Partially Reversible Capital Investment under Demand Ambiguity*

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Abstract

This paper investigates a firm's partially reversible capital investment problem when output demand is ambiguous. We adopt the Choquet–Brownian motion process to incorporate the ambiguity of demand. To solve the firm's problem, we formulate it as a singular stochastic control problem. Then, we use variational inequalities and derive the optimal investment strategy. It is described by two thresholds that respectively determine the capital expansion and reduction. Furthermore, we obtain useful insights for the firm's investment decision-making through a comparative static analysis. We find that higher volatility and ambiguity aversion discourage capital investment.

Keywords: capital investment; ambiguity; Choquet–Brownian motion; singular stochastic control; variational inequalities

1 Introduction

Capital investment has been a central issue in a firm's decision-making for decades, as it has an impact on firm value. Investment decision-making is influenced by future economic conditions. Thus, it is important for the firm's manager to manage the effects of uncertainty caused by future economic conditions. In particular, output demand/price uncertainty is a major uncertainty, and its effects have been investigated by many researchers. As representative studies, Hartman (1972), Able (1983), and Abel and Eberly (1994) investigated the effects of output price uncertainty on a firm's capital investment. They formulated a competitive firm's capital investment problem, which is to maximize the expected present value of operating profits minus total investment costs; that is, to maximize firm value by choosing the capital investment rate in each period. They showed that output price uncertainty promotes capital investment. See also Caballero (1991) for a discussion of capital investment under uncertainty.

Another important factor for capital investment is irreversibility, which comes from the sunk cost associated with investment. It is natural that irreversibility discourages capital investment.

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Real options analysis reveals that uncertainty postpones investment timing when the investment is irreversible. Because of irreversibility, a firm becomes more cautious about investing in capital under uncertainty. See Dixit and Pindyck (1994) for more detail on real options analysis.

The concept of uncertainty has received considerable attention from researchers. One of the most famous distinctions in this field is by Knight (1921), who provided separate definitions for risk and uncertainty. The probability of an outcome is not uniquely determined under uncertainty, whereas under risk, it is. The latter is termed Knightian uncertainty or ambiguity. In this paper, we use the term "ambiguity." See, for example, Camerer and Weber (1992), Etner et. al. (2012) and Guidolin and Rinaldi (2013) for a survey of decision-making under ambiguity. Nishimura and Ozaki (2007), Trojanowska and Kort (2010), Wang (2010), and Thijssen (2011) explored the irreversible capital investment problem under ambiguity.

In another line of the capital investment literature, many researchers have investigated the case in which the firm can invest in capital as needed. See, for example, Kobila (1993), Bertola and Caballero (1994), Chiarolla and Haussmann (2005), and Motairi and Zervos (2017). They investigated the firm's capital expansion problem under uncertainty. To solve this problem, they formulated it as a singular stochastic control problem. Furthermore, if there exists a secondary capital market or if the capital can be sold to another firm directly, investment expenditure is only partially irreversible. For partially irreversible investment, the firm's capital investment problem can be extended as a capital expansion and reduction problem. See, for example, Abel and Eberly (1996), Guo and Pham (2005), Merhi and Zervos (2007), Angelis and Ferrari (2014) and Federico and Pham (2014). They also formulated the firm's problem as a singular stochastic control problem. The capital expansion problem requires one boundary for investing, while the capital expansion and reduction problem requires two boundaries for investing.

This paper explores a firm's capital investment problem when future output demand is ambiguous. To this end, we formulate the firm's problem as a singular stochastic control problem. In this paper, output demand ambiguity is expressed by Choquet–Brownian motion, which was developed by Kast and Lapied (2010) and Kast et al. (2014). The previous studies mentioned above adopted the framework of κ-ignorance developed by Chen and Epstein (2002) in order to incorporate ambiguity. In this framework, the ambiguity affects only the drift terms of the associated processes. The framework provided by Kast et al., however, affects both the drift terms and diffusion terms of the associated processes. We solve the firm's problem by using variational inequalities and derive the optimal investment strategy. It is described by two thresholds that respectively determine the capital expansion and reduction. Furthermore, we conduct a comparative static analysis on some parameters. We obtain the following two main findings: higher volatility of output demand discourages capital investment; if the firm's manager is more ambiguity averse, capital investment is also postponed.

The rest of the paper is organized as follows. Section 2 describes the setup of the firm's problem. Section 3 solves the firm's problem. We present the numerical analysis in Section 4. Section 5 concludes the paper.

