Optimal Stopping and Real Options under Volatility Ambiguity

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Abstract

This paper studies an optimal stopping problem with an ambiguity-averse decision-maker who perceives ambiguously the volatility of the underlying process. Ambiguous volatility is modeled by a set of non-dominated probability measures and the analysis requires significant departure from the standard theory. We use recent advances in non-linear expectation theory to characterize the optimal stopping time of a general optimal stopping problem and to reduce this problem to a free-boundary problem, analogous to the standard case. We then apply these results to the canonical irreversible investment model and to the perpetual American Call. We show that an increase in ambiguity accelerates investment, contrary to the well-known result that uncertainty delays investment. This result is also in stark contrast with previous work on drift ambiguity.

Keywords: Optimal Stopping;Ambiguity;Real-Options;Continuous-time

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1 Introduction

In economics and finance, real option problems are typically treated as optimal stopping problems : the investor chooses the optimal time to exercise an option. This approach has been fruitfully applied to the study of optimal exercising rules of American options or the timing of an irreversible investment. Perhaps one of the most important insight from this approach is that uncertainty creates an incentive to wait for the investor, as he can wait for a positive shock to invest while being protected from negative shocks by simply not investing. Uncertainty is usually understood as the volatility of the underlying process and is most of the time captured by a single parameter. In this paper, we show that by considering a more "robust" notion of uncertainty, which we call ambiguous volatility, the uncertainty-investment relationship can hold in the opposite direction.

To do so, we develop a general framework of optimal stopping under volatility ambiguity. By volatility ambiguity, we mean that the decision maker only takes a stance on the upper and lower bounds in which the volatility of the underlying process lies, and considers *any* process within these bounds as a possible volatility scenario. This volatility ambiguity is modeled as the decision maker considering a *set* of probability measures on the state space and ambiguity aversion is modeled by the decision maker acting according to the most pessimistic belief in this set for a given prospect, as axiomatized by Gilboa and Schmeidler (1989). Importantly, the set of priors resulting from ambiguous volatility is *non-dominated*, in the sense that the probability measures contained in this set do not agree on null events. We see the contribution of this paper as double. First, we provide a flexible framework to study the role of volatility ambiguity in optimal stopping problems. Second, we apply these results to the study of option exercise under volatility ambiguity in different contexts.

Why consider ambiguous volatility? Almost all the economics literature on ambiguity in continuous-time focuses on the particular case of drift ambiguity developed by Chen and Epstein (2002) : a reference measure defines the null events and the set of priors is obtained via a Girsanov transformation. This restricts the nature of ambiguity in two ways. First, it implies that ambiguity only concerns the drift of the underlying process. Second, it implies that all measures in the set of priors are mutually equivalent, which has specific implications in terms of preferences, as Epstein and Marinacci (2007) point out. In the "robustness" or "model uncertainty" approached, developped by Hansen and Sargent (2001), model misspecification is taken as an absolutely continuous perturbation of the approximating model, which results in the different models entertained by the decision maker differing only in drift term. This restriction of ambiguity to the drift of the underlying process is motivated by the convenience of working with equivalent probability measures, rather than by economic reasons. In practice forecasting volatility is a difficult task, as it is sensible to data and the chosen forecasting model. The importance of considering stochastic and time-varying volatility in macroeconomics and finance has been underlined by several authors(see for example, Drechsler (2013) and Bloom (2009)). The modelling of stochastic volatility often assumes a specific parametric form, but the dependence of the forecast on the chosen volatility model is high in this case. In order to make a safe, or robust, decision, an investor may want to consider different possible models. In our setting, the investor only takes a stance on the upper and lower bound of the volatility process, and considers *any* model that lies within those bounds. The investment decision (optimal stopping), is made by considering the most pessimistic model. As such, our framework can be considered as an optimal stopping framework under robust stochastic volatility.

While there is by now a large body of evidence for ambiguity-averse behavior, both experimentally and empirically, the empirical relevance of ambiguous volatility remains a question. Recent evidence, however, seems to show that ambiguous volatility does matter for investment decisions. For example, Anderson et al. (2017), show that ambiguous volatility is crucial to understand investment decisions of US family owners, and in particular, what they call the "family firm puzzle". Similarly, Branger et al. (2016) construct a measure of ambiguity about macroeconomic volatility and show that an increase in volatility ambiguity predicts a significant increase in average excess equity returns.

The non-equivalence of priors that underlies the modelling of volatility ambiguity raises several technical difficulties. In particular, the standard techniques adopted in the probability space framework are of little use, as the set of priors contains mutually singular measures. Similarly, the definition of a multiple prior conditional expectation is not straightforward. To circumvent these difficulties, we use recent elements of non-linear expectation theory, in particular the G-expectation developed in Peng (2010). Starting from a non-linear expectation space,Peng (2010), develops a non-linear stochastic calculus theory that does not require the probability space framework and that accomodates ambiguous volatility. This framework is particularly useful as most of the tools of standard stochastic calculus and stochastic analysis have their non linear counterpart : the stochastic integral, Ito's formula, martingales, stopping times etc.

Our first task is to define a set of priors and a corresponding conditional expectation process under ambiguous volatility. We follow Epstein and Ji (2013), which guarantees that our expectation is dynamically consistent. This property is crucial as it allows to apply the principle of dynamic programming. In our case, it allows to directly apply the general characterization of the optimal stopping obtained by Nutz et al. (2015). Analogous to the standard theory, stopping is optimal when the payoff equals the value function of the problem.

Our next step is to use this result to characterize further the value function. This is done through a verification theorem that holds for stopping problems where the payoff is a function of a geometric (G-)Brownian motion.Using results in nonlinear stochastic calculus theory, we show that the value function of an optimal stopping problem under ambiguous volatility can be identified with the solution of a free boundary problem, analogous to the standard optimal stopping theory. The resulting Hamilton-Jacobi-Bellman equation is similar to the classical one, expect that it is non-linear in its second-order term, reflecting the decision maker's ambiguity aversion.

We then apply these results in several real option problems. We start with the irreversible investment model of McDonald and Siegel (1986), in which a firm has an investment opportunity, and must decide when to invest and pay some fixed cost, in order to start generating profit from the project. Using our HJB equation, we show that, analogous to the standard case, the firm follows a trigger strategy and invests when the cash flow reaches a certain threshold. In addition, the worst-case scenario for the firm is the lowest possible volatility, while the value of the investment once the investment option has been exercized is unaffected by ambiguity. Intuitively, since the firm is protected against downward risk pre-investment, the worst-case is low volatility before the option is exercized. Once the firm has invested, because the firm is risk neutral, volatility does not affect the value of the investment, and ambiguity does not play a role. A first consequence is that an increase in ambiguity erodes the value of investment, by lowering the value of investment before the investment option is exercized, and leaving the value post-investment unaffected. A second consequence is that the investment trigger becomes lower when ambiguity increases. Since the firm uses the prior that yields the lowest volatility, it expects less upward shocks, and the value of waiting decreases. These two reults on the effect of ambiguity on the value and timing of irreversible investment contrast with the standard results on the effect of uncertainty (taken in the sense of volatility) in the irreversible investment literature.

We then turn to the case where the firm is risk averse. In standard irreversible investment theory, the firm is assumed risk neutral as a consequence of financial market completeness. As several authors have shown, ambiguous volatility precludes the completeness of financial markets, which exposes firms to undiversifiable risk. Considering risk aversion is thus natural in an ambiguous volatility framework. We show that under risk-aversion, the worst-case prior becomes path dependent : the firm anticipates low volatility before investment and high volatility after investment.

This paper is organized as follows. The next section discusses the related literature. Section 3 presents a general theory for optimal stopping problems under volatility ambiguity and a verification theorem for markovian problems.

Section 4 applies these results to two real option problems : irreversible investment and the American call option. Section 5 provides some discussions. Section 6 concludes. The proofs are relegated to the Appendix, where we also provide a self-contained introduction to some results in the G-framework which we use in the proofs, as well as the mathematical construction of the set of priors.

2 Related literature

Several papers have provided a characterization of optimal stopping problems under ambiguity. Riedel (2009) and **?** provide a Snell Envelope characterization, when the set of priors is composed of mutually equivalent measures, in discrete and in continuous time. The more involved problem of optimal stopping with a set of non-dominated priors has been recently studied by Ekren et al. (2014), Bayraktar and Yao (2014) and Nutz et al. (2015). These paper provide a characterization of the optimal stopping problem in terms of a *non linear* Snell envelope and show that a characterization analogous to the standard one can be obtained. The results in these papers are technically involved and aim at obtaining general characterizations for a large class of optimal stopping problems. Our paper makes the link between these papers and economic theory. We restrict attention to a class of problems which includes most optimal stopping problems in economics, and we show how we can tractably analyze these problems and derive economic implications of volatility ambiguity.

