Irreversible Investment under Volatility Ambiguity

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Preliminary and incomplete draft : do not cite, do not distribute

Abstract

This paper studies the investment timing problem of a firm who possesses an irreversible investment option and perceives ambiguity about the volatility of the process driving her payoff. Ambiguous volatility is modeled through a set of probability measures which are non-equivalent, which requires significant departure from the standard theory. We exploit recent advances in non-linear expectation theory and a new theory of stochastic calculus to characterize the optimal investment strategy of the firm. Paralleling the standard theory, we show that the problem of the firm can be charachterized as the solution of a free boundary problem. We show that contrary to risk, ambiguity decreases the value of investment an accelerates investment. This result is also in contrast with previous work focusing on "drift-ambiguity". Extending the model to the case of a competitive equilibrium, we show that the optimality of myopic behavior, which holds in the standard framework, fails under ambiguity.

Keywords: Optimal Stopping;Ambiguity;Real-Options;Continuous-time

JEL Codes: C61, D81, G31

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1 Introduction

In economics and finance, irreversible investment problems are typically treated as real options problems : an investor possesses an investment option and chooses the optimal time to exercise this option. This approach has been fruitfully applied to the study of optimal timing of an irreversible investment in various contexts. Perhaps one of the most important insight from this approach is that uncertainty creates an incentive to wait for the investor, as he can wait for a positive shock to invest while being protected from negative shocks by simply waiting. As a consequence, when valuing the investment opportunity, uncertainty should be taken into account as having a positive value, which invalidates the traditional Net Present Value (NPV) approach. In this literature, uncertainty is usually understood as the volatility of the underlying process and is most of the time captured by a single parameter. In this paper, we consider a stronger notion of uncertainty, namely *volatility ambiguity*. We show that considering this alternative notion of uncertainty can significantly alter some standard results from the irreversible investment literature. Specifically, an increase in uncertainty, in the sense of volatility ambiguity, accelerates investment, contrary to the standard result the uncertainty delays investment.

To do so, we develop a general framework to study real options problem under volatility ambiguity. By volatility ambiguity, we mean that the decision maker only takes a stance on the upper and lower bounds in which the volatility of the underlying process lies, and considers *any* process within these bounds as a possible volatility scenario. This volatility ambiguity is modeled as the decision maker considering a *set* of probability measures on the state space and ambiguity aversion is modeled by the decision maker acting according to the most pessimistic belief in this set for a given prospect, as axiomatized by Gilboa and Schmeidler (1989). Importantly, the set of priors resulting from ambiguous volatility is *non-dominated*, in the sense that the probability measures contained in this set do not agree on null events.

Why consider ambiguous volatility? Virtually all the economics literature on ambiguity in continuous-time focuses on the particular case of drift ambiguity developed by Chen and Epstein (2002) : a reference measure defines the null events and the set of priors is obtained via a Girsanov transformation. This restricts the nature of ambiguity in two ways. First, it implies that ambiguity only concerns the drift of the underlying process. Second, it implies that all measures in the set of priors are mutually equivalent, which has specific implications in terms of preferences, as Epstein and Marinacci (2007) point out. In the "robustness" or "model uncertainty" approach, developped by Hansen and Sargent (2001), model misspecification is taken as an absolutely continuous perturbation of the approximating model, which results in the different models entertained by the decision maker differing only in drift term. This restriction of ambiguity to the drift of the underlying process is motivated by the convenience of working with equivalent probability measures, rather than by economic reasons. In practice forecasting volatility is a difficult task, as it is sensible to data and the chosen forecasting model. The importance of considering stochastic and time-varying volatility in macroeconomics and finance has been underlined by several authors(see for example, Drechsler (2013) and Bloom (2009)). The modelling of stochastic volatility often assumes a specific parametric form, but the dependence of the forecast on the chosen volatility model is high in this case. In order to make a safe, or robust to misspecifications, decision, an investor may want to consider different possible models. In our setting, the investor only takes a stance on the upper and lower bound of the volatility process, and considers *any* volatility model that lies within those bounds. The investment decision (optimal stopping), is made by considering the most pessimistic model. As such, our framework can be considered as a real option framework under robust stochastic volatility.

While there is by now a large body of evidence of ambiguity-averse behavior in a variety of contexts, both experimentally and empirically, the empirical relevance of ambiguous volatility remains an open question. Recent evidence, however, seems to show that ambiguous volatility does matter for investment decisions. For example, Anderson et al. (2017), show that ambiguous volatility is crucial to understand investment decisions of US family owners, and in particular, what they call the "family firm puzzle". Similarly, Branger et al. (2016) construct a measure of ambiguity about macroeconomic volatility and show that an increase in volatility ambiguity predicts a significant increase in average excess equity returns.

The non-equivalence of priors that underlies the modelling of volatility ambiguity raises several technical difficulties. In particular, the standard techniques adopted in the probability space framework are of little use, as the set of priors contains mutually singular measures. Similarly, the definition of a multiple prior conditional expectation is not straightforward. To circumvent these difficulties, we use recent elements of non-linear expectation theory, in particular the G-expectation developed in Peng (2010). Starting from a non-linear expectation space,Peng (2010), develops a non-linear stochastic calculus theory that does not require the probability space framework and that accomodates ambiguous volatility. This framework is particularly useful as most of the tools of standard stochastic calculus and stochastic analysis have their non linear counterpart : the stochastic integral, forward and backward stochastic differential equations, Ito's formula, martingales, stopping times etc.

Our first task is to define a set of priors and a corresponding conditional expectation process under ambiguous volatility. We follow Epstein and Ji (2013), which guarantees that our expectation is dynamically consistent. This property is crucial as it allows to apply the principle of dynamic programming. With a suitable definition of the set of priors and the domain of random variables and stochastic processes, the max-min expecation can be directly linked to Peng's G-expecation. We exploit this link to characterize the value function of an optimal stopping problem when the payoff is driven by a Markovian process. This is done through a verification theorem that holds for stopping problems where the payoff is a function of a geometric (G-)Brownian motion. Using results in nonlinear stochastic calculus theory, we show that the value function of an optimal stopping problem under ambiguous volatility can be identified with the solution of a free boundary problem, analogous to the standard optimal stopping theory. The resulting Hamilton-Jacobi-Bellman equation is similar to the classical one, expect that it is non-linear in its second-order term, reflecting the decision maker's ambiguity aversion.

We then apply these results in the irreversible investment context. We start with the irreversible investment model of McDonald and Siegel (1986), in which a firm has an investment opportunity, and must decide when to invest and pay some fixed cost, in order to start generating profit from the project. Using our HJB equation, we show that, analogous to the standard case, the firm follows a trigger strategy and invests when the cash flow reaches a certain threshold. In addition, the worst-case scenario for the firm is the lowest possible volatility, while the value of the investment once the investment option has been exercized is unaffected by ambiguity. Intuitively, since the firm is protected against downward risk pre-investment, the worst-case is low volatility before the option is exercized. Once the firm has invested, because the firm is risk neutral, volatility does not affect the value of the investment, and ambiguity does not play any role. A first consequence is that an increase in ambiguity erodes the value of investment, by lowering the value of investment before the investment option is exercized, and leaving the value post-investment unaffected. A second consequence is that the investment trigger becomes lower when ambiguity increases. Since the firm uses the prior that yields the lowest volatility, it expects less upward shocks, and the value of waiting decreases. These two results on the effect of ambiguity on the value and timing of irreversible investment contrast with the standard results on the effect of uncertainty (taken in the sense of volatility) in the irreversible investment literature. A consequence of this result is that as ambiguity increase, the optimal strategy of the firm gets closer to the NPV rule.

We then turn to the case where the firm is risk averse. In standard irreversible investment theory, the firm is assumed risk neutral as a consequence of financial market completeness. As several authors have shown, ambiguous volatility precludes the completeness of financial markets, which exposes firms to undiversifiable risk. Considering risk aversion is thus natural in an ambiguous volatility framework. We show that under risk-aversion, the worst-case prior is non-trivial in the sense that the firm anticipates low volatility before investing and high volatility right after the investment option is exercized. Intuitively, when investment is still an option, volatility has a positive value for the firm, as standard real option theory teaches us. But once the firm transforms this option into an asset, volatility has a negative value because the firm is risk averse. The resulting effect of ambiguity on the investment trigger is ambiguous. On the one hand the firm has a low option value, prompting her to invest earlier. On the other hand, investment transforms an option into an anticipated risky asset, which incentivizes the firm to wait. Which effect dominates depends on the power of risk aversion.

We then extend the model to a competitive equilibrium setting. In this setting, a continuum of firms observe a price process that is driven by the demand curve and an exogenous shock. Firms must decide when to enter the market, which will yield one unit of output forever, increasing total industry output. A fundamental result by Leahy (1993), shows that a firm deciding of its investment strategy without taking into account the effect of other firms' entry decisions on the price process("myopically"), actually follows the optimal strategy. We show that this optimality of myopic behavior fails under ambiguity, even in a world where all firms, including the myopic one, share the same set of priors. Computing the competitive equilibrium under ambiguity, we show that it is higher than the monopoly equilibrium under ambiguity (while Leahy (1993)'s result imply that they are the same under pure risk), in addition, the competitive equilibrium entry threshold is increasing in ambiguity, in contrast to the monopoly case. These results hinge on the very nature of the perfectly competitive equilibrium. Free-entry destroys option value

This paper is organized as follows. The next section discusses the related literature. Section 3 presents intuitively the notion of ambiguous volatility and a verification theorem for real otpions problems in this context. In section 4, we treat the monopoly irreversible investment problem, in complete and incomplete markets, and we discuss the irrelevance of lump sum as opposed to flow payoff in this context. In section 5 we extend the model to the perfectly competitive case. In section 6 we discuss some conceptual issues, inherent to the multiple prior model, on the interpretation of the set of priors. The proofs are relegated to the Appendix, where we also provide a self-contained introduction to some results in the G-framework which we use in the proofs, as well as the mathematically detailed construction of the set of priors.

2 Related literature

At the technical level, this paper is related to the literature on optimal stopping under ambiguity. Riedel (2009) and Cheng and Riedel (2013) provide a Snell Envelope characterization, when the set of priors is composed of mutually equivalent measures, in discrete and in continuous time, respectively. The more involved problem of optimal stopping with a set of non-dominated priors has been recently studied by Ekren et al. (2014), Bayraktar and Yao (2014) and Nutz et al. (2015). These papers provide a characterization of the optimal stopping problem in terms of a *non linear* Snell envelope and show that a characterization analogous to the standard one can be obtained. The results in these papers are technically involved and aim at obtaining general characterizations for a large class of optimal stopping problems and a general notion of non-linear expectations (including the G-expectation). Our paper makes the link between these papers and economic theory. We restrict attention to a class of problems which includes most optimal stopping problems in economics, and we show how we can tractably analyze these problems and derive economic implications of volatility ambiguity.

