
&-\frac{1}{2}\theta^2\lambda^2 G''(\lambda) - \lambda G'(\lambda)(\theta^2+\beta+\delta-r) + rG(\lambda)\\
&= \lambda^{-1/\gamma} + \frac{\theta^2}{2}(\alpha_\delta - \alpha_\delta^*)\lambda^{1-\alpha_\delta}\eta'(\lambda)\\
&= \lambda^{-1/\gamma} - \frac{\theta^2}{2}(\alpha_\delta - \alpha_\delta^*)\lambda^{1-\alpha_\delta^*}(\eta^*(\lambda))'.
\end{aligned}
$$

Then the differential equation (8) reduces

$$
\frac{\theta^2}{2}(\alpha_\delta - \alpha_\delta^*)\lambda^{1-\alpha_\delta}\eta'(\lambda) = -\delta K\Big(G(\lambda) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{-\gamma} G'(\lambda)
$$

and

$$
\frac{\theta^2}{2}(\alpha_\delta - \alpha_\delta^*)\lambda^{1-\alpha_\delta^*}(\eta^*(\lambda))' = \delta K\Big(G(\lambda) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{-\gamma} G'(\lambda),
$$

for $0 < \lambda < \overline{\lambda}$. Thus we get the following relationships:

$$
\eta(\lambda) = -\frac{2\delta K}{\theta^2(\alpha_\delta - \alpha_\delta^*)}\int_0^\lambda \mu^{\alpha_\delta - 1}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{-\gamma} G'(\mu)d\mu
$$

and

$$
\eta^*(\lambda) = \eta^*(\overline{\lambda}) - \frac{2\delta K}{\theta^2(\alpha_\delta - \alpha_\delta^*)}\int_\lambda^{\overline{\lambda}} \mu^{\alpha_\delta^* - 1}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{-\gamma} G'(\mu)d\mu.
$$

Hence, the general solution (23) of G is rewritten as

$$
\begin{aligned}
G(\lambda) = \frac{\gamma\lambda^{-1/\gamma}}{\gamma A + \delta} + \eta^*(\overline{\lambda})\lambda^{-\alpha_\delta^*} - \frac{2\delta K}{\theta^2(\alpha_\delta - \alpha_\delta^*)}\Big[\lambda^{-\alpha_\delta}\int_0^\lambda \mu^{\alpha_\delta - 1}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{-\gamma} G'(\mu)d\mu\\
+ \lambda^{-\alpha_\delta^*}\int_\lambda^{\overline{\lambda}} \mu^{\alpha_\delta^* - 1}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{-\gamma} G'(\mu)d\mu\Big].
\end{aligned}
$$

Note that

$$
\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{-\gamma} G'(\mu) = \frac{d}{d\mu}\Big\{\frac{1}{1-\gamma}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}\Big\}.
$$

Then using the integration by parts we obtain

$$G(\lambda) = \frac{\gamma\lambda^{-1/\gamma}}{\gamma A+\delta} + \Big\{\eta^*(\overline{\lambda}) + \overline{\lambda}^{\alpha_\delta^*-1}\frac{1}{1-\gamma}\Big(G(\overline{\lambda}) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}\Big\}\lambda^{-\alpha_\delta^*}$$
$$+ \frac{2\delta K}{\theta^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta - 1)\lambda^{-\alpha_\delta}\int_0^\lambda \mu^{\alpha_\delta-2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} d\mu$$
$$+ (\alpha_\delta^* - 1)\lambda^{-\alpha_\delta^*}\int_\lambda^{\overline{\lambda}} \mu^{\alpha_\delta^*-2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} d\mu\Big].$$

We define a constant $B^*(\overline{\lambda})$ as the following:

$$B^*(\overline{\lambda}) = \Big\{\eta^*(\overline{\lambda}) + \overline{\lambda}^{\alpha_\delta^*-1}\frac{1}{1-\gamma}\Big(G(\overline{\lambda}) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}\Big\}.$$

As a result, we derive the analytic solution given by (9) to the non-linear differential equation (8).

5.4.5 Proof of Lemma 5.2

We rewrite the first relationship given in (7) by using the convex-dual function G as the following:

$$(\beta+\delta)\phi(x) = rG(\lambda)\lambda - \frac{1}{2}\theta^2\lambda^2 G'(\lambda) + \frac{\gamma}{1-\gamma}\lambda^{1-1/\gamma} + \frac{\delta K}{1-\gamma}\Big(G(\lambda) - \frac{y}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}.$$

We define a function H by

$$H(\lambda) \equiv \frac{1}{(\beta+\delta)}\Big[rG(\lambda)\lambda - \frac{1}{2}\theta^2\lambda^2 G'(\lambda) + \frac{\gamma}{1-\gamma}\lambda^{1-1/\gamma} + \frac{\delta K}{1-\gamma}\Big(G(\lambda) - \frac{y}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}\Big]. \quad (24)$$

Then we get the relationship

$$\phi(x) = H(\lambda(x)).$$

From the equations (8) and (24), the following equality holds:

$$H'(\lambda) = \lambda G'(\lambda).$$

Therefore, we obtain that

$$\phi'(x) = H'(\lambda(x))\lambda'(x) = \frac{H'(\lambda(x))}{G'(\lambda(x))} = \lambda(x).$$

As a result, $\phi(x)$ is a solution to the differential equation (8). Using the boundary condition of $\phi(x)$ at $\underline{x}$ that $\phi(\underline{x}) = \widetilde{U}(\underline{x}; y)$, we obtain the value of H at $\overline{\lambda}$

$$H(\overline{\lambda}) = \frac{K^{1/\gamma}\overline{\lambda}^{1-1/\gamma}}{1-\gamma},$$

which is equivalent to the equality (28).