In a more applied literature, Nishimura and Ozaki (2004) study a job search problem under Choquet expected utility and find that ambiguity aversion accelerates acceptance of a job offer. Nishimura and Ozaki (2007) study an irreversible investment problem in continuous-time. In their model, the firm perceives ambiguously the drift of the stream of profit, and they show that an increase in what they call Knightian uncertainty lowers the value of investment and increases the incentives to wait for the firm. Our results complement theirs, by showing that Knightian uncertainty can in fact accelerate investment, if it is taken in the sense of volatility ambiguity. In a general model in discret time, Miao and Wang (2011) study option exercise under ambiguity. They show, using a specific parameterization, that the role of ambiguity on the value of waiting depends crucially on whether the payoff is a lump sum or a flow of profit. We discuss this result in relation to ours at the end of Section 3. Other papers that apply the drift ambiguity model to irreversible investment include Schröder (2011) in the α-maxmin framework and Asano (2010) in an application to the timing of environmental policies. See also Thijssen (2011) for irreversible investment under incomplete markets and Trojanowska and Kort (2010) for irreversible investment in a finite time horizon.

Recently, Epstein and Marinacci (2007) study optimal learning in an Ellsberg environment, where the decision maker is allowed costly sampling from the urn and decides when and how to bet. They show that learning opportunities may be rejected by an ambiguity averse decision maker, even when the cost of sampling is relatively low.

Our paper belongs to a fairly large literature on ambiguity averse preferences in continuous-time settings. As we have mentioned, the vast majority of this literature considers the case of drift ambiguity. Our paper belongs to a smaller set of papers that study the effect of ambigous volatility in economic or financial settings. The case of ambiguous volatility and its implications was studied in Epstein and Ji (2013) and Epstein and Ji (2014) for asset pricing, (Beissner, 2017) for general equilibrium theory, Riedel and Beißner (2014) for a study of Arrow-Debreu equilibria by continuous trading , Lin and Riedel (2014) for optimal consumption and portfolio choice, and Vorbrink (2014) for arbitrage theory.

3 Optimal stopping under volatility ambiguity

In what follows, we will consider that the process that drives the payoff of the decision maker follows the dynamic :

$$d\pi_t = \mu \pi_t dt + \pi_t dB_t \tag{1}$$

In the standard optimal stopping literature in economics and finance, B_t is a standard Brownian motion on a given probability space, and π_t is thus a geometric Brownian motion. In this paper, we want to study optimal stopping decisions of a decision maker who considers B as a process with ambiguious volatility. More precisely, the decision maker only knows that the quadratic variation of the process B lies in a given interval :

$$\langle B \rangle_t \in [\underline{\sigma}^2 t, \overline{\sigma}^2 t] \tag{2}$$

with $0 < \underline{\sigma} < \overline{\sigma}$. Contrary to the case where B is a standard Brownian motion, this quadratic variation needs not be deterministic or equal to elapsed time. The decision maker does not know the form of the volatility process of B motion and does not take a stand on this form, he only believes that this process lies in $[\underline{\sigma}^2, \overline{\sigma}^2]$. These beliefs are represented by a set of priors $\mathcal{P}^\Theta$ on Ω the set of continuous paths. Each P^σ in $\mathcal{P}^\Theta$ corresponds to a volatility process $(\sigma_t)_{t\geq 0}$ that lies in $[\underline{\sigma}^2, \overline{\sigma}^2]$.

The decision maker solves an optimal stopping problem and is ambiguity averse in the sense of Gilboa Schmeidler, that is, he solves the optimal stopping problem using the prior in $\mathcal{P}^\Theta$ that yields the lowest expected payoff. Formally, the problem is :

$$\sup_{\tau \in \Gamma} \inf_{P \in \mathcal{P}^\Theta} \mathbb{E}^P[f(\tau, \pi_\tau)] \tag{3}$$

where$f(t,.) : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\mathbb{E}^P$ is the standard expectation with respect to the probability measure P. When the set of priors is a singleton, $\mathcal{P}^\Theta = \{P^\sigma\}$, the problem becomes a classical optimal stopping problem under P^σ.

When the decision maker only considers upper and lower bounds for the volatility process, the associated set of priors becomes non-dominated, that is, somes measures in $\mathcal{P}^\Theta$ are mutually singular. To see this, consider the quadratic volatility under $P^{\underline{\sigma}}$ and $P^{\overline{\sigma}}$, we have :

$$P^{\underline{\sigma}}(\langle B \rangle_t = \underline{\sigma}^2 t) = 1 = P^{\overline{\sigma}}(\langle B \rangle_t = \overline{\sigma}^2 t) \tag{4}$$

Because of this singularity, instead of the standard filtered probability space, we will consider the tuple $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathcal{P})$ to model uncertainty and information. Since we consider priors that do not necessarily agree on null events, the almost-surely characterization of random variables and stochastic process becomes irrelevant, as we need to consider all priors. Following Denis and Martini, we adopt the *quasi-surely* terminology. A set $A \in \mathcal{F}$ is called *polar* if $P(A) = 0$ for all $P \in \mathcal{P}$. A property is said to hold *quasi-surely* if it holds outside of polar sets. In what follows, equations involving random variables and stochastic processes hold quasi-surely, unless otherwise noted. When a property holds quasi-surely, it holds almost-surely for all priors in $\mathcal{P}$.

A second specificity of this framework is that the domain of random variables and stochastic processes has to be modified to accomodate the nonequivalence of priors. Similarly, the domain of the stochastic integral has to be adapted. As these are essentially technical elements that only appear in the proofs, we detail these in the appendix.

Once the set of priors is fixed, the definition of a worst-case expectation is straightforward. The non-equivalence of priors is however problematic when defining a notion of conditional multiple prior expectation. The naive updating, by considering $\inf_{P\in\mathcal{P}^\Theta} \mathbb{E}^P[f(\pi_t)|\mathcal{F}_t]$ fails directly, as this expression is not defined P-almost surely for every P in $\mathcal{P}^\Theta$. As Epstein and Ji (2013) show, multiple-prior conditional expectation can be defined by suitably defining sets of conditional beliefs $\mathcal{P}^\Theta(\omega, t)$ for all $(\omega, t) \in \Omega \times [0, T]$. The conditional expectation is then defined as :

$$\mathcal{E}_t(\xi) = \inf_{P\in\mathcal{P}^\Theta(\omega,t)} \mathbb{E}^P[\xi] \;\forall(\omega, t) \in \Omega \times [0, T] \tag{5}$$

The details of the mathematical construction of $\mathcal{P}^\Theta$ and $\mathcal{P}^\Theta(\omega, t)$ can be found in Appendix A. It is important that under this construction of the multiple prior conditional expectation, the decision maker is dynamically consistent, and the law of iterated expectation holds, that is, for $s \le t \le T$:

$$\mathcal{E}_s(\mathcal{E}_t(\xi)) = \mathcal{E}_s(\xi) \tag{6}$$

This tower property is crucial to apply dynamic programming, which is the approach taken by Nutz et al. (2015). With the contruction of $\mathcal{P}^\Theta$ and $\mathcal{P}^\Theta(\omega, t)$, we can directly apply their results to get a general characterization of the optimal stopping time.

3.1 Optimal stopping time

In classical optimal stopping theory, the optimal stopping time is characterized as the first hitting time between the value function and the payoff process. Nutz et al. (2015) shows that, under suitable conditions, the analogous characterization holds true for a non-linear expectation that satisfies time consistency, which includes our expectation[1]. We consider optimal stopping problems of the form :

$$\sup_{\tau\in\Gamma} \inf_{P\in\mathcal{P}^{\Theta}} \mathbb{E}^P[f(\tau,\pi_\tau)] \tag{7}$$

where the set of priors is defined in the appendix.

Theorem 1 *Let*

$$V_t = \sup_{\tau\in\Gamma_t} \mathcal{E}_t[f(t,\pi_t)] \tag{8}$$

Where Γ_t is the set of stopping times with value in $[t,T]$. Consider our optimal stopping problem , then :

- *There exists an optimal stopping time τ^*, that is*

$$\inf_{P\in P^{\Theta}} \mathbb{E}[f(\tau^*,\pi_{\tau^*})] = \sup_{\tau\in\Gamma} \inf_{P\in P^{\Theta}} \mathbb{E}^P[f(\tau,\pi_\tau)] \tag{9}$$

- *τ^* is the first hitting time between V_t and the payoff process :*

$$\tau^* = \inf\{t\in[0,T] : V_t = f(t,\pi_t)\} \tag{10}$$

In fact, it can be shown (see the proof of Theorem 2 or (Nutz et al., 2015)), that the analogy with the classical case goes further. The value function of our problem is a supermartingale (with respect to $\mathcal{E}$), and a martingale until it is optimal to stop.

3.2 Hamilton-Jacobi-Bellman equation

We now proceed to charactherize the value function of our problem through a Hamilton-Jacobi-Bellman type equation. In problems without ambiguity, the value function V of the optimal stopping problem

$$\sup_{\tau\in\Gamma} \mathbb{E}[f(t,X_t)] \tag{11}$$

[1] In fact, their results are very general in the sense that they apply to situations where the bounds in which the volatility process lies are path and time dependent.

where $dX_t = X_t(\mu dt + \sigma dB_t)$ is an (unambiguous) Brownian motion, solves the following HJB equation :

$$\max_{t,x}\{f(t,x) - v(t,x), \frac{\partial v}{\partial t}(t,x) + \mu x\frac{\partial v}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2}(t,x)\} = 0 \qquad (12)$$

Heuristically, this equation says that at each time t, either stopping is optimal because we have $f(t,x) = V_t$, or waiting is optimal because the value function stays constant in expectation.