In a more applied literature, Nishimura and Ozaki (2004) study a job search problem under Choquet expected utility and find that ambiguity aversion accelerates acceptance of a job offer. Nishimura and Ozaki (2007) study an irreversible investment problem in continuous-time. In their model, the firm perceives ambiguously the drift of the stream of profit, and they show that an increase in what they call Knightian uncertainty lowers the value of investment and increases the incentives to wait for the firm. Our results complement theirs, by showing that Knightian uncertainty can in fact accelerate investment, if it is taken in the sense of volatility ambiguity. In a general model in discrete time, Miao and Wang (2011) study option exercise under ambiguity. They show, using a specific parameterization, that the role of ambiguity on the value of waiting depends crucially on whether the payoff is a lump sum or a flow of profit. Other papers that apply the drift ambiguity model to irreversible investment include **?** in the α-maxmin framework and **?** in an application to the timing of environmental policies. See also Thijssen (2011) for irreversible investment under incomplete markets and Trojanowska and Kort (2010) for irreversible investment in a finite time horizon. Recently, Epstein and Marinacci (2007) study optimal learning in an Ellsberg environment, where the decision maker is allowed costly sampling from the urn and decides when and how to bet. They show that learning opportunities may be rejected by an ambiguity averse decision maker, even when the cost of sampling is relatively low.

Our paper belongs to a now fairly large literature on ambiguity averse preferences in continuous-time settings. As we have mentioned, the vast majority of this literature considers the case of drift ambiguity. Our paper belongs to a smaller set of papers that study the effect of ambigous volatility in economic or financial settings. The case of ambiguous volatility and its implications was studied in Epstein and Ji (2013) and Epstein and Ji (2014) for asset pricing, (Beissner, 2017) for general equilibrium theory, Riedel and Beißner (2014) for a study of Arrow-Debreu equilibria by continuous trading , Lin and Riedel (2014) for optimal consumption and portfolio choice, and Vorbrink (2014) for arbitrage theory.

3 Optimal stopping under volatility ambiguity

In what follows, we will consider that the process that drives the payoff of the decision maker follows the dynamic :

$$dX_t^x = \mu X_t^x dt + X_t^x dB_t \tag{1}$$
$$X_0 = x \tag{2}$$

In the standard optimal stopping literature in economics and finance, B_t is a standard Brownian motion on a given probability space, and X_t^x is thus a geometric Brownian motion. In this paper, we want to study optimal stopping decisions of a decision maker who considers B as a process with ambiguious volatility[1]. More precisely, the decision maker only knows that the quadratic variation of the process B lies in a given interval :

$$\langle B \rangle_t \in [\underline{\sigma}^2 t, \overline{\sigma}^2 t] \tag{3}$$

with $0 < \underline{\sigma} < \overline{\sigma}$. Contrary to the case where B is a standard Brownian motion, this quadratic variation needs not be deterministic or equal to elapsed time. The decision maker does not know the form of the volatility process of B motion and does not take a stand on this form, he only believes that this process lies in $[\underline{\sigma}^2, \overline{\sigma}^2]$. These beliefs are represented by a set of priors $\mathcal{P}$ on Ω the set of continuous paths. Each P^σ in $\mathcal{P}$ corresponds to a volatility "scenario" $(\sigma_t)_{t\geq 0}$ that lies in $[\underline{\sigma}^2, \overline{\sigma}^2]$.

The decision maker solves an optimal stopping problem and is ambiguity averse in the sense of Gilboa Schmeidler, that is, he solves the optimal stopping problem using the prior in $\mathcal{P}$ that yields the lowest expected payoff. Formally, the problem is :

$$\sup_{\tau \in \Gamma} \inf_{P \in \mathcal{P}^\Theta} \mathbb{E}^P[e^{-\rho\tau} f(X_\tau^x)] \tag{4}$$

where$f(.) : \mathbb{R} \to \mathbb{R}$ is the payoff function, and $\mathbb{E}^P$ is the standard expectation with respect to the probability measure P. When the set of priors is a singleton, $\mathcal{P}^\Theta = \{P^\sigma\}$, the problem becomes a classical optimal stopping problem under P^σ.

When the decision maker only considers upper and lower bounds for the volatility process, the associated set of priors becomes non-dominated, that is, somes measures in $\mathcal{P}^\Theta$ are mutually singular. To see this, consider the quadratic volatility under $P^{\underline{\sigma}}$ and $P^{\overline{\sigma}}$, we have :

$$P^{\underline{\sigma}}(\langle B \rangle_t = \underline{\sigma}^2 t) = 1 = P^{\overline{\sigma}}(\langle B \rangle_t = \overline{\sigma}^2 t) \tag{5}$$

[1]See Epstein and Ji (2014) for an intuitive presentation of Brownian motion with ambiguous volatility, using discrete-time trees.

Because of this singularity, instead of the standard filtered probability space, we will consider the tuple $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathcal{P})$ to model uncertainty and information. Since we consider priors that do not necessarily agree on null events, the almost-surely characterization of random variables and stochastic process becomes irrelevant, as we need to consider all priors. Following Denis and Martini, we adopt the *quasi-surely* terminology. A set $A \in \mathcal{F}$ is called *polar* if $P(A) = 0$ for all $P \in \mathcal{P}$. A property is said to hold *quasi-surely* if it holds outside of polar sets. In what follows, equations involving random variables and stochastic processes hold quasi-surely, unless otherwise noted. When a property holds quasi-surely, it holds almost-surely for all priors in $\mathcal{P}$.

A second specificity of this framework is that the domain of random variables and stochastic processes has to be modified to accomodate the nonequivalence of priors. Similarly, the domain of the stochastic integral has to be adapted. As these are essentially technical elements that only appear in the proofs, we provide the technical details in the appendix.

3.1 Hamilton-Jacobi-Bellman equation

In many applications, the process driving the payoff of the stopping problem solves a stochastic differential equation and exhibits a markovian structure. We now proceed to charactherize the value function of our problem through a Hamilton-Jacobi-Bellman type equation. In problems without ambiguity, the value function V of the optimal stopping problem

$$V(x) = \sup_{\tau \in \Gamma} \mathbb{E}[e^{-\rho\tau} f(X_\tau^x)] \tag{6}$$

where $dX_t^x = X_t^x \mu dt + \sigma X_t^x dB_t \;\; X_0 = x$ is an (unambiguous) Brownian motion, solves the following HJB equation :

$$\max_{t,x} \{f(x) - v(x), -\rho v(x) + \mu x v'(x) + \frac{1}{2}\sigma^2 x^2 v''(x)\} = 0 \tag{7}$$

Heuristically, this equation says that at each time t, either stopping is optimal because we have $f(X_t^x) = v(X_t^x)$, or waiting is optimal because the value function stays constant in expectation.

Under ambiguous volatility and with our suitable set of priors, we can expect a similar equation, with the exception that the decision maker chooses the worst case for the volatility *for each* value of X_t^x. Intuitively, equation 7 should become :

$$\max_{x \in \mathbb{R}} \{f(x) - v(x), -\rho v(x) + \mu x v'(x) + \min_{\sigma^2 \in [\underline{\sigma^2}, \overline{\sigma^2}]} \{\frac{1}{2} v''(x) x^2 \sigma^2\}\} = 0$$

Our next theorem confirms this intuition. We show that if we can find a suitable function that solves the above equation, then we have found the value function of our optimal stopping problem.

Theorem 1 *Let $v(x)$ be a $\mathcal{C}^1$ function everywhere and $\mathcal{C}^2$ except on a finite number of points, that solves the following equation :*

$$\max_{x\in\mathbb{R}}\{f(x)-v(x),-\rho v(x)+\mu x v'(x)+\min_{\sigma^2\in[\underline{\sigma^2},\overline{\sigma^2}]}\{\frac{1}{2}v''(x)x^2\sigma^2\}\}=0 \tag{8}$$

Then $v(x)=V(x)$. In addition, $\tau^*=\inf\{t\geq 0: V(X_t^x)=f(X_t^x)\}$ is an optimal stopping time.

The interpretation of this equation remains the same : either stopping is optimal, or the value function is constant. The term on the right represents, loosely speaking, $\mathcal{E}_t(dV_t)$ and can be computed by using the G-Ito formula. The G-Ito formula is analgous to the standard one, except that the quadratic variation of the Brownian motion is ambiguous, which induces the non-linearity of the second-order term in the equation.

3.2 Worst-case prior

Hamilton-Jacobi-Bellman equation provides a way to establish the worst-case measure for a given stopping problem under volatility ambiguity, in the case where the payoff is driven by a geometric brownian motion. As can be readily seen from equation 8, the crucial element is the concavity or convexity of the value function. To see this, suppose that it can be established that the value function of a given problem under ambiguity is convex[2]. Denoting this function by $V(.)$, and using Theorem 2, then $V(.)$ is the solution to :

$$\max_{x\in\mathbb{R}}\{f(x)-v(x),-\rho v(x)+\mu x v'(x)+\min_{\sigma^2\in[\underline{\sigma^2},\overline{\sigma^2}]}\{\frac{1}{2}v''(x)x^2\sigma^2\}\}=0 \tag{9}$$

So we fall back on the typical linear HJB equation. By the standard verification theorem (see Oksendal ?), $v(.)$ is the value function of the stopping problem :

$$\sup_{\tau}\mathbb{E}^{P^{\underline{\sigma}}}[e^{-\rho\tau}f(X_\tau^x)] \tag{10}$$

Where $X_t^x=\mu X_t^x dt+\underline{\sigma}X_t^x d\overline{B}_t$, and $\overline{B}_t$ is a standard Brownian motion on $(\Omega,\mathcal{F},\{F\}_t,P^{\underline{\sigma}})$. Now, remember that we have $\langle B\rangle_t=\underline{\sigma}_t t \quad P^{\underline{\sigma}}-as$ and B_t is a martingale under $P^{\underline{\sigma}}$, so by Levy's characterization of Brownian motions, $\frac{B_t}{\underline{\sigma}}$ is a standard Brownian motion under $P^{\underline{\sigma}}$, so we have :

$$\sup_{\tau}\inf_{P\in P^{\Theta}}\mathbb{E}^{P}[e^{-\rho\tau}f(X_\tau^x)]=\sup_{\tau}\mathbb{E}^{P^{\underline{\sigma}}}[e^{-\rho\tau}f(X_\tau^x)] \tag{11}$$

[2]Note that establishing convexity is typically more intricate in the presence of multiple priors because the multiple prior expectation is sup-additive. Even if the payoff function is convex, and the process geometric, it is not straightforward to show convexity of the value function.

4 Irreversible investment: the monopoly case and accelerated investment

4.1 Setting

We start with the canonical irreversible investment model of McDonald and Siegel (1986). In the irreversible investment model, the firm possesses an investment opportunity (a real option), and can, at any time t, spend a fixed cost I to install the project and start generating profit. This model has a wide range of applications : macroeconomic dynamics (see Bertola and Caballero (1994), Caballero et al. (1996)), the timing of environmental policies (Pindyck (2000), Pindyck (2002)), optimal pricing and quality choice (Pennings (2004)). The project generates a stream of profit given by :

$$dX_t^x = \mu X_t^x dt + X_t^x dB_t \tag{12}$$

$$X_0 = x > 0 \tag{13}$$

where B_t is the G-Brownian motion as defined in section 2. The firm chooses the optimal time at which it decides to pay an irreversible cost I and starts generating profit from the project. The firm is ambiguity averse and has a discount factor ρ . We assume that $\rho > \mu$ [3]. The irreversible investment problem writes :

$$V(x) = \sup_{\tau \in \Gamma} \inf_{P \in \mathcal{P}^\Theta} \mathbb{E}^P[\int_\tau^T e^{-\rho t} X_t^x dt - e^{-\rho\tau} I] \tag{14}$$

Where $V(.)$ is the value function associated to this problem. In what follows, when we discuss the *value of investment*, that is, the value function of the firm, we distinguish between the value of the investment option, which corresponds to the value function *before* the investment has been operated, and the exercized investment option, which is the value function *after* the investment has been operated.

