Illiquidity and Indebtedness - Optimal Capital Structure under Realistic Default Triggers in a Double Barrier Option Framework

Abstract

Existing dynamic capital structure models are based on a single barrier determining bankruptcy, e.g. overindebtedness or illiquidity. However, it is observable that these approaches do not perform well empirically and omit a variety of constraints faced by equity and debt holders. This article incorporates these constraints examining corporate debt value and optimal capital structure in a double barrier world with knock-in and knock-out barrier options. The results elucidate why considering only one barrier distorts the estimates of risks for default and bankruptcy. In fact, the single barriers illiquidity and overindebtedness take the role of boundary conditions. Incorporating both conditions in this novel double barrier approach allows for capital structure estimations that are in better accordance with empirical findings. Beyond capital structure theory, other fields of economics and even medical science or the humanities are in context of problems that can be solved with such a double barrier approach.

Keywords: Dynamic models, Structural estimation, First hitting time, Second hitting time, Default Risk, Optimal Leverage

JEL classification: G12, G31, G32, G33

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1. Introduction

Barrier options play a central role in several fields of science, particulary in economics. In the study of corporate finance the equity value can be determined with the help of a call barrier option on the firm value that is activated only if the firm value touches a predetermined barrier, i.e. the bankruptcy trigger. Analogously, the same holds for the debt value with the help of a put barrier option. Certainly, debt values and capital structure are interlinked variables - it is virtually impossible to determine the debt value without knowing the firm's capital structure and vice versa. But both the debt value and the capital structure are constituent parts affecting the firm's risk for default and bankruptcy. Thus, for reasonable corporate valuation it is indispensable to characterize these influencing variables correctly. If we consider only a single barrier, e.g. illiquidity or overindebtedness, the underlying valuation problem is not very difficult (c.f. Merton (1974), Rubinstein and Reiner (1991)) and already solved. However, we observe that firms are exposed to a variety of constraints. Capturing illiquidity and overindebtedness in one single model leads directly into a double barrier approach. Its syndetic path-dependency owe a modus operandi that is less straightforward than dealing only with a single barrier. Valuing a double barrier option and thereby a firm that faces changing payouts whenever the underlying process hits either of two well-defined boundaries illiquidity and overindebtedness requires extensive results of the mathematical stochastic calculus.

This article examines corporate debt value and optimal capital structure in a double barrier world with knock-in and knock-out barrier options. Besides, it gets to the bottom of the discrepancy between theoretical forecasts and empirical observations in context of capital structure theory. Therefore, separate research approaches are combined into one single model. The upper boundary represents the illiquidity barrier and catches, e.g. the distinguished approaches of Couch et al. (2012) and Kim et al. (1993). The lower barrier stands for overindebtedness and thus includes the pioneering work of Leland (1994). The derived results show that the novel combination generates results in-between the particular single constraints.

Traditional capital structure theory states that insolvency triggers are an important determinant of optimal capital structure theory. Leland and Toft (1996) include the maturity of debt into the standard Leland model. Goldstein et al. (2001) extend the model further by basing it on a stochastic EBIT-process and allowing for an option to increase debt (dynamic capital structure). Hackbarth et al. (2007) dive deeper into the debt structure explaining the relation of bank loans and market debt. Titman and Tsyplakov (2007) present a model that allows for dynamic adjustment of both its capital structure and its investment choices. However, as all of these models consider only one single bankruptcy trigger risks for default and bankruptcy are either over- or underestimated. A pure illiquidity trigger overestimates the bankruptcy risk since in case of delay in payment only a minority declare bankruptcy[1]. On the other hand,

[1] Please note that in case of illiquidity we exclude the assumptions of *deep pockets* of the equity holders.

the trigger that bankruptcy occurs if and only if the firm is overindebted seems too weak since it is not always reasonable on the side of the equity holders to make some additional payments. These imprecise estimations lead to the influential effect that default risks of entire industries are wrongly ranked.

This article, based on the pricing formulas of Pelsser (2000), provides the first model that investigates corporate valuation and optimal capital structure decisions in a double barrier framework with knock-in and knock-out options. We are able to model both an illiquidity or covenant trigger for debt and an overindebtedness trigger determined endogenously by the equity holders. Hence, we provide a framework that best reflects realistic triggering events of default and bankruptcy. As expected our solutions to the optimal capital structure problem lie in-between the two classic approaches. By empirically testing our model for firms publicly listed in the US, we gain evidence that incorporating both triggers explains observable capital structures significantly better than existing models do.

Beyond that, we develop our double barrier framework in a general setting which is applicable in other fields of research where the object of investigation is faced with barriers. For instance, the problem of modeling optimal counter-cyclical policies (monetary policy and government investment programs) could be treated in such a framework. The diffusion of a flu is another example from biology: The flu stays normally within an endemic steady state but can suddenly become epidemic (or indeed pandemic like the 1918 flu pandemic) if its infection rate surpasses a special critical value, and it can also return to endemic state.

The generality of the application spectrum of double barrier options affects directly the composition of this article. Thus our structure is as follows: section 2 introduces the general model and depicts an intuitive access. General requirements of the two boundaries are provided. Followed by a profound analysis of the state prices this section ends with a general payout structure. In section 3 we follow the same structure with the difference that each subsection is applied to the special case of corporate valuation with illiquidity as the upper knock-in barrier and overindebtedness as the lower knock-out barrier. Thus, next to some specific mathematical requirements the exact barriers in case of illiquidity and overindebtedness are developed. Subsequently, the specific state prices are derived. Concluding this section, we develop debt value, tax benefits, bankruptcy costs, illiquidity expenses, net benefits, and equity value functions in vectorial writing. Section 4 deals with the analytic application of the model. Besides a differentiation between exogenous and endogenous variables, this section includes guidance in terms of derivations for calculating the optimal bankruptcy trigger and the optimal coupon payment. References and comparisons to the single barrier models are made and possible extensions of our model are highlighted. The section ends with an empirical test of our model results versus the observed leverage ratios of firms of all NAICS sectors publicly listed in the US. Section 5 concludes the article.

2. The General Double Barrier Model

2.1. Mathematical Requirements for the General Model

The assumptions we make about the nature of uncertainty are standard and we try to state them as general as possible. There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ supporting a standard Brownian motion W_t, where Ω is the sample space, $\mathcal{F}$ the σ-algebra and $\mathbb{P}$ the corresponding probability measure. We denote the available information at time t, with $t \in [0, \infty)$, by the filtration $\mathcal{F}_t \subset \mathcal{F}_s$ with $0 \leq t < s$ where $\mathcal{F}_t$ describes the augmented σ-algebra generated by W_t.

We consider a stochastic process $(R_t)_{t\in[0,\infty)}$, e.g. a revenue process that can be characterized by the following stochastic differential equation (SDE)

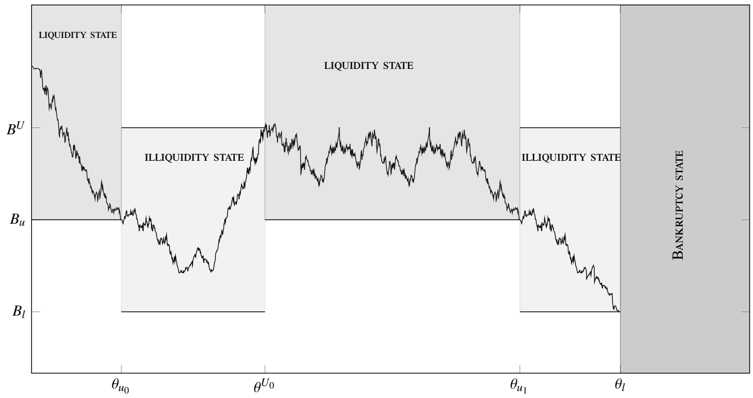
$$dR_t = \mu R_t dt + \sigma R_t dW_t \tag{2.1}$$

where $\mu \in \mathbb{R}$ is the (constant) growth rate, $\sigma \in \mathbb{R}_0^+$ is the corresponding (constant) volatility of the stochastic process R_t, and W_t is a standard Brownian motion for $t \in [0, \infty)$. The index t represents the time horizon, i.e. $t \in [0, \infty)$. The initial value R_0 needs to be positive, i.e. $R_0 \in \mathbb{R}^+$.

2.2. Default Triggers - An Intuitive Access to the General Model

Let us consider a stochastic process that is faced with different boundary conditions B^U, B_u and B_l. We assume that the starting point of the process R_0 is greater than B_u. The initial area is called *liquidity state* (LS). At the very moment when the process hits the initial barrier B_u the process leaves LS and enters *illiquidity state* (IS). When the process hits B_u for the first time we call the first hitting time θ_{u_0}. Continuing from the value B_u in time θ_{u_0} there are three basic options: (i) The process runs directly into the *bankruptcy state* (BS) in θ_l, hits the lower barrier B_l and ceases to exist. (ii) The process lives until infinity between the two boundaries B^U and B_l. Finally, (iii) The process leaves IS by hitting the upper-upper barrier B^U and reenters LS at time θ^{U_0}.

Figure 1: Introduction to the General Model



This figure depicts a stochastic process that starts in the *liquidity state* (LS). The process runs into *illiquidity state* (IS) at the moment θ_{u_0} when the lower-upper barrier B_u is hit. Continuing in IS, the process reenters LS in θ^{U_0} by hitting the upper-upper boundary B^U. In θ_{u_1} the process touches the lower-upper boundary B_u again and falls back into IS. Finally, the process is killed in θ_l, i.e. the process runs into *bankruptcy state* (BS) and hits thus the lower barrier B_l.

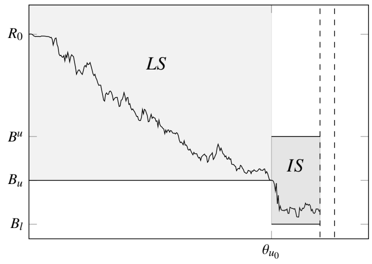
The instant of time where the process enters another state are mathematically known as stopping times[2]. Obviously, there is no need to subscript the hitting time θ_l due to the simple fact that the process is killed at the precise moment when it hits B_l. On the other hand, there is an obligation to subscript θ_{u_i} and θ^{U_i} with $i \in \mathbb{N}_0$, respectively because B_u and B^U could be hit countably infinite times almost surely without hitting B_l. In our framework the barriers B_u, B^U and B_l are constant in time. Please notice that there are only two possibilities: Either B_u is an valid barrier, i.e. B_u is *on* and this implies that B^U and B_l are both switched off or vice versa (c.f. figure 1).