5.4.6 Proof of Lemma 5.3

A straightforward calculation of the first derivative of G yields that

$$\begin{aligned} G'(\lambda) = & -\frac{\lambda^{-1/\gamma-1}}{\gamma A+\delta} - \alpha_\delta^* B^*(\overline{\lambda})\lambda^{-\alpha_\delta^*-1} + \frac{2\delta K}{\theta^2\lambda^2}\Big(G(\lambda) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} \\ & - \frac{2\delta K}{\theta^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}\Big[\alpha_\delta(\alpha_\delta - 1)\lambda^{-\alpha_\delta-1}\int_0^\lambda \mu^{\alpha_\delta-2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}d\mu \\ & + \alpha_\delta^*(\alpha_\delta^*-1)\lambda^{-\alpha_\delta^*-1}\int_\lambda^{\overline{\lambda}} \mu^{\alpha_\delta^*-2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}d\mu\Big]. \end{aligned}$$

Then the value of G' at $\overline{\lambda}$ follows

$$\begin{aligned} G'(\overline{\lambda}) = & -\frac{\overline{\lambda}^{-1/\gamma-1}}{\gamma A+\delta} - \alpha_\delta^* B^*(\overline{\lambda})\overline{\lambda}^{-\alpha_\delta^*-1} + \frac{2\delta K}{\theta^2\overline{\lambda}^2}\Big(G(\overline{\lambda}) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} \\ & - \frac{2\delta K\alpha_\delta(\alpha_\delta-1)\overline{\lambda}^{-\alpha_\delta-1}}{\theta^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\int_0^{\overline{\lambda}} \mu^{\alpha_\delta-2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}d\mu. \end{aligned}$$

By substituting $G(\overline{\lambda})$ and $B^*(\overline{\lambda})$ given in (11) and (29) into the above, we get

$$\begin{aligned} G'(\overline{\lambda}) = & -\Big(\frac{1-\alpha_\delta^*\gamma}{\gamma A+\delta} + \alpha_\delta^* K^{1/\gamma}\Big)\overline{\lambda}^{-1/\gamma-1} + \alpha_\delta^*\Big(\frac{y}{r} - \frac{\epsilon_L}{r}\Big)\overline{\lambda}^{-1} + \frac{2\delta K}{\theta^2\overline{\lambda}}\Big(K^{1/\gamma}\overline{\lambda}^{-1/\gamma} - \frac{y}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} \\ & - \frac{2\delta K(\alpha_\delta-1)\overline{\lambda}^{-\alpha_\delta-1}}{\theta^2(1-\gamma)}\int_0^{\overline{\lambda}} \mu^{\alpha_\delta-2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}d\mu. \end{aligned} \tag{25}$$

Putting (25) into the relationship (28) in Lemma 5.2 and rearranging give the equation (15).

5.4.7 Proof of Theorem 5.4

Because $\underline{M}_\delta(\overline{\lambda})$ and $\overline{M}_\delta(\overline{\lambda})$ are lower and upper bounds of $M_\delta(\overline{\lambda})$. Moreover, they are monotonically-decreasing and continuous functions with

$$\underline{M}_\delta(0) = \overline{M}_\delta(0) = +\infty, \quad \text{and}$$
$$\underline{M}_\delta(+\infty) = \overline{M}_\delta(+\infty) = 0.$$

Therefore, λ_δ^l and λ_δ^u satisfying

$$\underline{M}_\delta(\lambda_\delta^l) = N_\delta, \quad \textit{and} \;\; \overline{M}_\delta(\lambda_\delta^u) = N_\delta$$

become lower and upper bounds for $\overline{\lambda}$, respectively.

5.4.8 Proof of Theorem 5.4

Let G_1 and G_2 be the two solution satisfying (9). Then

$$\begin{aligned} G_1(\lambda) - G_2(\lambda) =& \frac{2\delta K}{\theta^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta - 1)\lambda^{-\alpha_\delta}\int_0^\lambda \mu^{\alpha_\delta - 2}\Big\{\Big(G_1(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} \\ & - \Big(G_2(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}\Big\}d\mu + (\alpha_\delta^* - 1)\lambda^{-\alpha_\delta^*}\int_\lambda^{\overline{\lambda}} \mu^{\alpha_\delta^* - 2}\Big\{\Big(G_1(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} \\ & - \Big(G_2(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}\Big\}d\mu\Big]. \end{aligned}$$

Since $\gamma > 1$,

$$\Big|\Big(G_1(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} - \Big(G_2(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}\Big| \leq |G_1(\mu) - G_2(\mu)|.$$

Hence,

$$|G_1(\lambda) - G_2(\lambda)| \le \frac{2\delta K}{\theta^2(\alpha_\delta - \alpha_\delta^*)(\gamma - 1)\overline{\lambda}} \sup_\mu |G_1(\mu) - G_2(\mu)|,$$

which completes the proof.

5.4.9 Proof of Theorem 5.5

Any solution to (8) satisfies the implicit equation (9). Using the condition (12) and the assumption in Theorem 5.5 yields that

$$\begin{aligned} B^*(\overline{\lambda}) &= \Big[K^{1/\gamma}\overline{\lambda}^{-1/\gamma} - \frac{y}{r} + \frac{\epsilon_L}{r} - \frac{\gamma\overline{\lambda}^{-1/\gamma}}{\gamma A + \delta} \\ &\quad - \frac{2\delta K(\alpha_\delta - 1)\overline{\lambda}^{-\alpha_\delta}}{\theta^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} \int_0^{\overline{\lambda}} \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} d\mu\Big]\overline{\lambda}^{\alpha_\delta^*} \\ &\le \Big[\overline{\lambda}^{-1/\gamma}\frac{\delta}{A}\Big\{\frac{1}{\gamma A + \delta} - \frac{2}{\theta^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}\Big\} - \frac{y}{r} + \frac{\epsilon_L}{r}\Big]\overline{\lambda^{\alpha_\delta^*}} < 0. \end{aligned}$$

Calculating the derivative of $G(\lambda)$ gives that

$$\begin{aligned} G'(\lambda) &= -\frac{\lambda^{-1-1/\gamma}}{\gamma A + \delta} - \alpha_\delta^* B^*(\overline{\lambda})\lambda^{-1-\alpha_\delta^*} + \frac{2\delta K}{\theta^2(1-\gamma)}\frac{1}{\lambda^2}\Big(G(\lambda) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} \\ &\quad - \frac{2\delta K}{\theta^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}\Big[\alpha_\delta(\alpha_\delta - 1)\lambda^{-1-\alpha_\delta}\int_0^\lambda \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} d\mu \\ &\quad + \alpha_\delta^*(\alpha_\delta^* - 1)\lambda^{-1-\alpha_\delta^*}\int_\lambda^{\overline{\lambda}} \mu^{\alpha_\delta^* - 2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} d\mu\Big] \\ &\le -\frac{\lambda^{-1-1/\gamma}}{-\gamma A + \delta} - \alpha_\delta^* B^*(\overline{\lambda})\lambda^{-1-\alpha_\delta^*} + \frac{2\delta K}{\theta^2(1-\gamma)\lambda^2}\Big[\Big(G(\lambda) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} \\ &\quad - \frac{\alpha_\delta\overline{\lambda}^2 + (\lambda^2 - \overline{\lambda}^2)\alpha_\delta^*}{(\alpha_\delta - \alpha_\delta^*)\overline{\lambda}^2}\Big(G(\overline{\lambda}) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}\Big], \end{aligned}$$

where the last inequality is derived by using the fact that $\gamma > 1$, accordingly

$$\Big(G(\lambda) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} \le \Big(G(\overline{\lambda}) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}.$$