2 The Model

Suppose that a firm produces a single output using capital K and sells it in a competitive market. The firm's manager faces ambiguity in output demand X. To analyze the ambiguity, we adopt the framework developed by Kast and Lapied (2010) and Kast et al. (2014). The dynamics of

X is governed by the following stochastic differential equation:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t^c, \quad X_{0^-} = x > 0, \tag{2.1}$$

where $\mu > 0$ and $\sigma > 0$ are constants. W_t^c is a generalized Wiener process with mean $m = 2c - 1$ and variance $s^2 = 4c(1-c)$:

$$\mathrm{d}W_t^c = m\mathrm{d}t + s\mathrm{d}W_t, \tag{2.2}$$

where W_t is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, where $\mathcal{F}_t$ is generated by W_t. c $(0 < c < 1)$ is the constant conditional Choquet capacity, which indicates the firm manager's attitude toward ambiguity. Then, output demand follows a Choquet–Brownian motion as follows:

$$\mathrm{d}X_t = (\mu + m\sigma)X_t \mathrm{d}t + s\sigma X_t \mathrm{d}W_t, \quad X_{0^-} = x > 0. \tag{2.3}$$

If $c < \frac{1}{2}$ (resp. $c > \frac{1}{2}$), the firm manager is ambiguity averse (resp. ambiguity loving). If $c = \frac{1}{2}$, then $m = 0$ and $s = 1$. This means that $\mathrm{d}W_t^c = \mathrm{d}W_t$ and ambiguity disappears and the firm manager has perfect confidence in the dynamics of X. The firm expands its business by accumulating capital, while the firm reduces its business by reducing capital. Let I_t^+ and I_t^- be the cumulative expansion and reduction of capital until t, respectively. They are right-continuous with left-hand limited adapted processes, nonnegative and nondecreasing, with $I_{0-}^+ = 0$ and $I_{0-}^- = 0$. The firm can purchase capital at the constant price $p > 0$ and sell it at the constant price $(1-\lambda)p > 0$, $\lambda \in (0,1)$. Then the dynamics of the capital are given by:

$$\mathrm{d}K_t = -\delta K_t \mathrm{d}t + \mathrm{d}I_t^+ - \mathrm{d}I_t^-, \quad K_{0^-} = k\ (>0), \tag{2.4}$$

where $\delta \in (0,1)$ is a constant depreciation rate. The parameter λ represents the degree of irreversibility of investment. The investment expenditure is a completely sunk cost, if λ goes to 1.

The firm's operating profit $\hat{\pi}$ at time t is given by:

$$\hat{\pi}(K_t, X_t) = K_t^\alpha X_t^\beta, \tag{2.5}$$

where $\alpha \in (0,1)$, $\beta > 0$. The firm's expected discounted profit $J(k, x; I^+, I^-)$ is given by:

$$\hat{J}(k, x; I^+, I^-) = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\hat{\pi}(K_t, X_t)\mathrm{d}t - p\int_0^\infty \mathrm{e}^{-rt}\mathrm{d}I_t^+ + (1-\lambda)p\int_0^\infty \mathrm{e}^{-rt}\mathrm{d}I_t^-\right], \tag{2.6}$$

where $r > 0$ is a discount rate, $I^+ := \{I_t^+\}_{t\geq 0}$ and $I^- := \{I_t^-\}_{t\geq 0}$ and the pair (I^+, I^-) is an investment strategy. The investment strategy is admissible when $(I^+, I^-) \in \mathcal{A}$, where $\mathcal{A}$ is the set of all admissible investment strategies. In this context, it is assumed that:

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\hat{\pi}(K_t, X_t)\mathrm{d}t\right] < \infty, \tag{2.7}$$

and

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\mathrm{d}I_t^+\right] < \infty. \tag{2.8}$$

Therefore, the firm's problem is to maximize the expected discounted profit over $\mathcal{A}$:

$$\hat{V}(k, x) = \sup_{(I^+, I^-)\in\mathcal{A}} \hat{J}(k, x; I^+, I^-) = \hat{J}(k, x; I^{+*}, I^{-*}), \tag{2.9}$$

where $\hat{V}$ is the value function and (I^{+*}, I^{-*}) is the optimal investment strategy. This firm's problem (2.9) is formulated as a singular stochastic control problem.

3 Variational Inequalities

From the formulation of the firm's problem (2.9), under an optimal investment strategy, we assume the following investment strategy for the firm. The firm maintains the capital stock level within a given region, so that whenever the capital stock is below (resp. above) the lower (resp. upper) boundary, the firm invests in (reduces) the capital stock. Note that the boundaries of the capital stock K depend on the demand level X. Then, the x-k plane is divided into three regions: continuation region, capital expansion region, and capital reduction region. In this paper, we change variables and replace the ratio of k and x with y for analytical tractability. Consequently, the three regions are defined by two thresholds, $\underline{y}$ and $\bar{y}$. In order to verify this conjecture regarding the investment strategy, we solve the firm's problem (2.9) using variational inequalities.

The variational inequalities of the firm's problem (2.9) are given as follows.

Definition 3.1 (Variational Inequalities)**.** *The following relations are called the variational inequalities for the firm's problem (2.9):*

$$\hat{\mathcal{L}}\hat{V}(k,x) + \hat{\pi}(k,x) \leq 0, \tag{3.1}$$

$$\hat{V}_K(k,x) \leq p, \tag{3.2}$$

$$\hat{V}_K(k,x) \geq (1-\lambda)p, \tag{3.3}$$

$$[\hat{\mathcal{L}}\hat{V}(k,x) + \hat{\pi}(k,x)][\hat{V}_K(k,x) - p][(1-\lambda)p - \hat{V}(k,x)] = 0, \tag{3.4}$$

where $\hat{\mathcal{L}}$ is the operator defined by:

$$\hat{\mathcal{L}} := -\delta k \frac{\partial}{\partial K} + (\mu + m\sigma)x\frac{\partial}{\partial X} + \frac{1}{2}s^2\sigma^2x^2\frac{\partial^2}{\partial X^2} - r. \tag{3.5}$$