4.1.1 Value of the exercized investment option

To make clear the role of volatility ambiguity, we start by discussing the value of the exercized investment option for the firm. The value of an investment option exercized at t is:

$$W(X_t^x, t) = \mathcal{E}_t[\int_t^T e^{\rho(s-t)} X_s^x ds] \tag{15}$$

[3] This assumption insures that the option is not held indefinitely

$W(X_t^x, t)$ represents the expected flow profit generated by the project from date t. Using the fact that the firm is risk neutral, we can characterize this value, as the next proposition shows.

Proposition 1 *The value of the utilized patent under volatility ambiguity is given by :*

$$W(X_t^x, t) = \frac{1 - e^{-(T-t)(\rho-\mu)}}{\rho - \mu} X_t^x \tag{16}$$

The intuition for this result is straightforward : the firm being risk-neutral, once the investment has been made, the value of the project is simply its net present value. In fact, ambiguity aversion does not impact this value, because the net present value is constant across any volatility process for the underlying Brownian motion.

This result contrasts with Nishimura and Ozaki (2007) who find that the value of the utilized patent is lower under what they call Knigthian uncertainty. The difference between the two results comes from the difference in the type of ambiguity considered. (Nishimura and Ozaki, 2007) consider a set of measures obtained by a Girsanov transformation with respect to some reference measure, and thus the firm is ambiguous about the drift of the profit stream. Under drift ambiguity, once the firm has invested, the value of the exercized investment option is the net present value under the lowest possible drift.

We would like to point out that Proposition 1 holds under a more general specification of the set of priors. The proof in Appendix shows that Proposition 1 is true for any set of priors $\mathcal{P}^\Theta$ generated by a possibly time and path dependent interval $[\overline{\sigma}_t(\omega), \underline{\sigma}_t(\omega)]$. In particular, the result is robust to any particular stochastic interval that accomodates learning.

4.1.2 Infinite-horizon case

It is well-known that the analysis of the irreversible investment problem becomes much more tractable when the time horizon (expiration date of the patent) becomes infinite. In fact, in infinite horizon, the problem can be solved analytically, which simplufies greatly the comparative statics exercise. Using Proposition 1, in infinite horizon, the value of the utilized patent becomes :

$$W(X_t^x) = \frac{1}{\rho - \mu} X_t^x \tag{17}$$

Our infinite-horizon problem can be reformulated as (see Lemma 1 in Appendix[4]) :

$$\sup_{\tau \in \Gamma} \inf_{P \in \mathcal{P}^{\Theta}} \mathbb{E}^P[e^{-\rho\tau}(\frac{1}{\rho - \mu} X_\tau^x - I)] \tag{18}$$

Denote by V the value function for this problem, then it solves the following HJB equation :

$$\max\{\frac{1}{\rho - \mu}x - I - V(x), -\rho + \mu x V'(x) + \min_{\sigma^2 \in [\underline{\sigma}^2, \overline{\sigma}^2]} \frac{1}{2}\sigma^2 \pi^2 V''(x)\} = 0 \tag{19}$$

This problem can be solved in a standard way (see Appendix) : the optimal investment strategy is a trigger strategy in which the firm invests when the profit process reaches a certain threshold. This result is summarized in the following proposition.

Proposition 2 *The optimal investment strategy for the firm is τ^* such that*

$$\tau^* = \inf\{t \in [0, \infty), X_t^x \geq X^*\} \tag{20}$$

where $X^* = \frac{\underline{\beta}(\rho - \mu)}{\underline{\beta} - 1} I$ and $\underline{\beta}$ is the positive root of the quadratic equation :

$$\frac{1}{2}\beta(\beta - 1)\underline{\sigma}^2 + \beta\mu - \rho = 0 \tag{21}$$

And the value function of the firm is given by :

$$V(x) = \begin{cases} (\frac{x}{X^*})^{\underline{\beta}}(\frac{X^*}{\rho - \mu} - I) & if \;\; x < X^* \\ \frac{1}{\rho - \mu}(x - I) & if \;\; x \geq X^* \end{cases} \tag{22}$$

As can be seen from equation 21, the ambiguity-averse firm behaves as a subjective expected utility firm, with prior belief $P^{\underline{\sigma}}$, the probability that induces the constant volatility process $\underline{\sigma}$. Remember that the firm being risk neutral, volatility gives the firm opportunities to invest as a positive shock occurs, thus, the worst-case for the firm is that those positive shocks do not occur (or occur with less magnitude). As a result, the worst-case prior is the lowest possible volatility.

4.1.3 Comparative statics

A major result of the real option literature is that uncertainty increases the value of an irreversible investment and creates an incentive to postpone investment. Using our analytical results, we show that this result is reversed if we take uncertainty to be in the sense of volatility ambiguity. In particular, we show that an increase in ambiguity *reduces* the value of an investment opportunity and *accerlerates* investment. We first define an increase in ambiguity.

[4]The inversible investment problem involves an integral, so we cannot apply directly our Theorem 1, we show in Appendix that by the Markovian nature of the profit flow (which holds in the G-framework), we can reduce the problem so as to apply directly our Theorem.

Definition 1 *Consider two decision makers, A and B who have subjective beliefs $\mathcal{P}^{\Theta^A}$ and $\mathcal{P}^{\Theta^B}$. We say that A perceives more ambiguity than B if* $\Theta^B \subset \Theta^A$ *with* $0 < \underline{\sigma}^A < \underline{\sigma}^B < \overline{\sigma}^B < \overline{\sigma}^A$.

Intuitively, ambiguity increases as the set of probability measures considered by the decision maker increases. With this definition in hand we can now compare the investment strategy and the value of investment for two firms who differ in the ambiguity they perceive about the investment.

Proposition 3 *Consider two decision makers, A and B facing our irreversible investment problem. If A perceives more ambiguity than B, then the value of the investment opportunity is higher for B than for A.*

To understand the role played by ambiguity here, one has to look at the effect of ambiguity on the value of investment *before* and *after* the investment is made. As we have seen in Proposition 2, the ambiguity averse firm that contemplates an investment opportunity behaves as a subjective expected utility firm with belief $P^{\underline{\sigma}}$. At the same time, we have seen that the value of an already-made investment is unaffected by ambiguity, and by extension, an increase in ambiguity. As a result, when ambiguity increases, before investing, the firm expects less "positive opportunities" (good shocks), which reduces the ex-ante value of the investment opportunity. Once the investment is made, the value is unaffected by ambiguity. As a consequence, the value of the investment opportunity decreases with an increase in ambiguity.

Proposition 4 *Consider two decision makers, A and B facing our irreversible investment problem. If A perceives more ambiguity than B, then* $X_A^* < X_B^*$.

As we have seen in Proposition 1, the firm's flow payoff after the investement option is exercized is unaffected by volatility ambiguity. As a consequence, for a given investment threshold, the exercized investment option is unaffected by an increase in ambiguity, which does not change the incentive to wait for the firm. On the other hand, before investment, the firm anticipates the lowest possible volatility, which decreases her incentives to wait as less upward shocks are expected. The combination of these two effects reduces the incentives to wait and thus the investment trigger.

A corollary of this result, or an alternative interpretation, is that when ambiguity increases, the NPV rule makes less mistake. In the limit, as $\underline{\sigma} \to 0$, then the NPV investment rule actually becomes optimal. Thus ambiguity or robustness concerns may provide a simple rationale for this widely use simple rule.

4.2 Irreversible investment under risk aversion

In the previous section, we carried the analysis under the assumption that the firm is risk-neutral. This assumption is common in the irreversible investment literature and follows from the underlying assumption of financial markets completeness. Facing complete markets, investors are able to diversify their portfolios, making them neutral to project-specific risk. There are several reasons why investors may face incomplete markets. Among them, is the presence of ambiguous volatility, see Vorbrink (2014) and Epstein and Ji (2013) [5]. When we consider irreversible investment under volatility ambiguity, it is thus natural to investigate the incomplete market case and thus to analyze what happens to the investment strategy when the firm is risk averse in addition to ambiguity averse.

We study the irreversible investment problem of section 4.1 under the assumption that the firm has additive expected utility risk preferences. The problem becomes

$$F(x) = \sup_{\tau \in \Gamma} \inf_{P \in P^{\Theta}} \mathbb{E}^P \Big[\int_{\tau}^{\infty} e^{-\rho t} U(X_t^x) dt - e^{-\rho \tau} U(I) \Big] \tag{23}$$

In particular, to facilitate the analysis, we assume CRRA risk preferences of the form[6] :

$$U : \mathbb{R} \to \mathbb{R} \quad x \to \frac{x^{1-R}}{1-R} \tag{24}$$

With R > 0 and logarithmic utility when $R = 1$. With this functional form assumption, we can derive the value of an exercized investment option under risk aversion, which is given in the next proposition.

Proposition 5 *Assume*

$$\delta = \rho + (R-1)(\mu - \frac{1}{2}\overline{\sigma}^2 R) > 0 \tag{25}$$

Under CRRA risk preferences, the value of an exercized investment option is given by :

$$W_R(X_t^x) = \frac{X_t^{1-R}}{(1-R)\delta} \tag{26}$$

In contrast to proposition 1, volatility ambiguity matters for the value of the exercized investment option when the firm is risk averse. Once the firm has exercized the investment option, it starts to receive a payoff flow which depends

[5] See also Mukerji and Tallon (2001) for incompleteness of financial markets when agents are Choquet expected utility maximizers

[6] Assuming this parametric form allows for analytical solutions. For a general treatment of irreversible investment inder risk aversion (without ambiguity), see Hugonnier and Morellec (2007)

on X_t^x, as the firm is risk-averse, the worst-case in terms of volatility is that this payoff flow is highly volatile. The worst-case prior is thus $P^{\overline{\sigma}}$, the prior that yields the highest possible volatility for X_t^x. As a consequence, volatility ambiguity decreases the value of the exercized investment option. We note here that this result is not a special feature of our functional form assumption, but carries to any concave utility function. Proceeding much as in section 4.1, we can then characterize the optimal strategy of the risk averse firm under volatility ambiguity as follows :

Proposition 6 *The optimal investment strategy for the irreversible investment problem under risk-aversion and ambiguous volatility is a trigger strategy:* $\tau^* = \inf\{t \geq 0 : X_t^x \geq \overline{X}\}$, *where:*

$$\overline{X} = I(\frac{\delta\beta}{\beta + (R-1)})^{\frac{1}{1-R}} \tag{27}$$

The value function of the firm is given by :

$$F(x) = \begin{cases} \frac{1}{\rho}U(I) + (\frac{1}{\delta}U(\overline{X}) - \frac{1}{\rho}U(I))(\frac{x}{\overline{X}})^{\beta} & \text{if } x < \overline{X} \\ \frac{1}{\delta}x - U(I) & \text{if } x \geq \overline{X} \end{cases} \tag{28}$$

Recall that :

$$\frac{1}{2}\beta(\beta-1)\underline{\sigma}^2 + \beta\mu - \rho = 0 \tag{29}$$

$$\delta = \rho + (R-1)(\mu - \frac{1}{2}\overline{\sigma}^2 R) \geq 0 \tag{30}$$

Thus, under the optimal strategy, the worst-case prior for the firm is the prior P^* associated to the path-dependent process :

$$\sigma_t^* = \mathbb{1}_{X_t^x < \overline{X}}\underline{\sigma} + \mathbb{1}_{X_t^x \geq \overline{X}}\overline{\sigma} \tag{31}$$

Before investing,the firm believes that the process has low volatility, thus anticipating low or few investment opportunities. However, the firm changes its belief about the volatility right after investing, and anticipates higly volatile payoffs, which is the worst case as the firm is risk averse.