Define

$$P(\lambda) = \Big(G(\lambda) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} - \frac{\alpha_\delta \overline{\lambda}^2 + (\lambda^2 - \overline{\lambda}^2)\alpha_\delta^*}{(\alpha_\delta - \alpha_\delta^*)\overline{\lambda}^2}\Big(G(\overline{\lambda}) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}.$$

Then

$$P(\lambda) \geq \Big(G(\lambda) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}\Big(\frac{-\lambda^2\alpha_\delta^*}{(\alpha_\delta - \alpha_\delta^*)\overline{\lambda}^2}\Big) > 0.$$

Therefore, $G'(\lambda) < 0$.

5.4.10 Proof of Theorem 5.6

Recall the following relationships:

$$M_\delta(\overline{\lambda}) = \Big[\Big\{\frac{\beta+\delta}{1-\gamma} - r - \frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma} - \frac{\gamma}{1-\gamma} - \frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta}\Big]$$
$$+ \frac{\delta K(\alpha_\delta - 1)\overline{\lambda}^{-\alpha_\delta}}{\gamma - 1}\int_0^{\overline{\lambda}} \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} d\mu,$$

$$N_\delta = \Big(1 + \frac{\theta^2}{2r}\alpha_\delta^*\Big)(-y + \epsilon_L),$$

$$\underline{M}_\delta(\overline{\lambda}) = \Big[\Big\{\frac{\beta+\delta}{1-\gamma} - r - \frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma} - \frac{\gamma}{1-\gamma} - \frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta}\Big]\overline{\lambda}^{-1/\gamma},$$

and

$$\overline{M}_\delta(\overline{\lambda}) = \underline{M}_\delta(\overline{\lambda}) + \frac{\delta K}{(\gamma-1)\overline{\lambda}}\Big(K^{1/\gamma}\overline{\lambda}^{-1/\gamma} - \frac{y}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}.$$

There exists at least one solution $\overline{\lambda}$ such that

$$M_\delta(\overline{\lambda}) = N_\delta,$$

because $M_\delta(\overline{\lambda}) \geq \underline{M}_\delta(\overline{\lambda})$, $M_\delta(\overline{\lambda}) \leq \overline{M}_\delta(\overline{\lambda})$, $\overline{M}_\delta(0) = \underline{M}_\delta(0) = +\infty$, $\overline{M}_\delta(+\infty) = \underline{M}_\delta(+\infty) = 0$, and $M_\delta(\overline{\lambda})$ is continuous. Furthermore, for $\lambda_\delta^l \leq \overline{\lambda} \leq \lambda_\delta^u$ the following inequalities

$$\frac{2\delta K}{\theta^2(\alpha_\delta - \alpha_\delta^*)(\gamma-1)\lambda_\delta^l} < 1, \tag{26}$$

$$(\lambda_\delta^l)^{-1/\gamma}\frac{\delta}{A}\Big\{\frac{1}{rA+\delta}-\frac{2}{\theta^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big\}-\frac{y}{r}+\frac{\epsilon_L}{r}<0, \tag{27}$$

give the assumptions in Theorem 5.4 and Theorem 5.5, respectively. The inequalities (26), (27) can be rewritten as

$$\underline{M}_\delta\Big(\frac{2\delta K}{\theta^2(\alpha_\delta-\alpha_\delta^*)(\gamma-1)}\Big)>N_\delta,$$

$$\underline{M}_\delta\Big(\Big[\Big(\frac{\delta}{A}\Big)^\gamma\Big\{\frac{1}{\gamma A+\delta}-\frac{2}{\theta^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big\}\Big/\Big(\frac{y-\epsilon_L}{r}\Big)\Big]^\gamma\Big)>N_\delta,$$

respectively. Then we obtain the following parameter conditions

$$\begin{aligned}&\Big[\Big\{\frac{\beta+\delta}{1-\gamma}-r-\frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma}-\frac{\gamma}{1-\gamma}-\frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta}\Big]\\ &\qquad\times\Big(\frac{2\delta K}{\theta^2(\alpha_\delta-\alpha_\delta^*)(\gamma-1)}\Big)^{-1/\gamma}>\Big(1+\frac{\theta^2}{2r}\alpha_\delta^*\Big)(-y+\epsilon_L),\end{aligned}$$

$$\begin{aligned}&\Big[\Big\{\frac{\beta+\delta}{1-\gamma}-r-\frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma}-\frac{\gamma}{1-\gamma}-\frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta}\Big]\\ &\qquad\times\frac{A}{\delta}\Big(\frac{y-\epsilon_L}{r}\Big)\Big/\Big\{\frac{1}{rA+\delta}-\frac{2}{\theta^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big\}>\Big(1+\frac{\theta^2}{2r}\alpha_\delta^*\Big)(-y+\epsilon_L).\end{aligned}$$

Hence, under the above conditions we can say that there exists a unique free boundary $\overline{\lambda}$ and a unique $G(\lambda)$ satisfying (12), (15).