See, for example, Harrison and Taksar (1983) and Merhi and Zervos (2007) for derivation of variational inequalities. The variational inequalities are summarized as:

$$\max\left\{\hat{\mathcal{L}}\hat{V}(k,x) + \hat{\pi}(k,x), \hat{V}_K(k,x) - p, (1-\lambda)p - \hat{V}_K(k,x)\right\} = 0. \tag{3.6}$$

Let $\hat{\mathcal{H}}$ be the continuation region given by:

$$\hat{\mathcal{H}} := \{(k,x); (1-\lambda)p < \hat{V}_K(k,x) < p, \hat{\mathcal{L}}\hat{V}(k,x) + \hat{\pi}(k,x) = 0\}. \tag{3.7}$$

In what follows, for tractability, we assume that $\beta = 1-\alpha$ as in Abel and Eberly (1996) and change variables as $Y_t := K_t/X_t$. Then, the profit function and the value function, respectively, can be rewritten as follows:

$$\hat{\pi}(K_t, X_t) = K_t^\alpha X_t^{1-\alpha} = Y_t^\alpha X_t = \pi(Y_t)X_t. \tag{3.8}$$

$$\hat{V}(k,x) = x\hat{V}\left(\frac{k}{x}, 1\right) = xV(y). \tag{3.9}$$

The dynamics of the variable Y are calculated as:

$$\mathrm{d}Y_t = -(\mu + m\sigma + \delta - s^2\sigma^2)Y_t\mathrm{d}t - s\sigma Y_t\mathrm{d}W_t + \mathrm{d}\zeta_t^+ - \mathrm{d}\zeta_t^-, \quad Y_{0-} = y\ (>0), \tag{3.10}$$

where $\mathrm{d}\zeta_t^{\pm} := \frac{1}{X_t}\mathrm{d}I_t^{\pm}$.

It follows from (3.9) that we have $\hat{V}_K(k,x) = V'(y)$, $\hat{V}_X(k,x) = V(y) - yV'(y)$ and $\hat{V}_{XX}(k,x) = (y^2/x)V''(y)$. The variational inequalities (3.1)–(3.4) can also be rewritten as:

$$\mathcal{L}V(y) + \pi(y) \le 0, \tag{3.11}$$

$$V'(y) \le p, \tag{3.12}$$

$$V'(y) \ge (1-\lambda)p, \tag{3.13}$$

$$[\mathcal{L}V(y) + \pi(y)][V'(y) - p][(1-\lambda)p - V'(y)] = 0, \tag{3.14}$$

where $\mathcal{L}$ is the operator defined by:

$$\mathcal{L} := -(\delta + \mu + m\sigma)y\frac{\mathrm{d}}{\mathrm{d}y} + \frac{1}{2}s^2\sigma^2y^2\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (r - \mu - m\sigma). \tag{3.15}$$

The continuation region (3.7) can be rewritten as:

$$\mathcal{H} := \{y; (1-\lambda)p < V'(y) < p, \mathcal{L}V(y) + \pi(y) = 0\}. \tag{3.16}$$

The capital expansion region $\mathcal{E}$ and the capital reduction region $\mathcal{R}$, respectively, are given by:

$$\mathcal{E} := \{y; (1-\lambda)p < V'(y), V'(y) = p, \mathcal{L}V(y) + \pi(y) < 0\}, \tag{3.17}$$

$$\mathcal{R} := \{y; (1-\lambda)p = V'(y), V'(y) < p, \mathcal{L}V(y) + \pi(y) < 0\}. \tag{3.18}$$

The following lemma is the well-known Skorohod Lemma, which implies that Y^* is a reflected diffusion at the boundaries, $\underline{y}$ and $\bar{y}$, and the nondecreasing process $\zeta^{\pm *}$ is the local time of Y^* at the boundaries.

Lemma 3.1. *For any $y > 0$ and given the boundaries $\underline{y}$ and $\bar{y}$ with $0 < \underline{y} < \bar{y}$, there exists a unique cadlag adapted process $Y^* = \{Y_t^*\}_{t \ge 0}$ and nondecreasing process $\zeta^{\pm *}$ satisfying the following Skorohod problem:*

$$\mathrm{d}Y_t^* = -(\mu + m\sigma + \delta - s^2\sigma^2)Y_t^*\mathrm{d}t - s\sigma Y_t^*\mathrm{d}W_t + \mathrm{d}\zeta_t^{+*} - \mathrm{d}\zeta_t^{-*}, \quad Y_{0-}^* = y\ (>0), \tag{3.19}$$

$$Y_t^* \in [\underline{y}, \bar{y}] \quad a.e., \quad t \ge 0, \tag{3.20}$$

$$\int_0^t \mathbf{1}_{\{Y_s^* > \underline{y}\}}\mathrm{d}\zeta_s^{+*} = 0 \ \text{and} \ \int_0^t \mathbf{1}_{\{Y_s^* < \bar{y}\}}\mathrm{d}\zeta_s^{-*} = 0. \tag{3.21}$$

Furthermore, if $y \in [\underline{y}, \bar{y}]$, then $\zeta^{\pm *}$ is continuous. If $y < \underline{y}$ (resp. $y > \bar{y}$), then $\zeta_0^{+*} = \underline{y} - y$ (resp. $\zeta_0^{-*} = y - \bar{y}$) and $Y_0^* = \underline{y}$ (resp. $Y_0^* = \bar{y}$).

