Financing Uncertain Growth Options

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Abstract

In this paper we study the financing of high uncertainty projects. High uncertainty is defined as the lack of knowledge of whether growth options exist. In this paper we will describe this uncertainty by a probability distribution which describes the arrival of a growth option at a deterministic time. Once the option arrives an additional uncertainty exists since it is not certain that it is profitable to exercise it. We value the corporate securities with contingent claims valuation both for a whole equity financed firm and a debt-equity financed firm. Unlike traditional capital structure models, we find non-convex value functions for the firm vis-a-vis the debt coupon under specific parametrizations. High and low leverage can yield similar firm value maximizing policies.

Keywords: Contingent claims, capital structure, real options, capital budgeting.

1 Introduction

Project financing is the cornerstone of financial decision making for corporations and governments. When the projects are very risky in nature due to inherent riskiness, the correct valuation and financing will determine its viability. In this paper we will deal with the financing of projects that are risky in the sense that the financial decision maker does not know whether the project can be realized.

To make a concrete example we assume the case of oil exploration. Suppose that a country wishes to explore for oil in a particular area. The country has some information about the appropriateness of the rock formations regarding their ability to contain oil. However, it is not certain that the exploration will yield any oil. Therefore, one layer of uncertainty is the existence of the opportunity. Once the uncertainty of the existence of oil has been resolved, another layer of uncertainty exists. The next uncertainty is whether it is profitable to extract oil.

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Summing up, the riskiness is two-fold. First, does an opportunity exist? Second, if it exists is it profitable to exploit it? Projects of this nature are ubiquitous in modern economies. Research and development in pharmaceutical industry is such a field where pharmaceutical companies face staged decision approval from the FDA. It is unknown ex-ante whether a drug will be approved and if it is approved, are the market conditions appropriate for a profitable sale in the market?

Traditionally, the real options framework has been used to valuate projects with uncertainty. In real options model the option to wait allows firms with growth options to time their strategy in order to undertake projects when market conditions are favorable. However, the implicit assumption is that the growth opportunity exists. This paper abstracts from this simplification by assuming that a growth opportunity arrives as a particular point in time with a specific known probability.

In reality, projects arrive at random times and not at specific times. Even though this is true, the modelling of the random arrival of the growth opportunity would be difficult. Modelling the arrival stochastically for a debt-equity financed project, the analysis would rely solely on numerical analysis and simulation. In this paper we are able to derive largely analytic solutions and use some numerical analysis by assuming that the growth option arrives at a known time. Given this setting, it is not far-fetched to assume that there are projects for which we know when they can be realized. For example, oil exploration has largely, a known time when the engineers know if there is oil. Therefore our assumption is not that far-fetched.

The literature in the field of real options and corporate financing has been rich. Real options and their value was first acknowledged by the seminal paper of McDonald and Siegel (1986) who consider an investment problem of stochastic costs and stochastic cash flows. They solve a profit maximization problem in a stochastic control framework that was solved by John McKean in the 60s and derive useful results and comparative statics and their message is that the option to wait has value. Leland (1994) takes as a starting point Brennan and Schwartz (1978) and by assuming perpetual debt in the capital structure he is able to provide closed form solutions for a debt and equity values. The infinite life assumption allows him to extract closed form solutions by rendering obsolete the calendar time partial derivative in the valuation partial differential equation (PDE). Leland examines mostly credit spreads and optimal leverage but he does not consider investment and real options. Leland studies mostly firms with stable earnings and not growth ones.

More close to this model is the paper by Sundaresan and Wang (2006) who study a staged investment problem. They assume a firm holding two real options at its inception and no debt or assets in place. An exogenous state process determines the cash flows and when this exogenous process hits a specific threshold, real options are exercised sequentially (given some technology assumptions) and the financing of the exercise is implemented through the debt market and equity. That is, each real option is financed with a mixture of equity and debt. Sundaresan and Wang study the implications of this staged investment problem to the financing of the firm, credit spreads and the existence of debt overhang as it has been documented in Myers (1977). They find severe debt overhang problems and show that preexisting debt induces excessive risk taking. They also solve the model such that the initial capital structure is optimized in such a way to mitigate future debt overhang problems. Nevertheless they do not consider any sort of uncertainty regarding the existence of those real options.

Asvanunt, Broadie and Sundaresan (2011) solve a real options investment problem with a role for retained earnings but they also do not consider any sort of uncertainty in the existence of real options. Mauer and Sarkar (2005) examine equityholder-debtholder conflicts over the exercise decision of a firm's investment option on corporate financing decisions. Although this paper does not study such conflicts, we use insights from their solution method for this paper. This paper contributes to the literature by modelling explicitly for the uncertainty of the existence of the growth option, therefore it captures better the life cycle of a R&D or exploration firm. In a nutshell this paper lies between of those of Leland (1994) and Sundaresan and Wang (2006) with one real option.

In this paper, between the initial contracting of the firm and the fixed time when the uncertainty regarding the appearance of the real option is resolved the firm can only default. If the real option shows up and provided that the firm has not defaulted up to that fixed time, the manager waits until the state process of the demand reaches an upper level to exercise the option to expand. If the real option does not show up the firm continues with its assets in place and the only event that can disrupt it is default. So in essence we have from time zero up to that fixed time where the option may show up or not, a quasi-Leland (1994) firm. Afterwards the firm can either be a growth firm or stay a Leland firm. The growth firm after exercising its expansion option becomes a Leland (1994) firm again. Therefore we see in this setting the life cycle of the firm starting from being potentially a growth firm and then either turning up a value firm or simply not fulfilling up to its promises.