Our result on the effect of an increase in ambiguity on the value of investment carries to the case of the risk averse firm. In fact, the erosion of the value of investment is magnified by the presence of risk aversion, because an increase in ambiguity now decreases the value of the investment option *and* the value of the exercized investment option. This can be seen easily when considering the value function above the investment trigger :

$$\frac{x^{1-R}}{\rho(1-R) + (1-R)(1+r)(\mu - \frac{1}{2}\overline{\sigma}^2 R)} \tag{32}$$

which decreases with the upper bound $\overline{\sigma}$: as the firm dislikes risk, an increase in ambiguity is perceived as an increase in risk.

The effect of an increase in ambiguity on the investment threshold when the firm is risk averse is less clear. The reason is that the investment trigger now depends on both the lower and the upper bound of possible volatilities. To deal with this issue we will restrict attention to *homogeneous* increases in ambiguity.

Definition 2 *A perceives homogeneously more ambiguity than B if* $\underline{\sigma}^B - \underline{\sigma}^A = \overline{\sigma}^A - \overline{\sigma}^B$

This definition is more restictive than definition, as we require the increase in perceived ambiguity to be the same at the upper and lower bound of the interval. This definition allows to quantify the increase in ambiguity, which makes comparative statics easier.

Proposition 7 *Suppose A and B share the same level of risk aversion, and A perceives homogeneously more ambiguity than B, then there exists a threshold level of risk aversion above which A invests later than B and below which B invests later than A*

When the firm is risk averse, the effect of an increase in ambiguity is ambiguous. This is the result of two conflicting effects, which are due to the path-dependence of the worst-case prior. Before investment, the worst-case is the lowest constant volatility, which gives an incentive to the firm to accelerate investment. On the other hand, as soon as the firm invests, the worst-case becomes the highest volatility, and by investing, the firm exposes itself to this high volatility. Thus, the higher the ambiguity level the higher the volatility anticipated after investment and as a consequence, the higher the investment threshold required to reach the same exercized investment option value. Hence the incentive to delay investment. When risk aversion is relatively low, the effect of ambiguity before investment dominates, and investment is accelerated. When risk aversion is relatively high, the effect after investment dominates, which delays investment.

4.3 Lump-sum versus flow payoff: the case of the American option

We now treat the case of the American Call and show that the intuition about the effect of volatility ambiguity on the value of waiting is analogous to the case of irreversible investment. The main difference between the timing of an American option and an irreversible investment lies in the payoff structure.

An American option yields a lump-sum payoff, while an irreversible investment yields a flow payoff. As Miao and Wang (2011) showed, ambiguity can have a different effect on stopping decisions, depending on whether stopping yields a lump-sum or a flow payoff. Trojanowska and Kort (2010) prove a similar result in a continuous-time framework under drift ambiguity. Through the example of the American option, we show that this does not necessarily hold under ambiguous volatility.

The payoff of an American call option is given by :

$$e^{-rt}(X_t^x - K)^+ \tag{33}$$

where $r > 0$ is the interest rate and $K > 0$ the strike price. Again, for analytical convenience and to allow comparative statics, we consider the case of the American perpetual call. The optimal stopping problem of the investor is then :

$$\sup_{\tau \in \Gamma} \inf_{P \in \mathcal{P}^\Theta} \mathbb{E}^P[e^{-r\tau}(X_\tau^x - K)^+] \tag{34}$$

The analysis can be carried out much in the same way as in the previous section. The investment strategy is given in the next proposition :

Proposition 8 *The optimal investment strategy is τ^* such that*

$$\tau^* = \inf\{t \in [0, \infty), X_t^x \geq X^*\} \tag{35}$$

where $X^* = \frac{\beta}{1-\beta} I$ and β is the positive root of the quadratic equation :

$$\frac{1}{2}\beta(\beta - 1)\underline{\sigma}^2 + \beta\mu - \rho = 0 \tag{36}$$

As in the irreversible investment problem, the investor acts as a subjective expected utility investor who holds the belief associated to the constant lowest volatility $\underline{\sigma}$. As result, the value of the option is lower under ambiguity, and ambiguity lowers the value of waiting, resulting in a lower stopping threshold. This result comes from the fact that under ambiguous volatility, even if the firm is exposed to a flow-payoff, all priors yield the same pay-off, thus the firm is not exposed to ambiguity, even if she receives a flow-payoff[7].

5 Competitive equilibrium and the non-optimality of myopic behavior

In a perfectly competitive setting, a remarkable result, first shown by Leahy (1993), is that a firm deciding of its investment strategy without taking into

[7]We note here that contrary to Trojanowska and Kort (2010), the time horizon does not matter. Fixing any $T > 0$ as an expiration date would not alter the result that higher ambiguity accelerates investment.

account the effect of other firms' decisions on the price process actually follows the optimal strategy. This somehow surprising result has important implications, as the lessons drawn from the analysis of the monopoly firm carry on to the competitive equilibrium. Methodologically, this result provides a simple way of computing industry equilibrium, which can be quite intricate otherwise. In this section, we model a perfectly competitive equilibrium in which all firms hold the same set of priors. We show that contrary to the standard case, a myopic firm who holds the same set of priors as all other firms in the industry has a non-optimal investment strategy.

We consider a setting in which there is a non-atomic continuum of firms, all of which possessing an option to irreversibly enter a market, by paying the fix cost I. When a firm enters the market, it yields one unit of output forever and thus receives the flow payoff P_t, where P_t is the market price of one unit of output. P_t evolves stochastically according to the market clearing condition :

$$P_t = X_t^x D(Q_t) \tag{37}$$

where Q_t is the total industry output, D is the inverse demand function such that $D'(Q) < 0$, and X_t^x an exogenous shock such that :

$$dX_t^x = \mu X_t^x dt + X_t^x dB_t \quad X_0 > 0 \tag{38}$$

The price process is influenced by the total industry output, and each firm acts like a price-taker and decides of its investment strategy given this price process. Formally, each firm solves :

$$\sup \inf_{P \in \mathcal{P}} \mathbb{E}^P[\int_\tau^\infty e^{-rs} P_s ds - e^{-\rho\tau} I] \tag{39}$$

Following Leahy (1993), we formally define a competitive equilibrium as follows :

Definition 3 *A competitive equilibrium is a process P_t associated to an upper reflecting boundary $\overline{P}$ such that :*

1. $P_t = X_t^x D(Q_t)$
2. $P_t \leq \overline{P}$
3. *Entry occurs only when* $P_t = \overline{P}$
4. *Each firm chooses a strategy so as to maximize 39*
5. *The value of an idle firm is zero*

In technical terms, the price process solves a stochastic differential equation with an upper reflecting barrier, meaning that whenever the price reaches $\overline{P}$, it is automatically pushed down to stay under the barrier. This is modelled by adding to P_t an additional process that is strictly decreasing and decreases whenever the barrier is reached[8]. Economically, when $\overline{P}$ is reached, a positive measure of firms enter the market, increasing total industry output and shifting down the price along the demand curve. The presence of the upper reflecting boundary affects the value of the exercized investment option. To see this, we compute in the next proposition the value of an active firm in the competitive market.

Proposition 9 *The value of an option exercized at P in the competitive equilibrium is given by :*

$$w(P) = \frac{P}{\rho - \mu} - \frac{1}{(\rho - \mu)\overline{\beta}} P^{\overline{\beta}} \overline{P}^{1-\overline{\beta}} \tag{40}$$

Where $\overline{\beta}_1$ is the positive solution to :

$$\beta(\beta - 1)\overline{\sigma}^2 + \beta\mu - \rho \tag{41}$$

The first term on the right hand side of equation 40 corresponds to the value of an exercized investment option, if the price process is not bounded from above, that is, in the absence of competition. It is the value of the exercized investment option that we found when dealing with a monopoly firm. The second term corresponds to the penalty suffered by the firm in the presence of the reflecting barrier. Clearly, the presence of this barrier reduces the value of the exercized investment option as the payoff is bounded from above. As a consequence, volatility impacts negatively the payoff of the firm as the boundary limits upward, but not downward shocks. The worst-case prior is thus the highest possible volatility, and an increase in ambiguity reduces the value of the exercized investment option.

To see why the investment strategy of the myopic firm cannot be optimal, remember that the difference between the competitive firm and the myopic firm is that the myopic firm ignores the effect of competition, which amounts to ignoring the presence of the barrier. Thus, a myopic firm chooses its investment strategy as if the price process evolved as :

$$P_t = \mu P_t dt + P_t dt \tag{42}$$

As we have seen in Section 4, the investment trigger is then given by :

$$\tilde{P} = \frac{\beta}{\beta - 1}(\rho - \mu)I \tag{43}$$

[8] Formally, we have: $P_t = \mu P_t dt + P_t dBt - dK_t$, $P_t \leq \overline{P}$ and $\int_0^\infty (P_t - \overline{P}) dK_t = 0$. The existence and uniqueness of the pair $P_t, \overline{P}$ in the G-framework was proven by Lin 2013, in addition to an Itô formula for this class of processes, which we exploit in the proof of the next proposition.

Can $\tilde{P}$ be the equilibrium entry price ? If it is the case, then the value of an exercized option at $\tilde{P}$ should be I, given that the equilibrium barrier is $\tilde{P}$, that is, we shoulde have $w(\tilde{P},\tilde{P}) = I$, by the definition of equilibrium. Plugging the myopic threshold 43 into the value of the exercized option in a competitve equilibrium 40, we get :

$$w(\tilde{P},\tilde{P}) = \frac{\underline{\beta}}{\underline{\beta}-1}(\frac{\overline{\beta}-1}{\overline{\beta}})I \neq I \tag{44}$$

Implying that $\tilde{P}$ cannot be an equilibrium. Note that when $\underline{\beta} = \overline{\beta}$, that is when $\underline{\sigma} = \overline{\sigma}$ and the set of priors is a singleton, then we have $w(\tilde{P},\tilde{P}) = I$ and $\tilde{P}$ is indeed an equilibrium.

Proposition 10 *The myopic strategy given by the threshold 43 is not a perfectly competitive equilibrium under ambiguity.*

To get a sense of this result, let us recall the intuition behind Leahy's optimality of myopic behavior. Introducing competition has two effects. The first one is to reduce the value of the exercized investment option, by imposing a barrier on the price process, as we have seen in Proposition 8. As such, this decrease in value gives an incentive to delay investment. On the other hand, free-entry completely kills the option value of waiting, thus accelerating investment. As it turns out, these two effects exactly cancel each other when the firm holds a single prior. Under ambiguity, what happens is that the myopic firm and the competitive firm have a different worst-case prior. For a myopic firm, ambiguity matters for the value of investment before the option is exercized and in this case, as we have seen, the worst-case prior is the lowest volatility. For the competitive firm, since equilibrium requires value zero for non-active firm, this value can be supported by any prior, but the worst-case after investment is the highest possible volatility. Thus, the perfect cancelation cannot happen, because we need to consider more than one prior here. The equilibrium trigger under ambiguity can be computed using our definition of equilibrium and our verification theorem. The result is presented in the following proposition.