2.3. State Prices in the General Model

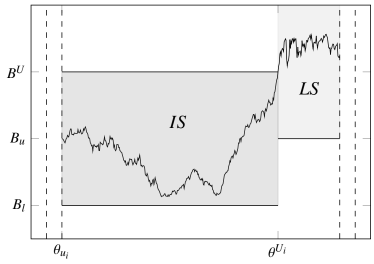
Before we can adapt the aforementioned framework to an optimal capital structure model, we need to derive the state prices of our defined states. State prices reflect the present value of an asset that pays 1$ if a certain state is reached. In other words, state prices represent the probability of entering a certain state discounted back to today. Figure 2 illustrates our methodology.

2 In the following named as hitting times. For a formal definition c.f. Definition 2.1.

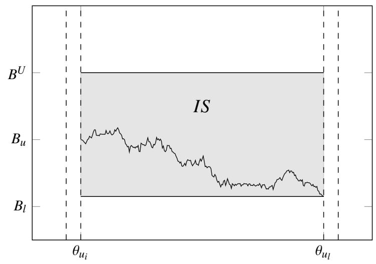
Figure 2: State Prices p_0, p_1, p_2, and p_3 in the General Model



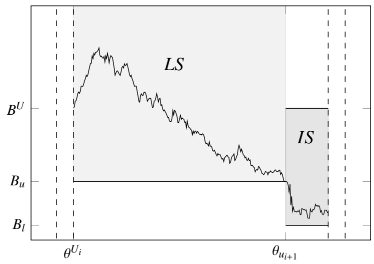
a.) p_0 from liquidity state to illiquidity state, starting in R_0



b.) p_1 from illiquidity state to liquidity state



c.) p_2 from illiquidity state to bankruptcy



d.) p_3 from liquidity state to illiquidity state

θ_{u_i} represents an arbitrary point in time at which the firm runs into *illiquidity state* (IS) coming from *liquidity state* (LS). θ^{U_i} represents an arbitrary point in time at which the firm runs into LS coming from IS. θ_l is the exact point in time at which bankruptcy occurs. R_0 is the starting point of the stochastic process. B^U and B_u is the upper knock in barrier option, respectively. B_l is the lower knock out barrier. The field on a lighter grey background LS represents the LS. In contrast the field on a darker grey background IS symbolizes IS. The parallel dashed lines indicate that the given figure is only an excerpt of the underlying process.

The first graph sketches a firm that runs from LS into IS. This is abbreviated by p_0. The second shows the path of a firm that runs from IS into LS, denoted by p_1. The third picture represents the path of a firm that goes bankrupt entering BS, labeled with the state price p_2. Note that having been in IS is a crucial prerequisite for running into BS. Obviously, the firm is bankrupt at the very moment when the stochastic process R_t equals B_l for an arbitrary $t \in [0, \infty)$[3]. Finally, p_3 is represented in the last figure that shows again a firm running from LS to IS. The difference to the first picture is that the last represents the behaviour of one path in the middle of a firm's life, while the first illustrates only a possible path development at the beginning of a firm's life. Without loss of generality the following figure comprises all possible development opportunities of a firm in a double barrier option frame work, i.e. a framework with a changing upper knock-in barrier characterized by the lower-upper barrier B_u and the upper-upper barrier B^U, respectively and a lower knock-out barrier B_l.

Having given an intuitive access to the state prices, it is indispensable to provide a proper definition of $p_0, ..., p_3$. We start with formally defining the hitting times $\theta_{u_i}, \theta^{U_i}$ and θ_l

[3] To be more precise, this happens if and only if $t = \theta_l$ (c.f. Def. 2.1).

Definition 2.1 (**Hitting Times**). *Given three boundary constraints B_l, B_u, B^U with $B_l \leq B_u < B^U$, the corresponding hitting times are defined as follows for $i \in \mathbb{N}_0$:*

$$\begin{aligned}
\theta_l &:= \inf\{t \geq 0 \,|\, R_t = B_l\} \\
\theta_{u_0} &:= \inf\{t \geq 0 \,|\, R_t = B_u\} \\
\theta^{U_0} &:= \inf\{t \geq \theta_{u_0} \,|\, R_t = B^U \wedge R_s > B_l \text{ for all } s \in [\theta_{u_0}, t]\} \\
\theta_{u_1} &:= \inf\{t \geq \theta^{U_0} \,|\, R_t = B_u \wedge R_s > B_l \text{ for all } s \in [\theta^{U_0}, t]\} \\
&\ldots \\
\theta_{u_i} &:= \inf\{t \geq \theta^{U_{i-1}} \,|\, R_t = B_u \wedge R_s > B_l \text{ for all } s \in [\theta^{U_{i-1}}, t]\} \\
\theta^{U_i} &:= \inf\{t \geq \theta_{u_i} \,|\, R_t = B^U \wedge R_s > B_l \text{ for all } s \in [\theta_{u_i}, t]\}.
\end{aligned}$$

Remark 2.2. *Technically speaking, for the definition of θ_{u_i} we can omit the constraint $R_s > B_l$ for all $s \in [\theta^{U_{i-1}}, t]$. So the following remains*

$$\theta_{u_i} := \inf\{t \geq \theta^{U_{i-1}} \,|\, R_t \leq B_u\}. \tag{2.2}$$

Owing to readability we do not suppress this constraint, since we want to make sure that the above given nonempty stopping times θ_{u_i} and θ^{U_i} for $i \in \mathbb{N}_0$ exclude bankruptcy.

Remark 2.3. *If $\theta_{u_i} \leq \theta_l \leq \theta^{U_i}$, then $\theta_{u_{i+1}} = \theta^{U_i} = \emptyset$.*

Proof. Assume that $\theta_{u_i} \leq \theta_l \leq \theta^{U_i}$. This yields that $\theta^{U_i} = \emptyset$. Simply applying the definition for $\theta_{u_{i+1}}$ we have

$$\begin{aligned}
\theta_{u_{i+1}} &= \inf\{t \geq \theta^{U_i} \,|\, R_t = B_u \wedge R_s > B_l \,\forall s \in [\theta^{U_i}, t]\} \\
&= \inf\{t \geq \theta^{U_i} \,|\, R_t = B_u\} \\
&= \emptyset.
\end{aligned}$$

The last equality holds due to the simple fact that R_t for all $t \geq \theta_l$ and $\theta^{U_i} \geq \theta_l$ owing to the above mentioned assumption. □

Based on the aforementioned insights, we define the state prices p_0, p_1, p_2, p_3 as follows:

Definition 2.4 (**State Prices** $p_0, ..., p_3$).
p_0 is the price of a knock out barrier option that pays 1 \$ in θ_{u_0} starting in $t = 0$ (with the corresponding ordinate value R_0) when the stochastic process $(R_t)_{t\in[0,\infty)}$ hits the lower-upper barrier B_u, i.e. p_0 represents the discounted probability of hitting B_u in θ_{u_0}.

Analogously, p_1 is the price of 1 \$ in θ^{U_i} starting in θ_{u_i} for all $i \in \mathbb{N}_0$ (with the corresponding ordinate value B_u) when the stochastic process $(R_t)_{t\in[0,\infty)}$ hits the upper-upper barrier B^U without hitting the lower barrier B_l.

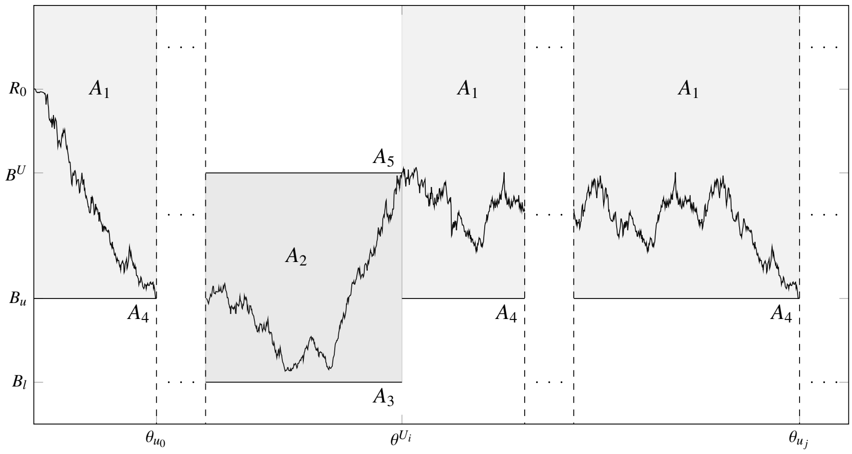
p_2 is the price of 1 \$ in θ_l starting in θ_{u_i} for alle $i \in \mathbb{N}_0$ (with the corresponding ordinate value B_u) when the stochastic process $(R_t)_{t\in[0,\infty)}$ hits the lower barrier B_l without hitting the upper-upper barrier B^U.

Finally, p_3 is the price of a knock out barrier option that pays 1 \$ in $\theta_{u_{i+1}}$ starting in θ^{U_i} for all $i \in \mathbb{N}_0$ (with the corresponding ordinate value B^U) when the stochastic process $(R_t)_{t\in[0,\infty)}$ hits the lower-upper barrier B_u.

2.4. Contingent Claims in the General Setting

Now we have the instruments to consider a general quantifiable model generating the following payout structure. Without loss of generality this excerpt shows all possible states of a firm that has not hit B_l yet.

Figure 3: General Payout Structure of a Stochastic Process



The figure depicts a general payout structure that can be generated in a double barrier framework with *liquidity state* (LS) , *illiquidity state* (IS) and *bankruptcy state* (BS). If the underlying process is in LS the payout equals A_1. In case of IS the generated payout is A_2.Hitting the lower boundary B_l the payout accords with A_3. The same holds for the lower-upper barrier B_u and the payout A_4 and the upper-upper barrier B^U with the payout A_5, respectively.

The capital letters A_j, $j = 1, ..., 5$ are place holders for an arbitrary payout subject to the stochastic process $(R_t)_{t\in[0,\infty)}$. The area A_1 comprises an arbitrary payout of R_t with $t \in [0, \theta_{u_0}] \cup [\theta^{U_i}, \theta_{u_{i+1}}]$ with $i \in \mathbb{N}_0$. This is the payout in LS. A_2 represents the payout in IS that is realized if and only if the stochastic process lies in the middle of the barriers B_l and B^U until the process hits one of them, i.e. A_2 is given if and only if $t \in [\theta_{u_i}, \theta^{U_i}] \cup [\theta_{u_j}, \theta_l]$ with $i < j \in \mathbb{N}$. Note that there is no need that B_l equals A_3 and B^U equals A_5, respectively. The payout A_3 is given if and only if $t = \theta_l$. This is equivalent to the condition that $R_t = B_l$ for an arbitrary $t \in [0, \infty)$. Analogously A_4 is generated if and only if $R_t = B_u$ for all $t \in [0, \infty)$, i.e. $t = \theta_{u_i}$ with $i \in \mathbb{N}_0$. Finally, the payout A_5 is realized if and only if $t = \theta^{U_i}$ with $i \in \mathbb{N}_0$.