Proof. See Rogers and Williams (2000, pp. 117-118). □

The condition (3.21) means that $\zeta^{\pm *}$ increases only when Y^* reaches $\underline{y}$ or $\bar{y}$. Then, the continuation region, the capital expansion region, and the capital reduction region, respectively, are replaced as follows:

$$\mathcal{H} := \{y; \underline{y} < y < \bar{y}\}, \quad \mathcal{E} := \{y; y \le \underline{y}\}, \quad \mathcal{R} := \{y; y \ge \bar{y}\}. \tag{3.22}$$

Let $\phi \in C^2$ be a function, and let $T < \infty$ be a stopping time. From the Ito formula for cadlag semimartingales, we have:

$$\begin{aligned}\mathrm{e}^{-rT} X_T \phi(Y_T) =& x\phi(y) + \int_0^T \mathrm{e}^{-rT} X_t \mathcal{L}\phi(Y_t)\mathrm{d}t + \int_0^T \mathrm{e}^{-rT} X_t[s\sigma\phi(Y_t) - s\sigma Y_t \phi'(Y_t)]\mathrm{d}W_t \\ &+ \int_0^T \mathrm{e}^{-rT}\phi'(Y_t)[\mathrm{d}I_t^{+c} - \mathrm{d}I_t^{-c}] + \sum_{0\le t\le T} \mathrm{e}^{-rT}[\phi(Y_t) - \phi(Y_{t-})].\end{aligned} \tag{3.23}$$

where $I_t^{\pm c} = I_t^{\pm} - \sum_{0\le u\le t} \Delta I_s^{\pm}$ are the continuous and discontinuous parts of $I^{\pm}$. We are now in a position to prove that a solution to variational inequalities is optimal. The following theorem is the well-known verification theorem. We prove the theorem by following Pham (2006, Proposition 1.3.1) and Yang and Liu (2004, Theorem 1) in Appendix A.

Theorem 3.1. *(I) Let ϕ be a solution of the variational inequalities (3.11)–(3.14) that satisfies the following:*

$$\lim_{t\to\infty} \mathrm{e}^{-rt}\phi(Y_t) = 0. \tag{3.24}$$

Then, we obtain

$$\phi(y) \ge V(y), \qquad y > 0. \tag{3.25}$$

(II) ϕ also satisfies the following:

$$\mathcal{L}\phi(y) + \pi(y) = 0, \qquad \underline{y} < y < \bar{y}, \tag{3.26}$$

$$\phi(y) = p(\underline{y} - y) + d^+, \qquad y \le \underline{y}, \tag{3.27}$$

$$\phi(y) = (1-\lambda)p(y - \bar{y}) + d^-, \qquad y \ge \bar{y}, \tag{3.28}$$

where $d^{\pm}$ are constants. Then, there exists an optimal policy $(\zeta^{+*}, \zeta^{-*}) \in \mathcal{A}$ such that

$$\phi(y) = V(y). \tag{3.29}$$

That is, ϕ is the value function, and (ζ^{+*}, ζ^{-*}) is the corresponding optimal policy.

Proof. See Appendix A. □

For $y \in \mathcal{H}$, the variational inequalities (3.11)–(3.14) lead to the following ordinary differential equation:

$$\frac{1}{2}s^2\sigma^2 y^2\phi''(y) - (\mu + m\sigma + \delta)y\phi'(y) - (r - \mu - m\sigma)\phi(y) + y^{\alpha} = 0. \tag{3.30}$$

The general solution of the homogeneous part of (3.30) is given by:

$$\phi(y) = A_1 y^{\gamma_1} + A_2 k^{\gamma_2}, \qquad y \in \mathcal{H}, \tag{3.31}$$

where A_1 and A_2 are constants to be determined. γ_1 and γ_2 are the solutions to the following characteristic equation:

$$\frac{1}{2}s^2\sigma^2\gamma^2 - \left(\mu + m\sigma + \delta + \frac{1}{2}s^2\sigma^2\right)\gamma - (r - \mu - m\sigma) = 0. \tag{3.32}$$

They are calculated with:

$$\begin{aligned}\gamma_1 &= \frac{1}{2} + \frac{\mu + m\sigma + \delta}{s^2\sigma^2} + \left[\left(\frac{1}{2} + \frac{\mu + m\sigma + \delta}{s^2\sigma^2}\right)^2 + \frac{2(r - \mu - m\sigma)}{s^2\sigma^2}\right]^{\frac{1}{2}} > 1,\\ \gamma_2 &= \frac{1}{2} + \frac{\mu + m\sigma + \delta}{s^2\sigma^2} - \left[\left(\frac{1}{2} + \frac{\mu + m\sigma + \delta}{s^2\sigma^2}\right)^2 + \frac{2(r - \mu - m\sigma)}{s^2\sigma^2}\right]^{\frac{1}{2}} < 0.\end{aligned} \tag{3.33}$$