The structure of the paper goes as following, first we study the benchmark case of only equity-financed firm and then we solve for the more general case of a debt-equity financed firm. An interesting extension to this model would be to consider stochastic arrival time for the real option, where the time can be modeled by an exponential distribution, however there are no closed form solutions for the debt-equity financed case but only for the whole equity financed firm. The modeling of business cycle effects where the probabilities of having or not the real option are correlated with the cash flow process is beyond the scope of this paper.

2 The model

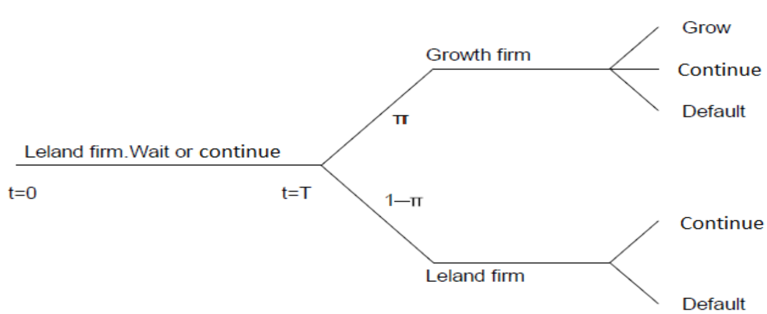
Consider a probability space $(\Omega, \mathcal{F}, P)$ such that the known usual conditions hold. In this probability space the filtration $\mathcal{F}_t$ is generated by a standard Wiener process $\widetilde{W_t}$ under the real measure P. There is a cash flow process that follows a Geometric Wiener process under P. In this economy there exists an additional asset which is traded in liquid markets, perfectly correlated with the cash flow process. Therefore we can find a unique martingale measure $Q \sim P$ such that derivatives upon X_t can be priced according to standard financial derivatives pricing. The form of the process under Q is:

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{1}$$

where $\mu = r - q$, q is the payout ratio, $W_t = \int_0^t \gamma(s)\, ds + \widetilde{W}_t$, $\gamma(t)$ is the Girsanov kernel and r is the constant risk free rate. Assume a firm which is debt-equity financed having assets in place. Debt is perpetual with coupon c_0 and the firm is subject to a constant corporate tax rate τ. Choosing debt is incentivized by the tax shields that debt generates and constraints to the availability of equity capital. At $t = 0$ the firm raises debt and equity. At a fixed deterministic time $t = T$ a real option with growth opportunities may show up with probability $\pi < 1$. The growth option allows the firm to increase its profits by a growth factor $K > 1$, by investing the amount I.The cost I is paid by the equityholders and no additional debt is issued. The firm waits until it is profitable to exercise its growth option.

Since $\pi < 1$ the firm may end up without that growth option at all and continues generating cash flows from its assets in place. Schematically the life cycle of the firm is given by the following figure:

Figure 1. Life Cycle of the Firm



At the beginning, the firm starts as a typical Leland (1994) firm, having two corporate securities outstanding. That is debt and equity. At this stage the firm either continues generating cash flows from its assets in place, or defaults if equityholders don't find it profitable any more to continue operations. One can assume that this is the front investment which is required to explore the uncertain growth opportunity.

When $t = T$ the firm realizes if the real option has shown up or not (henceforward I shall use the term real option or growth option meaning the growth opportunity of the firm) and shifts its corporate decision making accordingly. If the option has not shown up, the firm remains a Leland firm. Again the firm can either continue or default. If the option has shown up, the firm is not a Leland firm, but a growth firm. Now, besides choosing whether to continue or default, the firm can decide whether to expand. Once more, equityholders will adjust the default policy to take into account the growth opportunity. After the firm expands, it returns to the Leland type, by generating cash flows from assets in place.

As we see this model describes the life-cycle of a firm starting from an adolescent growth aspiring firm and culminating into either a growth story or a regular value firm. Our paper deals with the optimal financing of this firm. To solve this problem we need to find the debt and equity values at time $t = 0$. Contrary to traditional corporate financing models with existing growth opportunities, we must solve this problem contingent on whether the real option has shown up or not. Having found all the possible values of the corporate securities, we then find the value of the firm by adding debt and equity value. Plotting the firm value with respect to the coupon, we can find the optimal coupon policy that maximizes the firm value.

3 Benchmark case. Whole equity financing

Before we proceed to the debt-equity financed firm it is necessary to solve for the simpler case of a wholly equity financed firm with a fixed date $t = T$ when the firm realizes if there is a growth option or not. We introduce the following notation table:

Table 1. Notation

$V(X)$	$\triangleq$	Firm value.
X_I	$\triangleq$	Investment threshold.
τ	$\triangleq$	Tax rate.
I	$\triangleq$	Cost of exercising the option.
K	$\triangleq$	Growth factor of profits after option exercise.
π	$\triangleq$	Probability of option showing up.

Assuming that the firm has no variable costs and the cash flow process follows a Geometric Wiener process, the firm will never liquidate. Up to time $t = T$ the firm has only assets in place which generate cash flows. Later on, the uncertainty regarding the existence of the real option resolves. If the option shows up, the firm has a growth opportunity and waits up to the point where it is optimal to exercise its growth option. If the option does not show up, the assets in place keep generating the cash flows.