Next we derive the expected values of the payouts we introduced, and we start with A_1.

$$\begin{aligned}
\mathbb{E}[A_1] = A_1[&(1-p_0)+ && \text{, value until the first liquidity crisis } \theta_{u_0}\\
&p_0p_1(1-p_3)+ && \text{, value after leaving first IS } \theta^{U_0} \text{ and until } \theta_{u_1}\\
&p_0p_1p_3p_1(1-p_3)+ && \text{, value after } \theta^{U_1} \text{ and until } \theta_{u_2}\\
&...]\\
= A_1&[(1-p_0)+p_0p_1(1-p_3)\sum_{i=0}^{\infty} p_1^i p_3^i]\\
= A_1&[(1-p_0)+\frac{p_0p_1(1-p_3)}{1-p_1p_3}].
\end{aligned}$$

From now on we say $pr^0_{A_1} := (1-p_0)+\frac{p_0p_1(1-p_3)}{1-p_3p_1}$ is the state price of the payout A_1 starting in $t=0$. Analogously, we calculate the expected value for the payout A_2.

$$\begin{aligned}
\mathbb{E}[A_2] = A_2[&p_0(1-p_1-p_2)+ && \text{, value after } \theta_{u_0} \text{ until } \theta^{U_0}\\
&p_0p_1p_3(1-p_1-p_2)+ && \text{, value after } \theta_{u_1} \text{ until } \theta^{U_1}\\
&...]\\
= A_2&[p_0(1-p_1-p_2)\sum_{i=0}^{\infty} p_1^i p_3^i]\\
= A_2&[\frac{p_0(1-p_1-p_2)}{1-p_1p_3}].
\end{aligned}$$

So the state price of the payout A_2 starting in $t=0$ is given by $pr^0_{A_2} := \frac{p_0(1-p_1-p_2)}{1-p_1p_3}$. Analogously, we calculate the expected value for the payout A_3.

$$\begin{aligned}
\mathbb{E}[A_3] = A_3[&p_0p_2+ && \text{, going bankrupt in } [\theta_{u_0}, \theta^{U_0}]\\
&p_0p_1p_3p_2+ && \text{, going bankrupt in } [\theta_{u_1}, \theta^{U_1}]\\
&...]\\
= A_3&[p_0p_2\sum_{i=0}^{\infty} p_1^i p_3^i]\\
= A_3&[\frac{p_0p_2}{1-p_1p_3}].
\end{aligned}$$

From now on we say $pr^0_{A_3} := \frac{p_0p_2}{1-p_1p_3}$ is the state price of the payout A_3 starting in $t=0$. Calculating the expected value for the payout A_4 yields

$$\begin{aligned}
\mathbb{E}[A_4] &= A_4[p_0 + && \text{, touching } B_u \text{ in } \theta_{u_0} \\
&\quad p_0 p_1 p_3 + && \text{, touching } B_u \text{ in } \theta_{u_1} \\
&\quad ...] \\
&= A_4[p_0 \sum_{i=0}^{\infty} p_1^i p_3^i] \\
&= A_4[\frac{p_0}{1 - p_1 p_3}]
\end{aligned}$$

where $pr_{A_4}^0 := \frac{p_0}{1-p_1 p_3}$ is the state price of the payout A_4 starting in $t = 0$. Finally, we calculate the expected value for the payout A_5.

$$\begin{aligned}
\mathbb{E}[A_5] &= A_5[p_0 p_1 + && \text{, touching } B^U \text{ in } \theta^{U_0} \\
&\quad p_0 p_1 p_3 p_1 + && \text{, touching } B^U \text{ in } \theta^{U_1} \\
&\quad ...] \\
&= A_5[p_0 p_1 \sum_{i=0}^{\infty} p_1^i p_3^i] \\
&= A_5[\frac{p_0 p_1}{1 - p_1 p_3}].
\end{aligned}$$

From now on we say $pr_{A_5}^0 := \frac{p_0 p_1}{1-p_1 p_3}$ is the state price of the payout A_4 starting in $t = 0$. Now we are able to convert the one dimensional setup into vectorial calculus. Next to a better readability this brings the advantage that we can compress our notation to a minimum. Therefore, let $\vec{PO}$ denote the general payout structure and $\vec{pr}_0$ the according state prices starting in $t = 0$. The first row represents the payout A_1 and the state price $pr_{A_1}^0$, respectively. In conclusion, we have

$$\vec{PO} := \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix} \qquad \vec{pr}_0 := \begin{pmatrix} pr_{A_1}^0 \\ pr_{A_2}^0 \\ pr_{A_3}^0 \\ pr_{A_4}^0 \\ pr_{A_5}^0 \end{pmatrix} = \begin{pmatrix} (1 - p_0) + \frac{p_0 p_1 (1-p_3)}{1-p_1 p_3} \\ \frac{p_0(1-p_1-p_2)}{1-p_1 p_3} \\ \frac{p_0 p_2}{1-p_1 p_3} \\ \frac{p_0}{1-p_1 p_3} \\ \frac{p_0 p_1}{1-p_1 p_3} \end{pmatrix}. \tag{2.3}$$

To illustrate our approach let us consider the following example: If a firm is liquid it distributes dividends of 5\$ to the owner, i.e. $A_1 = 5\$$. If it has got any pecuniary difficulties the dividends will be reduced to 2\$ (payout in A_2). In case of bankruptcy no dividends will be distributed anymore ($A_3 = 0\$$). In the very moment the firm runs from LS to IS and vice versa, no payments to the owner are made.

Summarizing yields [4]

$$\vec{PO}^{\intercal} \quad = (5\$ \quad 2\$ \quad 0\$ \quad 0 \quad 0\$)\,. \tag{2.4}$$

If we want to calculate the expected value of the dividend of the owner, all we have to do is to calculate $\vec{P0}\vec{pr}_0$, i.e.

$$\mathbb{E}[\vec{P0}^{\intercal}\vec{pr}_0] = A_1 \cdot pr^0_{A_1} + A_2 \cdot pr^0_{A_2} + ... + A_5 \cdot pr^0_{A_5} \tag{2.5}$$

$$= 5pr^0_{A_1} + 2 \cdot pr^0_{A_2}. \tag{2.6}$$

3. The Capital Structure Model reflecting Illiquidity and Overindebtedness (IO-Model)

3.1. Basic Framework of the IO-Model

We assume the mathematical requirements stated in section 2.1 are fulfilled. The market is free of arbitrage opportunities, and for each subjective probability measure $\mathbb{P}$ there exists an equivalent measure $\mathbb{Q}$ called the risk-neutral probability measure.

We consider a firm whose instantaneous revenues $(R_t)_{t\in[0,\infty)}$ follow a geometric Brownian motion under the risk neutral pricing measure, i.e.

$$dR_t = \mu R_t dt + \sigma R_t dW^{\mathbb{Q}}_t, \tag{3.1}$$

where μ is the revenue's growth rate, σ is the corresponding volatility, and W_t is a standard Brownian motion under the risk-neutral measure. The initial value of revenue is $R_0 > 0$.

The firm faces variable costs captured by a deterministic ratio of revenues γ and deterministic fixed costs F independent of revenues. Thus, earnings before interest and taxes $EBIT_t$ in our setting are defined by,

$$EBIT_t = R_t(1-\gamma) - F \quad \forall t \in [0, \infty). \tag{3.2}$$

The risk free rate is captured by r. Moreover, we assume a flat corporate tax rate τ and do not consider personal taxes. Similar to other dynamic models (e.g., Hackbarth et al., 2007), we presuppose the unlevered cash flow to be $EBIT_t(1-\tau)$ for all $t \in [0, \infty)$ and ignore other cash-relevant items (e.g. depreciations, capital expenditures or changes in net working capital) for simplicity.[5]

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4 $^{\intercal}$ is the symbol for the vector transpose $\vec{v}^{\intercal}$ of $\vec{v}$

5 We do so without a loss of generality. The inclusion of these items in our model is simple but inflates the cash flow equation without adding further insights to our underlying research questions.

The conditional expected unlevered firm value subject to $\mathcal{F}_t$ $\mathbb{E}[V_t \,|\, \mathcal{F}_t]$ in such a setting is

$$\mathbb{E}[V_t \,|\, \mathcal{F}_t] = \int_t^\infty e^{-r(s-t)} \left(R_s(1-\gamma) - F\right)(1-\tau)ds \tag{3.3}$$

$$= \frac{R_t(1-\gamma)(1-\tau)}{r-\mu} - \frac{F(1-\tau)}{r}. \tag{3.4}$$

Please note that we will suppress the conditional lettering $\mathcal{F}_t$ due to readability. Whenever we will consider an expected value we deal with a conditional expected value. The corresponding σ-algebra is given by the context and indicated by R_t[6]. We need to split the variable part $(R_t(1-\gamma)(1-\tau))$ and the fixed part $(F(1-\tau))$ of the cash flow in (3.4) as the fixed part is not expected to grow with μ over time but to remain constant.

In our model we denote the market value of debt as $D(V)$ and follow the classic assumption of debt being issued as a console bond with constant coupon payment C to infinity (c.f. Leland (1994), Goldstein, Ju, and Leland (2001), Strebulaev (2007) et al.).

3.2. Default Triggers in the IO-Model

Existing dynamic models in corporate finance involve only one lower boundary for the underlying stochastic process. In Leland (1994) bankruptcy is triggered if the discounted conditional expected asset value V_t falls to a certain level V_B which is endogenously derived by the investors in order to maximize their equity value (endogenous default trigger). The second type of default trigger is exogenously determined by a covenant within the debt contract or by liquidity constraints. In such a setup the firm defaults either because it violates a certain debt covenant or because the firm and equityholders have no spare cash to pay their current cash obligations (i.e., redemption payments and/or interest payments).

The exogenous trigger is less often applied in literature (see e.g., Kim et al., 1993; Couch et al., 2012). Usually it is argued that it causes firms to cease their operations although the equity value is still positive. However, rationale equityholders would be ready to fund the firm as long as the market value of their investment exceeds the debt obligation. Only if the described condition is not fulfilled, equityholders will file for bankruptcy (Leland, 2006).[7] Thus, the vast majority of existing dynamic models relies on the endogenous trigger and ignores the exogenous one (see e.g., Leland and Toft, 1996; Goldstein et al., 2001; Hackbarth et al., 2007).