Under ambiguous volatility and with our suitable set of priors, we can expect a similar equation, with the exception that the decision maker chooses the worst case for the volatility *for each* value of π_t. Intuitively, equation 12 should become :

$$\max\{f(t,x) - v(t,x), \frac{\partial v}{\partial t}(t,x) + \mu x\frac{\partial v}{\partial x}(t,x) + \min_{\sigma^2 \in [\underline{\sigma^2}, \overline{\sigma^2}]}\{\frac{1}{2}\frac{\partial^2 v}{\partial x2}(t,x)x^2\sigma^2\}\} = 0$$

Our next theorem confirms this intuition. We show that if we can find a suitable function that solves the above equation, then we have found the value function of our optimal stopping problem.

Theorem 2 *Let $v(t,x)$ be a $\mathcal{C}^1$ function everywhere and $\mathcal{C}^2$ except on a finite number of points that solves the following equation :*

$$\max_{(t,x)\in[0,T]\times\mathbb{R}}\{f(t,x)-v(t,x), \frac{\partial v}{\partial t}(t,x)+\mu x\frac{\partial v}{\partial x}(t,x)+\min_{\sigma^2 \in [\underline{\sigma^2}, \overline{\sigma^2}]}\{\frac{1}{2}\frac{\partial^2 v}{\partial x2}(t,x)x^2\sigma^2\}\} = 0 \qquad (13)$$

And $v(T,\pi_T) = f(T,\Pi_T)$. Then $V_t = v(\pi_t, t)$.

The interpretation of this equation remains the same : either stopping is optimal, or the value function is constant. The term on the right represents, loosely speaking, $\mathcal{E}_t(dV_t)$ and can be computed by using the G-Ito formula. The G-Ito formula is analgous to the standard one, except that the quadratic variation of the Brownian motion is ambiguous, which induces the non-linearity of the second-order term in the equation.

4 Application : real option problems

4.1 Irreversible investment

We start with the canonical irreversible investment model of McDonald and Siegel (1986). In the irreversible investment model, the firm possesses an investment opportunity (a real option), and can, at any time t, spend a fixed cost

I to install the project and start generating profit. This model has a wide range of applications : macroeconomic dynamics (see Bertola and Caballero (1994), Caballero et al. (1996)), the timing of environmental policies (Pindyck (2000), Pindyck (2002)), optimal pricing and quality choice ((Pennings, 2004)). The project generates a stream of profit given by :

$$d\pi_t = \mu dt + \pi_t dB_t \tag{14}$$

$$\pi_0 > 0 \tag{15}$$

where B_t is the G-Brownian motion as defined in section 2. The firm chooses the optimal time at which it decides to pay an irreversible cost I and starts generating profit from the project. The firm is ambiguity averse and has a discount factor ρ . We assume that $\rho > \mu$. The irreversible investment problem writes :

$$\sup_{\tau \in \Gamma} \inf_{P \in \mathcal{P}^\Theta} \mathbb{E}^P [\int_\tau^T e^{-\rho t} \pi_t dt - e^{-\rho \tau} I] \tag{16}$$

The value function for the firm is given by :

$$\sup_{\tau \in \Gamma} \mathcal{E}_t [\int_\tau^T e^{-\rho t} \pi_t dt - e^{-\rho \tau} I] \tag{17}$$

In what follows, when we discuss the *value of investment*, that is, the value function of the firm, we distinguish between the value of the investment option, which corresponds to the value function *before* the investment has been operated, and the exercized investment option, which is the value function *after* the investment has been operated.

4.1.1 Value of the exercized investment option

To make clear the role of volatility ambiguity, we start by discussing the value of the exercized investment option for the firm. The value of an investment option exercized at time t is given by :

$$W(\pi_t, t) = \mathcal{E}_t [\int_t^T e^{\rho(s-t)} \pi_s ds] \tag{18}$$

$W(\pi_t, t)$ represents the expected profit generate by the project from date t. Using the fact that the firm is risk neutral, we can characterize this value, as the next proposition shows.

Proposition 1 *The value of the utilized patent under volatility ambiguity is given by :*

$$W(\pi_t, t) = \frac{1 - e^{-(T-t)(\rho - \mu)}}{\rho - \mu} \pi_t \tag{19}$$

The intuition for this result is straightforward : the firm being risk-neutral, once the investment has been made, the value of the project is simply its net present value. In fact, ambiguity aversion does not impact this value, because the net present value is constant across any volatility process for the underlying Brownian motion.

This result contrasts with Nishimura and Ozaki (2007) who find that the value of the utilized patent is lower under Knigthian uncertainty. The difference between the two results come from the difference in the type of ambiguity considered. (Nishimura and Ozaki, 2007) consider a set of measures obtained by a Girsanov transformation with respect to some reference measure, and thus the firm is ambiguous about the drift of the profit stream. Under drift ambiguity, once the firm has invested, the value of the exercized investment option is the net present value under the lowest possible drift.

We would like to point out that Proposition 1 holds under a more general specification of the set of priors. The proof in Appendix shows that Proposition 1 is true for any set of priors $\mathcal{P}^{\Theta}$ generated by a possibly time and path dependent interval $[\overline{\sigma}_t(\omega), \underline{\sigma}_t(\omega)]$. In particular, the result is robust to any particular stochastic interval that accomodates learning.

4.1.2 Infinite-horizon case

It is well-known that the analysis of the irreversible investment problem becomes much more tractable when the time horizon (expiration date of the patent) becomes infinite. In fact, in infinite horizon, the problem can be solved analytically, which simplufies greatly the comparative statics exercise. As our results on optimal stopping have been fomulated for a finite horizon T, we let, for the remaining of this section, T go to infinity, and we assume that the relationship between the different variable holds in infinite horizon, specifically, our HJB equation.

Using Proposition 1, in infinite horizon, the value of the utilized patent becomes :

$$W(\pi_t) = \frac{1}{\rho - \mu}\pi_t \tag{20}$$

Our infinite-horizon problem can be reformulated as (see Lemma 1 in Appendix) :

$$\sup_{\tau \in \Gamma} \inf_{P \in \mathcal{P}^{\Theta}} \mathbb{E}^P[e^{-\rho\tau}(\frac{1}{\rho - \mu}\pi_\tau - I)] \tag{21}$$

Denote by V the value function for this problem, then it solves the following HJB equation :

$$\max\{\frac{1}{\rho - \mu}\pi - I - V(\pi), -\rho + \mu\pi V'(\pi) + \min_{\sigma^2 \in [\underline{\sigma}^2, \overline{\sigma}^2]} \frac{1}{2}\sigma^2\pi^2 V''(\pi)\} = 0 \tag{22}$$

This problem can be solved in a standard way (see Appendix) : the optimal investment strategy is a trigger strategy in which the firm invests when the profit process reaches a certain threshold. This result is summarized in the following proposition.

Proposition 2 *The optimal investment strategy for the firm is τ^* such that*

$$\tau^* = \inf\{t \in [0, \infty), \pi_{\tau^*} \geq \pi^*\} \tag{23}$$

where $\pi^* = \frac{\beta(\rho-\mu)}{\beta-1} I$ and β is the positive root of the quadratic equation :

$$\frac{1}{2}\beta(\beta-1)\underline{\sigma}^2 + \beta\mu - \rho = 0 \tag{24}$$

And the value function of the firm is given by :

$$V(\pi_t) = \begin{cases} (\frac{I}{\beta-1})^{1-\beta}\beta^{-\beta}\pi_t^\beta & if \;\; \pi_t < \pi^* \\ \frac{1}{\rho-\mu}\pi_t - I \;\; if \;\; \pi_t \geq \pi^* \end{cases} \tag{25}$$

As can be seen from equation 24, the ambiguity-averse firm behaves as a subjective expected utility firm, with prior belief $P^{\underline{\sigma}}$, the probability that induces the constant volatility process $\underline{\sigma}$. Remember that the firm being risk neutral, volatility gives the firm opportunities to invest as a positive shock occurs, thus, the worst-case for the firm is that those positive shocks do not occur (or occur with less magnitude). As a result, the worst-case prior is the lowest possible volatility.

4.1.3 Comparative statics

A major result of the real option literature is that uncertainty increases the value of an irreversible investment and creates an incentive to postpone investment. Using our analytical results, we show that this result is reversed if we take uncertainty to be in the sens of volatility ambiguity. In particular, we show that an increase in ambiguity *reduces* the value of an investment opportunity and *accerlerates* investment. We first define an increase in ambiguity.

Definition 1 *Consider two decision makers, A and B who have subjective beliefs $\mathcal{P}^{\Theta^A}$ and $\mathcal{P}^{\Theta^B}$. We say that A perceives more ambiguity than B if $\Theta^B \subset \Theta^A$ with $0 < \underline{\sigma}^A < \underline{\sigma}^B < \overline{\sigma}^B < \overline{\sigma}^A$.*

Intuitively, ambiguity increases as the set of probability measures considered by the decision maker increases. With this definition in hand we can now compare the investment strategy and the value of investment for two firms who differ in the ambiguity they perceive about the investment.