To value the firm we will compute the present value of cash flows up to time $t = T$. After $t > T$ we will valuate the expected present value of the firm given that it is a growth firm with probability π. There is a delicate valuation argument for the calculation of the expected present for $t > T$. Remember that we are valuing at $t = 0$ but the uncertainty regarding the existence or inexistence of the real option will be known at $t = T$, therefore we have to take both into account the value of the state process X_t at $t = T$ which is unknown and the time value of money. Then we discount the cash flows back to time zero. Considering the above we have the following proposition:

Proposition 1 *Under all equity financing the value of the firm is:*

$$V(X) = (1-\tau)\frac{X}{r-\mu} + \pi\frac{Ie^{-T(r-\lambda)}}{b_1 - 1}\left(\frac{X}{X_I}\right)^{b_1} \tag{2}$$

with

$$\begin{aligned} X_I &= \frac{r-\mu}{1-\tau}\frac{b_1}{b_1-1}\frac{I}{K-1} && (3) \\ b_1 &= \frac{1}{\sigma^2}\left[-\left(\mu-\frac{\sigma^2}{2}\right)+\sqrt{\left(\mu-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}\right] && (4) \\ \lambda &= \mu b_1 + \frac{1}{2}b_1(b_1-1)\sigma^2 && (5) \end{aligned}$$

Proof. *Complete and detailed proof is in the appendix.* ■

The above formula summarizes all the characteristics of the firm. The formula merely states that the value of the firm is the expected present value of assets in place given by $(1-\tau)\frac{X}{r-\mu}$ plus the expected benefits if there is a real option. Assets in place do not require any probability weight because regardless whether the option shows up, the firm will always generate profits from assets in place. In fact the formula is quite similar to that of a firm holding a real option with expansion capability, with the difference being the discount factor $\pi e^{-T(r-\lambda)}$ which discounts for two facts. First the probability of having the option and second the timing of the late arrival of the growth option.

4 Debt and equity financing

This is the more general case since now we allow the firm to issue debt at its inception to finance potential future opportunities and current assets. The firm anticipates that there may be a real option in the future and adjusts its capital structure at inception accordingly. We will decompose the value of the firm into three parts. First, the firm is oblivious of whether a real option may show up or not and that is for $t < T$. This is a Leland firm with debt and equity only. Second, equityholders realize that the option has shown up and it decides whether to either continue and not exercise the option, default or expand by paying the amount of I. Third, equityholders realize that the option has not shown up and the firm continues as a regular Leland firm.

Contrary to the benchmark case, closed form solutions do not exist due to the fact that up to time T corporate securities are time dependent. Therefore the optimal default policy that decides the optimal capital structure is extracted by solving a time dependent optimal stopping problem (i.e. American option). The PDE which describes the value process cannot be solved analytically. In this case we will have to resort into some sort of discrete time approximation.

4.1 After $t = T$.

After time $t = T$ the firm has realized whether the real option has shown up or not. Given this information, the firm either remains a Leland type firm or it is a growth firm. To find the value of the firm at $t = 0$, we employ dynamic programming, that is we start from the end and work backwards in time. This technique will be used for both cases, i.e. whether the option has shown up or not. The following sections describe the value functions conditional on the arrival of the option.

4.1.1 Case 1. Debt-equity financed firm with a real option showing up at $t = T$.

We know that the real option has shown up at $t = T$ and the value of the state process is $X_T = X$. The following notation will be used.

Table 2. Notation

$E_1(X)$	$\triangleq$	Equity value before growth option exercise.
$E_2(X)$	$\triangleq$	Equity value after growth option exercise.
$D_1(X)$	$\triangleq$	Debt value before growth option exercise.
$D_2(X)$	$\triangleq$	Debt value after growth option exercise.
X_{D2}	$\triangleq$	Default threshold after growth option exercise.
X_{D1}	$\triangleq$	Default threshold before growth option exercise.
X_I	$\triangleq$	Investment (growth) threshold.
τ	$\triangleq$	Tax rate.
α	$\triangleq$	Bankruptcy costs.
c_0	$\triangleq$	Debt coupon.
I	$\triangleq$	Cost of exercising the option.
K	$\triangleq$	Growth factor of profits after option exercise.

Since this is a "before-after investment" situation, again dynamic programming is going to be used for finding the values of debt and equity. We will first study the firm after the growth option has been exercised and having done that we will see how the expansion capability affects the firm before the exercise of the option.

4.1.2 After the option is exercised

After the option is exercised there are two corporate securities outstanding, equity and debt that was raised at $t = 0$ to finance the firm at inception. After the option is exercised only one event can occur that affects the firm. This is default. We consider the stopping time $T_{D2} = \inf\{t \geq T_I : X_t = X_{D2}\}$ where X_{D2} is the default threshold and T_I is the time at which the growth option was exercised. The equity and debt values $E_2(X)$ and $D_2(X)$ post-expansion are described by the following relationships:

$$E_2(X) = (1-\tau)\,\mathbb{E}\left[\int_{T_I}^{T_{D2}} e^{-r(s-T_I)}(KX_s - c_0)\,ds\right] \tag{6}$$

$$D_2(X) = \mathbb{E}\left[\int_{T_I}^{T_{D2}} e^{-r(s-T_I)}c_0 ds + e^{-r(T_{D2}-T_I)}D_2(X_{D2})\right] \tag{7}$$

The equity value satisfies two boundary conditions namely:

$$E_2(X_{D2}) = 0 \tag{8}$$

$$E_{2x}(X_{D2}) = 0 \tag{9}$$

Value matching condition 8 states that equityholders at default get nothing, whereas the smooth pasting condition 9 merely states that the default threshold will be chosen optimally by the equityholders as to maximize the equity value. On the other hand, debt has to satisfy another boundary condition:

$$D_2(X_{D2}) = (1-\alpha)(1-\tau)\frac{X_{D2}}{r-\mu} \tag{10}$$

This value matching condition merely states that at default debtholders are senior to equityholders and thus they will receive the residual firm value net of bankruptcy costs. Implicitly it is assumed that the value of the growth option after it is exercised should zero regardless of default. Note that we do not need a smooth pasting condition for the debt, since equityholders optimize the default threshold in order to maximize equity value.