Proposition 11 *The investment trigger in the competitive equilibrium is $\frac{\overline{\beta}}{\overline{\beta}-1}(\rho-\mu)I$, it is higher than the entry threshold of the monopoly and it increases with ambiguity.*

This result follows from the fact that, as we have seen in the previous proposition, the firm "switches" its worst-case prior when switching from a monopoly to a perfectly competitive equilibrium. In a perfectly competitive equilibrium, free-entry kills any option value, so that ambiguity plays no role before investment. However, after investment, the presence of the reflecting boundary corresponding to the competitive entry threshold, creates a negative value for volatility. As a consequence, the value of the exercized investment option at a given threshold decreases with ambiguity, which prompts firms to wait longer before entering the market.

We conclude this section with a remark on drift and volatility ambiguity. Although it has, to the best of our knowledge, not been formally shown, myopic and competitive behavior are the same under drift ambiguity. As shown by Cheng and Riedel (2013), what matters under drift ambiguity is the monocity of the payoff with respect to the driving process. In both the monopoly and the competitive case, the payoff is monotone and increasing in the price process, so that the competitive firm and the myopic firm both use the prior that yields the lowest drift when choosing their optimal strategy. In this case, the intuition behind Leahy (1993)'s result holds and the two investment triggers coincide. This observation shows that not only does ambiguity produce different predictions from risk, but different notions of ambiguity can also produce very different results[9].

5.1 Discussion

5.1.1 Ambiguity, ambiguity aversion

In our comparative statics exercise, we have interpreted an expansion of the set of priors, and thus an expansion of the interval $[\overline{\sigma}^2, \underline{\sigma}^2]$, as representing an increase in the ambiguity perceived by the decision maker. This interpretation facilitates discussions on the role of ambiguity as we can compare an increase in perceived ambiguity and an increase in what is typically called uncertainty in the real-options literature, that is, volatility. This interpretation is however problematic as the multiple-prior theory of Gilboa and Schmeidler (1989) is purely subjective, which entails that the set of priors represents perceived ambiguity *and* attitude towards ambiguity. A rationale for interpreting $[\overline{\sigma}^2, \underline{\sigma}^2]$ is provided in Gajdos et al. (2008), where the decision maker has access to some objective probabilistic information, the set $\mathcal{P}$, and chooses according to a maxmin criterion with respect to a *subset* of this set, so that the expectation operator becomes :

$$\inf_{P \in \Phi(\mathcal{P})} \mathbb{E}^P[X] \tag{45}$$

where $\Phi(\mathcal{P}) \subset \mathcal{P}$, represents the DM's ambiguity aversion. When $\Phi(\mathcal{P})$ is a singleton, the decision maker is ambiguity-neutral, and when $\Phi(\mathcal{P}) = \mathcal{P}$, the DM exhibits extreme ambiguity aversion. If we interpret the interval $[\underline{\sigma}, \overline{\sigma}]$ as the upper and lower volatility interval resulting from the selection

[9] A similar idea can be found in Lang (2016) who notes that different ambiguity preferences can generate different predictions in various economic applications, and establishes a typology of ambiguity preferences accordingly. Our paper shows that even within a preference class, we can obtain different predictions by considering different specifications.

$\Phi(\mathcal{P})$, then widening this interval corresponds to increase ambiguity aversion [10]. In this case, it would perhaps be fairer to compare the effect of ambiguity aversion to that of risk aversion. As shown by Hugonnier and Morellec (2007), the presence of risk aversion delays investment in the standard irreversible investment model. Thus our results show that, at least in the monopoly case, risk aversion and ambiguity aversion have opposite effect on the timing of investment.

6 Conclusion

This paper develops a framework to study real option problems under volatility ambiguity. Modelling ambiguous volatility requires a set of non equivalent priors, which requires departure from the standard probability space framework. To circumvent this difficulty, we exploit recent results in non-linear expectation theory, in particular, we use the G framework developed by Shige Peng and the associated G-stochastic calculus, which provides the non linear counterpart to the standard theory.

In an irreversible investment model, we show that volatility ambiguity accelerates option exercise. This result contrasts with the standard result in real option theory, according to which uncertainty increases to value of waiting. Our result complement previous finding on the effect of ambiguity on option exercise, which were restricted to drift ambiguity. We show that drift and volatility ambiguity have opposite effect on the timing of investment, a hint of the rich domain of possibilities brought by ambiguity. At a more general level, a unifying feature of the literature on ambiguity aversion since the seminal contribution of Dow and Werlang is the idea that ambiguity creates inertia. The main result in this paper shows that in some situations, ambiguity can create the opposite of inertia, as our ambiguity averse firm accelerates investment when exposed to ambiguity.

Being a first step, the paper has focused on canonical applications in which the worst-case could be easily identified. In spite of this simplicity, the worst-case prior varies with the context of the irreversible investment, which makes for interesting results showing that many lessons on the role of uncertainty in competitive markets do not survive under ambiguity. In addition we saw that adding risk aversion, a natural assumption under ambiguous volatility, to the monopoly case suffices to generate path dependent minimizing priors. We strongly suspect that in more complex applications, for example some exotic options, the worst-case prior has a more complex structure, corresponding to a specific stochastic volatility model. This question is left for future research.

[10] This supposes, of course, that the objective information has "nice" properties, in the sense that it can be constructed as we describe in appendix, which need not be the case, as this information is objective and exogenous.

Our results can now be used to tackle more complex economic problems involving real options. We mention here some possible applications. As we have studied the monopoly and perfect competition cases, a natural next step is to study the effect of ambiguous volatility on investment timing in irreversible investment games (see for instance **?**). A challenge of such an extension is to find a suitable notion of equilibrium, which may not be straightforward in a framework with a set of non-equivalent probability measures. A second possible application is the interaction between ambiguous volatility, investment timing and agency. Many investment decisions are taken in the presence of moral-hazard (e.g. IPOs, venture capital), which arises precisely because the payoff process is uncertain. For this reason, the interaction between agency and investment has recently received attention.[11]. We expect that the role of ambiguity on the timing of investment may be less straightforward in the presence of moral-hazard. In the same vein, investment decisions are often taken jointly with capital structure decisions, studying the interactions between these decisions and ambiguous volatility would be a step towards a robust theory of corporate financial decisions[12].

A The G- framework

In this appendix, we present briefly some elements related to the G-framework and some results that we use further in the text. We refer the reader to Peng (2007) for the seminal paper on this subject. Peng (2010) provides an exhaustive survey and a general exposition of the G-expectation framework.

A.1 Sublinear expectation, G-Brownian motion, G-Expectation

Definition 4 *Let Ω be a given set. Let $\mathcal{H}$ be a linear space of real-valued functions defined on Ω. A sublinear expactation $\hat{\mathbb{E}}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ such that for any $X, Y \in \mathcal{H}$:*

1. *If $X \geq Y$ then $\hat{\mathbb{E}}(X) \geq \hat{\mathbb{E}}(Y)$ (Monotonicity)*
2. *$\hat{\mathbb{E}}(c) = c$(Constant preserving)*
3. *$\hat{\mathbb{E}}(X + Y) \leq \hat{\mathbb{E}}(X) + \hat{\mathbb{E}}(Y)$ (sub-linearity)*
4. *$\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}(X)$ $\forall \lambda \geq 0$ (positive homogeneity)*

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space. Note that the sub-additivity is equivalent to $\hat{\mathbb{E}}(X) - \hat{\mathbb{E}}(Y) \leq \hat{\mathbb{E}}(X - Y)$.

[11]See for example Gryglewicz and Hartman-Glaser (2014)

[12]For recent work in this direction, see Miao and Rivera (2015)

In what follows, $\mathcal{C}_{l,Lip}(\mathbb{R}^n)$ $n \geq 1$ is the space of real-valued continuous functions ϕ such that $|\phi(x) - \phi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|$ $\forall x, y \in \mathbb{R}^n$ where k is an integer depending on ϕ.

Definition 5 ***Independence***- *Let $X, Y \in \mathcal{H}$. Y is said to be independent from X under $\hat{\mathbb{E}}$ if for any test function phi $\in \mathcal{C}_{l,Lip}(\mathbb{R}^2$ we have :*

$$\hat{\mathbb{E}}[\phi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\phi(x, Y)_{x=X}]] \tag{46}$$

Definition 6 ***Identical Distribution***- *Let $X_1, X_2 \in \mathcal{H}$ two random variables on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. They are called identically distributed, denoted $X_1 \sim X_2$ is*

$$\hat{\mathbb{E}}[\phi(X_1)] = \hat{\mathbb{E}}[\phi(X_2)] \quad \forall \phi \in \mathcal{C}_{l,Lip}(\mathbb{R}) \tag{47}$$

We say $\overline{X}$ is an independent copy of X if $X \sim \overline{X}$ and $\overline{X}$ is independent from X.

Definition 7 ***G-distribution***- *A random variable X on a sublinear expactation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called centralized G-normal distributed if for any $a, b \geq 0$:*

$$aX + B\overline{X} \sim \sqrt{a^2 + b^2}X \tag{48}$$

where $\overline{X}$ is an independent copy of X. Here, the function G denotes the function $G(y) = \frac{1}{2}\hat{\mathbb{E}}[yX^2]$. The following holds : $G(y) = \frac{1}{2}\overline{\sigma}^2 y^+ - \frac{1}{2}\underline{\sigma}^2 y^-$ where $\overline{\sigma}^2 = \hat{\mathbb{E}}[X^2]$ and $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-X^2]$. We write X is $N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$.

Definition 8 *In a sublinear expectation space, $(X_t^x)_{t \geq 0}$ is called a stochastic process if X_t^x is a random variable in $\mathcal{H}$ for each $t \geq 0$.*

Definition 9 *A process $(B_t)_{t \geq 0}$ on a sublinear expectation space is called a G-Brownian motion if the following properties are satisfied :*

1. *$B_0 = 0$*
2. *For each $t, s \geq 0$, $(B_{t+s} - B_t)$ is $N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$ distributed and independent from $(B_{t_1}, ..., B_{t_n})$ for each $n \in \mathbb{N}$, $0 \leq t_1 \leq ... \leq t_n \leq t$.*

Let $\Omega = \mathcal{C}([0, T]; \mathbb{R})$ be the space of all real valued continuous functions $(\omega_t)_{t \in [0,T]}$ with $\omega_0 = 0$, eqquiped with the distance :

$$\rho(\omega^1, \omega^2) = \sum_i^{\infty} 2^{-i}[(\max |\omega_t^1 - \omega_t^2|) \wedge 1] \quad \omega^1, \omega^2 \in \Omega \tag{49}$$

For each $\phi \in C_{l,lip}(\mathbb{R})$ let u_ϕ be the unique viscosity solution of the following partial differential equation :

$$\begin{cases} \frac{\partial u}{\partial t} = G(\frac{\partial^2 u}{\partial x^2}, \ (t, x) \in [0, T] \times \mathbb{R} \\ u(0, x) = \phi(x) \end{cases} \tag{50}$$

Finir avec dédinition de G-E sur L_2

A.2 G-stochastic calculus

In what follows, $M_G^{p,0}(0,T)$, $p \geq 0$ denotes the set of simple processes of the form : given a partition $\{t_0, ..., t_N\}$, $N \in \mathbb{N}$ of $[0,T]$, $\xi_i \in L_G^p(\Omega_{t_i}$, $i = 0, 1, ..., N-1$, for any $t \in [0,T]$, the process η is defined as :

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{t_j, t_{j+1}}(t) \tag{51}$$

We denote by $M_G^p(0,T)$ the completion of $M_G^{p,0}(0,T)$ under the norm $||.||_{M_G^P}$ defined as $||\eta||_{M_G^P} = (\mathbb{E}_G \int_0^T |\eta_s|^P ds)^{\frac{1}{p}}$.