Proposition 3 *Consider two decision makers, A and B facing our irreversible investment problem. If A perceives more ambiguity than B, then the value of the investment opportunity is higher for B than for A.*

To understand the role played by ambiguity here, one has to look at the effect of ambiguity on the value of investment *before* and *after* the investment is made. As we have seen in Proposition 2, the ambiguity averse firm that contemplates an investment opportunity behaves as a subjective expected utility firm with belief $P^{\underline{\sigma}}$. At the same time, we have seen that the value of an already-made investment is unaffected by ambiguity, and by extension, an increase in ambiguity. As a result, when ambiguity increases, before investing, the firm expects less "positive opportunities" (good shocks), which reduces the ex-ante value of the investment opportunity. Once the investment is made, the value is unaffected by ambiguity. As a consequence, the value of the investment opportunity decreases with an increase in ambiguity.

Proposition 4 *Consider two decision makers, A and B facing our irreversible investment problem. If A perceives more ambiguity than B, then* $\pi_A^* < \pi_B^*$.

Since the firm's payoff is not affected by ambiguity after the investment trigger is reached, the firm has no incentive to delay investment to avoid ambiguity. Before the investment threshold is reached, the firm is affected by ambiguity, which leads the firm to anticipate less upward shocks, diminishing the incentives to wait.

4.2 Irreversible investment under risk aversion

In the previous section, we carried the analysis under the assumption that the firm is risk-neutral. This assumption is common in the irreversible investment literature and follows from the underlying assumption of financial markets completeness. Facing complete markets, investors are able to diversify their portfolios, making them neutral to project-specific risk. There are several reasons why investors may face incomplete markets. Among them, is the presence of ambiguous volatility, see Vorbrink (2014) and Epstein and Ji (2013) [2]. It thus seems natural to analyze the interaction between volatility ambiguity and risk aversion, and its effect on investment.

We study the irreversible investment problem of section 4.1 under the assumption that the firm has additive expected utility risk preferences. The problem becomes

$$\sup_{\tau\in\Gamma}\inf_{P\in P^{\Theta}}\mathbb{E}^P\Big[\int_{\tau}^{\infty}e^{-\rho t}U(\pi_t)dt-e^{-\rho\tau}U(I)\Big] \tag{26}$$

[2] See also Mukerji and Tallon (2001) for incompleteness of financial markets when agents are Choquet expected utility maximizers

In particular, to facilitate the analysis, we assume CRRA risk preferences of the form[3] :

$$U : \mathbb{R} \to \mathbb{R} \quad x \to \frac{x^{1-R}}{1-R} \tag{27}$$

With R $>$ 0 and logarithmic utility when $R = 1$. With this functional form assumption, we can derive the value of an exercized investment option under risk aversion, which is given in the next proposition.

Proposition 5 *Assume*

$$\delta = \rho + (R-1)(\mu - \frac{1}{2}\overline{\sigma}^2 R) > 0 \tag{28}$$

Under CRRA risk preferences, the value of an exercized investment option is given by :

$$W(\pi_t) = \frac{\pi_t^{1-R}}{(1-R)\delta} \tag{29}$$

In contrast to proposition 1, volatility ambiguity matters for the value of the exercized investment option when the firm is risk averse. Once the firm has exercized the investment option, it starts to receive a payoff flow which depends on π_t, as the firm is risk-averse, the worst-case in terms of volatility is that this payoff flow is highly volatile. The worst-case prior is thus $P^{\overline{\sigma}}$, the prior that yields the highest possible volatility for π_t. As a consequence, volatility ambiguity decreases the value of the exercized investment option.

Proposition 6 *The optimal investment strategy for the irreversible investment problem under risk-aversion and ambiguous volatility is a trigger strategy, under which the firm invests if $\pi \geq \pi^*$, where :*

$$\pi^* = I(\frac{\delta\beta}{\beta + (R-1)})^{\frac{1}{1-R}} \tag{30}$$

The value function of the firm is given by :

$$F(\pi_t) = \begin{cases} \frac{1}{\rho}U(I) + (\frac{1}{\delta}U(\pi^*) - \frac{1}{\rho}U(I))(\frac{\pi_t}{\pi^*})^{\beta} & \text{if} \quad \pi_t < \pi^* \\ \frac{1}{\delta}\pi_t - U(I) & \text{if} \quad \pi_t \geq \pi^* \end{cases} \tag{31}$$

Recall that :

$$\frac{1}{2}\beta(\beta-1)\underline{\sigma}^2 + \beta\mu - \rho = 0 \tag{32}$$

$$\delta = \rho + (R-1)(\mu - \frac{1}{2}\overline{\sigma}^2 R) \geq 0 \tag{33}$$

[3]Assuming this parametric form allows for analytical solutions. For a general treatment of irreversible investment inder risk aversion (without ambiguity), see Hugonnier and Morellec (2007)

Thus, under the optimal strategy τ^*, the worst-case prior for the firm is the prior P^* associated to the path-dependent process :

$$\sigma_t^* = \mathbb{1}_{t<\tau^*}\underline{\sigma} + \mathbb{1}_{t\geq\tau^*}\overline{\sigma} \tag{34}$$

Before investing,the firm believes that the process has low volatility, thus anticipating low or few investment opportunities. However, the firm changes its belief about the volatility right after investing, and anticipates higly volatile payoffs, which is the worst case as the firm is risk averse.

Our result on the effect of an increase in ambiguity on the value of investment carries to the case of the risk averse firm. In fact, the erosion of the value of investment is magnified by the presence of risk aversion, because an increase in ambiguity decreases the value of the investment option *and* the value of the exercized investment option. This can be seen easily when considering the value function above the investment trigger :

$$\frac{\pi_t^{1-R}}{\rho(1-R)+(1-R)(1+r)(\mu-\frac{1}{2}\overline{\sigma}^2 R)} \tag{35}$$

which decreases with the upper bound $\overline{\sigma}$: the risk-averse firm dislikes increases in risk.

The effect of an increase in ambiguity on the investment threshold when the firm is risk averse is less clear. The reason is that the investment trigger now depends on both the lower and the upper bound of possible volatilities. To deal with this issue we will restrict attention to *homogeneous* increases in ambiguity.

Definition 2 *A perceives homogeneously more ambiguity than B if* $\underline{\sigma}^B - \underline{\sigma}^A = \overline{\sigma}^A - \overline{\sigma}^B$

This definition is more restictive than definition, as we require the increase in perceived ambiguity to be the same at the upper and lower bound of the interval. This definition allows to quantify the increase in ambiguity, which makes comparative statics easier.

Proposition 7 *Suppose A and B share the same level of risk aversion, and A perceives homogeneously more ambiguity than B, then there exists a threshold level of risk aversion above which A invests later than B and below which B invests later than A*

When the firm is risk averse, the effect of an increase in ambiguity is ambiguous. This is the result of two conflicting effects, which are due to the path-dependence of the worst-case prior. Before investment, the worst-case is the lowest constant volatility, which gives an incentive to the firm to accelerate investment. On the other hand, as soon as the firm invests, the worst-case becomes the highest volatility, and by investing, the firm exposes itself to this high volatility. Hence the incentive to delay investment. When risk aversion is high, the second effect dominates, and the firm delays investment when ambiguity increases. When risk aversion is relatively low, the first effect dominates, and the firm accelerates investment.

4.3 American options

We now treat the case of the American Call and show that the intuition about the effect of volatility ambiguity on the value of waiting is analogous to the case of irreversible investment. The main difference between the timing of an American option and an irreversible investment lies in the payoff structure. An American option yields a lump-sum payoff, while an irreversible investment yields a flow payoff. As Miao and Wang (2011) showed, ambiguity can have a different effect on stopping decisions, depending on whether stopping yields a lump-sum or a flow payoff. Through the example of the American option, we show that this does not necessarily hold under ambiguous volatility.

The payoff of an American call option is given by :

$$f(t, \pi_t) = e^{-rt}(\pi_t - K)^+ \tag{36}$$

where $r > 0$ is the interest rate and $K > 0$ the strike price. Again, for analytical convenience and to allow comparative statics, we consider the case of the American perpetual call. The optimal stopping problem of the investor is then :

$$\sup_{\tau \in \Gamma} \inf_{P \in \mathcal{P}^{\Theta}} \mathbb{E}^P[e^{-r\tau}(\pi_\tau - K)^+] \tag{37}$$

The analysis can be carried out much in the same way as in the previous section. The investment strategy is given in the next proposition :

Proposition 8 *The optimal investment strategy is τ^* such that*

$$\tau^* = \inf\{t \in [0, \infty), \pi_{\tau^*} \geq \pi^*\} \tag{38}$$

where $\pi^* = \frac{\beta}{1-\beta} I$ and β is the positive root of the quadratic equation :

$$\frac{1}{2}\beta(\beta - 1)\underline{\sigma}^2 + \beta\mu - \rho = 0 \tag{39}$$