However, in reality we frequently observe that debtholders protect their claims with well-defined financial covenants allowing them to cancel the debt (and request a full redemption) whenever the covenant is triggered. While the option to cancel the debt is usually not exercised, the triggering event provides the opportunity to adjust (or to renegotiate if not pre-specified) the promised yield of debt and to influence

6 It should be remembered here that $\mathbb{E}[V_0 \,|\, \mathcal{F}_0] = \mathbb{E}[V_0]$

7 A crucial assumption for this policy is that equityholders can access external funds whenever the firm is threatened by illiquidity, i.e., they have "deep pocket". This assumption opens the field for arguments preferring the exogenous trigger (no external funds available or if it may be costly or difficult due to timing constraints or covenants in the debt contract).

strategic decisions regarding the firm (Achleitner et al., 2012). Moreover, entering this state, which we call *illiquidity state* , generates additional direct costs (e.g., lawyer or advisory expenses, discounts when selling assets) and indirect costs (e.g. loss of clients, disproportionate dilution by additionally raised equity) to the firm.

As additional covenant restrictions and liquidity constraints are ignored by traditional dynamic trade-off models it is not surprising that these models imply excessively high optimal leverage ratios compared to reality. Strebulaev (2007) emphasizes this fact and proposes the so far only known model combining both boundaries. He does not attempt to solve the model analytically and to derive general theoretical proofs but to calibrate the model for simulating firms' capital structure paths. His results are of particular importance for empirical tests of dynamic capital structure models.

We are able to model both, the exogenous covenant or liquidity boundary (B_u smaller upper barrier from above and B^U greater upper barrier from below) and the endogenous bankruptcy boundary (B_l from above), and to derive a closed-form analytic solution allowing us to draw general theorems regarding the choice of optimal capital structures. To the best of the authors knowledge this is the first attempt to model the optimal capital structure in a double barrier option framework. We state our first model-specific assumption:

Assumption 3.1. *The stochastic revenue process of our firm $R_t)_{t\in[0,\infty)}$ starts in liquidity state LS at R_0 above the lower- upper boundary B_u. When R_t hits B_u for some $t \in [0,\infty)$ the firm switches into illiquidity state IS, and R_s continues facing an upper-upper boundary B^U as well as a lower boundary B_l for some $t < s$. The firm reenters LS if and only if R_s hits B^U before it hits B_l for $t < s$. The number of switching events between LS and IS is not restricted. Given the firm stays in IS, the bankruptcy state BS is triggered if and only if R_s hits B_l before it hits B^U for $t < s$. At the time where $R_s = B_l$ for $t < s$ the firm files bankruptcy and the stochastic process R_s stops, i.e. R_s is not defined for $t < s$,.*

Figure 1 in section 2.2 illustrates the general setting of default triggers in our model. An important prerequisite in this setting is the relation $B_l \leq B_u < B^U$ which we prove in Lemma 3.6 after having derived explicit expressions of the boundaries.

We base the covenant boundary on the interest coverage ratio, unlevered cash flow to firm $EBIT(1-\tau)$ divided by coupon payments C, which must not fall below the covenant value δ and state B_u:

Lemma 3.2. *The firm will enter illiquidity state (IS) if $EBIT(1-\tau) \leq \delta C$, which corresponds to $R_t \leq B_u$ where $B_u = (\delta C + F(1-\tau)) \,/\, ((1-\gamma)(1-\tau))$.*

Proof. We substitute Equation (3.2) into the covenant definition from above and rearrange for R_t:

$$\begin{aligned} EBIT(1-\tau) &= \delta C \\ (R_t(1-\gamma) - F)(1-\tau) &= \delta C \\ R_t &= \frac{\delta C + F(1-\tau)}{(1-\gamma)(1-\tau)}. \end{aligned}$$

Since the covenant definition $(1-\tau)EBIT_t = \delta C$ corresponds to $R_t = B_u$, we have:

$$B_u := \frac{\delta C + F(1-\tau)}{(1-\gamma)(1-\tau)} \qquad \square$$

The starting point of the revenue process R in *illiquidity state* (IS) is R_{θ_u} which can be substituted by B_u: $R_{\theta_u} = B_u$. We capture the consequences for a firm entering (IS) in our second model-specific assumption.

Assumption 3.3. *When the firm enters illiquidity state (IS), certain default expenses occur, e.g. due to customers that stop buying the firms' products, which we assume to be a proportion ϵ of $\mathbb{E}[V_{\theta_{u_i}} | \mathcal{F}_{\theta_{u_i}}]$. Moreover, as long as the firm remains in IS ($B_l < R_t < B^U$ with $t \geq \theta_{u_i}$) the debtholders demand penalty interest C_{il} with $C_{il} > C$. Consequently, the covenant boundary B^U for the revenue process coming from below is greater than the covenant boundary B_u for the revenue process coming from above, i.e. $B_u < B^U$. If the firm returns from IS to liquidity state LS, the penalty interest payments will stop and the regular coupon payment C will be enforced.*

Assumption 3.3 allows us to derive B^U explicitly in our setting:

Lemma 3.4. *The firm will reenter liquidity state LS if $EBIT(1-\tau) = \delta C_{il}$ with $t \geq \theta_{u_i}$, which corresponds to $R_t = B^U$ where $B^U = (\delta C_{il} + F(1-\tau)) / ((1-\gamma)(1-\tau))$ with $t \geq \theta_{u_i}$.*

Proof. We substitute Equation (3.2) into the adjusted covenant definition from above and rearrange for R_t:

$$\begin{aligned} EBIT(1-\tau) &= \delta C_{il} \\ (R_t(1-\gamma) - F)(1-\tau) &= \delta C_{il} \\ R_t &= \frac{\delta C_{il} + F(1-\tau)}{(1-\gamma)(1-\tau)}. \end{aligned}$$

Since the covenant definition $EBIT(1-\tau) = \delta C_{il}$ corresponds to $R_t = B^U$, we have:

$$B^U := \frac{\delta C_{il} + F(1-\tau)}{(1-\gamma)(1-\tau)}. \qquad \square$$

Note that for $\delta = 1-\tau$ the boundaries B_u and B^U do not only represent covenant triggers but, indeed, illiquidity triggers, i.e., the firm is not able to pay its cash obligations.

The last possibility to be detailed is when the firm runs from IS to *bankruptcy state* (BS). In triggering bankruptcy we follow the classic assumption of Leland (1994) which is used in many more models (e.g., Leland and Toft, 1996; Goldstein et al., 2001; Hackbarth et al., 2007; Danis et al., 2014): If the expected asset value $\mathbb{E}[V_t]$ falls to a certain level V_B where liquidating the firm is optimal, i.e., value maximizing for the equityholders, the firm will file for bankruptcy. V_B is endogenously chosen by maximizing the equity value. In section 4 we demonstrate how to derive V_B. For now we consider it a constant parameter. The difference of our setting compared to existing models is that our underlying stochastic process regards the revenue and, thus, we need to transfer the classic bankruptcy condition $\mathbb{E}[V_t] = V_B$ to the condition $R_t = B_l$. Lemma 3.5 presents the transformation.

Lemma 3.5. *The firm will file for bankruptcy if* $\mathbb{E}[V_t] = V_B$ *with* $t \geq \theta_{u_i}$*, which corresponds to* $R_t = B_l$ *where* $B_l = \left(\left(V_B + \frac{F(1-\tau)}{r}\right)(r-\mu)\right) / ((1-\gamma)(1-\tau))$ *with* $t \geq \theta_{u_i}$*.*

Proof. We substitute equation (3.4) into the bankruptcy trigger definition from above and rearrange for R_t:

$$\mathbb{E}[V_t \,|\, \mathcal{F}_t] = V_B$$
$$\frac{R_t(1-\gamma)(1-\tau)}{r-\mu} - \frac{F(1-\tau)}{r} = V_B.$$

Since the bankruptcy definition $\mathbb{E}[V_t] = V_B$ corresponds to $R_t = B_l$, we have by simple rearrangements:

$$B_l := \frac{\left(V_B + \frac{F(1-\tau)}{r}\right)(r-\mu)}{(1-\gamma)(1-\tau)} \qquad \square$$

Finally, we prove the necessary relationship of our triggers in Lemma 3.6.

Lemma 3.6. *The covenant boundary* B^U*, upper-upper boundary to the revenue process* R_t *if the firm stays in illiquidity state (IS), is strictly greater than the covenant boundary* B_u*, lower-upper boundary to* R_t *if the firm stays in liquidity state (LS). Moreover,* B_u *is greater than or equal to the bankruptcy boundary* B_l*, lower boundary to* R_t *if the firm stays in IS. Thus, we have* $B_l < B_u < B^U$*.*

Proof.

$$B^U > B_u \tag{3.5}$$
$$\frac{\delta C_{il} + F(1-\tau)}{(1-\gamma)(1-\tau)} > \frac{\delta C + F(1-\tau)}{(1-\gamma)(1-\tau)} \tag{3.6}$$
$$C_{il} > C. \qquad \square$$

This holds since $C_{il} > C$ by definition.

$$B_u > B_l \tag{3.7}$$
$$\frac{\delta C + F(1-\tau)}{(1-\gamma)(1-\tau)} > \frac{\left(V_B + \frac{F(1-\tau)}{r}\right)(r-\mu)}{(1-\gamma)(1-\tau)} \tag{3.8}$$
$$\delta C + F(1-\tau) > \left(V_B + \frac{F(1-\tau)}{r}\right)(r-\mu) \tag{3.9}$$
$$V_B < \frac{\delta C + F(1-\tau)}{r-\mu} - \frac{F(1-\tau)}{r}. \tag{3.10}$$

The last inequality proves the statement. Considering in a first step F to be equal to zero the upper limit for considering bankruptcy V_B on the part of the equity holders is simply $\frac{\delta C}{r-\mu}$. They have to subtract C on their cash flow and add in case of tax advantages τC to their cash flow in a continuous setting. This equals $\frac{\delta C}{r-\mu}$ in $t = 0$. So δ covers the tax advantage. Its lower limit is given by $1-\tau$ just simply owing that no more tax benefits can be generated in our model. For $\delta > (1-\tau)$ the tax effect is strengthened. The same holds for the fixed term $\frac{F(1-\tau)}{r-\mu} - \frac{F(1-\tau)}{r}$ which has the function of an additive term.

As pointed out in section 2, *LS* and *IS* can alternate infinite times but the process will stop immediately as soon as the bankruptcy trigger B_l is hit. While the starting point of R_t in the first *LS* is special

(R_0), the starting points of R_t for the subsequent IS and LS are repetitive (R_{θ_u} and R_{θ^U}, respectively). This is an important feature for valuing the levered firm in section 3.4.