Proposition 2 *Debt and equity values after the real option has been exercised are given by:*

$$E_2(X) = K(1-\tau)\frac{X}{r-\mu} - (1-\tau)\frac{c_0}{r} - \frac{1-\tau}{b_2-1}\frac{c_0}{r}\left(\frac{X}{X_{D2}}\right)^{b_2} \tag{11}$$

$$D_2(X) = \frac{c_0}{r} - \left(\frac{c_0}{r} - D_1(X_{D2})\right)\left(\frac{X}{X_{D2}}\right)^{b_2} \tag{12}$$

with

$$X_{D2} = \frac{r-\mu}{rK}\frac{b_2}{b_2-1}c_0 \tag{13}$$

$$D_2\left(X_{D2}\right) = (1-\alpha)(1-\tau)\frac{KX_{D2}}{r-\mu} \tag{14}$$

$$b_2 = -\frac{1}{\sigma^2}\left[\left(\mu-\frac{\sigma^2}{2}\right)+\sqrt{\left(\mu-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}\right] \tag{15}$$

Proof. *Complete and detailed proof is in the appendix.* ■

4.1.3 Before the option is exercised.

Before the option is exercised the firm still has two corporate securities outstanding namely equity $E_1(X)$ and debt $D_1(X)$ with coupon c_0. Two events can occur to this firm, first default and second option exercise that boosts profits by a factor K at a cost I. Consider the stopping times $T_I = \inf\{t \geq T : X_t = X_I\}$ and $T_{D1} = \inf\{t \geq T : X_t = X_{D1}\}$ where X_I, X_{D1} are the optimal investment and default thresholds respectively. Debt and equity values are given by:

$$E_1(X) = (1-\tau)\mathbb{E}\left[\int_T^{T_{D1}\wedge T_I} e^{-r(s-T)}\left(X_s-c_0\right)ds + e^{-r(T_I-T)}\left(E_2\left(X_I\right)-I\right)1_{\left(T_{D1}>T_I\right)}\right] \tag{16}$$

$$D_1(X) = \mathbb{E}\left[\int_T^{T_{D1}\wedge T_I} e^{-r(s-T)}c_0 ds + e^{-r(T_{D0}-T)}D_1\left(X_{D1}\right)1_{\left(T_{D1}\leq T_I\right)} + e^{-r(T_I-T)}D_2\left(X_I\right)1_{\left(T_{D1}>T_I\right)}\right] \tag{17}$$

Both equity and debt values will satisfy a number of boundary conditions. Equity satisfies:

$$E_1\left(X_{D1}\right) = 0 \tag{18}$$

$$E_{1x}\left(X_{D1}\right) = 0 \tag{19}$$

$$E_1\left(X_I\right) = E_2\left(X_I\right) - I \tag{20}$$

$$E_{1x}\left(X_I\right) = E_2\left(X_I\right) \tag{21}$$

Conditions $18-21$ state that before the exercise of the option, equityholders optimize the default decision as to maximize equity value. At the same time if the firm defaults, equityholders receive nothing due to limited liability. Since equityholders cover the cost of the investment outlay I, what they get after they exercise the option, is the equity value of the firm with boosted capacity minus the cost that they paid for the investment. The investment timing is done optimally in a way to maximize shareholder value.

Debtholders on the other hand, face two possible events. First, the firm may default thus due to seniority they receive the scrap value of the firm. Second, the firm boosts capacity and there is a jump of debt value at no extra cost for them. These two possibilities are described by the two following conditions:

$$D_1\left(X_{D1}\right) = (1-\alpha)(1-\tau)\frac{X_{D1}}{r-\mu} \tag{22}$$

$$D_1\left(X_I\right) = D_2\left(X_I\right) \tag{23}$$

Note that there are no optimally conditions whatsoever for debt since decisions are taken from equityholders only. We now state proposition 2.

Proposition 3 *Under debt and equity financing the manager before the real option is exercised maximizes equity value. Equity value itself is a linear combination of assets in place and default and growth possibilities. Debt is a linear combination of a risk free perpetual debt plus expected gains given expansion plus expected value if default occurs.*
Proof. *Complete and detailed proof is in the appendix.* ■

Unfortunately closed form solutions do not exist and the proposition states obvious results from the corporate finance literature. Corporate securities are discounted present values of cash flows. Cash flows are distinguished between those that are generated by assets in place and those that are contingent on various trigger events. Therefore in total the value of corporate securities is the sum of profits generated by assets in place plus gains contingent on the realization of an event, weighted by the respective Arrow-Debreu security that pays a unit of money if that event is realized.

4.1.4 Case 2. Debt-equity financed firm without a real option.

At $t = T$ the value of the cash flow process is $X_T = X$ and the following notation will be used:

Table 3. Notation

$E_l(X)$	$\triangleq$	Equity value.
$D_l(X)$	$\triangleq$	Debt value.
X_{Dl}	$\triangleq$	Default threshold.
τ	$\triangleq$	Tax rate.
α	$\triangleq$	Bankruptcy costs.
c_0	$\triangleq$	Debt coupon.