As in the irreversible investment problem, the investor acts as a bayesian investor who holds the belief associated to the constant lowest volatility $\underline{\sigma}$. As result, the value of the option is lower under ambiguity, and ambiguity lowers the value of waiting, resulting in a lower stopping threshold. The same intuition holds for the perpetual American put, for which the payoff is given by :

$$f(t, \pi_t) = e^{-rt}(K - \pi_t)^+ \tag{40}$$

We can also show that the American straddle option yields the same result on ambiguity and the value of waiting, even if the payoff is non-monotonic. This is clear intuitively, as an American straddle combines the American put and the American call :

$$f(t, \pi_t) = e^{-\rho t}|\pi_t - K| \tag{41}$$

4.4 Discussion

4.4.1 Ambiguity, ambiguity aversion

In our comparative statics exercise, we have interpreted an expansion of the set of priors, and thus an expansion of the interval $[\overline{\sigma}^2, \underline{\sigma}^2]$, as representing an increase in the ambiguity perceived by the decision maker. This interpretation facilitates discussions on the role of ambiguity as we can compare an increase in perceived ambiguity and an increase in what is typically called uncertainty in the real-options literature, that is, volatility. This interpretation is however problematic as the multiple-prior theory of Gilboa and Schmeidler (1989) is purely subjective, which entails that the set of priors represents perceived ambiguity *and* attitude towards ambiguity. A rationale for interpreting $[\overline{\sigma}^2, \underline{\sigma}^2]$ is provided in Gajdos et al. (2008), where the decision maker has access to some probabilistic information, in the form of a set of priors, and chooses according to a maxmin criterion with respect to a *subset* of this set. The subset is given by a contraction function that represents ambiguity aversion : the more contracted the subset, the less the DM is ambiguity averse. If we are to interpret an expansion of $[\overline{\sigma}^2, \underline{\sigma}^2]$ as an increase in ambiguity aversion, then our model predicts that in the presence of ambiguous volatility, an increase in ambiguity aversion accelerates option exercise. In this case, ambiguity aversion acts in the opposite direction of risk aversion : as Hugonnier and Morellec (2007) show, the presence of risk aversion in an irreversible investment model delays investment.

5 Conclusion

This paper develops a framework to study optimal stopping problems under volatility ambiguity. Modelling ambiguous volatility requires a set of non equivalent priors, which requires departure from the standard probability space framework. To circumvent this difficulty, we exploit recent results in non-linear expectation theory, in particular, we use the G framework developed by Peng and the associated G-stochastic calculus, which provides the non linear counterpart to the standard theory.

We apply our results to option exercise. In an irreversible investment model, we show that volatility ambiguity accelerates option exercise. This result contrasts with the standard result in real option theory, according to which uncertainty increases to value of waiting. Our result complement previous finding on the effect of ambiguity on option exercise, which were restricted to drift ambiguity. We show that drift and volatility ambiguity have opposite effect on the timing of investment, a hint of the rich domain of possibilities brought by ambiguity. Being a first step, the paper has focused on rather simple and canonical applications in which the worst-case could be easily identified. We saw that adding risk aversion, a natural assumption under ambiguous volatility, to these vanilla cases suffices to generate path dependent minimizing priors. We strongly suspect that for options with non-monotone or path-dependent payoffs (e.g. barrier options and other exotic options), the worst-case prior has a more complex structure, corresponding to a specific stochastic volatility model. The study of more complex options is left for future research.

Our results can now be used to tackle more complex economic problems involving optimal stopping. We mention here two possible applications. One possible next step is to study the effect of ambiguous volatility on investment timing in a real-option game (Grenadier (2002)). A possible challenge of such an extension is to find a suitable notion of equilibrium, which may not be straightforward with singular measures. A second possible application is the interaction between ambiguous volatility, investment timing and agency. Many investment decisions are taken in the presence of moral-hazard (e.g. IPOs, venture capital), which arises precisely because the payoff process is uncertain. For this reason, the interaction between agency and investment has recently received attention.[4]. We expect that the role of ambiguity on the timing of investment may be less straightforward in the presence of moral-hazard.

A The G- framework

In this appendix, we present briefly some elements related to the G-framework and some results that we use further in the text. We refer the reader to Peng (2007) for the seminal paper on this subject. Peng (2010) provides an exhaustive survey and a general exposition of the G-expectation framework.

[4]See for example **?**, Gryglewicz and Hartman-Glaser (2014)

A.1 Sublinear expectation, G-Brownian motion, G-Expectation

Definition 3 *Let Ω be a given set. Let $\mathcal{H}$ be a linear space of real-valued functions defined on Ω. A sublinear expactation $\hat{\mathbb{E}}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ such that for any $X, Y \in \mathcal{H}$:*

1. *If $X \geq Y$ then $\hat{\mathbb{E}}(X) \geq \hat{\mathbb{E}}(Y)$ (Monotonicity)*
2. *$\hat{\mathbb{E}}(c) = c$(Constant preserving)*
3. *$\hat{\mathbb{E}}(X+Y) \leq \hat{\mathbb{E}}(X) + \hat{\mathbb{E}}(Y)$ (sub-linearity)*
4. *$\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}(X)\ \forall \lambda \geq 0$ (positive homogeneity)*

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space. Note that the sub-additivity is equivalent to $\hat{\mathbb{E}}(X) - \hat{\mathbb{E}}(Y) \leq \hat{\mathbb{E}}(X - Y)$.

In what follows, $\mathcal{C}_{l,Lip}(\mathbb{R}^n)$ $n \geq 1$ is the space of real-valued continuous functions ϕ such that $|\phi(x) - \phi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|\ \forall x, y \in \mathbb{R}^n$ where k is an integer depending on ϕ.

Definition 4 ***Independence***- *Let $X, Y \in \mathcal{H}$. Y is said to be independent from X under $\hat{\mathbb{E}}$ if for any test function phi $\in \mathcal{C}_{l,Lip}(\mathbb{R}^2$ we have :*

$$\hat{\mathbb{E}}[\phi(X,Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\phi(x,Y)_{x=X}]] \tag{42}$$

Definition 5 ***Identical Distribution***- *Let $X_1, X_2 \in \mathcal{H}$ two random variables on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. They are called identically distributed, denoted $X_1 \sim X_2$ is*

$$\hat{\mathbb{E}}[\phi(X_1)] = \hat{\mathbb{E}}[\phi(X_2)] \quad \forall \phi \in \mathcal{C}_{l,Lip}(\mathbb{R}) \tag{43}$$

We say $\overline{X}$ is an independent copy of X if $X \sim \overline{X}$ and $\overline{X}$ is independent from X.

Definition 6 ***G-distribution***- *A random variable X on a sublinear expactation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called centralized G-normal distributed if for any $a, b \geq 0$:*

$$aX + B\overline{X} \sim \sqrt{a^2 + b^2} X \tag{44}$$

where $\overline{X}$ is an independent copy of X. Here, the function G denotes the function $G(y) = \frac{1}{2}\hat{\mathbb{E}}[yX^2]$. The following holds : $G(y) = \frac{1}{2}\overline{\sigma}^2 y^+ - \frac{1}{2}\underline{\sigma}^2 y^-$ where $\overline{\sigma}^2 = \hat{\mathbb{E}}[X^2]$ and $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-X^2]$. We write X is $N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$.

Definition 7 *In a sublinear expectation space, $(X_t)_{t \geq 0}$ is called a stochastic process if X_t is a random variable in $\mathcal{H}$ for each $t \geq 0$.*

Definition 8 *A process $(B_t)_{t\geq 0}$ on a sublinear expectation space is called a G-Brownian motion if the following properties are satisfied :*

1. $B_0 = 0$
2. *For each $t, s \geq 0$, $(B_{t+s} - B_t)$ is $N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$ distributed and independent from $(B_{t_1}, ..., B_{t_n})$ for each $n \in \mathbb{N}$, $0 \leq t_1 \leq ... \leq t_n \leq t$.*

Let $\Omega = \mathcal{C}([0,T];\mathbb{R})$ be the space of all real valued continuous functions $(\omega_t)_{t\in[0,T]}$ with $\omega_0 = 0$, eqquiped with the distance :

$$\rho(\omega^1, \omega^2) = \sum_{i}^{\infty} 2^{-i}[(\max|\omega_t^1 - \omega_t^2|) \wedge 1] \quad \omega^1, \omega^2 \in \Omega \tag{45}$$

For each $\phi \in C_{l,lip}(\mathbb{R})$ let u_ϕ be the unique viscosity solution of the following partial differential equation :

$$\begin{cases} \frac{\partial u}{\partial t} = G(\frac{\partial^2 u}{\partial x^2}, \ (t,x) \in [0,T] \times \mathbb{R} \\ u(0,x) = \phi(x) \end{cases} \tag{46}$$

Finir avec dédinition de G-E sur L_2

A.2 G-stochastic calculus

In what follows, $M_G^{p,0}(0,T)$, $p \geq 0$ denotes the set of simple processes of the form : given a partition $\{t_0, ..., t_N\}$, $N \in \mathbb{N}$ of $[0,T]$, $\xi_i \in L_G^p(\Omega_{t_i}$, $i = 0, 1, ..., N-1$, for any $t \in [0,T]$, the process η is defined as :

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{t_j, t_{j+1}}(t) \tag{47}$$

We denote by $M_G^p(0,T)$ the completion of $M_G^{p,0}(0,T)$ under the norm $||.||_{M_G^p}$ defined as $||\eta||_{M_G^p} = (\mathbb{E}_G \int_0^T |\eta_s|^P ds)^{\frac{1}{p}}$.