3.3. State Prices in the IO-Model

In this subsection we investigate the specific state prices p_0, p_1, p_2, and p_3, which we introduced in Definition 2.4, for our IO-model. As a reminder, p_0 and p_3 can be seen as assets, or more specifically as perpetual, down-and-in, cash-at-hit-or-nothing, single-barrier options which pay \$1 when the stochastic process R_t hits the barrier B_u which is below the initial value of the stochastic process. p_0 and p_3 only differ with respect to its initial values which are R_0 and $R_{\theta^U} = B^U$, respectively. The pricing formula for such an option type is well known[8] and, thus, can be applied to

$$p_0 = \left(\frac{B_u}{R_0}\right)^y \tag{3.11}$$

and analogously to

$$p_3 = \left(\frac{B_u}{B^U}\right)^y, \tag{3.12}$$

where

$$a := \mu - \frac{1}{2}\sigma^2, \qquad b := \sqrt{a^2 + 2\sigma^2 \cdot r}, \qquad y := \frac{a+b}{\sigma^2}. \tag{3.13}$$

Explicitly pricing p_1 and p_2 is less trivial as we deal with perpetual, cash-at-hit-or-nothing, double barrier options. The lower barrier is the bankruptcy boundary B_l and the upper barrier is the covenant boundary B^U. p_1 and p_2 differ with respect to its payout structure as the latter pays \$1 when the lower barrier is hit before the upper barrier has been hit and vice versa. Pelsser (2000) provides a pricing formulas for both structures in finite time which can be easily extended to a perpetual setting and applied to our specific problem. Thus, we have

$$p_1 = \exp\left\{\frac{a(l-x)}{\sigma^2}\right\} \frac{\sinh(\frac{b}{\sigma^2}x)}{\sinh(\frac{b}{\sigma^2}l)} \tag{3.14}$$

and analogously

$$p_2 = \exp\left\{\frac{-ax}{\sigma^2}\right\} \frac{\sinh(\frac{b}{\sigma^2}(l-x))}{\sinh(\frac{b}{\sigma^2}l)}, \tag{3.15}$$

8 Rubinstein and Reiner (1991) provide a very intuitive access to valuing such options. Moreover, in their compendium of exotic options (Rubinstein and Reiner, 1992) they investigate the pricing of many more option types.

where

$$x := \log\left(\frac{B_u}{B_l}\right) := \log\left(\frac{\delta C + F(1-\tau)}{V_B + \frac{F(1-\tau)}{r}(r-\mu)}\right), \tag{3.16}$$

$$l := \log\left(\frac{B^U}{B_l}\right) := \log\left(\frac{\delta C_{il} + F(1-\tau)}{V_B + \frac{F(1-\tau)}{r}(r-\mu)}\right), \tag{3.17}$$

and a as well as b are as defined in (3.13). Please note that x and l are functions of V_B.

3.4. Contingent Claims in the IO-Model

With the individual state prices p_0, p_1, p_2, and p_3 at hand, we are able to develop a framework a firm usually faces when generating a capital structure consisting of debt and equity. We start by deriving the value of debt, continue with benefits and costs of debt, and conclude the subsection by stating the resulting levered firm value and equity value. For each of these value components we first discuss its payoff structure and link it to the payoffs of the general model (A_1 to A_5). Subsequently, we show how to arrive at the expected value for each of them applying the state price vector $\vec{pr}_0$ as derived in section 2.4. Note that the individual state prices p_0 to p_3 defined in the previous section 3.3 provide the input for $\vec{pr}_0$.

The value of debt is defined by $D(V, C, C_{il})$. Due to readability we suppress the coupon payments C and penalty coupon payments C_{il}, and simply write $D(V)$. Debt promises a perpetual coupon payment C whose level remains constant unless the firm enters IS, i.e. the stochastic process R_t hits the covenant barrier B_u. Thus, in LS the debt value equals $\frac{C}{r}$ (c.f. A_1). As long as the firm remains in IS it needs to pay a permanent penalty coupon C_{il} unless the firm reenters LS or declares bankruptcy, i.e. enters BS. The debt value in IS is equal to $\frac{C_{il}}{r}$ (c.f. A_2). Let V_B denote the level of the asset value at which the firm runs into bankruptcy. If bankruptcy occurs, a fraction $0 \leq \alpha \leq 1$ of value will be lost to bankruptcy costs, including direct and indirect costs. This leaves the debtholders with value $(1-\alpha)V_B$ (c.f. A_3) and the equityholders with nothing. Note that we will not take any taxes in cases of bankruptcy into consideration, such as taxes on cancellation of debt. As already mentioned, bankruptcy occurs if and only if the firm ran into IS previously. In the very moment the firm hits the barrier B_u or B^U the value of the debt does not change (c.f. $A_4 = A_5 = 0$). Summarizing, we have the following payout structure $\vec{D}$ for the debt value:

$$\vec{D}^\intercal = \begin{pmatrix} \frac{C}{r} & C_{il} & (1-\alpha)V_B & 0 & 0 \end{pmatrix}. \tag{3.18}$$

To obtain the expected debt value $\mathbb{E}[DV(V)]$ we need to multiply the payout vector $\vec{D}$ with the state price vector $\vec{pr}_0$ derived in section 2.4. Due to readability, we suppress the expected value notation, so it simply remains $DV(V)$:

$$D(V) = \vec{D}^\intercal \vec{pr}_0. \tag{3.19}$$

Now we consider the value of tax benefits associated with the debt financing. These benefits resemble a security that pays a constant coupon equal to the tax-sheltering value of interest payments τC as long as the firm is in LS, τC_{il} in case of IS and nothing in BS. In the very moment the stochastic process hits a barrier B^U, B_u or B_l no tax benefits are generated. As we are concerned with a continuous framework A_4 and A_5 equal zero. Thus, we have the following payout structure $\vec{TB}$:

$$\vec{TB}^\intercal = \begin{pmatrix} \frac{\tau C}{r} & \frac{\tau C_{il}}{r} & 0 & 0 & 0 \end{pmatrix}. \tag{3.20}$$

Suppressing the expected value notation and the coupon payment, C, and multiplying the appropriate probability vector yields the following value of tax benefits $TB(V)$:

$$TB(V) = \vec{TB}^\intercal \vec{pr}_0. \tag{3.21}$$

Bankruptcy costs $BC(V)$ occur if and only if the firm goes bankrupt. This implies that the stochastic process R_t equals B_l. Thus, the unlevered firm value at θ_l is represented by $V_B = \frac{B_l(1-\gamma)(1-\tau)}{r-\mu} - \frac{F(1-\tau)}{r}$ and αV_B reflects the bankruptcy costs if bankruptcy is triggered (A_3). In no other states bankruptcy costs occur leaving us with a bankruptcy cost payout structure.

$$\vec{BC}^\intercal = \begin{pmatrix} 0 & 0 & \alpha V_B & 0 & 0 \end{pmatrix}. \tag{3.22}$$

In vectorial writing, we represent the value of bankruptcy costs $BC(V)$ as

$$BC(V) = \vec{BC}^\intercal \vec{pr}_0 \tag{3.23}$$

Finally, illiquidity expenses IE may occur whenever the firm enters IS. This can ultimately be ascribed to two key causes: on the one hand, direct costs of lawyers, banking fees and so on and on the other hand indirect costs, such as loss of investors' or customers' confidence. This will be priced with a fee in portion ϵ to the then prevailing unlevered firm value $\mathbb{E}[V_{\theta_{u_i}}]$. Thus, we have the following payout structure for IE

$$\vec{IE}^\intercal = \begin{pmatrix} 0 & 0 & 0 & \varepsilon \cdot \mathbb{E}[V_{\theta_{u_i}}] & 0 \end{pmatrix}. \tag{3.24}$$

Again, multiplication with the state price vector yields the value of the illiquidity expenses $IE(V)$

$$IE(V) = \vec{IE}^\intercal \vec{pr}_0. \tag{3.25}$$

The total firm value $V^L(V)$ (this equals the levered L firm value), is the sum of the four previous terms:

the firms' asset value (V), less the bankruptcy costs ($BC(V)$) and illiquidity expenses ($IE(V)$), plus value of tax benefits ($TB(V)$). For the payout structure of the net benefits NB value we consider in the next step all terms except of the firms' asset value V:

$$
\begin{aligned}
\vec{NB} &= \vec{TB} - \vec{IE} - \vec{BC} \\
&= \begin{pmatrix} \frac{\tau C}{r} \\ \frac{\tau C_{il}}{r} \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ \varepsilon \cdot \mathbb{E}[V_{\theta_{u_i}}] \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \alpha V_B \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\tau C}{r} & \frac{\tau C_{il}}{r} & -\alpha V_B & -\varepsilon \cdot \mathbb{E}[V_{\theta_{u_i}}] & 0 \end{pmatrix}^{\intercal}.
\end{aligned}
\tag{3.26}
$$

Taking the conditional expected value V into consideration we have the following total firm value:

$$V^L(V) = V + \vec{NB}^{\intercal} \vec{pr}_0. \tag{3.27}$$

The value of equity is the total value of the levered firm less the value of debt.

$$EV(V) = V + \vec{NB}^{\intercal} \vec{pr}_0 - \vec{D}^{\intercal} \vec{pr}_0. \tag{3.28}$$

The contingent claims of our IO-model developed in this section provide safe grounds for exploring solutions to the optimal capital structure problem in the next section.

4. Analysis of the Optimal Capital Structure in the IO-Model

In general, we are concerned with maximizing the levered firm value with respect to the coupon payments C subject to certain constraints. The classic constraint introduced by Leland (1994) is that equityholders choose V_B, the asset value where the firm files for bankruptcy, in order to maximize the equity value. We denote this optimal level of bankruptcy asset value with V_B^* which is not exogenously determined but endogenously obtained by setting the first derivative of the equity value with respect to V_B equal to zero. An additional constraint in our setting is that C_{il} needs to reflect a certain risk spread φ above the risk free rate r. Thus, our optimization problem can be formally stated as follows:

$$V^L(V, C, C_{il}) \to \max \tag{4.1}$$
$$\text{s.t. } \frac{\partial EV(V, C, C_{il})}{\partial V_B} = 0$$
$$C_{il} - \varphi r DV(V, C, C_{il}) = 0.$$

All other parameters in our model are exogenously set and can be either observed in reality or empirically estimated. Table 1 summarizes these parameters, suggests how to determine them, and provides an idea with respect to reasonable value assumptions.