The particular solution of (3.30), however, is calculated as the expected discounted value of the profit function $\pi(y)$ when the firm will not invest in capital forever:

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\pi(Y_t)\mathrm{d}t\right] = \frac{y^\alpha}{\rho}, \tag{3.34}$$

where $\rho := r + (\mu + m\sigma + \delta)\alpha + \frac{1}{2}s^2\sigma^2\alpha(1-\alpha)$. It follows from the assumption (2.7) that $\rho > 0$. The general solution of (3.30) is:

$$\phi(y) = A_1 y^{\gamma_1} + A_2 y^{\gamma_2} + \frac{y^\alpha}{\rho}, \qquad y \in \mathcal{H}. \tag{3.35}$$

It follows from the definition of the firm's problem that the function ϕ satisfies the following inequality:

$$\phi(y) > \frac{y^\alpha}{\rho}. \tag{3.36}$$

The first and second terms of (3.35) represent the option value to invest in capital or to reduce capital. This implies that both constants to determine A_1 and A_2 must be positive.

Let ϕ be redefined as a candidate function of the value function and be given by:

$$\phi(y) = \begin{cases} \psi(\underline{y}) - p(\underline{y} - y), & y \le \underline{y},\\ \psi(y) := A_1 y^{\gamma_1} + A_2 y^{\gamma_2} + \frac{y^\alpha}{\rho}, & y \in (\underline{y}, \bar{y}),\\ \psi(\bar{y}) + (1-\lambda)p(y - \bar{y}), & y \ge \bar{y}. \end{cases} \tag{3.37}$$

Four unknowns A_1, A_2, $\underline{y}$ and $\bar{y}$ are determined by the following simultaneous equations:

$$\psi'(\underline{y}) = p, \tag{3.38}$$

$$\psi'(\bar{y}) = (1-\lambda)p, \tag{3.39}$$

$$\psi''(\underline{y}) = 0, \tag{3.40}$$

$$\psi''(\bar{y}) = 0. \tag{3.41}$$

Equations (3.38) and (3.39) are the smooth-pasting conditions, and equations (3.40) and (3.41) are the super contact conditions (see Dumas (1991) for details).

It follows from (3.38) and (3.40) that we have the following two equations:

$$\gamma_1(\gamma_2 - \gamma_1)A_1\underline{y}^{\gamma_1 - 1} + \frac{\alpha(\gamma_2 - \alpha)\underline{y}^{\alpha-1}}{\rho} = p(\gamma_2 - 1), \tag{3.42}$$

$$\gamma_2(\gamma_1-\gamma_2)A_2\underline{y}^{\gamma_2-1}+\frac{\alpha(\gamma_1-\alpha)\underline{y}^{\alpha-1}}{\rho}=p(\gamma_1-1). \tag{3.43}$$

Similarly, from (3.39) and (3.41), we have the following two equations:

$$\gamma_1(\gamma_2-\gamma_1)A_1\bar{y}^{\gamma_1-1}+\frac{\alpha(\gamma_2-\alpha)\bar{y}^{\alpha-1}}{\rho}=(1-\lambda)p(\gamma_2-1), \tag{3.44}$$

$$\gamma_2(\gamma_1-\gamma_2)A_2\bar{y}^{\gamma_2-1}+\frac{\alpha(\gamma_1-\alpha)\bar{y}^{\alpha-1}}{\rho}=(1-\lambda)p(\gamma_1-1). \tag{3.45}$$

From (3.42) and (3.44), we have:

$$p(\gamma_2-1)\left[\left(\frac{\bar{y}}{\underline{y}}\right)^{\gamma_1}-(1-\lambda)\left(\frac{\bar{y}}{\underline{y}}\right)\right]=\frac{\alpha(\gamma_2-\alpha)\underline{y}^{\alpha-1}}{\rho}\left[\left(\frac{\bar{y}}{\underline{y}}\right)^{\gamma_1}-\left(\frac{\bar{y}}{\underline{y}}\right)^{\alpha}\right]. \tag{3.46}$$

We follow the method of Guo and Pham (2005, Section 6.3) and put $z:=\frac{\bar{y}}{\underline{y}}$ (>1). Then, (3.46) can be rewritten as:

$$p(\gamma_2-1)\left[z^{\gamma_1}-(1-\lambda)z\right]=\frac{\alpha(\gamma_2-\alpha)\underline{y}^{\alpha-1}}{\rho}\left[z^{\gamma_1}-z^{\alpha}\right]. \tag{3.47}$$

Similarly, from (3.43) and (3.45), we obtain that:

$$p(\gamma_1-1)\left[z^{\gamma_2}-(1-\lambda)z\right]=\frac{\alpha(\gamma_1-\alpha)\underline{y}^{\alpha-1}}{\rho}\left[z^{\gamma_2}-z^{\alpha}\right]. \tag{3.48}$$