5.4.11 Proof of Theorem 5.7

Define

$$Q(x)\equiv\phi(x)-\widetilde{U}(x;y).$$

Then $Q(\underline{x})=0$ because $\phi(\underline{x})=\widetilde{U}(x;y)$. Firstly, we show that $Q'(x)\geq 0$ for $x\geq\underline{x}$. Then we can conclude that the second inequality in (6) holds. It is enough to verify that

$$G(\lambda)\geq K^{1/\gamma}\lambda^{-1/\gamma}+\frac{\epsilon_L-y}{r},$$

for $0 < \lambda \leq \overline{\lambda}$. If we assume that

$$\Big[\Big\{\frac{\beta+\delta}{1-\gamma} - r - \frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma} - \frac{\gamma}{1-\gamma} - \frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta}\Big] \\ \times\frac{A}{\delta}\Big(\frac{y-\epsilon_L}{r}\Big)\Big/\Big\{\frac{1}{rA+\delta} - \frac{2}{\theta^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big\} > \Big(1+\frac{\theta^2}{2r}\alpha_\delta^*\Big)(-y+\epsilon_L),$$

then by the proof of Theorem 5.6, $G(\lambda)$ is monotonically decreasing. Since $G(\overline{\lambda}) = K^{1/\gamma}\lambda^{-1/\gamma} + \frac{\epsilon_L - y}{r}$, accordingly

$$G(\lambda) \geq K^{1/\gamma}\lambda^{-1/\gamma} + \frac{\epsilon_L - y}{r},$$

for $0 < \lambda \leq \overline{\lambda}$.

Secondly, we show that the first inequality in (6) holds. For $\underline{x} < x$, the free boundary problem (7) gives that the equality of the first relationship in (6) holds. Hence, it remains to verify whether or not the solution to (7) satisfies the first inequality in (6) for $0 < x \leq \underline{x}$. In fact, we obtain

$$(\beta+\delta)\phi(x) - (rx+\epsilon_L)\phi'(x) + \frac{1}{2}\theta^2\frac{\phi'(x)^2}{\phi''(x)} - \frac{\gamma}{1-\gamma}\{\phi'(x)\}^{1-1/\gamma} - \delta\widetilde{U}(x;\epsilon_H) \\ = K\Big(x+\frac{y}{r}\Big)^{1-\gamma}\Big[\frac{\beta+\delta}{1-\gamma} - r\frac{x+\epsilon_L/r}{x+y/r} - \frac{\theta^2}{2\gamma} - \frac{\gamma}{1-\gamma}K^{-1/\gamma} - \delta\frac{1}{1-\gamma}\Big(\frac{x+\epsilon_H/r}{x+y/r}\Big)^{1-\gamma}\Big].$$

Define

$$R(x) \equiv \frac{\beta+\delta}{1-\gamma} - r\frac{x+\epsilon_L/r}{x+y/r} - \frac{\theta^2}{2\gamma} - \frac{\gamma}{1-\gamma}K^{-1/\gamma} - \delta\frac{1}{1-\gamma}\Big(\frac{x+\epsilon_H/r}{x+y/r}\Big)^{1-\gamma}.$$

Since $R'(x) < 0$, $R(x)$ is monotonically decreasing. The following parameter conditions

$$\lambda_\delta^l \geq K\Big[\Big(-\frac{\beta+\delta}{1-\gamma} + \frac{\theta^2}{2\gamma} + \frac{\gamma}{1-\gamma}K^{-1/\gamma} + r\Big)\Big/(y-\epsilon_L)\Big]^\gamma \tag{28}$$

for $\lambda_\delta^l \leq \overline{\lambda}$ yield

$$R(\underline{x}) \geq 0.$$

We have shown that $R(x)$ is monotonically decreasing, hence under the condition (28)

$$R(x) \geq 0,$$

for $0 < x \leq \underline{x}$. Finally, the condition (28) is equivalent to

$$\underline{M}_\delta\Big(\lambda_\delta^l \geq K\Big[\Big(-\frac{\beta+\delta}{1-\gamma}+\frac{\theta^2}{2\gamma}+\frac{\gamma}{1-\gamma}K^{-1/\gamma}+r\Big)\Big/(y-\epsilon_L)\Big]^\gamma\Big) \geq N_\delta, \tag{29}$$

where

$$\underline{M}_\delta(\overline{\lambda}) = \Big[\Big\{\frac{\beta+\delta}{1-\gamma}-r-\frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma}-\frac{\gamma}{1-\gamma}-\frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta}\Big]\overline{\lambda}^{-1/\gamma},$$

and

$$N_\delta = \Big(1+\frac{\theta^2}{2r}\alpha_\delta^*\Big)(-y+\epsilon_L).$$

Rewriting the condition (29) gives that

$$\begin{aligned}&\Big[\Big\{\frac{\beta+\delta}{1-\gamma}-r-\frac{1}{2}\theta^2\alpha_\delta^*\Big\}K^{1/\gamma}-\frac{\gamma}{1-\gamma}-\frac{1}{2}\theta^2(1-\gamma\alpha_\delta^*)\frac{1}{\gamma A+\delta}\Big]\\ &\quad\times\frac{A}{\delta}\Big(\frac{y-\epsilon_L}{r}\Big)\Big/\Big\{\frac{1}{rA+\delta}-\frac{2}{\theta^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big\} > \Big(1+\frac{\theta^2}{2r}\alpha_\delta^*\Big)(-y+\epsilon_L).\end{aligned}$$

Therefore, if we take the above parameter conditions, then the solution to the free boundary problem (7) satisfies the first inequality in (6) for $0 < x \leq \underline{x}$.

5.5 A Consumption-Saving Model

We have quantified a hedging effect of a market portfolio against undiversifiable idiosyncratic risk by using an economic concept of the certainty equivalent wealth. We have compared two value functions where one is allowed to participate in the stock market and the other is not. In this section, we construct a consumption-saving model for an entrepreneur who has limited access to the stock market. The wealth process for the entrepreneur with initial wealth x ($x > -\frac{1}{r+\delta}(\epsilon_L + \epsilon_H \frac{\delta}{r})$) follows

$$dX_t = (rX_t - c_t + \epsilon_t)dt, \quad t \geq 0.$$

The entrepreneurial business plan is to maximize her CRRA lifetime utility by controlling per-period consumption c and the time τ to exit from her risky business and accept a safe job in the presence of undiversifiable idiosyncratic risk. That is, she would like to find the following value function:

$$\Psi(x) \equiv \max_{(c,\tau)} E\Big[\int_0^{\tau\wedge\tau_\delta} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-\beta(\tau\wedge\tau_\delta)}\int_{\tau\wedge\tau_\delta}^{\infty} e^{-\beta(t-\tau\wedge\tau_\delta)}\frac{c_t^{1-\gamma}}{1-\gamma}dt\Big]. \qquad (30)$$

Firstly, we consider the following maximization problem:

$$U(x;a) \equiv \max_c E\Big[\int_0^\infty e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\Big],$$

provided that the entrepreneur receives incomes at the rate equal to a infinitely. Then we obtain the closed form solution given by

$$U(x;a) = F\frac{(x+a/r)^{1-\gamma}}{1-\gamma}, \quad F = \Big(\frac{\gamma-1}{\gamma}r + \frac{\beta}{\gamma}\Big)^{-\gamma},$$

solving the associated Hamilton-Jacobi-Bellman equation.