Table 1: Exogenous Parameters of the IO-Model

Parameter	**Description**	**Rationale**	**Exemplary reasonable values**
r	risk free rate	Average of 10-year Treasury rate (1/1989-7/2016) Approach similar to Leland (2004), Huang and Huang (2012)	0.05
τ	corporate tax rate	Federal corporate income tax rate in the US for bigger companies Approach similar to similar to Leland and Toft (1996), Strebulaev (2007)	0.35
R_0	initial value of the revenue process	Firm individual observable parameter	\$25 bn
μ	risk-neutral drift of the revenue process	Firm individual empirical estimation of the real drift μ_P and risk-neutral adjustment by $\mu = \mu_P - (r_A - r)$ Adjustment similar to Goldstein et al. (2001), Couch et al. (2012)	0.02
σ	volatility of the revenue process	Firm individual empirical estimation of the revenue's volatility	0.25
γ	variable cost ratio	Firm individual empirical estimation of the costs of goods sold ratio	0.70
F	fixed costs	Firm individual empirical estimation of selling, general and administrative expenses	0.00
δ	interest coverage ratio	Firm or debt tranche individual covenant defined in the debt contract. Natural lower boundary: $1 - \tau$ as this reflects illiquidity.	$1 - \tau$
φ	spread factor for illiquid firms vs. r	Estimation based on average spread between the promised yield of Caa-rated firms (highly vulnerable to nonpayment) and the risk free rate with 10 years maturity (source: Moody's)	2.50
α	bankruptcy cost ratio	Firm or industry-specific estimation based on empirical models We use findings of Glover (2016)	e.g. 0.39 (Food) 0.49 (machinery)
ϵ	illiquidity cost ratio	Firm or industry-specific estimation based on emprical models with respect to technical defaults We use findings of Ertan and Karolyi (2016)	0.04

This table contains all exogenously set parameters of the IO-model. It also provides suggestions how to observe or estimate the parameters and gives indications with respect to reasonable values.

In the subsequent subsection we develop a solution to our general optimization problem outlined in Eq. (4.1) and compare the results of our IO-model to the results of pure illiquidity and overindebtedness models. Thereafter, we discuss a possible extension to our optimization framework by endogenizing the covenant ratio δ. This allows us to investigate not only the influence of δ on the optimal solution but also whether optimal δ values may exist. Finally, we apply the IO-model to publicly listed companies in the US in order to judge whether our model may explain observed leverage ratios.

4.1. Identification of the Optimal Bankruptcy Trigger V_B^*

This subsection investigates the optimal bankruptcy trigger V_B^* via maximizing the equity value, i.e.

$$E(V) \rightarrow \max \tag{4.2}$$

$$\Leftrightarrow \frac{\partial E(V)}{\partial V_B} = 0. \tag{4.3}$$

Technically, we calculate the first derivative of the the equity value with respect to V_B. As we face a long complex value function we present the result based on the modular principle. We benefit from this technique since the *differentiation is linear*. Additionally, beyond reducing complexity, this method allows for investigating some boundary constraints, e.g. fixed costs equal to zero $F = 0$. Our proceeding is related to the equity value function $E(V)$ (c.f. Eq. (3.28)) consisting of the vector $\vec{NB}$ and $\vec{D}$, the unlevered firm value V, and the state price vector $\vec{pr}_0$. In turn, the state price vector $\vec{pr}_0$ consists of the single state prices p_0 to p_3 derived in section 3.3 for the IO-model. The place holders a, b and y of p_0 to p_3 are constants. However, the place holders x and l of p_0 to p_3 are functions of V_B (c.f. Eq. (3.16)-(3.17)). We start with their first derivatives. The following holds:

$$\frac{\partial x}{\partial V_B} = \frac{-1}{V_B + F(1-\tau)r^{-1}} \tag{4.4}$$

$$\frac{\partial l}{\partial V_B} = \frac{-1}{V_B + F(1-\tau)r^{-1}} \tag{4.5}$$

$$\frac{\partial (l-x)}{\partial V_B} = 0. \tag{4.6}$$

Please note that the first derivative x' of x with respect to V_B equals the first derivative l' of l with respect to V_B. In the next step we want to calculate the derivatives of the state prices p_0 to p_3. Since p_0 is independent of V_B we obtain:

$$\frac{\partial p_0}{\partial V_B} = 0.$$

The same holds for p_3. Thus, we have

$$\frac{\partial p_3}{\partial V_B} = 0.$$

Consequently, it remains to calculate the derivatives of p_1 and p_2 which we do by applying $\frac{\partial \sinh(x)}{\partial x}$ =

$\cosh(x)$:

$$
\begin{aligned}
p_1' = \frac{\partial p_1}{\partial V_B} =& 0 + e^{\frac{a}{\sigma^2}(l-x)} \frac{\frac{b}{\sigma^2} x' \sinh'(\frac{b}{\sigma^2} x)\sinh(\frac{b}{\sigma^2} l) - \frac{b}{\sigma^2} l' \sinh(\frac{b}{\sigma^2} x)\sinh'(\frac{b}{\sigma^2} l)}{\sinh^2(\frac{b}{\sigma^2} l)} \\
=& e^{\frac{a}{\sigma^2}(l-x)} \frac{b}{\sigma^2} x' \frac{\cosh(\frac{b}{\sigma^2} x)\sinh(\frac{b}{\sigma^2} l) - \sinh(\frac{b}{\sigma^2} x)\cosh(\frac{b}{\sigma^2} l)}{\sinh^2(\frac{b}{\sigma^2} l)} \\
=& e^{\frac{a}{\sigma^2}(l-x)} \frac{b}{\sigma^2} x' \frac{\sinh(\frac{b}{\sigma^2}(x-l))}{\sinh^2(\frac{b}{\sigma^2} l)} \\
=& \frac{-e^{\frac{a}{\sigma^2}(l-x)} \frac{b}{\sigma^2}}{V_B + F(1-\tau)r^{-1}} \cdot \frac{\sinh(\frac{b}{\sigma^2}(x-l))}{\sinh^2(\frac{b}{\sigma^2} l)}.
\end{aligned}
$$

For the derivative of the state price p_2 we receive the following:

$$
\begin{aligned}
p_2' = \frac{\partial p_2}{\partial V_B} =& \frac{-a}{\sigma^2} x' e^{\frac{-a}{\sigma^2}(l-x)} + e^{\frac{-a}{\sigma^2}(l-x)} \frac{\frac{b}{\sigma^2}(l-x)' \sinh'(\frac{b}{\sigma^2}(l-x))\sinh(\frac{b}{\sigma^2} l) - \frac{b}{\sigma^2} l' \sinh(\frac{b}{\sigma^2}(l-x))\sinh'(\frac{b}{\sigma^2} l)}{\sinh^2(\frac{b}{\sigma^2} l)} \\
=& \frac{-a}{\sigma^2} x' e^{\frac{-a}{\sigma^2}(l-x)} + e^{\frac{-a}{\sigma^2}(l-x)} \frac{-\frac{b}{\sigma^2} l' \sinh(\frac{b}{\sigma^2}(l-x))\cosh(\frac{b}{\sigma^2} l)}{\sinh^2(\frac{b}{\sigma^2} l)} \\
=& x' e^{\frac{-a}{\sigma^2}(l-x)} [\frac{-a}{\sigma^2} - \frac{b}{\sigma^2}] \frac{\sinh(\frac{b}{\sigma^2}(l-x))\cosh(\frac{b}{\sigma^2} l)}{\sinh^2(\frac{b}{\sigma^2} l)} \\
=& \frac{e^{\frac{-a}{\sigma^2}(l-x)}}{\sigma^2[V_B + F(1-\tau)r^{-1}]} [a+b] \frac{\sinh(\frac{b}{\sigma^2}(l-x))\cosh(\frac{b}{\sigma^2} l)}{\sinh^2(\frac{b}{\sigma^2} l)}.
\end{aligned}
$$

Hence, we obtain the first derivative of the state price vector $\vec{pr}_0$ with respect to V_B by using the product and quotient rule, i.e.

$$
\frac{\partial \vec{pr}_0}{\partial V_B} = \begin{pmatrix} 1 + \frac{p_0 p_1 (1-p_3)}{(1-p_1 p_3)^2} \\ \frac{-p_0(1-p_1-p_2)}{(1-p_1 p_3)^2} \\ \frac{-p_0 p_2}{(1-p_1 p_3)^2} \\ \frac{-p_0}{(1-p_1 p_3)^2} \\ \frac{-p_0 p_1}{(1-p_1 p_3)^2} \end{pmatrix}. \tag{4.7}
$$

For the derivative of the net benefit vector $\vec{NB}$ and the debt vector $\vec{D}$ we have

$$
\frac{\partial \vec{NB}}{\partial V_B} = \begin{pmatrix} 0 \\ 0 \\ -\alpha \\ 0 \\ 0 \end{pmatrix}, \qquad \frac{\partial \vec{D}}{\partial V_B} = \begin{pmatrix} 0 \\ 0 \\ 1-\alpha \\ 0 \\ 0 \end{pmatrix}. \tag{4.8}
$$

In summary, we have to solve the following equation with the help of the product rule and linearity

$$\frac{\partial E(V)}{\partial V_B} = \frac{\partial}{\partial V_B}[V + ((\vec{NB}) - (\vec{D}))p\vec{r}_0]. \tag{4.9}$$

Thus, we arrive at

$$0 \overset{!}{=} 1 + \frac{p_0 p_2}{1 - p_1 p_3} + (\vec{NB} - \vec{D})p\vec{r}_0{}'. \tag{4.10}$$

This implicit equation can be solved with the help of mathematical software such as Matlab and using some known methods, e.g. Newton's method.

Before we can compare the optimal bankruptcy trigger $V^*_{B_{IO}}$ from the IO-model with the optimal bankruptcy trigger in a single barrier world, such as the model of Leland (1994) (overindebtedness) or Couch et al. (2012) (illiquidity), we need to match the assumptions. As mentioned in section 3.1 we refer to a revenue process. Thus, we have to transfer the firm's asset approach in a single barrier world into a revenue's approach in a single barrier world[9]. Furthermore, two famous bankruptcy triggers are known in literature. On the one hand bankruptcy is triggered when the firm is overindebted. Leland (1994) investigates the implications to the optimal capital structure given this constraint. On the other hand bankruptcy can be declared when the firm is illiquid or breaks a covenant. Couch et al. (2012) base their investigations of valuing tax shields on this barrier. Adjusting the Leland model (overindebtedness) to the revenue process yields the following optimal bankruptcy trigger $V^*_{B_{over}}$:

$$V^*_{B_{over}} = \frac{y}{1+y}\frac{C(1-\tau)}{r} - (1 - \frac{y}{1+y})\frac{F(1-\tau)}{r}. \tag{4.11}$$

When the fixed costs F equal zero we generate the standard Leland solution. The appropriate optimal bankruptcy trigger $V^*_{B_{illiquid}}$ given illiquidity as the bankruptcy criterion with fixed costs F equal to zero resemble the standard solution given inCouch et al. (2012).