We solve the equations (3.47) and (3.48) in terms of $\frac{\alpha\underline{y}^{\alpha-1}}{\rho}$ and obtain an equation for z:

$$F(z):=\frac{\Gamma_2 z^{\gamma_1-1}(z^{\gamma_2}-z^{\alpha})-\Gamma_1 z^{\gamma_2-1}(z^{\gamma_1}-z^{\alpha})}{\Gamma_2(z^{\gamma_2}-z^{\alpha})-\Gamma_1(z^{\gamma_1}-z^{\alpha})}=1-\lambda, \tag{3.49}$$

where $\Gamma_1:=(\gamma_1-1)(\gamma_2-\alpha)$ and $\Gamma_2:=(\gamma_2-1)(\gamma_1-\alpha)$. It is obvious that:

$$\lim_{z\to1}F(z)=1, \qquad \lim_{z\to\infty}F(z)=0. \tag{3.50}$$

Equations (3.49) and (3.50) mean that $z_\lambda>1$ such that (3.49) holds. Therefore, the four unknowns—A_1, A_2, $\underline{y}$ and $\bar{y}$—can be expressed in the following forms:

$$A_1=\frac{\underline{y}^{1-\gamma_1}}{\gamma_1(\gamma_2-\gamma_1)}\left[p(\gamma_2-1)-\frac{\alpha(\gamma_2-\alpha)}{\rho}\underline{y}^{\alpha-1}\right], \tag{3.51}$$

$$A_2=\frac{\underline{y}^{1-\gamma_2}}{\gamma_2(\gamma_1-\gamma_2)}\left[p(\gamma_1-1)-\frac{\alpha(\gamma_1-\alpha)}{\rho}\underline{y}^{\alpha-1}\right], \tag{3.52}$$

$$\underline{y}=\left[\frac{\rho(\gamma_1-1)(z_\lambda^{\gamma_2}-(1-\lambda)z_\lambda)}{\alpha(\gamma_1-\alpha)(z_\lambda^{\gamma_2}-z_\lambda^{\alpha})}p\right]^{\frac{1}{\alpha-1}}, \tag{3.53}$$

$$\bar{y}=z_\lambda\underline{y}. \tag{3.54}$$

We will solve (3.49) numerically and obtain the four unknowns in the next section.

4 Numerical Analysis

In this section, we numerically calculate the four unknowns—A_1, A_2, $\underline{y}$ and $\bar{y}$—and investigate the effects of changes in the parameters on the thresholds, $\underline{y}$ and $\bar{y}$. The basic parameter values are set out as follows: $r = 0.05$, $\delta = 0.1$, $\mu = 0.01$, $\sigma = 0.15$, $\alpha = 0.6$, $p = 10$, $\lambda = 0.5$, and $c = 0.4$. Then, we obtain $A_1 = 0.00014$, $A_2 = 0.49589$, $\underline{y} = 0.31654$, and $\bar{y} = 4.13044$. Figure 1 illustrates the continuation, capital investment, and capital reduction regions, $\mathcal{H}$, $\mathcal{E}$ and $\mathcal{R}$ in the x-k plane, respectively.

We provide the results of the comparative static analysis of the thresholds, $\underline{y}$ and $\bar{y}$, in Figures 2–6. Figure 2 shows that the reduction region $\mathcal{R}$ is decreasing in the volatility of demand, σ, while the continuation region $\mathcal{H}$ and the capital investment region $\mathcal{E}$ are increasing in the volatility. Overall, this result implies that the incentive to wait for new information about demand becomes stronger as the volatility increases. This is consistent with the standard result of real options analysis.

Next, Figure 3 illustrates the impact of the manager's attitude toward the ambiguity regarding investment decision-making. If the firm's manager is more ambiguity averse, the ambiguity attitude parameter c is smaller. Figure 3 shows that if the firm's manager is more ambiguity averse, the capital reduction region is smaller, while the continuation region is larger and the capital expansion region is slightly larger. Overall, the firm's manager has an incentive to wait for new information and is more cautious in investment decision-making. Notice that when $c = 0.5$, the firm's manager is ambiguity neutral. In this case, the four unknown values are $A_1 = 0.00118$, $A_2 = 0.97722$, $\underline{y} = 0.12004$, and $\bar{y} = 1.87207$.

Next, Figure 4 depicts how the output elasticity of capital, α, affects investment decision-making. Recall that output per capita increases in α. Figure 4 shows that the continuation region is increasing in α. The reduction region is decreasing in α, while the capital investment region changes marginally.

Next, Figure 5 explains how the price of capital has an impact on investment decision-making. Figure 5 shows that the price of capital has a negative impact on the continuation region and the capital investment region. However, the price has a positive effect on the capital reduction region.

Finally, Figure 6 shows the impact of the degree of irreversibility of investment, λ, on investment decision-making. If λ goes to 1, the investment expenditure is a completely sunk cost. The numerical example shows that $\underline{y}$ is slightly decreasing in λ, while $\bar{y}$ is increasing in λ. Furthermore, $\underline{y}$ goes to 0.3165 and $\bar{y}$ goes to ∞ as λ goes to 1. These implies that the degree of irreversibility of investment discourages investment. The firm's problem reduces to a capital investment decision-making problem except for selling capital as λ goes to 1.

These results provide useful insights into investment decision-making under demand ambiguity.