By using the conditional expectation of τ_δ and the principle of dynamic programming, the value function (30) is reformulated as the following:

$$\Psi(x) = \max_{(c,\tau)} E\Big[\int_0^\tau e^{-(\beta+\delta)t}\Big\{\frac{c_t^{1-\gamma}}{1-\gamma} + \delta U(X_t;\epsilon_H)\Big\}dt + e^{-(\beta+\delta)\tau}U(X_\tau;y)\Big].$$

Then the value function satisfies the following optimal stopping problem:

$$\begin{cases} (\beta+\delta)\psi(x) - (rx+\epsilon_L)\psi'(x) - \dfrac{\gamma}{1-\gamma}\{\psi'(x)\}^{1-1/\gamma} = \delta U(x;\epsilon_H), \quad x^* < x, \\ \psi(x) = U(x;y), \quad < x \le x^*, \\ \psi(x^*) = U(x^*;y), \\ \psi'(x^*) = U'(x^*;y), \end{cases} \tag{31}$$

derived similarly as in the optimal stopping problem (7). Here, x^* is the critical wealth level under which it is optimal for the entrepreneur who has limited access to the stock market to quit her risky business and accept a safe job. To solve the optimal stopping problem (31), we employ the modified convex-duality approach developed by Bensoussan *et al.* (2016). More specifically, we introduce a dual variable ρ defined by the marginal value of the value function ψ. It follows that

$$\rho(x) \equiv \psi'(x), \quad \text{and} \quad \bar{\rho} \equiv F(x^* + y/r)^{-\gamma}.$$

By differentiating the first equality in (31) with respect to x, we obtain

$$(\beta+\delta)\rho(x) - r\rho(x) - (rx+\epsilon_L)\rho'(x) + \{\rho(x)\}^{-1/\gamma}\rho'(x) = \delta F(x+\epsilon_H/r)^{-\gamma}.$$

We also introduce a function H satisfying

$$H\big(\rho(x)\big) \equiv x + \frac{\epsilon_L}{r}.$$

Then the first relationship in (31) is rewritten in terms of newly defined variables ρ, H as the following:

$$(\beta+\delta-r)H'(\rho)\rho - rH(\rho) + \rho^{-1/\gamma} = \delta F\Big(H(\rho) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{-\gamma} H'(\rho), \quad 0 < \rho < \bar{\rho}, \tag{32}$$

where $\bar{\rho}$ is a free boundary to be determined according to the value-matching and smooth-pasting conditions.

We can always write a general solution to the non-linear differential equation (32) as the following:

$$H(\rho) = C(\rho)\rho^{r/(\beta+\delta-r)} + \frac{1}{\{r + (\beta+\delta-r)/\gamma\}}\rho^{-1/\gamma},$$

where $C(\rho)$ is an arbitrary function of ρ. By substituting the general solution $H(\rho)$ into the equation (32), we get

$$\begin{aligned} H(\rho) =& D(\overline{\rho})\rho^{r/(\beta+\delta-r)} + \frac{\delta F}{(\beta+\delta-r)(1-\gamma)}\rho^{-1}\Big(H(\rho) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} + \frac{1}{\{r+(\beta+\delta-r)/\gamma\}}\rho^{-1/\gamma} \\ & - \frac{\delta F(\beta+\delta)}{(1-\gamma)(\beta+\delta-r)^2}\rho^{r/(\beta+\delta-r)}\int_{\rho}^{\overline{\rho}} \xi^{-r/(\beta+\delta-r)-2}\Big(H(\xi) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} d\xi, \end{aligned}$$

where $D(\overline{\rho})$ is a constant to be determined and satisfies

$$D(\overline{\rho}) = C(\overline{\rho}) - \frac{\delta F}{(\beta+\delta-r)(1-\gamma)}\overline{\rho}^{-r/(\beta+\delta-r)-1}\Big(H(\overline{\rho}) - \frac{\epsilon_L}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma}.$$

Using the smooth-pasting condition of $\psi'(x^*) = U'(x^*; y)$, we obtain

$$\begin{aligned} F^{1/\gamma}\overline{\rho}^{-1/\gamma} - \frac{y}{r} + \frac{\epsilon_L}{r} = & D(\overline{\rho})\overline{\rho}^{r/(\beta+\delta-r)} + \frac{\delta F}{(\beta+\delta-r)(1-\gamma)}\overline{\rho}^{-1}\Big(F^{1/\gamma}\overline{\rho}^{-1/\gamma} - \frac{y}{r} + \frac{\epsilon_H}{r}\Big)^{1-\gamma} \\ & + \frac{1}{\{r+(\beta+\delta-r)/\gamma\}}\overline{\rho}^{-1/\gamma}. \end{aligned} \tag{33}$$

Furthermore, using the value-matching condition of $\psi(x^*) = U(x^*; y)$ we get

$$\Big[\frac{(\beta+\delta-r)F^{1/\gamma} - \gamma}{1-\gamma}\Big]\overline{\rho}^{1-1/\gamma} + (y - \epsilon_L)\overline{\rho} = \frac{\delta F(F^{1/\gamma}\overline{\rho}^{-1/\gamma} - y/r + \epsilon_H/r)^{1-\gamma}}{1-\gamma}. \tag{34}$$

The free boundary $\overline{\rho}$ is easily determined from (34) numerically, accordingly, the relationship (33) yields the constant $D(\overline{\rho})$.