Definition 10 *For $\eta \in M_G^2(0,T)$, the integral mapping $I : M_G^2(0,T) \to L_G^2(\Omega_T)$ is defined by :*

$$I(\eta) = \int_0^T \eta_s dB_s = \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}) \tag{52}$$

The quadratic variation of a G-Brownian motion is given by :

$$\langle B \rangle_t = B_t^2 - 2 \int_0^t B_s ds \; \forall t \leq T \tag{53}$$

Importantly, this process is *not* deterministic, as it contains the ambiguity of the G-Brownian motion. We can define the stochastic integral with respect to the quadratic variation of the G-Brownian motion in an analogous way

Definition 11 *For $\eta \in M_G^1(0,T)$, the integral mapping $I : M_G^1(0,T) \to L_G^1(\Omega_T)$ is defined by :*

$$I(\eta) = \int_0^T \eta_s dB_s = \sum_{j=0}^{N-1} \xi_j (\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}) \tag{54}$$

In what follows, we use the following propositions :

Proposition 12 *Let $f \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$ and $X_t^x = X_0 + \int_0^t \alpha_s ds + \int_0^t \eta_s d\langle B \rangle_s + \int_0^t \beta_s dB_s$ where $\alpha, \eta \in M_G^1(0,T), \beta \in M_G^2(0,T)$. Then for each $t \in [0,T]$ we have :*

$$f(t, X_t^x) = f(0, X_0) + \int_0^t \partial_x f(u, X_u) \beta_u dB_u + \int_0^t [\partial_t f(u, X_u) + \partial_x f(u, X_u) \alpha_u] du \tag{55}$$

$$+ \int_0^t [\partial_x f(u, X_u) \eta_u + \frac{1}{2} \partial_{xx}^2 f(u, X_u) \beta_u^2] d\langle B \rangle_u \tag{56}$$

In fact, this formula holds for $\mathcal{C}^1$ functions that are $\mathcal{C}^2$ everywhere except on a set of zero measure. This result follows directly from the generalized Itô formula for $\mathcal{C}^1$ functions in Yan, Sun, Gao (2015). As this is the case that is most often faced in economic applications (including the applications in this paper), we use this result for our verification theorem. Greater generality might be achievable using viscosity solutions, but in practice, it is often easier to check piecewise differentiability.

Proposition 13 *Let $\eta \in M_G^1(0,T)$ and $\alpha \in M_G^2(0,T)$, the process :*

$$M_t = M_0 + \int_0^t \alpha_s dB_s + \int_0^t \alpha_s d\langle B\rangle_s - \int_0^t 2G(\alpha_s)ds \tag{57}$$

is a G-martingale.

It is behind the scope of this paper to make an exhaustive presentation of the G-framework, we mention here some references that we use in our proofs :

1. (Li and Peng, 2011) for a definition of stopping times, a definition and properties of the Itô integral on $[0,\tau]$ where τ is a stopping time, and the related spaces.

2. (Hu and Peng, 2013) for a definition of stopping times on larger spaces, the property of conditional G-Expectation on these spaces (Monotonicity, constantpreserving, sublinearity and positive homogenity are preserved), and a version of the optional sampling theorem, that we use in the proof of Theorem 1.

3. (Hu et al., 2017) extend the conditional G-expectation to optional times and prove a strong markov property for solutions of stochastic differential equations driven by G-brownian motions.

4. (Hu et al., 2014a) provide a theory of Backward Stochastic Differential Equations driven by G-Brownian motions, (Lin et al., 2013) proves existence and uniqueness of reflected stochastic differential equations in the G-framework.

B Ambiguous volatility

B.1 The set of priors

Let $C([0,\infty))$ be the set of all continuous functions with values in $\mathbb{R}$. Define $\Omega = \{\omega : \omega \in C([0,\infty)), \omega_0 = 0\}$ and the coordinate process $B_t(\omega) = \omega(t)$.

Let P_0 be the classical Wiener measure, under which the coordinate process is a standard Brownian Motion and $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by $(B_t)_{t\geq 0}$ augmented by the P_0-null sets.

Define $\Sigma = [\underline{\sigma}, \overline{\sigma}] \subset \mathbb{R}$ with $0 < \underline{\sigma} < \overline{\sigma}$. For any process $(\sigma_t)_{t\geq 0}$ with values in Σ, the stochastic differential equation (SDE) :

$$\begin{cases} dX_t = \sigma_t dB_t \\ X_0 = 0 \end{cases} \tag{58}$$

Has a unique solution under P_0, denoted X^σ. Define the probability measure P^σ as :

$$P^\sigma(A) = P_0(X^\sigma \in A) \quad A \in \mathcal{F}_T \tag{59}$$

Denote by $\mathcal{P}_0$ the set of all probability measures that an be constructed in this way and $\mathcal{P}$ the closure of $\mathcal{P}_0$ under the topology of weak convergence.

With the set of priors $\mathcal{P}$, we define :

$$L^0(\Omega) = \{X : \Omega \to [-\infty, \infty] \quad X \text{ is } \mathcal{F}_t - \text{measurable}\} \tag{60}$$

$$\mathcal{L}(\Omega) = \{X \in L^0(\Omega) : \mathbb{E}^P[X] \text{ exists for each } P \in \mathcal{P}\} \tag{61}$$

The worst-case multiple priors expectation is then defined on $\mathcal{L}(\Omega)$ as :

$$\inf_{P\in\mathcal{P}^\Theta} \mathbb{E}^P[X] \tag{62}$$

For $X \in \mathcal{L}(\Omega)$. The corresponding upper expectation

$$\hat{\mathbb{E}}[X] = \sup_{P\in\mathcal{P}^\Theta} \mathbb{E}^P[X] \tag{63}$$

is the G-expectation, as introduced in Peng (2007) and extended to $\mathcal{L}(\Omega)$ in Hu and Peng (2013). The extension to $\mathcal{L}(\Omega)$ is important in our context, as it allows stopped processes (under a suitable notion of stopping times, see bellow) to remain in the domain of the G-expectation. The G-expectation satisfies monotonicity, constant preserving, sublinearity (and thus, self dominance) and positive homogeneity. Peng (2010) develops a rich theory of stochastic calculus that provides the non-linear counterpart to the standard theory. We exploit this theory, noting that :

$$\inf_{P\in\mathcal{P}^\Theta} \mathbb{E}^P = -\hat{\mathbb{E}}[-\xi] \tag{64}$$

B.2 Conditional expectation

To define a notion of conditional expectation on the suitable spaces, we follow Hu and Peng (2013) and define :

$$\mathbb{L}^p(\mathcal{F}) = \{X \in L^0(\mathcal{F}) : \hat{\mathbb{E}}[|X|^p] < \infty\} \text{ for } p \geq 1 \tag{65}$$

$$L_G^{1^*}(\mathcal{F}) = \{X \in \mathbb{L}^1 : \exists X_n \in L_G^1 \text{ such that } X_n \downarrow X, q.s\} \tag{66}$$

$$L_G^{1^*_*}(\mathcal{F}) = \{X \in \mathbb{L}^1 : \exists X_n \in L_G^1 \text{ such that } X_n \uparrow X, q.s\} \tag{67}$$

$$\overline{L}_G^{1^*_*}(\mathcal{F}) = \{X \in \mathbb{L}^1 : \exists X_n \in L_G^1 \text{ such that} \hat{\mathbb{E}}[|X_n - X|] \to 0\} \tag{68}$$

We can define analogously $\mathbb{L}^p(\mathcal{F}_t), L_G^{1^*}(\mathcal{F}_t), L_G^{1^*_*}(\mathcal{F}_t), \overline{L}_G^{1^*_*}(\mathcal{F}_t)$. The conditional expectation $\hat{\mathbb{E}}_t$ is then given by :

$$\mathbb{E}_t = \text{ess sup}_{Q \in \mathcal{P}(t,P)} \mathbb{E}^Q[X|\mathcal{F}_t] \ P - \text{as} \tag{69}$$

And the worst-case multiple prior conditional expectation as :

$$\underline{\mathbb{E}}_t = -\hat{\mathbb{E}}_t[-X] \tag{70}$$

Note that this expectation is defined on a larger space than the "classical" G-expectation space L_G^1, again, this is because we have stopping times in mind. The concept of G-martingale is defined analogously to the standard case :

Definition 12 *A process $(M_t)_{t\geq 0} \in L_G^1(\Omega_t)$ is a G-martingale if for each $t \geq 0$ and for each $s \in [0,t]$, we have $\hat{\mathbb{E}}_t[M_t] = M_s$*

B.3 Stopping times

As shown by (Hu and Peng, 2013), a suitable notion of stopping times in the G-framework is given by :

Definition 13 *Let τ be a random time. τ is a $*$-stopping time if $I_{\tau \geq t} \in L_G^{1^*}(\mathcal{F}_t)$ for each $t \geq 0$.*

An important example of $*$-stopping time is the first exist time of a process $X_t \in L_G^1(\mathcal{F}_t)$:

$$\tau = \inf\{t \geq 0 : X_t \in F\} \tag{71}$$

In the main text, we refer to Γ as the set of finite $*$-stopping times, and we assume that the firm chooses her optimal strategy within Γ. Although formally a restriction, as we do not consider the set of all stopping times, it is merely a technical restriction, and does not alter the economic analysis. Finally, we have the important optional sampling theorem for $*$-stopping times :

Proposition 14 *Let M_t be a G-martingale and τ a $*$-stopping time, then $M_\tau \in \overline{L}_G^{1^*_*}(\mathcal{F})$ and $\hat{\mathbb{E}}[M_\tau] = M_0$*

C Proofs

C.1 Theorem 1

To obtain the desired result, our first step is to characterize the value function $\overline{v}$ of the stopping problem :

$$\overline{v}(x) = \inf_{\tau\in\Gamma} \sup_{P\in\mathcal{P}} \mathbb{E}^P[-e^{-\rho\tau} f(X_\tau^x)] \tag{72}$$

$\overline{v}$ will immediately yield the solution to our problem since :

$$-\overline{v}(x) = -\inf_{\tau\in\Gamma} \sup_{P\in\mathcal{P}} \mathbb{E}^P[-e^{-\rho\tau} f(X_\tau^x)] = -\inf_{\tau\in\Gamma}(-\inf_{P\in\mathcal{P}} \mathbb{E}^P[e^{-\rho\tau} f(X_\tau^x)]) \tag{73}$$

$$= \sup_{\tau\in\Gamma} \inf_{P\in\mathcal{P}} \mathbb{E}^P[-e^{-\rho\tau} f(X_\tau^x)] \tag{74}$$

Consider the solution to the Hamilton-Jacobi-Bellman equation :

$$\max_x\{\rho w(x) - \mu x w'(x) - \sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]}\{\frac{1}{2}\sigma^2 x^2 w''(w)\}, w(x) + f(x)\} = 0 \tag{75}$$