This case is the typical Leland (1994) firm, a firm holding two corporate securities namely debt and equity. Since the firm holds no growth options, the only event that can affect the value of the firm is default. Equityholders will again optimize the default decision as to maximize equity value, while debtholders facing no seniority constraints will receive at default the unlevered asset value reduced by bankruptcy costs. Equity value $E_l(X)$ will be equal to $E_2(X)$ with the adjustment of $K = 1$:

$$E_l(X) = (1-\tau)\frac{X}{r-\mu} - (1-\tau)\frac{c_0}{r} - \frac{1-\tau}{b_2-1}\frac{c_0}{r}\left(\frac{X}{X_{Dl}}\right)^{b_2} \tag{24}$$

while debt is given by the formula:

$$\begin{aligned} D_l(X) &= \frac{c_0}{r} - \left(\frac{c_0}{r} - (1-\alpha)(1-\tau)\frac{X_{Dl}}{r-\mu}\right)\left(\frac{X}{X_{Dl}}\right)^{b_2} && (25) \\ X_{Dl} &= \frac{r-\mu}{r}\frac{b_2}{b_2-1}c_0 && (26) \end{aligned}$$

4.2 Before $t = T$

Before we proceed, we introduce the final notation table.

Table 4. Notation

$V_t(X)$	$\triangleq$	Firm value at time $t \leq T$.
$E_t(X)$	$\triangleq$	Equity value at time $t \leq T$.
$D_t(X)$	$\triangleq$	Debt value at time $t \leq T$.

Before the firm realizes the existence or the absence of the growth option, cash flows are generated only from assets in place. Default is the only event that can disrupt the operations of the firm. Corporate securities in this interval $[0, T)$ are time dependent and no closed form solutions are available. Therefore we need to employ a numerical algorithm to value debt and equity. Following Broadie and Kaya (2007), we set up a binomial tree model for the interval $[0, T)$. In order to do that we first discretize the state process, using the Cox-Ross-Rubinstein method, which consists of matching the mean and the variance of the Geometric Wiener process, with that of a binomial process. Given the process in (1) and a time interval dt, the discretized binomial process is:

$$X_{t+dt} = \begin{cases} X^u_{t+dt} = X_t e^{\sigma\sqrt{dt}} & p = \dfrac{e^{\mu dt} - e^{-\sigma\sqrt{dt}}}{e^{\sigma\sqrt{dt}} - e^{-\sigma\sqrt{dt}}} \\ \\ X^d_{t+dt} = X_t e^{-\sigma\sqrt{dt}} & 1-p = \dfrac{e^{\sigma\sqrt{dt}} - e^{\mu dt}}{e^{\sigma\sqrt{dt}} - e^{-\sigma\sqrt{dt}}} \end{cases}$$

Having now the discretized process, we work backwards in time for all corporate securities. First we condition on what happens at $t = T$. At this point in time the firm knows whether the growth option exists or not. But at $t = 0$ the firm has an expectation about the existence of the option. From the perspective of $t = 0$, the value of corporate securities at $t = T$ is the following:

$$
\begin{aligned}
E_T(X_T) &= (1-\pi) E_l(X_T) 1_{(X_T > X_{Dl})} + \pi E_1(X_T) 1_{(X_T > X_{D1})} 1_{(X_T < X_I)} \\
&\quad + \pi (E_2(X_T) - I) 1_{(X_T > X_{D1})} 1_{(X_T \geq X_I)} && (27) \\
D_T(X_T) &= (1-\pi) \left(D_l(X_T) 1_{(X_T > X_{Dl})} + (1-\alpha)(1-\tau) \frac{X_{Dl}}{r-\mu} 1_{(X_T \leq X_{Dl})} \right) \\
&\quad + \pi \left(D_1(X_T) 1_{(X_T > X_{D1})} + (1-\alpha)(1-\tau) \frac{X_{D1}}{r-\mu} 1_{(X_T \leq X_{D1})} \right) 1_{(X_T < X_I)} \\
&\quad + \pi \left(D_2(X_T) 1_{(X_T > X_{D2})} + (1-\alpha)(1-\tau) \frac{K X_{D2}}{r-\mu} 1_{(X_T \leq X_{D2})} \right) 1_{(X_T \geq X_I)} && (28) \\
V_T(X_T) &= E_T(X_T) + D_T(X_T) && (29)
\end{aligned}
$$

The indicator functions take into account the various events that can happen in the binomial tree. First if the growth option does not show up and according to the value of the state process on the binomial tree, the firm may or may not default. This is controlled by the indicator function $1_{(X_T > X_{Dl})}$. If there is default, equity holders get nothing and debtholders get the scrap value. If the real option shows up there are three possibilities. First, the firm defaults immediately if the value of the state process is too low. This is controlled by the indicator function $1_{(X_T > X_{D1})}$. Second, the state process is high enough for the firm not to default, but too low for the firm to expand. The product $1_{(X_T > X_{D1})} 1_{(X_T < X_I)}$, controls for that event. Third, the state process could be so high, that the firm expands immediately, while debtholders experience an increase in debt value. We control for this event with the indicator function $1_{(X_T \geq X_I)}$.