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δ	μ			σ			γ		
	0.1023	0.1123	0.1223	0.1854	0.1954	0.2054	1.5	2	2.5
0.20	−33.1610	−33.3528	−33.5374	−33.4288	−33.3528	−33.2831	−36.0946	−33.3528	−30.4592
0.15	−29.9395	−30.2410	−30.5190	−30.3569	−30.2410	−30.1331	−34.0770	−30.2410	−26.2180
0.10	−20.9649	−21.5240	−22.0136	−21.7307	−21.5240	−21.3276	−28.2844	−21.5240	−14.5062

Table 1: **Critical wealth level $\underline{x}$ for various parameter values of μ, σ, and γ.** Default parameter values: $\beta = 0.0371$, $r = 0.0371$, $\epsilon_L = 0.25$, $y = 1.5$, and $\epsilon_H = 2.5$. The borrowing limits for three value of $\delta \in \{0.20, 0.15, 0.10\}$ are computed as the following: -57.8958, -55.3598, -50.9741, respectively. Note that the values of $\underline{x}$ are negative and above the borrowing limits.

x	μ			σ		
	0.1023	0.1123	0.1223	0.1854	0.1954	0.2054
$\underline{x}$	0.9931	1.0515	1.1210	1.0784	1.0515	1.0288
$\underline{x}+5$	1.2152	1.2971	1.3933	1.3344	1.2971	1.2654
$\underline{x}+10$	1.4593	1.5649	1.6880	1.6128	1.5649	1.5242
$\underline{x}+15$	1.7191	1.8481	1.9978	1.9063	1.8481	1.7985
$\underline{x}+20$	1.9891	2.1410	2.3168	2.2094	2.1410	2.0826

x	μ			σ		
	0.1023	0.1123	0.1223	0.1854	0.1954	0.2054
$\underline{x}$	20.2924	22.3104	24.2082	24.3378	22.3104	20.5369
$\underline{x}+5$	22.1575	24.7621	27.2861	27.1852	24.7621	22.6599
$\underline{x}+10$	24.6234	27.8606	31.0440	30.7298	27.8606	25.3837
$\underline{x}+15$	27.6163	31.5061	35.3581	34.8562	31.5061	28.6223
$\underline{x}+20$	31.0442	35.5901	40.1062	39.4434	35.5901	32.2777

Table 2: **The sensitivity of optimal consumption (top panel) and risky portfolio (bottom panel) strategies to changes in investment opportunity.** Default parameter values: $\delta = 0.10$, $\beta = 0.0371$, $r = 0.0371$, $\gamma = 2$, $\epsilon_L = 0.25$, $y = 1.5$, and $\epsilon_H = 2.5$.

x	μ			σ		
	0.1023	0.1123	0.1223	0.1854	0.1954	0.2054
$x^* + 5$	8.9447	11.0834	12.9441	11.8534	11.0834	10.2833
$x^* + 10$	9.9501	12.3184	14.5741	13.2528	12.3184	11.4517
$x^* + 15$	10.8978	13.5411	16.1320	14.6036	13.5411	12.5744
$x^* + 20$	11.8029	14.7287	17.6173	15.9104	14.7287	13.6564

Table 3: **The sensitivity of hedging effect $HE(x)$ to changes in investment opportunity.** Default parameter values: $\delta = 0.10$, $\beta = 0.0371$, $r = 0.0371$, $\gamma = 2$, $\epsilon_L = 0.25$, $y = 1.5$, and $\epsilon_H = 2.5$.

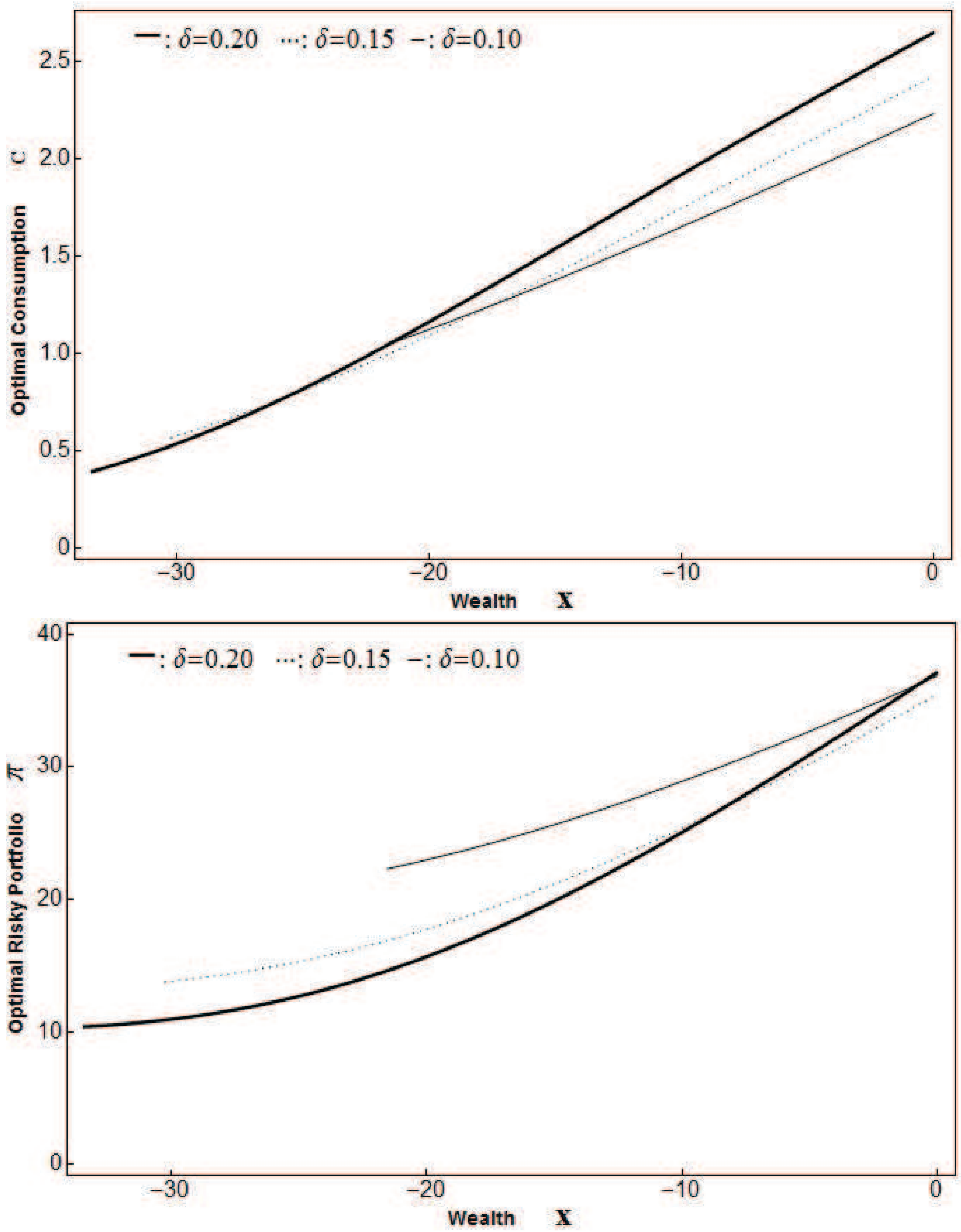

Figure 1: **Optimal consumption and risky portfolio strategies which are functions of initial wealth x**. Default parameter values: $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\gamma = 2$, $\epsilon_L = 0.25$, $y = 1.5$, and $\epsilon_H = 2.5$.