4.2. Identification of the Optimal Coupon Payment C

With the help of section 4.1 we are able to maximize our total firm value V^L given the optimal bankruptcy trigger $V^*_{B_{IO}}$. This is done by endogenizing the coupon payments C. Thus, the coupon payment is no longer fixed and considered as a constant. Rather, we compute the first derivative of the total firm value V^L subject to C. Finally, we set the first derivative of the total firm value equal to zero, i.e.

$$\frac{\partial V^L(V)}{\partial C} = 0. \tag{4.12}$$

[9] The firm's asset approach is given by the diffusion process $\frac{dV}{V} = \mu dt + \sigma dW$, where V represents the value of the firm's activities, μ the constant growth rate, σ the constant volatility, and W a standard Brownian motion. V is usually known as the *asset value* of the firm.

Solving this equation for the optimal coupon C^*_{IO} maximizes the total firm value. We will now compare the firm's maximizing coupon payment C^*_{IO} in a double barrier world with the firm maximizing coupon payment C^*_{over} and $C^*_{illiquid}$ that are generated when either overindebtedness or illiquidity are the bankruptcy triggers. The following figure illustrates the findings graphically.

Figure 4: Optimal Capital Structure under the IO-Model, pure Illiquidity Model and pure Overindebtedness Model

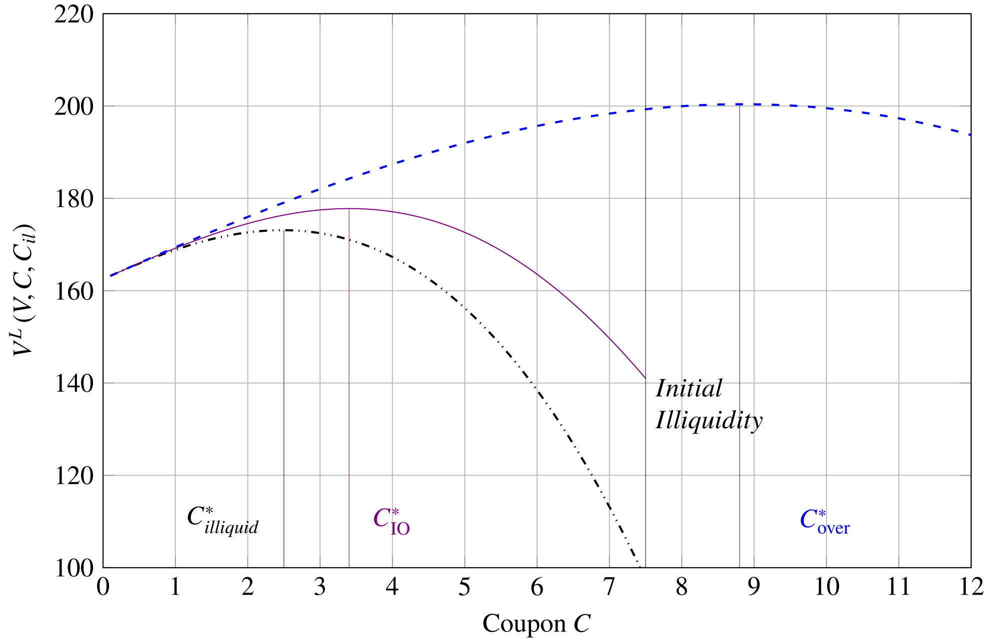


Figure 5: This figure analyzes the firm value ($V^L(V)$)-maximizing choice of coupon payments C for the IO-model in comparison to the classic models of illiquidity and overindebtedness. The blue, dashed line represents $V^L(V)$ for different C with overindebtedness as a bankruptcy trigger. The violet, solid line represents the IO-model and the black, dashed-dotted line depicts the case of illiquidity. The chosen model parameters are as follows: $r = 0.05$, $\tau = 0.35$, $R_0 = 25$, $\mu = 0.02$, $\sigma = 0.20$, $\gamma = 0.70$, $F = 0$, $\delta = 1 - \tau$, $\epsilon = 0.00$, and $\varphi = 2.5$.

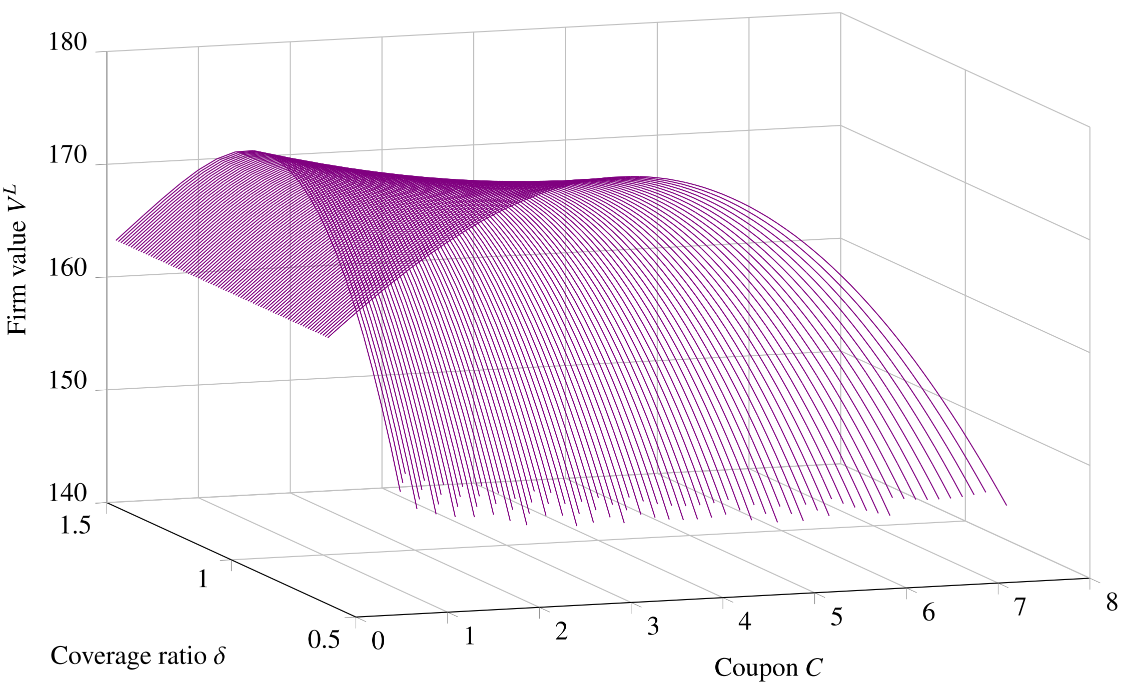
The figure shows the coupon level C subject to the total firm value V^L. Each of the three parables represent a different bankruptcy trigger. The curve on top is the function that arises if and only if overindebtedness is the only bankruptcy trigger. Analogously, the curve on bottom is generated if and only if illiquidity creates bankruptcy. The curve in the middle combines both approaches and represents the total firm value function with respect to C of the IO-model. The figure depicts four main aspects. (i) We can observe that all three curves are concave, i.e. there exists a global maximum. (ii) In case of overindebtedness the total firm value with respect to C is greater than in case of illiquidity. Taking both barriers into consideration provides a curve that lies in-between. (iii) The same holds true for the optimal coupon payments, i.e. $C^*_{illiquid} < C^*_{IO} < C^*_{over}$. Finally, (iv) if there is only overindebtedness as the bankruptcy trigger, the optimal coupon payment C^*_{over} is in the area of illiquidity. Thus, optimizing the total firm value with overindebtedness as the bankruptcy criterion provokes directly illiquidity.

As the figure shows, the double barrier approach provides solution that are in-between the rough constraints of overindebtedness and illiquidity. This is in accordance with the intuition. Moreover, the optimal coupon payment C^*_{IO} of the IO-model is in the area of liquidity.

4.3. Extensions to the Optimization Framework - Endogenizing Debt Contract Parameters

The IO-model provides insights beyond the discussed framework where the optimal capital structure is derived with an endogenously obtained V^*_B but otherwise given parameters. For instance, it allows for analyzing some standard debt contract parameters like the covenant ratio δ. We are able to determine its impact on the optimal capital structure choice and to investigate whether an optimal δ exists. Figure 6 depicts the analysis results when C and δ can be freely chosen.

Figure 6: Levered Firm Value $V^L(V)$ in dependence of Covenant Ratio δ and Coupon Payment C



The graph depicts how changing δ and C impacts $V^L(V)$. For lower *delta* values the maximum levered firm value $V^{L,*}(V)$ is achieved with higher choices of C^* and vice versa. The global optimum is at the minimum δ of $1-\tau$. The chosen model parameters are as follows: $r = 0.05$, $\tau = 0.35$, $R_0 = 25$, $\mu = 0.02$, $\sigma = 0.20$, $\gamma = 0.70$, $F = 0$, $\epsilon = 0.00$, and $\varphi = 2.5$.

As Figure 6 reveals, a higher δ causes lower optimal choices of C^* and also reduces the optimal levered firm value $V^{L,*}(V)$. The results may surprise as we usually observe δ values between 1 and 2 in corporate debt contracts. Two reasons for the discrepancy are identified:

(i) Debtholders in our setting are risk-neutral, i.e. they are only interested in an expected net present value of zero and do not discount riskier payoff structures. We demonstrate the effect of higher δ values on the state price (discounted probability) of the BS in Table 2. Clearly, the state price $pr^0_{(1-\alpha)V_B}$ decreases with increasing δ. Risk-averse debtholders value this fact while risk-neutral debtholders are indifferent. Thus, we may have found an indication for risk-aversion of debtholders.

Table 2: Bankruptcy State Prices $pr^0_{(1-\alpha)V_B}$ in dependence of the covenant ratio δ

δ	C^*	$V^{L,*}$	$L^* = D(V)/V^{L,*}$	$pr^0_{(1-\alpha)V_B}$
0.65	3.40	177.79	0.37	0.21
0.75	3.00	176.56	0.35	0.20
0.85	2.70	175.59	0.32	0.20
0.95	2.40	174.80	0.30	0.18
1.05	2.20	174.15	0.29	0.18
1.15	2.00	173.59	0.27	0.17

The table illustrates how increasing δ values lead to a lower bankruptcy risk (represented by a lower bankruptcy state price). This shows that debt holders which are not risk-neutral may actually insist on a δ greater than $1-\tau$ depending on their risk appetite.

(ii) Information are symmetrically distributed in our setting, i.e. debtholders know the true V_B where equityholders file for bankruptcy. However, in reality this information is most likely only known to the equityholders themselves. Pretending a higher V_B may result in better debt contracts. Debtholders shield themselves against such behavior with increased covenant ratios. Please note that the analysis of (ii) will be detailed in the next version of the working paper.