Let $\tau \in \Gamma$, by the G-Ito formula applied to $e^{-\rho\tau} w(X_\tau^x)$, we have :

$$e^{-\rho\tau} w(X_\tau^x) = w(x) + \int_0^\tau -\rho e^{-\rho t} w(X_t^x) + \mu X_t^x w'(X_t^x) dt \tag{76}$$

$$+ \int_0^\tau e^{-\rho t} X_t^x w'(X_t^x) dB_t + \int_0^\tau e^{-\rho t} \frac{1}{2} X_t^2 w''(X_t^x) d\langle B\rangle_t \tag{77}$$

$$= w(x) + \int_0^\tau -\rho e^{-\rho t} w(X_t^x) + \mu X_t^x w'(X_t^x) + G(\frac{1}{2} X_t^2 w''(X_t^x)) dt \tag{78}$$

$$+ \int_0^\tau e^{-\rho t} X_t^x w'(X_t^x) dB_t + \int_0^\tau e^{-\rho t} \frac{1}{2} X_t^2 w''(X_t^x) d\langle B\rangle_t \tag{79}$$

$$- \int_0^\tau G(\frac{1}{2} X_t^2 w''(X_t^x)) dt \tag{80}$$

By the Hamilton-Jacobi-Bellman equation 75, $w(X_\tau^x$ is dominated by $f(X_\tau^x)$ so we have :

$$e^{-\rho\tau} f(X_\tau^x) \geq e^{-\rho\tau} w(X_\tau^x) \geq w(x) + \int_0^\tau e^{-\rho t} X_t^x w'(X_t^x) dB_t + \int_0^\tau e^{-\rho t} \frac{1}{2} X_t^2 w''(X_t^x) d\langle B\rangle_t - \int_0^\tau G(\frac{1}{2} X_t^2 w'' \tag{81}$$

where the second inequality comes from the HJB equation. By Proposition 1.4 Chapter IV in Peng (2010), the process :

$$\int_0^t e^{-\rho t} X_t^x w'(X_t^x) dB_t + \int_0^t e^{-\rho t} \frac{1}{2} X_t^2 w''(X_t^x) d\langle B\rangle_t - \int_0^t G(\frac{1}{2} X_t^2 w''(X_t^x)) dt \tag{82}$$

is a G-martingale, so taking expectation in 81 yields, by the optional sampling theorem in the G-framework :

$$\hat{\mathbb{E}}[-e^{-\rho\tau} f(X_\tau^x)] \geq w(x) \tag{83}$$

which yields $\overline{v}(x) \geq w(x)$. To prove the converse inequality, let :

$$D = \inf\{t \geq 0 : w(X_t^x) = -f(X_t^x)\} \tag{84}$$

By the HJB equation, we have :

$$\int_0^D -\rho e^{-\rho t} w(X_t^x) + \mu X_t^x w'(X_t^x) + G(\frac{1}{2} X_t^2 w''(X_t^x)) dt = 0 \tag{85}$$

Since $D \in \Gamma$, we have by the optional sampling theorem :

$$\hat{\mathbb{E}}[\int_0^D e^{-\rho t} X_t^x w'(X_t^x) dB_t + \int_0^D e^{-\rho t} \frac{1}{2} X_t^2 w''(X_t^x) d\langle B \rangle_t - \int_0^D G(\frac{1}{2} X_t^2 w''(X_t^x)) dt] = 0 \tag{86}$$

Thus, noting that $w(X_D) = -f(X_D)$ and taking the G-expectation yields :

$$w(x) = \hat{\mathbb{E}}[e^{-\rho D} w(X_D)] = \hat{\mathbb{E}}[e^{-\rho D} f(X_D)] \geq \overline{v}(x) \tag{87}$$

Thus we have $w(x) = \overline{v}(x)$ and D is an optimal stopping time for the problem 72. We directly obtain the desired verification theorem by the fact that $v(x) = \overline{v}(x)$ and it follows that $\tau^* = \inf\{t \geq 0 :\}$

C.2 Proposition 1

Proof. By the Markov property for solutions of G-SDEs, it suffices to compute:

$$\min_{P \in \mathcal{P}} \mathbb{E}^P[\int_t^T e^{-\rho(s-t)} X_s^{t,y} ds]_{y = X_t^x} \tag{88}$$

Where $X_s^{t,y}$ refers to the solution of SDE 12 starting at date t with value y. Take $P \in \mathcal{P}$, then we have :

$$\mathbb{E}^P[\int_t^T e^{-\rho(s-t)} X_s^{t,y} ds] = \mathbb{E}^P[\int_t^T e^{-\rho(s-t)} X_t^x \exp(\int_t^s \mu du + \int_t^s dB_u - \int_t^s \frac{1}{2} d\langle B \rangle_u) ds] \tag{89}$$

$$= \mathbb{E}^P[\int_t^T \exp((\mu - \rho)(s-t)) X_t^x \exp(\int_t^s dB_u - \int_t^s \frac{1}{2} d\langle B \rangle_u) ds] \tag{90}$$

$$= [\int_t^T \exp((\mu - \rho)(s-t)) X_t^x \mathbb{E}^P \exp(\int_t^s dB_u - \int_t^s \frac{1}{2} d\langle B \rangle_u) ds] \tag{91}$$

$$= \int_t^T \exp((\mu - \rho)(s-t)) X_t^x ds \tag{92}$$

$$= \frac{1 - e^{-(T-t)(\rho - \mu)}}{\rho - \mu} X_t^x \tag{93}$$

Where the first equality is an application of Ito's lemma, the second separates the deterministic and random terms in the exponential, the third is an application of Fubini's Theorem[13] and the fourth comes from the fact that:

$$Z_s = \exp(\int_t^s dB_u - \int_t^s \frac{1}{2} d\langle B\rangle_u) \tag{94}$$

is a martingale under any $P \in \mathcal{P}$, because B is a martingale under any prior in $\mathcal{P}$ and Novikov's condition holds :

$$\mathbb{E}^P[\exp(\frac{1}{2}\langle B\rangle_T])] < \infty \tag{95}$$

For all $P \in \mathcal{P}$, since $\langle B\rangle_t = \int_0^T \sigma_s^2 ds \ - \ P^\sigma a.s$ where $\sigma_s \in [\underline{\sigma}, \overline{\sigma}]$ ■

C.3 Lemma 1

Lemma 1 *For any $\tau \in \Gamma$, we have :*

$$\inf_{P\in\mathcal{P}^\Theta} \mathbb{E}^P[\int_\tau^T e^{-\rho s} X_s^x ds - e^{-\rho\tau} I] = \inf_{P\in\mathcal{P}^\Theta} \mathbb{E}^P[e^{-\rho\tau}(\frac{1}{\rho-\mu} X_\tau^x - I)] \tag{96}$$

Proof. Let $\tau \in \Gamma$, then by the consistency property of the conditional G-expecation $\hat{\mathbb{E}}_\tau$ as defined by Hu et al. (2017), we have:

$$\hat{\mathbb{E}}[\int_\tau^\infty e^{-\rho s} X_s^x ds - e^{-r\tau} I] = \hat{\mathbb{E}}[\hat{\mathbb{E}}_\tau[\int_\tau^\infty e^{-\rho s} X_s^x ds - e^{-\rho\tau} I]] \tag{97}$$

$$= \hat{\mathbb{E}}[\hat{\mathbb{E}}_\tau[\int_\tau^\infty e^{-\rho s} X_s^x ds] - e^{-r\tau} I] \tag{98}$$

$$= \hat{\mathbb{E}}[\hat{\mathbb{E}}[\int_\tau^\infty e^{-\rho s} X_s^y ds]_{y=X_\tau^x} - e^{-\rho\tau} I] \tag{99}$$

$$\tag{100}$$

Where the first equality is the consistency property for the conditional G-expectation extended to optional times (see Hu et al. (2017) Proposition 3.13), the second equality comes from the fact that $-e^{-\rho\tau} I$ is $\mathcal{F}_\tau$- measurable (Proposition 3.19 in Hu et al. (2017)), where $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}$.The last equality is the strong Markov property for G-SDEs (Theorem 4.4 in Hu et al. (2017)). To complete the proof, we can compute $\hat{\mathbb{E}}[\int_\tau^\infty e^{-\rho s} X_s^y ds]_{y=X_\tau^x}$ as in Proposition 1, replacing the deterministic time by τ. ■

[13] We have $\int_t^T \mathbb{E}^P[|\exp(-\rho(s-t)\pi_s)|ds] < \infty$.

C.4 Proposition 2

Proof. We look for a candidate solution of the form : $v(t,x) = e^{-\rho t}v(x)$. By the nature of the problem, we look for a solution v to the following free boundary problem, with parameter x_0 such that :

$$v(x) = \frac{1}{\rho - \mu}x - I \quad \text{for } x \in [x_0, \infty) \tag{101}$$

$$-\rho v(x) + \mu x v'(x) + \frac{1}{2}\min_{\sigma^2 \in [\underline{\sigma}^2, \overline{\sigma}^2]} \sigma^2 x^2 v''(x) = 0 \quad \text{for } x \in [0, x_0) \tag{102}$$

$$v(0) = 0 \tag{103}$$

$$v(x_0) = \frac{1}{\rho - \mu}x_0 - I \tag{104}$$

$$v'(x_0) = \frac{1}{\rho - \mu} \tag{105}$$

For now, let's assume that v is convex on $[0, x_0)$, equation 102 becomes :

$$-\rho v(x) + \mu x v'(x) + \frac{1}{2}\underline{\sigma}^2 x^2 v''(x) = 0 \tag{106}$$

This a second-order Cauchy-Euler equation, thus we seek a solution of the form :

$$v(x) = Ax^{\beta_1} + Bx^{\beta_2} \tag{107}$$

where $\beta_2 < 0 < \beta_1$ are the roots of the quadratic equation

$$\frac{1}{2}\beta(\beta - 1)\underline{\sigma} + \beta\mu - \rho = 9 \tag{108}$$

From condition 103 we get $B = 0$. By plugging 107 in 104 (value matching condition) and 105 (smooth pasting), we obstain the candidate solution :

$$x_0 = \frac{\beta(\rho - \mu)}{\beta - 1}I \tag{109}$$

$$v(x) = \mathbb{1}_{[0,x_0)}[(\frac{x_0}{\rho - \mu} - I)(\frac{x}{x_0})^{\beta}] + \mathbb{1}_{[x_0,\infty)}[\frac{x}{\rho - \mu} - I] \tag{110}$$

The function is convex on $[0, x_0)$ and $\mathcal{C}^2$ on $\mathbb{R}\backslash\{x_0\}$, by our verification Theorem 1, we have $V(x) = v(x)$ and in addition Theorem 1, $\tau^* = \inf\{t \in [0, \infty), X_t^x \geq X^*\}$ is the optimal stopping time. ■

C.5 Proposition 3

Straightforward calculations show that for any $\sigma > 0$:

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{(\frac{\mu}{\sigma^2} - \frac{1}{2})^2 + \frac{2\rho}{\sigma^2}} \tag{111}$$

It can be checked that $\frac{\partial \beta}{\partial \sigma} < 0$ and $\frac{\partial \pi^*}{\partial \beta} < 0$ from which we get $\frac{\partial \pi^*}{\partial \sigma} > 0$