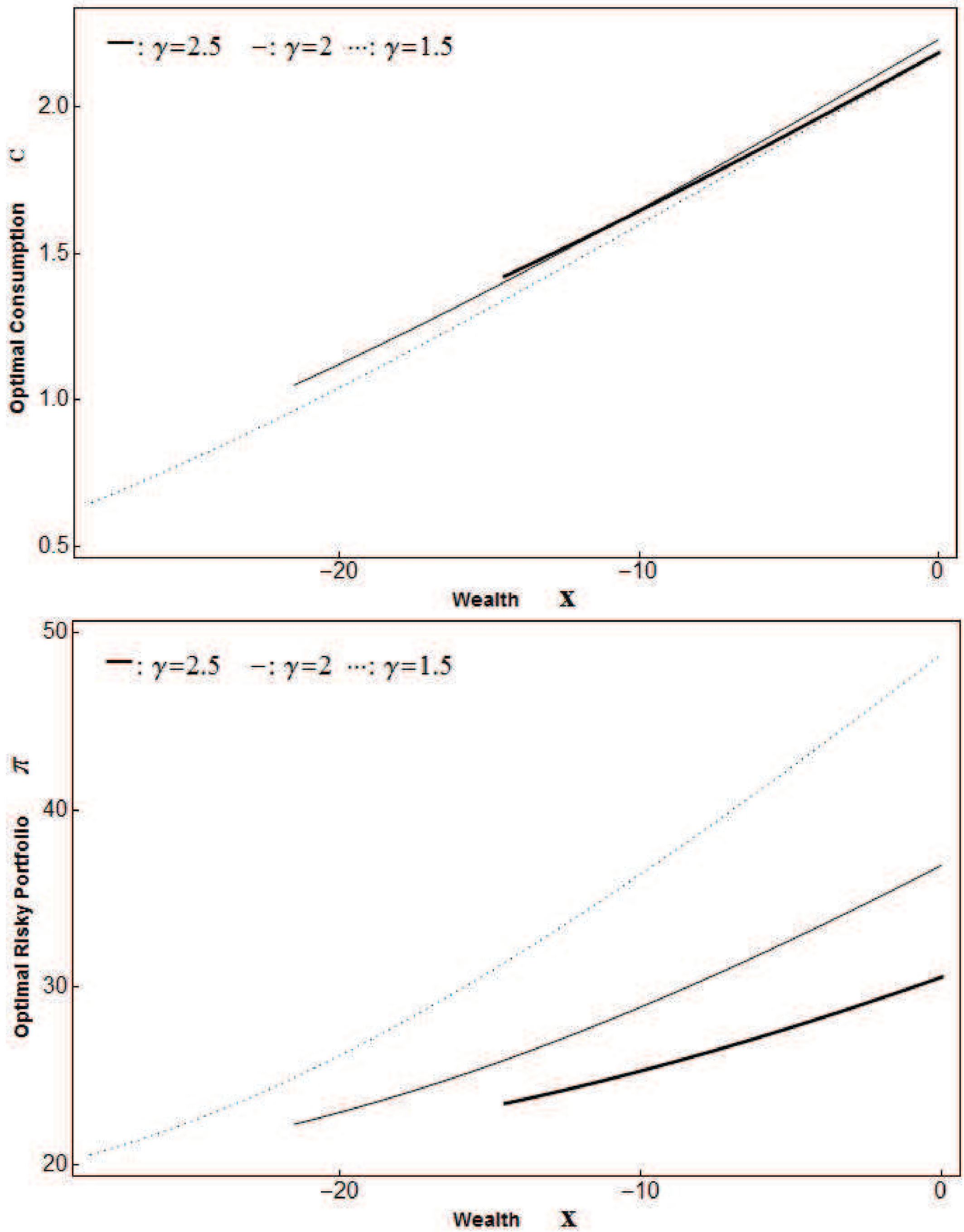

Figure 2: **Optimal consumption and risky portfolio strategies which are functions of initial wealth x**. Default parameter values: $\delta = 0.10$, $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\epsilon_L = 0.25$, $y = 1.5$, and $\epsilon_H = 2.5$.

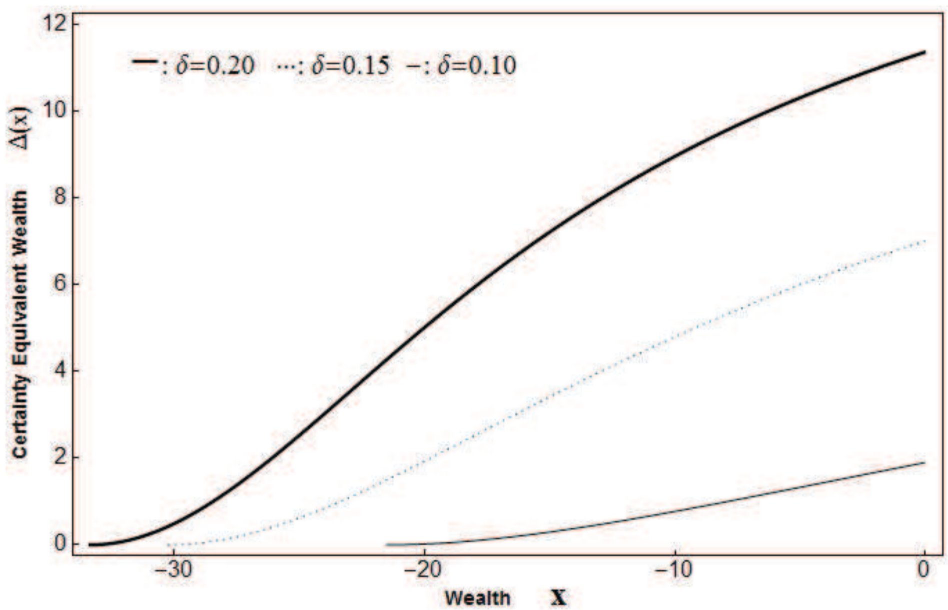

Figure 3: **Certainty equivalent wealth induced by running a risky business as a function of initial wealth x for various values of** δ. Default parameter values: $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\gamma = 2$, $\epsilon_L = 0.25$, $y = 1.5$, and $\epsilon_H = 2.5$..