4.4. Empirical Application of the IO-Model

Finally, we test our model for firms publicly listed in the US. Our dataset, retrieved from Thomson-Reuters EIKON, is based on the logic of the Center for Research in Security Prices (CRSP). We consider all firms that have been listed on the NYSE, NASDAQ, NYSE MKT and NYSE ARCA between 1981 and 2016 including all leavers and joiners of this period. We exclude firms from finance and insurance (NAICS sector code 52) as well as firms with inconsistent data (e.g. constantly negative revenues) or not sufficient time series (less than 10 firm years). After these exclusions, our sample contains 4,845 firms and 97,001 firm-year observations with non-missing values for revenues, costs of goods sold (COGS), selling, general and administrative expenses (SGA), debt, total assets, and market capitalization.

In a first step we estimate the parameters of the stochastic revenue process, drift rate μ and standard deviation σ. Moreover, we test whether the observed revenue paths could follow a geometric Brownian motion (gBm) by applying the Jarque-Bera (JB) test for normal distribution. In total, at the 5% interval we cannot reject the null hypothesis of the JB-test postulating that the considered process is not following a gBm for 47.5% of the firms. Thus, our basic model requirement is valid for almost half of the publicly listed firms in the US. Table 3 summarizes the test results for all NAICS sectors.

For the firms where the revenues follow a gBm we proceed with estimating the other parameters and calculate the average observed leverage $L = D(V)/V^L(V)$. While we can retrieve the estimates for the variable cost ratio γ and the fixed costs F from our dataset, we have to rely on other studies for the other missing parameters. We follow Glover (2016) in his industry-specific estimates of the expected

Table 3: Normal-Distribution Test of the log-changes of R_t

		Jarque-Bera Test					
		N, Norm.-Dist.		in %, Norm.-Dist.			
NAICS Sector	**No. Of Firms (N)**	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.10$	μ	σ
Accommodation and Food Services	96	44	36	0.4583	0.3750	0.0264	0.1030
Administrative, Support, Waste, Remediation	105	42	31	0.4000	0.2952	0.0253	0.2175
Construction	77	33	29	0.4286	0.3766	0.0110	0.2190
Health Care and Social Assistance	103	31	27	0.3010	0.2621	0.0074	0.1824
Information	529	256	216	0.4839	0.4083	0.0199	0.1776
Manufacturing	1988	948	785	0.4769	0.3949	-0.0026	0.1526
Mining, Quarrying, and Oil and Gas Extraction	264	157	126	0.5947	0.4773	0.0117	0.2622
Professional, Scientific, and Technical Services	408	221	181	0.5417	0.4436	0.0014	0.1454
Real Estate and Rental and Leasing	204	77	66	0.3775	0.3235	0.0392	0.1912
Retail Trade	242	122	107	0.5041	0.4421	0.0438	0.1270
Transportation and Warehousing	143	56	46	0.3916	0.3217	0.0188	0.1607
Utilities	103	41	33	0.3981	0.3204	-0.0054	0.1780
Wholesale Trade	147	69	55	0.4694	0.3741	0.0269	0.2304
Others	110	48	41	0.4364	0.3727	0.0274	0.2014

The table depicts the results of the Jarque-Bera test for normal distribution which we apply to examine the log-changes of the stochastic process R_t. The null hypothesis of the test is that the underlying process is normally distributed. Thus, choosing a higher significance level α leads to a higher number of firms for which normal distribution is ruled out. The last two columns provide our estimations of the risk-neutral drift of the revenue process μ and its standard deviation σ.

bankruptcy costs α. For the illiquidity expenses ϵ and the average covenant ratio δ industry-specific estimates are not yet available. Thus, we apply the general estimates of Ertan and Karolyi (2016) to all industries. Please note that we have indexed the initial level of the stochastic process R_0 to 100 in order to make all firms comparable. Table 4 summarizes our input choices.

To conclude, we obtain the optimal leverage based on the IO-model as well as for the pure illiquidity and pure overindebtedness model. These results are compared to the observed leverage ratios. The results are shown in Table 5.

The leverage ratios estimated by the IO-model show the lowest absolute deviation (Abs. Dev.) from the observed leverage except for the sector "Real Estate and Rental and Leasing" where the overindebtedness model performs slightly better. The IO-estimates lie within the one standard error range for 3 of the sectors and within a two standard error range for another 3 sectors. The pure illiquidity model underestimates optimal leverage consistently in all sectors while the pure overindebtedness model leads consistently to overestimation. None of the two models achieves results within one or two standard errors from the observed leverage. The results prove that the IO-model is a major step in explaining observed leverage ratios and delivers a new unique contribution to the capital structure literature.

5. Conclusion

This article establishes the first dynamic corporate valuation model incorporating an illiquidity trigger and a bankruptcy trigger in a double barrier framework.

First, we introduce a general model of our framework which we carefully develop towards definitions of state prices and payout structures. Subsequently, we apply the general model to corporate valuation and

Table 4: Input Parameters of the IO-Model and Observed Leverage

NAICS Sector	α	ϵ	δ	γ	F	R_0	$L = D(V)/V^L(V)$
Accommodation and Food Services	0.3890	0.04	1.00	0.6195	14.00	100	0.4594
Administrative, Support, Waste, Remediation	0.4740	0.04	1.00	0.5110	23.04	100	0.1994
Construction	0.3740	0.04	1.00	0.7220	18.83	100	0.4405
Health Care and Social Assistance	0.4740	0.04	1.00	0.2483	51.35	100	0.5497
Information	0.4740	0.04	1.00	0.3941	25.37	100	0.2927
Manufacturing	0.3970	0.04	1.00	0.6915	15.85	100	0.3071
Mining, Quarrying, and Oil and Gas Extraction	0.4630	0.04	1.00	0.5165	11.42	100	0.2535
Professional, Scientific, and Technical Services	0.4740	0.04	1.00	0.5131	33.90	100	0.2225
Real Estate and Rental and Leasing	0.4740	0.04	1.00	0.4066	14.27	100	0.5412
Retail Trade	0.4420	0.04	1.00	0.7026	19.19	100	0.2714
Transportation and Warehousing	0.4130	0.04	1.00	0.4513	27.16	100	0.4286
Utilities	0.4740	0.04	1.00	0.3518	33.02	100	0.4531
Wholesale Trade	0.4420	0.04	1.00	0.7382	23.74	100	0.2923
Others	0.4598	0.04	1.00	0.5824	23.69	100	0.2683

The table provides an overview of the chosen input parameters for each NAICS sector. For the bankruptcy costs α we follow the estimates of Glover (2016). Regarding the illiquidity expenses ϵ and the average covenant ratio δ industry-specific estimates are not yet available. Thus, we apply the general estimates of Ertan and Karolyi (2016) to all industries. The starting point of the stochastic revenue process R_0 is indexed to 100. The estimates for the variable cost ratio γ and the fixed costs F are based on all normally distributed firms in our sample from NASDAQ, NYSE, NYSE ARCA, and NYSE MKT. F has been related to the index of R_0. The leverage ratio $L = D(V)/V^L(V)$ is based on our sample, too.

Table 5: Optimal Capital Structure Estimates versus Observed Leverage for NAICS Sectors

	Observed		**Illiquidity**		**IO-Model**		**Overindebtedness**	
NAICS Sector	L	**1 Std. Err.**	L^*	**Abs. Dev.**	L^*	**Abs. Dev.**	L^*	**Abs. Dev.**
Accommodation and Food Services	0.4594	0.0742	0.1861	0.2734	0.3096	0.1498	0.7942	0.3348
Administrative, Support, Waste, Remediation	0.1994	0.0380	0.1599	0.0394	0.2985	0.0991	0.8064	0.6071
Construction	0.4405	0.0376	0.1442	0.2962	0.3658	0.0747	0.6620	0.2215
Health Care and Social Assistance	0.5497	0.0430	0.0662	0.4835	0.4627	0.0871	0.6554	0.1057
Information	0.2927	0.0160	0.0316	0.2611	0.2236	0.0691	0.8629	0.5703
Manufacturing	0.3071	0.0084	0.0472	0.2599	0.3252	0.0181	0.6746	0.3675
Mining, Quarrying, and Oil and Gas Extraction	0.2535	0.0142	0.0125	0.2410	0.2450	0.0085	0.6359	0.3824
Professional, Scientific, and Technical Services	0.2225	0.0127	0.0974	0.1251	0.2227	0.0003	0.6897	0.4672
Real Estate and Rental and Leasing	0.5412	0.0268	0.0149	0.5263	0.1584	0.3828	0.7198	0.1786
Retail Trade	0.2714	0.0170	0.0412	0.2302	0.3017	0.0303	0.6859	0.4145
Transportation and Warehousing	0.4286	0.0252	0.0452	0.3834	0.2446	0.1840	0.7138	0.2852
Utilities	0.4531	0.0259	0.0104	0.4427	0.4583	0.0051	0.6323	0.1791
Wholesale Trade	0.2923	0.0337	0.0459	0.2464	0.1329	0.1594	0.6626	0.3703
Others	0.2683	0.0242	0.0936	0.1747	0.1689	0.0994	0.6880	0.4197

This table summarizes the optimal leverage ratios $L^* = D(V)/V^{L,*}(V)$ generated by the IO-model, and for a pure illiquidity or overindebtedness trigger. The results are compared to the observed average leverage L for all NAICS sectors. The absolute deviation towards the observed leverage is depicted for each of the three models (Abs. Dev.).

the problem of optimal capital structure. Thereby, we create the illiquidity-overindebtedness (IO-) model which allows us to price all components of debt and equity value. Finally, we compare our solution to the two classic cases of only considering one of the two boundaries. The results we obtain lie in-between and explain observed capital structure choices much better than the existing models as we demonstrate by an empirical study of the US market.

Moreover, our general model proves to be relevant in many other research areas. Two examples may be mentioned: (i) The problem of modeling optimal counter-cyclical policies (monetary policy and government investment programs) could be treated in such a framework where the lower-upper boundary (illiquidity) triggers e.g. an investment program for a specific industry. Hitting the lower boundary (bankruptcy) could lead to a stop of the program as there is no positive prospect for the industry and the upper-upper boundary could represent a stop of the program as the industry has recovered. (ii) In a biological application, the flu diffusion can be described with our model as it stays normally within an endemic steady state but can suddenly become epidemic (or indeed pandemic like the 1918 flu pandemic) if its infection rate surpasses a special critical value, and it can also return to endemic state.

The model also provides a good base for further extensions. For instance, it is sometimes observed in reality that the stochastic process jumps whenever the illiquidity boundary is hit which is easily implementable into the existing framework. Additionally, adjustments in the payout structure can be simply executed as we provide a general framework for all kinds of payout. Beyond that further empirical studies in the field of corporate finance (e.g. cost of capital, probability of default) can be based upon the model.

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