5 Conclusion

This paper investigates a firm's partially reversible capital investment problem under ambiguous demand. We express the ambiguous demand as a Choquet–Brownian motion process. To solve the firm's problem, we formulate it as a singular stochastic control problem. Then, we use variational inequalities and derive the optimal investment strategy. It is described by two thresholds that respectively determine the capital expansion and reduction. Furthermore, we obtain useful insights for the firm's investment decision-making through a comparative static analysis. We find that higher volatility and ambiguity aversion discourage capital investment.

To conclude the paper, we suggest some possible extensions to our model. First, we could adopt another type of output demand such as in Caballero (1991), where the firm faces an isoelastic demand function. Next, we could explore the effect of uncertainty about investment costs on the firm's investment decision-making. We also could incorporate a fixed investment cost. In this case, the firm's problem is formulated as an impulse control problem. Finally, we could consider the firm's investment decision-making in a duopoly market. We leave these important topics to future research.

Appendix A.

Proof of Theorem 3.1. (I) For $(\zeta^+, \zeta^-) \in \mathcal{A}$, let $T_n = \inf\{t \geq 0; Y_t \geq n\} \wedge n$, $n \in \mathbb{N}$ be the finite stopping time. We apply (3.23) between $t = 0$ and $t = T_n$ and take expectations. Then, we obtain that:

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-rT_n}\phi(Y_{T_n})] =&\phi(y) + \mathbb{E}\left[\int_0^{T_n} \mathrm{e}^{-rt}\mathcal{L}\phi(Y_t)\mathrm{d}t\right] + \mathbb{E}\left[\int_0^{T_n} \mathrm{e}^{-rt}\phi'(Y_t)[\mathrm{d}\zeta_t^{+c} - \mathrm{d}\zeta_t^{-c}]\right] \\ &+ \mathbb{E}\left[\sum_{0\leq t\leq T_n} \mathrm{e}^{-rt}[\phi(Y_t) - \phi(Y_t^-)]\right],\end{aligned} \tag{A.1}$$

As $p(1-\lambda) \leq \phi'(y) \leq p$ and $Y_t - Y_{t-} = \Delta\zeta_t^+ - \Delta\zeta_t^-$, the mean-value theorem implies that:

$$\phi(Y_t) - \phi(Y_{t-}) = \phi'(\theta)(\Delta\zeta_t^+ - \Delta\zeta_t^-) \leq p\Delta\zeta_t^+ - (1-\lambda)p\Delta\zeta_t^-, \tag{A.2}$$

where $\theta \in (Y_t - Y_{t-})$. It follows from (3.11) that (A.1) can be rewritten as:

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-rT_n}\phi(Y_{T_n})] \leq&\phi(y) - \mathbb{E}\left[\int_0^{T_n} \mathrm{e}^{-rt}\pi(Y_t)\mathrm{d}t\right] \\ &+ \mathbb{E}\left[\int_0^{T_n} \mathrm{e}^{-rt}\left[p\mathrm{d}\zeta_t^{+c} - (1-\lambda)p\mathrm{d}\zeta_t^{-c}\right]\right] \\ &+ \mathbb{E}\left[\sum_{0\leq t\leq T_n} \mathrm{e}^{-rt}\left[p\Delta\zeta_t^+ - (1-\lambda)p\Delta\zeta_t^-\right]\right].\end{aligned} \tag{A.3}$$

Furthermore it follows from $\zeta_t^{\pm c} = \zeta_t^{\pm} - \sum_{0\leq s\leq t}\Delta\zeta_s^{\pm}$ that we have:

$$\mathbb{E}[\mathrm{e}^{-rT_n}\phi(Y_{T_n})] \leq \phi(y) - \mathbb{E}\left[\int_0^{T_n} \mathrm{e}^{-rt}\pi(Y_t)\mathrm{d}t - \int_0^{T_n} \mathrm{e}^{-rt}\left[p\mathrm{d}\zeta_t^+ - (1-\lambda)p\mathrm{d}\zeta_t^-\right]\right]. \tag{A.4}$$

Taking $\lim_{n\to\infty}$ and using (3.24) and the dominated convergence theorem, we obtain that:

$$\phi(y) \geq \mathbb{E}\left[\int_0^{\infty} \mathrm{e}^{-rt}\pi(Y_t)\mathrm{d}t - \int_0^{\infty} \mathrm{e}^{-rt}\left[p\mathrm{d}\zeta_t^+ - (1-\lambda)p\mathrm{d}\zeta_t^-\right]\right] = J(y;\zeta^+,\zeta^-). \tag{A.5}$$

From the arbitrariness of (ζ^+, ζ^-), we have:

$$\phi(y) \geq \sup_{(\zeta^+,\zeta^-)\in\mathcal{A}} J(y;\zeta^+,\zeta^-) = V(y), \tag{A.6}$$

which completes the proof of (I).