Working backwards in time requires a default check at each node. At $t = T - dt$ we check have the following conditions:

$$
\begin{aligned}
&\text{If } e^{-rdt} \left(pE_T(X_T^u) + (1-p) E_T\left(X_T^d\right) \right) + (1-\tau)(X_{T-dt} - c_0) dt > 0 : \\
&E_{T-dt}(X_{T-dt}) = e^{-rdt} \left(pE_T(X_T^u) + (1-p) E_T\left(X_T^d\right) \right) + (1-\tau)(X_{T-dt} - c_0) dt \\
&D_{T-dt}(X_{T-dt}) = e^{-rdt} \left(pD_T(X_T^u) + (1-p) D_T\left(X_T^d\right) \right) + c_0 dt \\
&V_{T-dt}(X_{T-dt}) = E_{T-dt}(X_{T-dt}) + D_{T-dt}(X_{T-dt}) \\
&\text{Else} \\
&E_{T-dt}(X_{T-dt}) = 0 \\
&D_{T-dt}(X_{T-dt}) = (1-\alpha)(1-\tau) \frac{X_{T-dt}}{r-\mu} \\
&V_{T-dt}(X_{T-dt}) = E_{T-dt}(X_{T-dt}) + D_{T-dt}(X_{T-dt}) && (30)
\end{aligned}
$$

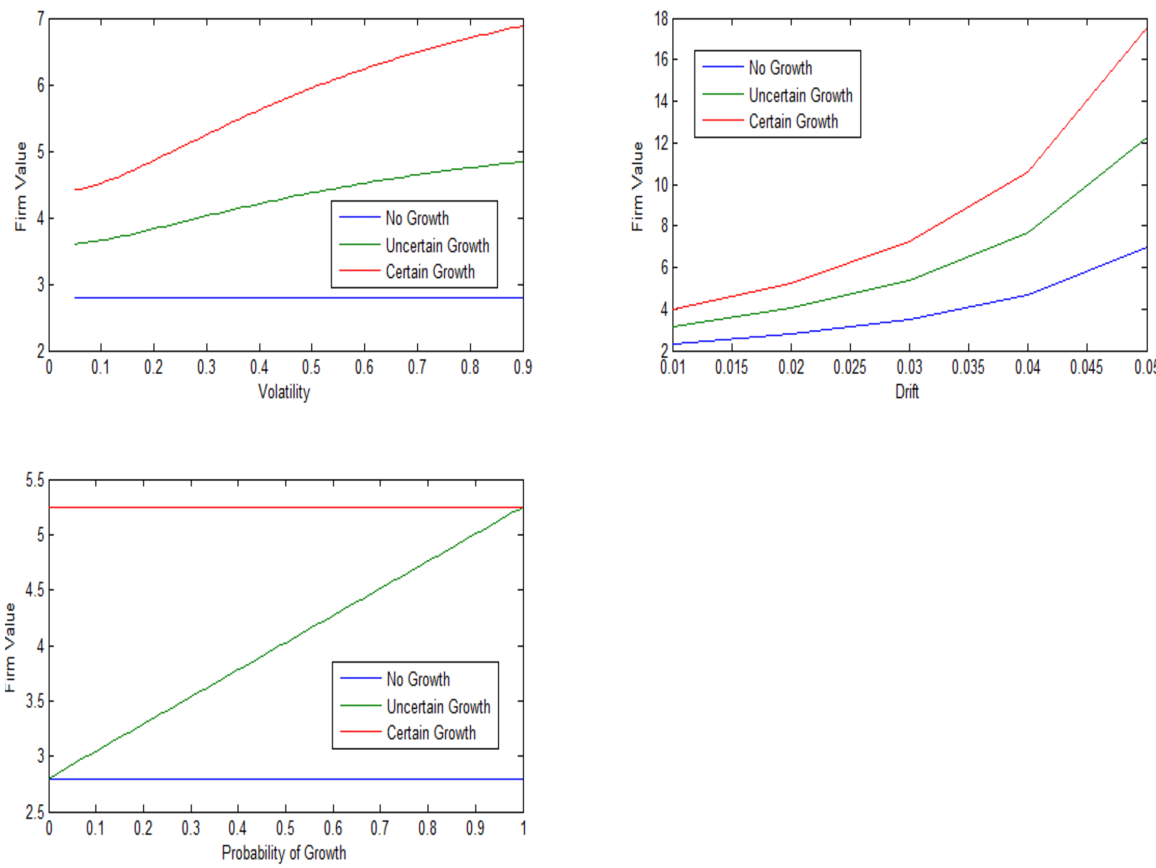
Repeating this process until $t = 0$, we extract the values of all corporate securities. In order to find the optimal debt policy for the firm, we plot the value of the firm with respect to the coupon rate. The optimal coupon rate is the one that maximizes firm value.

5 Model Analysis

5.1 Whole equity financing

In this section we perform comparative statics for the whole equity financed firm. In the following panel we compare three cases. A firm without growth opportunities i.e. $\pi = 0$, a firm with uncertain growth with probability $\pi = 0.5$ and a firm with certain growth or $\pi = 1$.The parameters are, $X = 0.2, T = 4, \mu = 2\%, \tau = 30\%$ and $r = 7\%$.

Panel 1. Comparative Statics for Whole Equity Financing



We generally observe that the value of the firm increases with the drift and the volatility of the state process. It makes intuitive sense since higher drift implies higher expected profits since $E[X_t|X_0 = x] = x \exp(\mu t)$ which is an increasing function of μ.Likewise the higher the volatility, the higher the probability of higher profits. Moreover, the call option nature of the option to expand increases in value with volatility. The probability π of having a growth option, increases linearly the value of the firm with uncertain growth.

Even though the case of a whole equity financed firm does not convey very interesting insights, it is the baseline case and shows that potential growth opportunities affect the value of the firm in a linearly functional form. Moreover, the value of the growth threshold will be used in the numerical algorithm for the more complicated case of a debt-equity financed firm.