Definition 9 *For $\eta \in M_G^2(0,T)$, the integral mapping $I : M_G^2(0,T) \to L_G^2(\Omega_T)$ is defined by :*

$$I(\eta) = \int_0^T \eta_s dB_s = \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}) \tag{48}$$

The quadratic variation of a G-Brownian motion is given by :

$$\langle B \rangle_t = B_t^2 - 2 \int_0^t B_s ds \ \forall t \leq T \tag{49}$$

Importantly, this process is *not* deterministic, as it contains the ambiguity of the G-Brownian motion. We can define the stochastic integral with respect to the quadratic variation of the G-Brownian motion in an analogous way

Definition 10 *For $\eta \in M_G^1(0,T)$, the integral mapping $I : M_G^1(0,T) \to L_G^1(\Omega_T)$ is defined by :*

$$I(\eta) = \int_0^T \eta_s dB_s = \sum_{j=0}^{N-1} \xi_j (\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}) \tag{50}$$

In what follows, we use the following propositions :

Proposition 9 *Let $f \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$ and $X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \eta_s d\langle B \rangle_s + \int_0^t \beta_s dB_s$ where $\alpha, \eta \in M_G^1(0,T), \beta \in M_G^2(0,T)$. Then for each $t \in [0,T]$ we have :*

$$f(t, X_t) = f(0, X_0) + \int_0^t \partial_x f(u, X_u) \beta_u dB_u + \int_0^t [\partial_t f(u, X_u) + \partial_x f(u, X_u) \alpha_u] du \tag{51}$$

$$+ \int_0^t [\partial_x f(u, X_u) \eta_u + \frac{1}{2} \partial_{xx}^2 f(u, X_u) \beta_u^2] d\langle B \rangle_u \tag{52}$$

In fact, this formula holds for $\mathcal{C}^1$ functions that are $\mathcal{C}^2$ everywhere except on a set of zero measure. This result follows directly from the generalized Itô formula for $\mathcal{C}^1$ functions in Yan, Sun, Gao (2015). As this is the case that is most often faced in economic applications (including the applications in this paper), we use this result for our verification theorem. Greater generality might be achievable using viscosity solutions, but in practice, it is often easier to check piecewise differentiability.

Proposition 10 *Let $\eta \in M_G^1(0,T)$ and $\alpha \in M_G^2(0,T)$, the process :*

$$M_t = M_0 + \int_0^t \alpha_s dB_s + \int_0^t \alpha_s d\langle B \rangle_s + \int_0^t 2G(\alpha_s) ds \tag{53}$$

is a G-martingale.

It is behind the scope of this paper to make an exhaustive presentation of the G-framework, we mention here two important references on stopping times for the interested reader :

1. Li and Peng (2011) for a definition of stopping times, a definition and properties of the Itô integral on $[0, \tau]$ where τ is a stopping time, and the related spaces.

2. Hu and Peng (2014) for a definition of stopping times on larger spaces, the property of conditional G-Expectation on these spaces (Monotonicity, constantpreserving, sublinearity and positive homogenity are preserved), and a version of the optional sampling theorem, that we use in the proof of Theorem 2.

B Conditioning and the set of priors

In this appendix, we provide a mathematical description of the construction of the set of priors. We follow Epstein and Ji (2014). Their construction guarantees that wa can obtain a notion of conditional expectation and dynamic consistency, which is crucial for our results. In particular, Theorem 1 holds because our set of priors satisfies the assumtpions in Nutz et al. (2015), which allows to directly apply their results.

B.1 Defining expectation

Let $C([0,\infty))$ be the set of all continuous functions with values in $\mathbb{R}$. Define $\Omega = \{\omega : \omega \in C([0,\infty)), \omega_0 = 0\}$ and the coordinate process $B_t(\omega) = \omega(t)$. Let P_0 be the classical Wiener measure, under which the coordinate process is a standard Brownian Motion and $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by $(B_t)_{t\geq 0}$ augmented by the P_0-null sets.

Define $\Sigma = [\underline{\sigma}, \overline{\sigma}] \subset \mathbb{R}$ with $0 < \underline{\sigma} < \overline{\sigma}$. For any process $(\sigma_t)_{t\geq 0}$ with values in Σ, the stochastic differential equation (SDE) :

$$\begin{cases} dX_t = \sigma_t dB_t \\ X_0 = 0 \end{cases} \tag{54}$$

Has a unique solution under P_0, denoted X^σ. Define the probability measure P^σ as :

$$P^\sigma(A) = P_0(X^\sigma \in A) \quad A \in \mathcal{F}_T \tag{55}$$

Denote by $\mathcal{P}_0$ the set of all probability measures that an be constructed in this way and $\mathcal{P}$ the closure of $\mathcal{P}_0$ under the topology of weak convergence.

With this construction, the multiple-prior expectation is given by :

$$\inf_{P\in\mathcal{P}^\Theta} \mathbb{E}^P[\xi] \tag{56}$$

Where ξ is a random variable on $(\Omega, \mathcal{F}_T)$ and $\mathbb{E}^P$ is the standard expectation under the measure P. Note that the upper expectation defined as :

$$\hat{\mathbb{E}} = \sup_{P\in\mathcal{P}^\Theta} \mathbb{E}^P[\xi] \tag{57}$$

Corresponds to the G-expectation parameterized by Θ, as introduced in Peng (2007). This representation result for the G-expectation can be found in (?). The G-expectation satisfies monotonicity, constant preserving, sublinearity (and thus, self dominance) and positive homogeneity. The G-Expectation is defined on the space $L^2(\Omega)$, which is the completion of the set of all bounded continuous functions on Ω, under the norm : $||\xi|| = (\hat{\mathbb{E}}[|\xi|^2])^{\frac{1}{2}})$. Peng (2010) develops a rich theory of stochastic calculus that provides the non-linear counterpart to the standard theory. We exploit this theory, noting that :

$$\inf_{P \in \mathcal{P}^{\Theta}} \mathbb{E}^P = -\hat{\mathbb{E}}[-\xi] \tag{58}$$

B.2 Conditioning

The approach to conditioning consists in specifying conditional beliefs at any (ω, t) deemed possible by a measure in $\mathcal{P}^{\Theta}$. Intuitively, if a volatility scenariofrom t onwards along path ω is seen as plausible ex-ante, then it is also seen as possible ex-post, conditional on $(t\omega)$. This procedure rules out unanticipated changes in possible scenarios. Consider the following stochastic differential equation :

$$\begin{cases} dX_s = \sigma_s dB_s \quad t \leq s \leq T \\ X_s = \omega_s \quad t \leq s \leq t \end{cases} \tag{59}$$

For all $(t, \omega) \in [0, T] \times \mathbb{R}$. Denote by $X^{\sigma,\omega,t}$ the strong solution to this stochastic differential equation. Again, through P^0, this solution induces a probability measure $P^{\sigma,t,\omega}$ on Ω. Denote by $\mathcal{P}^{\Theta}(t, \omega)$ the set of all probability measures that can be constructed this way, by taking all the processes $(\sigma_s)_{s \geq t}$ that lie in Θ. $\mathcal{P}^{\Theta}(t, \omega)$ is the set of priors, conditional on (t, ω).

Define

$$\hat{\mathbb{E}}_t[X] = \sup_{P \in \mathcal{P}^{\Theta}(t,\omega)} E^P[X] \tag{60}$$

Then by Soner et al. (2011), $\hat{\mathbb{E}}$ is the conditional G-expectation. The conditional G-expectation satisfies monotonicity, constant preserving, sublinearity (and self-domination) and positive homogeneity.

C Proofs

C.1 Theorem 1

Proof. Consider the following optimal stoppig problem :

$$\inf_{\tau \in \Gamma} \sup_{P \in \mathcal{P}^{Theta}} \mathbb{E}^P[-f(\pi_\tau)] \tag{61}$$

And the associated value function :

$$Z_t = \inf_{\tau \in \Gamma_t} \sup_{P \in P^{\Theta}(\omega,t)} \mathbb{E}^P[-f(\pi_{tau})] \tag{62}$$

Where P^{Θ} and $P^{\Theta}(\omega, t)$ are defined in Appendix A. By Theorem 3.4 in Nutz et al. (2015) (it can be checked that their assumptions (3.1), (3.2), (3.3) are verified in our framework) we have : $\tau^* = \inf\{t \in [0,T] : Z_{\tau^*} = -f(\tau^*)$ is an optimal stopping time for problem 61, that is : $\inf_{\tau\in\Gamma}\sup_{P\in\mathcal{P}^\Theta}\mathbb{E}^P[-f(\pi_\tau)] = \sup_{P\in\mathcal{P}^\Theta}\mathbb{E}^P[-f(\pi_{\tau^*})]$. Coming back to our problem, we have :

$$\sup_{\tau\in\Gamma}\inf_{P\in\mathcal{P}^\Theta}\mathbb{E}^P[f(\pi_\tau)] = \sup_{\tau\in\Gamma} - \sup_{P\in\mathcal{P}^\Theta}\mathbb{E}^P[-f(\pi_\tau)] = -\inf_{\tau\in\Gamma}\sup_{P\in\mathcal{P}^\Theta}\mathbb{E}^P[-f(\pi_\tau)] \tag{63}$$

$$= -\sup_{P\in\mathcal{P}^\Theta}\mathbb{E}^P[-f(\pi_{\tau^*})] = \inf_{P\in\mathcal{P}^\Theta}\mathbb{E}^P[f(\pi_{\tau^*})] \tag{64}$$