C.6 Proposition 5

We first show that the function :

$$F\hat{(\pi)} = \hat{\mathbb{E}}[\int_0^\infty -e^{\rho t} U(X_t^x)] \tag{112}$$

is strictly convex. Let $\lambda \in [0,1]$, $x_1, x_2 > 0$ and X_1^t, X_t^2 the processes :

$$X_t^1 = \mu X_t^1 dt + X_t^1 dB_t \quad X_0^1 = x_1 \tag{113}$$

$$X_t^2 = \mu X_t^2 dt + X_t^2 dB_t \quad X_0^2 = x_2 \tag{114}$$

Then by the geometric nature of X_t^1 and X_t^2, we have, denoting $x_\lambda = \lambda x_1 + (1-\lambda)x_2$:

$$F(x_\lambda) = \hat{\mathbb{E}}[\int_0^\infty -e^{\rho t} U(\lambda X_t^1 + (1-\lambda) X_t^2) dt] \tag{115}$$

$$< \hat{\mathbb{E}}[\int_t^\infty e^{-\rho t}(-\lambda U(\lambda X_t^1) - (1-\lambda) U(X_t^2) dt] \tag{116}$$

$$\leq \lambda \hat{\mathbb{E}}[\int_0^\infty -e^{\rho t} U(X_t^1) dt + (1-\lambda)\hat{\mathbb{E}}[\int_0^\infty -e^{-\rho t} U(X_t^2) dt \tag{117}$$

$$= \lambda \hat{W}_R(x_1) + (1-\lambda)\hat{W}_R(x_2) \tag{118}$$

where in the first strict inequality we have used the strict convexity of $-U$, and in the second inequality we use the sub-additivity and positive homogeneity properties of $\hat{\mathbb{E}}$, and we have that $\hat{W}_R$ is strictly concave. Now, consider the following infinite horizon G-FBSDE :

$$Y_t = Y_T + \int_t^T -\rho Y_s - U(X_s^x) ds - \int_t^T Z_s dB_s - (K_T - K_t) \quad t \leq T \tag{119}$$

Where K is a decreasing G-martingale. From Hu and Wang (2017), this system has a unique solution. Moreover, this solution is explicit and can be expressed at time 0 as:

$$Y_0 = \hat{\mathbb{E}}[e^{\rho T} Y_T + \int_0^T -e^{-\rho t} U(X_t^x) dt] \tag{120}$$

Letting $T \to \infty$ and using the fact that Y_T is uniformly bounded (Theorem 3.3 in Hu and Wang (2017), we have :

$$Y_0 = \hat{W}_R(x) = \hat{\mathbb{E}}[\int_0^\infty -e^{-\rho t} U(X_t^x) dt] \tag{121}$$

In addition, by Theorem 5.4 in Hu and Wang (2017)[14] Y^π, and thus by unicity $\hat{W}_R$, is the unique viscosity solution to :

$$\rho \hat{W}_R(x) + \mu x \hat{W}_R'(x) + \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} [\frac{1}{2}\sigma^2 x^2 \hat{W}_R''(x)] - U(x) = 0 \tag{122}$$

[14]More precisely, since Hu and Wang (2017)'s focus is on optimal control of infinite horizon G-BSDEs, we apply their results to the degenerate case where the set of admissible controls is a singleton, thus obtaining an infinite horizon Feynman-Kac formula in the G-Framework.

Since $\hat{W}_R$ is strictly concave, the supremum in the above ODE is reached at $\overline{\sigma}$, thus, $\hat{W}_R$ is the unique viscosity solution to :

$$\rho\hat{W}_R(x) + \mu x\hat{W}'_R(x) + \frac{1}{2}\overline{\sigma}^2 x^2\hat{W}''_R(x) - U(x) = 0 \tag{123}$$

Noting that by Levy's charachterization of Brownian motions, $\overline{B}_t = \frac{B_t}{\overline{\sigma}}$ is a standard Brownian motion under $P^{\overline{\sigma}}$ and thus $X_t^x = \mu X_t^x dt + \overline{\sigma} d\overline{B}_t \;\; P^{\overline{\sigma}} - as$,we have, by the standard Feynman-Kac representation theorem :

$$\hat{W}_R(\pi) = \mathbb{E}^{P^{\overline{\sigma}}}[\int_0^\infty -e^{-\rho t}U(X_t^x)dt] \tag{124}$$

Thus we have :

$$W_R(x) = \min_{P\in\mathcal{P}} \mathbb{E}^P[\int_0^\infty e^{s-t}U(X_t^x)dt] = -\hat{W}_R(x) = \mathbb{E}^{P^{\overline{\sigma}}}[\int_0^\infty e^{-\rho t}U(X_t^x)dt] \tag{125}$$

To compute $F(x)$, we apply Theorem 9.18 in Karatzas et al. (1998) to the case $U(x) = \frac{x^{1-R}}{1-R}$, which yields the desired result.

C.7 Proposition 6

We proceed analogously to the proof of Proposition 2. Consider the free-boundary problem with parameter x_0 :

$$v(x) = \frac{x^{1-R}}{\delta(1-R)} - I \quad \text{for } x \in [x_0, \infty) \tag{126}$$

$$\rho v(x) + \mu x v'(x) + \frac{1}{2}\min_{\sigma^2\in[\underline{\sigma}^2,\overline{\sigma}^2]} \sigma^2 x^2 v''(x) \quad \text{for } x \in [0, x_0) \tag{127}$$

$$v(0) = 0 \tag{128}$$

$$v(x_0) = \frac{x_0^{1-R}}{\delta(1-R)} - \frac{I^{1-R}}{1-R} \tag{129}$$

$$v'(x_0) = \frac{x^{-R}}{\delta} \tag{130}$$

Using the same arguments as in the proof of Proposition 2, we obtain the following candidate solution :

$$x_0 = I(\frac{\delta\beta}{\beta + R - 1})^{1-R} \tag{131}$$

$$v(x) = \mathbb{1}_{[0,x_0)}[\frac{1}{\rho}U(I) + (\frac{U(x_0)}{\delta} - \frac{U(I)}{\rho})(\frac{x}{x_0})^\beta] + \mathbb{1}_{[x_0,\infty)}[U(x) - U(I)] \tag{132}$$

Since v is C^2 on $\mathbb{R}\backslash\{x_0\}$, by Theorem 1 we have $F(x) = v(x)$. In addition, $\overline{X}$ is the optimal investment threshold.

C.8 Proposition 9

Let us define $\hat{H}$:

$$\hat{H}(x) = \hat{\mathbb{E}}[\int_0^\infty -e^{-\rho(s)} P_s ds] \tag{133}$$

where :

$$P_s = \mu P_s ds + P_s dB_s - dK_s \tag{134}$$
$$P_0 = x \tag{135}$$
$$P_s \leq \overline{p} \tag{136}$$
$$\int_0^T (\overline{P} - P_s) dK_s = 0 \quad \forall T > 0 \tag{137}$$

Lin et al. (2013) proves existence and uniqueness of the couple (P, K). Using a similar argument as in the proof of proposition 5, we can show that $\hat{W}$ is convex. We now show that $\hat{H}$ can be identified with the solution of the boundary problem :

$$\rho v(x) - \mu x v'(x) - \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \frac{1}{2} \sigma^2 x^2 v''(x) = -x \quad x \in [0, \overline{P}] \tag{138}$$
$$v'(\overline{P}) = 0 \tag{139}$$
$$v(0) = 0 \tag{140}$$

Let $v \in \mathcal{C}^2$ be a solution to the above problem and fix $T \geq 0$, then by the extended G-Ito formula of Lin et al. (2013) applied to $v(P_T)$, we have, for all $t \leq T$:

$$v(P_t) = v(P_T) + \int_t^T [-\mu P_s v'(P_s)] ds \tag{141}$$
$$- \int_t^T [\frac{1}{2} P_s^2 v''(P_s) d\langle B \rangle_s - \int_t^T v'(P_s) dK_s - \int_t^T \mu P_s v'(P_s) dB_s \tag{142}$$
$$= v(P_T) + \int_t^T [\rho v(P_s) - \mu P_s v'(P_s) - \frac{1}{2} G(\sigma^2 P_s^2 v''(P_s))] ds \tag{143}$$
$$- \int_t^T [\frac{1}{2} P_s^2 v''(P_s) d\langle B \rangle_s + \int_t^T G(\frac{1}{2} \sigma^2 P_s^2 v''(P_s)) ds \tag{144}$$
$$- \int_t^T v'(P_s) dK_s - \int_t^T \mu P_s v'(P_s) dB_s \tag{145}$$

From condition 135 and the fact that $v'(\overline{P}) = 0$, using the fact that v solves the boundary problem, we have :

$$v(P_t) = v(P_T) + \int_t^T -\rho v(P_s) - P_s ds - \int_t^T \mu P_s v'(P_s) dB_s \tag{146}$$
$$- \int_t^T \frac{1}{2} P_s^2 v''(P_s) d\langle B \rangle_s + \int_t^T G(\frac{1}{2} \sigma^2 P_s^2 v''(P_s)) ds \tag{147}$$

Let :

$$Y_s = v(P_s) \tag{148}$$

$$Z_s = \mu P_s v'(P_s) \tag{149}$$

$$K_s = \int_0^s [\frac{1}{2}P_r^2 v''(P_r) d\langle B\rangle_r + \int_0^s G(\frac{1}{2}\sigma^2 P_r^2 v''(P_r))dr \tag{150}$$

Since K is a decreasing G-martingale (Peng (2010) Proposition 1.4 Chapter V), the triple (Y,Z,K) is the solution to the G-Backward Stochastic Differential Equation (G-BSDE) [15] :

$$Y_t = Y_T + \int_t^T f(Y_s, P_s)ds - \int_t^T Z_s dB_s - (K_T - K_t) \tag{151}$$

With driver $f(y, x) = -\rho y - x$. Using the property of G-BSDEs, we know that this BSDE has a unique solution. Moreover, by linearity of the driver, this BSDE has an explicit solution (Remark 3.3 in Hu et al. (2014b)):

$$Y_t = \hat{\mathbb{E}}_t[e^{-\rho(T-t)}Y_T + \int_t^T -e^{-\rho(s-t)}P_s ds] \tag{152}$$

Thus, by uniqueness, we have:

$$v(P_t) = \hat{\mathbb{E}}_t[e^{-\rho(T-t)}v(P_T) + \int_t^T -e^{-\rho(s-t)}P_s ds] \tag{153}$$

Since P_s has an upper reflecting barrier and v is continuous, we can let $T \to \infty$ and we obtain :

$$v(p) = \mathbb{E}[\int_0^\infty -e^{-\rho s}P_s ds] = \hat{H}(p) \tag{154}$$

Using the fact that $\hat{H}(p)$ is convex and solves the boundary problem, we get (Corollary 6.3 in Harrison (1985)):

$$\hat{H}(x) = \mathbb{E}[\int_0^\infty -e^{-\rho s}P_s ds] = \mathbb{E}^{P^{\underline{\sigma}}}[\int_0^\infty -e^{-\rho s}P_s ds] \tag{155}$$

Thus, coming back to $H(.)$, we have :

$$H(p) = -\hat{H}(p) = \mathbb{E}^{P^{\underline{\sigma}}}[\int_0^\infty e^{-\rho(s-t)}P_s ds] \tag{156}$$

which can be computed using Proposition 6.4 in Harrison (1985).

[15]see Hu et al. (2014a)

C.9 Proposition 11

Let us denote by G the value of an idle firm. Using Theorem 1, our definition of equilibrium, and the strong Markov property of (P, K) under $P^{\overline{\sigma}}$ (see Proposition 6.1 in Harrison (1985)), together with Proposition 9, the equilibrium threshold must satisfy :

$$-\rho + \mu p G'(p) + \min_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \frac{1}{2}\sigma^2 p^2 G''(p) = 0 \quad p \in [0, \overline{P}] \tag{157}$$

$$G(\overline{P}) = H(\overline{P}) - I \tag{158}$$

$$G'(\overline{P}) = H'(\overline{P}) \tag{159}$$

In addition $G(p) = 0$ on $[0, \overline{P}]$. Our result follows.

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