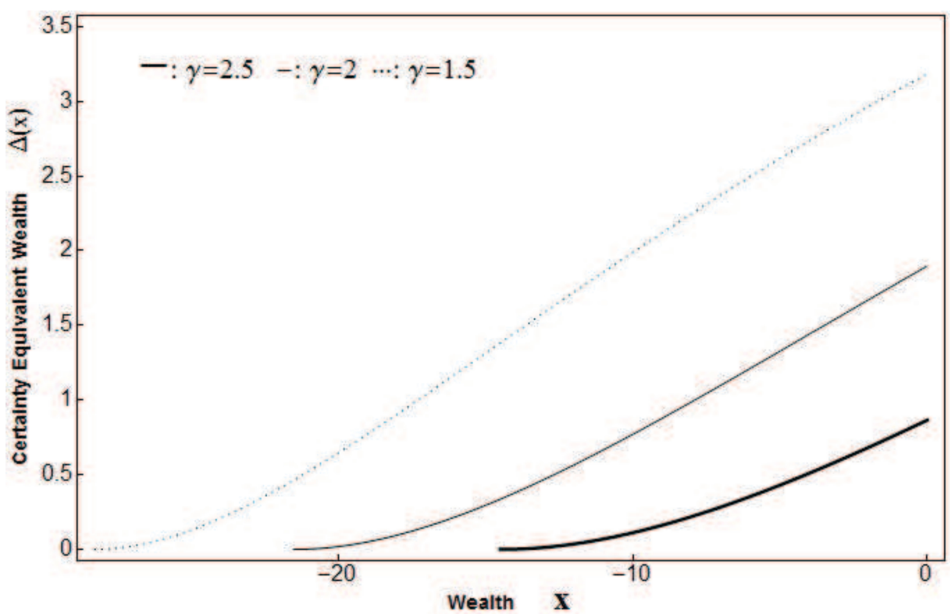

Figure 4: **Certainty equivalent wealth induced by running a risky business as a function of initial wealth x for various values of γ.** Default parameter values: $\delta = 0.10$, $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\epsilon_L = 0.25$, $y = 1.5$, and $\epsilon_H = 2.5$.

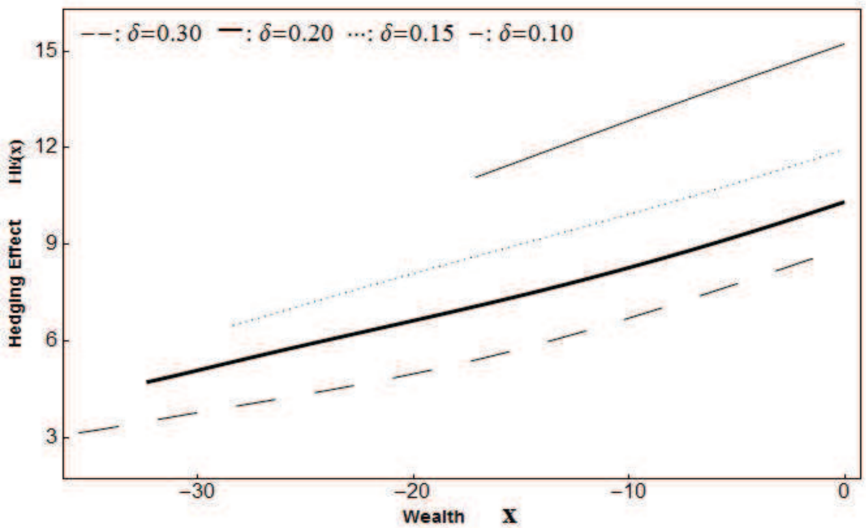

Figure 5: **The hedging effect $HE(x)$ of risky portfolio against undiversifiable idiosyncratic risk as a function of initial wealth x for various values of δ.** Default parameter values: $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\gamma = 2$, $\epsilon_L = 0.25$, $y = 1.5$, and $\epsilon_H = 2.5$.

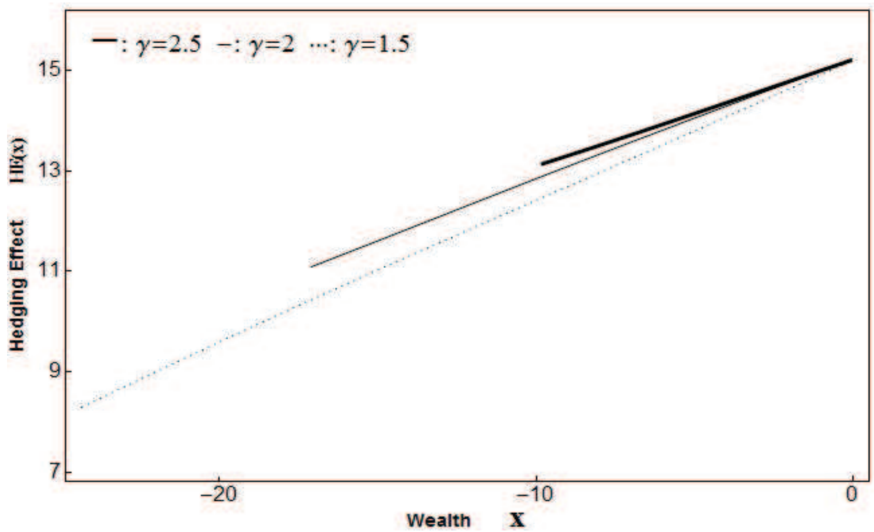

Figure 6: **The hedging effect $HE(x)$ of risky portfolio against undiversifiable idiosyncratic risk as a function of initial wealth x for various values of γ.** Default parameter values: $\delta = 0.10$, $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\epsilon_L = 0.25$, $y = 1.5$, and $\epsilon_H = 2.5$.