Thus τ^* is an optimal stopping time for our problem. Now, we have :

$$V_t = \sup_{\tau\in\Gamma_t}\inf_{P\in\mathcal{P}^\Theta(\omega,t)}\mathcal{E}[f(\pi_t)] = \sup_{\tau\in\Gamma_t} - \sup_{P\in\mathcal{P}^\Theta(\omega,t)}\mathcal{E}[-f(\pi_t)] \tag{65}$$

$$= -\inf_{\tau\in\Gamma_t}\sup_{P\in\mathcal{P}^\Theta(\omega,t)}\mathcal{E}[f(\pi_t)] = -Z_t \tag{66}$$

Thus, $\tau^* = \inf\{t \in [0,T] : V_t = f(\pi_t)\}$, which completes the proof. ∎

C.2 Theorem 2

Proof. Using the Itô formula[5], we have

$$v(t,\pi_t) = v(0,\pi_t) + \int_0^t \frac{\partial v}{\partial \pi_u}(u,\pi_u)dB_u + \int_0^t (\frac{\partial v}{\partial u}(u,\pi_u) + \mu\frac{\partial v}{\partial \pi_u}(u,\pi_u))du \tag{67}$$

$$+ \int_0^t \frac{1}{2}\frac{\partial^2}{\partial \pi_u^2}\pi_u^2 d\langle B\rangle_u \tag{68}$$

$$-v(t,\pi_t) = -v(0,\pi_t) + \int_0^t -\frac{\partial v}{\partial \pi_u}(u,\pi_u)dB_u + \int_0^t -\frac{1}{2}\frac{\partial^2}{\partial \pi_u^2}\pi_u^2 d\langle B\rangle_u \tag{69}$$

$$- \int_0^t 2G(-\frac{1}{2}\frac{\partial^2}{\partial \pi_u^2}\pi_u^2)du \tag{70}$$

$$- (\int_0^t (\frac{\partial v}{\partial u}(u,\pi_u) + \mu\frac{\partial v}{\partial \pi_u}(u,\pi_u))du - \int_0^t 2G(-\frac{1}{2}\frac{\partial^2}{\partial \pi_u^2}\pi_u^2)du) \tag{71}$$

Where

$$G : \mathbb{R} \to \mathbb{R} \;\text{ such that }\; G(x) = \frac{1}{2}\max_{\sigma^2\in[\underline{\sigma}^2,\overline{\sigma}^2]} x\sigma^2 \tag{72}$$

Denote :

$$A_t = \int_0^t (\frac{\partial v}{\partial u}(u,\pi_u) + \mu\frac{\partial v}{\partial \pi_u}(u,\pi_u))du - \int_0^t 2G(-\frac{1}{2}\frac{\partial^2}{\partial \pi_u^2}\pi_u^2)du \tag{73}$$

$$M_t = \int_0^t -\frac{\partial v}{\partial \pi_u}(u,\pi_u)dB_u + \int_0^t -\frac{1}{2}\frac{\partial^2}{\partial \pi_u^2}\pi_u^2 d\langle B\rangle_u - \int_0^t 2G(-\frac{1}{2}\frac{\partial^2}{\partial \pi_u^2}\pi_u^2)du \tag{74}$$

[5]The Itô formula applies when the function is $\mathcal{C}^1$ everywhere and $\mathcal{C}^2$ except on a set of zero measure.

Then we have $-v(t, \pi_t) = M_t - A_t$. By the subadditivity of the conditional linear expectation (see (Peng, 2010)), we have, for $s < t < T$:

$$\hat{\mathbb{E}}_s[-v(t, \pi_t)] = \hat{\mathbb{E}}_s[M_t - A_t] \geq \hat{\mathbb{E}}_s[M_t] - \hat{\mathbb{E}}_s[A_t] \tag{75}$$

From Peng (2010) chapter IV Proposition 1.4, M_t is a G-martingale, thus $\hat{\mathbb{E}}_s[M_t] = M_s$. By the Hamilton-Jacobi-Bellman equation, we have that :

$$\frac{\partial v}{\partial u}(u, \pi_u) + \mu \frac{\partial v}{\partial \pi_u}(u, \pi_u) - 2G(-\frac{1}{2}\frac{\partial^2}{\partial \pi_u^2}\pi_u^2) \leq 0 \quad \forall u \in [0, T] \tag{76}$$

Which implies that $\hat{\mathbb{E}}_s[A_t] \leq A_s$. Thus we have :

$$\hat{\mathbb{E}}_s[-v(t, \pi_t)] \geq \hat{\mathbb{E}}_s[M_t] - \hat{\mathbb{E}}_s[A_t] \geq M_s - A_s = -v(s, \pi_s) \tag{77}$$

Thus $-v_t$ is a G-submartingale and thus an $\mathcal{E}-$supermartingale. For a given t, let Γ_t be the set of stopping times taking values in $[0, T]$. Then we have for all $\tau \in \Gamma_t$ we have, since $v(\pi_t, t)$ dominates $f(\pi_t, t)$ by virtue of our HJB equation :

$$\hat{\mathbb{E}}_t[-f(\pi_\tau, \tau)] \geq \hat{\mathbb{E}}_t[-v(\pi_\tau, \tau)] = \hat{\mathbb{E}}_t[M_\tau - A_\tau] \geq \hat{\mathbb{E}}_t[M_\tau] - \hat{\mathbb{E}}_t[A_\tau] \tag{78}$$

Where the second inequality holds by the self domination property of conditional G-expectation (see Hu and Peng (2013), Proposition 38). By Theorem 2.9 in Gao (2009), $M_T \in L_G^2(\mathcal{F}_t)$. In addition, since $(M_t)_{t\geq 0}$ is a martingale, we have for all t, $M_t = \hat{\mathbb{E}}[M_T]$. By the optional sampling theorem for the conditional G-expectation (Hu and Peng (2013) Theorem 48) [6]:

$$\hat{\mathbb{E}}_t[M_\tau] = M_t \tag{79}$$

Thus, by 76, we have :

$$\hat{\mathbb{E}}_t[-f(\pi_\tau, \tau)] \geq \hat{\mathbb{E}}_t[-v(\pi_\tau, \tau) = \hat{\mathbb{E}}_t[M_\tau - A_\tau] \geq \hat{\mathbb{E}}_t[M_\tau] + \hat{\mathbb{E}}_t[A_\tau] \geq M_t - A_t = -v_t \tag{80}$$

From which we get :

$$V_t \leq v_t \tag{81}$$

We now prove the opposite inequality. Define $\tau_t = \inf\{u \in [t, T], v(u, \pi_u) = f(u, \pi_u)\}$. Then, by our HJB equation, $(-v(u \wedge \tau_t, \pi_{u\wedge\tau(t)})$ is a G-martingale and a $\mathcal{E}-$martingale. By continuity of f and v on the sample path of π, we have :$-v(\tau_t, \pi_{\tau_t}) = -f(\tau_t, \pi_{\tau_t})$. We get :

$$v_t = \mathcal{E}_t[v(\tau_t, \pi_{\tau_t})] = \mathcal{E}_t[f(\tau_t, \pi_{\tau_t})] \leq V_t \tag{82}$$

And we conclude that : $V_t = v(\pi_t, t)$ ■

[6] The optional sampling theorem of Hu and Peng (2013) applies to so-called *-stopping times. A sufficient condition for τ to be a *-stopping time is that τ is the first exit time of $(\pi_t)_{t\geq 0}$ from a given bounded subset of $\mathbb{R}$, which is the case in most applications, including all examples in this paper.