5.2 Debt-equity financing

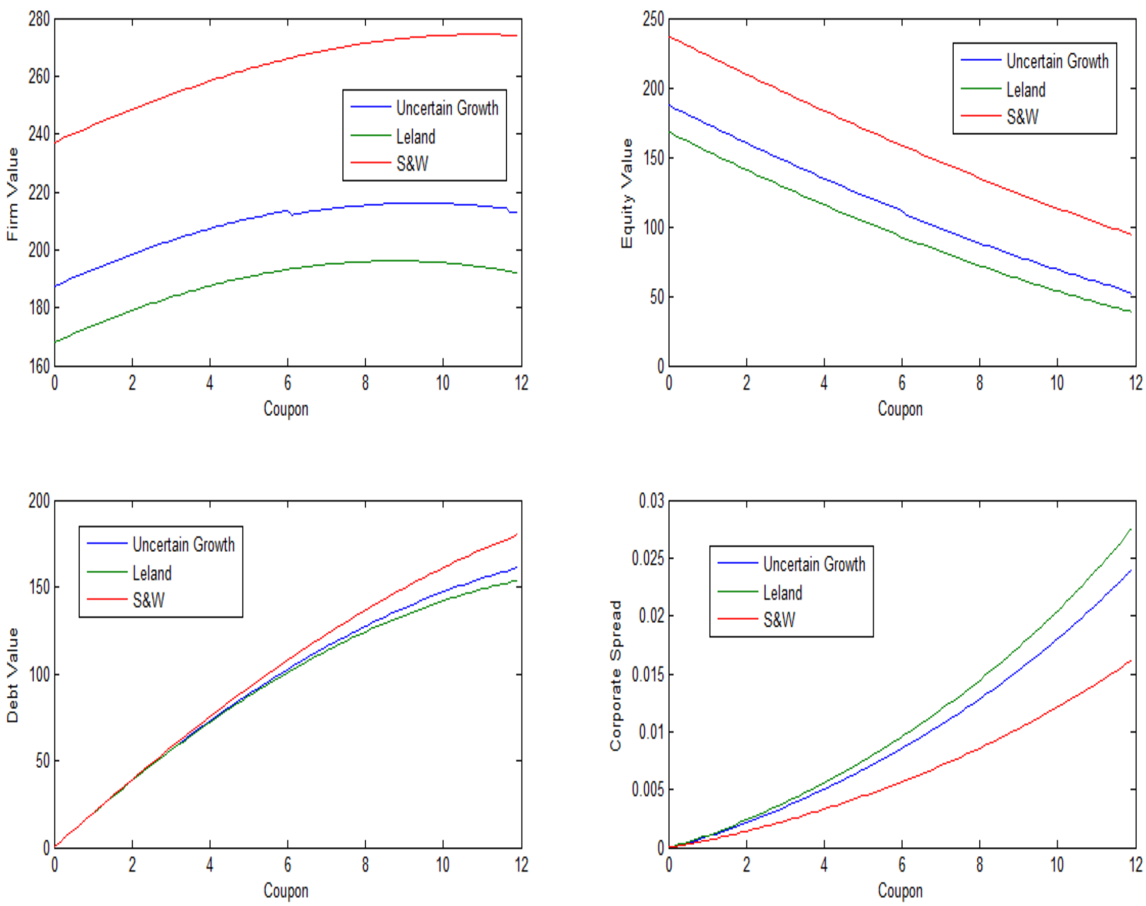
In this section we will show the comparative statics of firm value, equity value, debt value and the credit spread for three firms. The first firm is a firm without a growth option which we call a Leland firm from Leland's (1994) paper. The second firm is a firm with a growth option which we call S&W firm from the Sundaresan and Wang (2006) paper. The third firm is the firm with uncertain growth. We will use a base specification with $X_0 = 12, \pi = 10\%, \sigma = 20\%, \mu = 0\%, \alpha = 30\%, \tau = 30\%, I = 100, K = 2$ and $r = 5\%$.The real option arrives within one year, that is $T = 1$.

In order to perform the binomial approximation to the value of debt and equity prior to $t = T$, we discretize with fifteen steps so that we have in total 2^{15} paths of the binomial tree which are enough for the purposes of this paper. More paths did not alter too much the results and they added more computational time.

To implement the numerical algorithm for the firm with uncertain growth and the S&W firm, we will need to solve for the values of the default and investment threshold X_{D1} and X_I. We use an iterative procedure. We first assume a very low coupon rate ε, an almost debt-free firm. Then the investment threshold should be very close to $X_I = \frac{r-\mu}{1-\tau} \frac{b_1}{b_1-1} \frac{I}{K-1}$,which is the investment threshold for a whole equity financed firm. Using these starting values, we solve using a higher coupon rate $\varepsilon' > \varepsilon$ for the default and investment threshold. We iterate for all coupons ε and we extract the functions $X_{D1}(\varepsilon)$ and $X_I(\varepsilon)$.

Once the default and investment thresholds have been calculated, the values of debt and equity are extracted. These are post exercise values. The algorithm then, evaluates at the binomial grid at $t = T$ the values of debt, equity and firm value by augmenting them with the probability π of having a growth option. The algorithm also checks whether the state process X_t is at levels where there can be default, expansion or continuation given the expected values for debt, equity and firm augmented again by the probability π.Once the algorithm has identified the expected values at the end of the grid, it then discounts all these values back to $t = 0$. These are the expected present values that we present in panel 2.