(II) For $y \in (\underline{y}, \bar{y})$, $\zeta^{\pm *}$ is continuous and increases only when $y = \underline{y}$ or $\bar{y}$. Then, for $\zeta^{\pm} = \zeta^{\pm *}$ (A.5) becomes equality:

$$\begin{aligned} \phi(y) &= \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\pi(Y_t)\mathrm{d}t - \int_0^\infty \mathrm{e}^{-rt}\left[p\mathrm{d}\zeta_t^+ - (1-\lambda)p\mathrm{d}\zeta_t^-\right]\right] \\ &= J(y;\zeta^{+*},\zeta^{-*}) \\ &= V(y). \end{aligned} \tag{A.7}$$

For $y \leq \underline{y}$, it follows from Lemma 3.1 that we have:

$$\phi(y) = p(\underline{y} - y) + \phi(\underline{y}). \tag{A.8}$$

However, for $y \geq \bar{y}$, we have:

$$\phi(y) = (1-\lambda)p(y - \bar{y}) + \phi(\bar{y}). \tag{A.9}$$

From (A.7) we have $\phi(\underline{y}) = V(\underline{y})$ and $\phi(\bar{y}) = V(\bar{y})$. From the continuous property of $\phi(y)$, $\phi(\underline{y}) = d^+$ and $\phi(\bar{y}) = d^-$. Thus, for all $y \leq \underline{y}$ we have:

$$V(y) = p(\underline{y} - y) + \phi(\underline{y}) = \phi(y). \tag{A.10}$$

for all $y \geq \bar{y}$ we have:

$$V(y) = (1-\lambda)p(y - \bar{y}) + \phi(\bar{y}) = \phi(y). \tag{A.11}$$

This completes the proof of (II).

□

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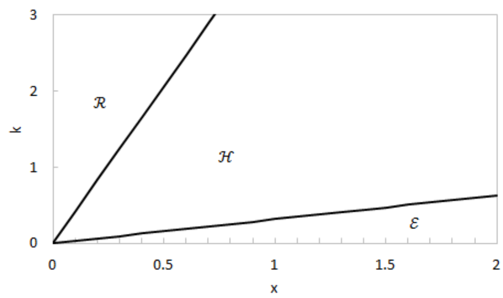

Figure 1: Continuation, expansion, and reduction regions.

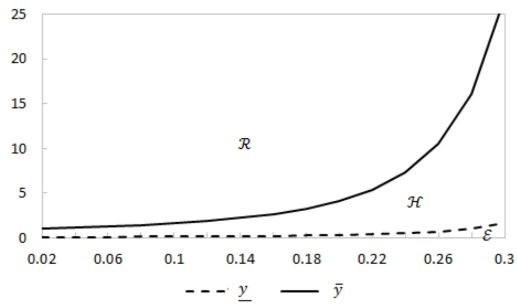

Figure 2: The effect of changing the volatility, σ, on the thresholds $\underline{y}$ and $\bar{y}$.

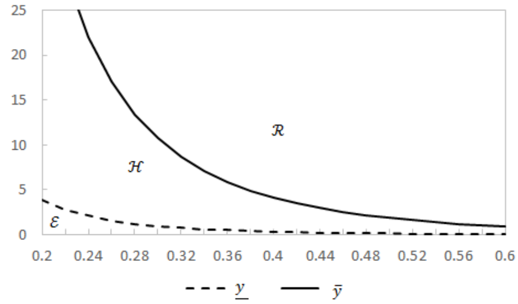

Figure 3: The effect of changing the ambiguity attitude parameter, c, on the thresholds $\underline{y}$ and $\bar{y}$.

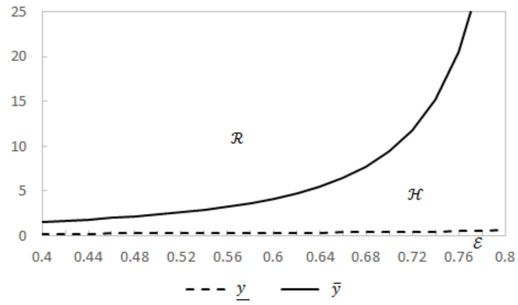

Figure 4: The effect of changing the output elasticity of capital, α, on the thresholds $\underline{y}$ and $\bar{y}$.

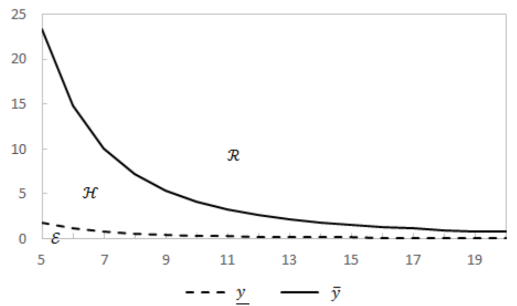

Figure 5: The effect of changing the price of capital, p, on the thresholds $\underline{y}$ and $\bar{y}$.

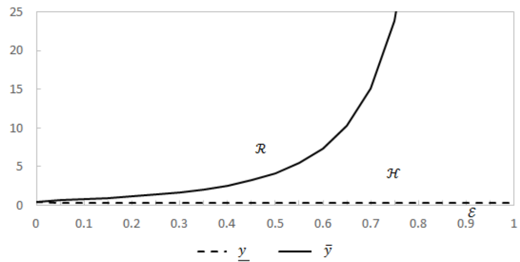

Figure 6: The effect of changing the degree of irreversibility of investment, λ, on the thresholds $\underline{y}$ and $\bar{y}$.