C.3 Proposition 1

Proof. Take any P in $\mathcal{P}^{\Theta}(t,\omega)$. Then, by Ito's formula we have for π_t given :

$$\mathbb{E}^P[\int_t^T e^{-\rho(s-t)}pi_s ds] = \mathbb{E}^P[\int_t^T e^{-\rho(s-t)}\pi_t \text{exp}(\int_t^s \mu du + \int_t^s dB_u - \int_t^s \frac{1}{2}d\langle B\rangle_u)ds] \tag{83}$$

$$= \mathbb{E}^P[\int_t^T \text{exp}((\mu-\rho)(s-t))\pi_t \text{exp}(\int_t^s dB_u - \int_t^s \frac{1}{2}d\langle B\rangle_u)ds] \tag{84}$$

$$= [\int_t^T \text{exp}((\mu-\rho)(s-t))\pi_t \mathbb{E}^P \text{exp}(\int_t^s dB_u - \int_t^s \frac{1}{2}d\langle B\rangle_u)ds] \tag{85}$$

$$= \int_t^T \text{exp}((\mu-\rho)(s-t))\pi_t ds \tag{86}$$

$$= \frac{1-e^{-(T-t)(\rho-\mu)}}{\rho-\mu} \tag{87}$$

Where the first equality is a simplification, the second is from Fubini's theorem[7], the third by noting that :

$$Z_s = \text{exp}(\int_t^s dB_u - \int_t^s \frac{1}{2}d\langle B\rangle_u) \tag{88}$$

is a martingale under any $P \in P(\omega,t)$, because B is a martingale under any prior in $\mathcal{P}(t,\omega)$ and Novikov's condition holds :

$$\mathbb{E}^P[\text{exp}(\frac{1}{2}\langle B\rangle_T])] < \infty \tag{89}$$

For all P in $P(\omega,t)$, since $\langle B\rangle_t = \int_0^T \sigma_s^2 ds \ - \ P^{\sigma} a.s$ where $\sigma_s \in [\underline{\sigma},\overline{\sigma}]$ ■

C.4 Lemma 1

Lemma 1 *For anny stopping time τ, we have :*

$$\inf_{P\in\mathcal{P}^{\Theta}} \mathbb{E}^P[\int_\tau^T e^{-\rho s}\pi_s ds - e^{-\rho\tau}I] = \inf_{P\in\mathcal{P}^{\Theta}} \mathbb{E}^P[e^{-\rho\tau}(\frac{1}{\rho-\mu} - I)] \tag{90}$$

Proof. Let τ be a stopping time, we have :

$$\mathcal{E}_0[\int_\tau^\infty e^{-rs}\pi_s ds - e^{-r\tau}I] = \mathcal{E}_0[\mathcal{E}_\tau[\int_\tau^\infty e^{-rs}\pi_s ds - e^{-r\tau}I]] \tag{91}$$

$$= \mathcal{E}_0[\mathcal{E}_\tau[\int_\tau^\infty e^{-rs}\pi_s ds] - e^{-r\tau}I] \tag{92}$$

[7] We have $\int_t^T \mathbb{E}^P[|\text{exp}(-\rho(s-t)\pi_s)|ds] < \infty$.

Where the first equality is due to the tower property of our conditional expectation (which is linked to dynamic consistency), and the second is due to the fact that $e^{-\rho\tau}I$ is $\mathcal{F}_\tau$-measurable. The computation of $\mathcal{E}_\tau[\int_\tau^\infty e^{-rs}\pi_s ds]$ follows from Proposition 1, by replacing the deterministic time t by the stopping time τ. By the strong Markov property for solutions of SDE driven by a G-Brownian motion (Theorem 4.4 in Hu et al. (2017)), we obtain our result. ■

C.5 Proposition 2

Proof. We look for a candidate solution of the form : $v(t,x) = e^{-\rho t}v(x)$. By the nature of the problem, we look for a solution v to the following free boundary problem, with parameter x_0 such that :

$$v(x) = \frac{1}{\rho-\mu}x - I \quad \text{for } x \in [x_0,\infty) \tag{93}$$

$$\rho v(x) + \mu x v'(x) + \frac{1}{2}\min_{\sigma^2\in[\underline{\sigma}^2,\overline{\sigma}^2]}\sigma^2 x^2 v''(x) = 0 \quad \text{for } x \in [0,x_0) \tag{94}$$

$$v(0) = 0 \tag{95}$$

$$v(x_0) = \frac{1}{\rho-\mu}x_0 - I \tag{96}$$

$$v'(x_0) = \frac{1}{\rho-\mu} \tag{97}$$

For now, let's assume that v is convex on $[0,x_0)$, equation 94 becomes :

$$\rho v(x) + \mu x v'(x) + \frac{1}{2}\underline{\sigma}^2 x^2 v''(x) \tag{98}$$

This a second-order Cauchy-Euler equation, thus we seek a solution of the form :

$$v(x) = Ax^{\beta_1} + Bx^{\beta_2} \tag{99}$$

where $\beta_2 < 0 < \beta_1$ are the roots of the quadratic equation

$$\frac{1}{2}\beta(\beta-1)\underline{\sigma} + \beta\mu - \rho = 9 \tag{100}$$

From condition 95 we get $B = 0$. By plugging 99 in 96 (value matching condition) and 97 (smooth pasting), we obstain the candidate solution :

$$x_0 = \frac{\beta(\rho-\mu)}{\beta-1}I \tag{101}$$

$$v(x) = \mathbb{1}_{[0,x_0)}[(\frac{x_0}{\rho-\mu} - I)(\frac{x}{x_0})^\beta] + \mathbb{1}_{[x_0,\infty)}[\frac{x}{\rho-\mu} - I] \tag{102}$$

The function is convex on $[0,x_0)$ and $\mathcal{C}^2$ on $\mathbb{R}\backslash\{x_0\}$, by our verification Theorem 2, we have $V_t = v(\pi_t)$ and by Theorem 1, $\tau^* = \inf\{t \in [0,\infty), \pi_t \geq \pi^*\}$ is the optimal stopping time. ■

C.6 Proposition 3

Straightforward calculations show that for any $\sigma > 0$:

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{(\frac{\mu}{\sigma^2} - \frac{1}{2})^2 + \frac{2\rho}{\sigma^2}} \tag{103}$$

It can be checked that $\frac{\partial \beta}{\partial \sigma} < 0$ and $\frac{\partial \pi^*}{\partial \beta} < 0$ from which we get $\frac{\partial \pi^*}{\partial \sigma} > 0$

C.7 Proposition 5

Let $\overline{W}_t = \hat{\mathbb{E}}_t[-\int_t^T e^{-\rho(s-t)} U(\pi_s)]$, then by the nonlinear Feynman Kac Formula in Peng (2010) (Chapter 5, Theorem 3.7):

$$\frac{\partial \overline{W}_t}{\partial t} + \sup_{\sigma^2 \in [\underline{\sigma}^2, \overline{\sigma}^2]} \frac{1}{2}\pi^2\sigma^2 \frac{\partial^2 \overline{W}_t}{\partial^2 \pi} + \mu\pi \frac{\partial \overline{W}_t}{\partial \pi} = 0 \tag{104}$$

$$W(\pi_T) = 0 \tag{105}$$

Since $U(.)$ is strictly concave, by the subadditivity and the positive homogeneity of $\hat{\mathbb{E}}_t$, $\overline{W}(.)$ is convex in π_t, so the suprema in the above equation is reached at $\overline{\sigma}^2$. The equation becomes :

$$\frac{\partial \overline{W}_t}{\partial t} + \frac{1}{2}\pi^2\overline{\sigma}^2 \frac{\partial^2 \overline{W}_t}{\partial^2 \pi} + \mu\pi \frac{\partial \overline{W}_t}{\partial \pi} = 0 \tag{106}$$

$$\overline{W}(\pi_T) = 0 \tag{107}$$

Therefore,

$$\overline{W}_t = \mathbb{E}^{P^{\overline{\sigma}}}[-\int_t^T e^{-\rho(s-t)} U(\pi_s)|\mathcal{F}_t] \tag{108}$$

Where $P^{\overline{\sigma}}$ is the measure induced by the constant volatility process $\sigma_t = \overline{\sigma}$. We have :

$$W_t = -\overline{W}_t = -\mathbb{E}^{P^{\overline{\sigma}}}[-\int_t^T e^{-\rho(s-t)} U(\pi_s)|\mathcal{F}_t] = \mathbb{E}^{P^{\overline{\sigma}}}[\int_t^T e^{-\rho(s-t)} U(\pi_s)|\mathcal{F}_t] \tag{109}$$

To obtain the results in the proposition, note that by Levy's characterization of Brownian motion, $(\frac{B_t}{\overline{\sigma}})_{t\geq 0})$ is a standard Brownian motion under $P^{\overline{\sigma}}$. Now let $T \to \infty$ and apply Theorem 9.18 in Karatzas et al. (1998) to obtain the result.

C.8 Proposition 6

We proceed analogously to the proof of Proposition 2. Consider the free-boundary problem with parameter x_0 :

$$v(x) = \frac{x^{1-R}}{\delta(1-R)} - I \quad \text{for } x \in [x_0, \infty) \tag{110}$$

$$\rho v(x) + \mu x v'(x) + \frac{1}{2} \min_{\sigma^2 \in [\underline{\sigma}^2, \overline{\sigma}^2]} \sigma^2 x^2 v''(x) \quad \text{for } x \in [0, x_0) \tag{111}$$

$$v(0) = 0 \tag{112}$$

$$v(x_0) = \frac{x_0^{1-R}}{\delta(1-R)} - \frac{I^{1-R}}{1-R} \tag{113}$$

$$v'(x_0) = \frac{x^{-R}}{\delta} \tag{114}$$

Using the same arguments as in the proof of Proposition 2, we obtain the following candidate solution :

$$x_0 = I(\frac{\delta\beta}{\beta + R - 1})^{1-R} \tag{115}$$

$$v(x) = \mathbb{1}_{[0,x_0)}[\frac{1}{\rho}U(I) + (\frac{U(x_0)}{\delta} - \frac{U(I)}{\rho})(\frac{x}{x_0})^{\beta}] + \mathbb{1}_{[x_0,\infty)}[U(x) - U(I)] \tag{116}$$

Since v is C^2 on $\mathbb{R}\backslash\{x_0\}$, by Theorem 2 we have $F_t = v(\pi_t)$. By Theorem 1, π^* is the optimal investment threshold.

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