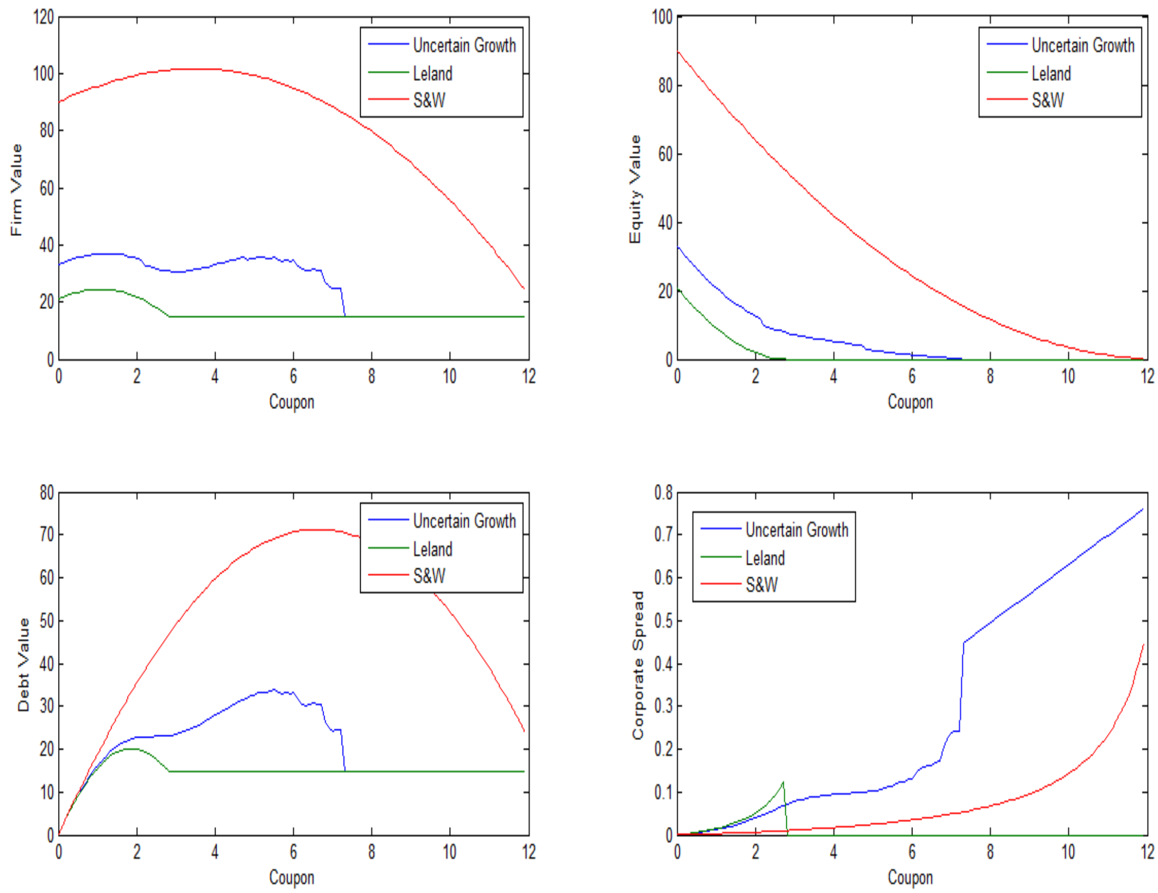
Panel 2. Comparative Statics for Debt-Equity Financing with Relatively Low Growth ($K = 2$)



We observe that the uncertain growth firm lies in between the Leland firm without growth options and the S&W firm with a certain growth option. It makes intuitive sense since certain growth is more valuable than uncertain growth. Likewise the values of debt and equity follow the same pattern. It is interesting to observe that the firm value for the firm with uncertain growth has a concavity near the optimal coupon rate which in general is lower that those for the Leland and the S&W firm. Authors such as Smith and Watts (1992), Barclay, Smith and Watts (1995) find a negative relation between market leverage and the market-to-book ratio which is a proxy for growth options. In general the firm with uncertain growth is a combination of the two firms (Leland and S&W) as in the case of a whole equity financed firm.

We also wanted ti experiment with some other parameter specification. We increased the growth factor from $K = 2$ to $K = 9$ and the initial value of the state process X_0 in order to have meaningful figures. The results are shown in panel 3.

Panel 3. Comparative Statics for Debt-Equity Financing with High Growth ($K = 9$)



For this specification the parameters are $X_0 = 1.5, \pi = 10\%, \sigma = 20\%, \mu = 0\%, a = 30\%, \tau = 30\%, I = 100$ and $r = 5\%$.This high growth scenario shows a rather remarkable result. Notice the value of the debt with uncertain growth vis-a-vis the coupon rate. The schedule is non-convex unlike all traditional models of capital structure. Moreover there seem to be multiple optima in the schedule, indicating that there are multiple optimum capital structures. This result is in contrast to Leland (1994) who finds a unique optimal capital structure and also to the literature where a single optimum is derived.

Under this model the firm can form financing habits where the manager chooses low or high leverage based on idiosyncratic factors. If we can derive a testable hypothesis that would be that firms with high grow potential and debt financing should have a larger variance of their debt ratios compared to firms financed with debt and have lower growth potential. As in the baseline case, the value of the firm is in between a firm with certain growth (S&W firm) and a firm without growth (Leland firm). Yet the debt of the firm with uncertain growth is far riskier with very high credit spreads when the coupon rate increases.

There are two economic forces that drive the result. On the one hand the probability of large growth provides an incentive for the firm to issue debt. On the other hand the additional leverage increases the risk of default. Nevertheless the two effects are so finely tuned that the trade-off between default and growth leads to multiple optima and financing habits. And this is the surprising result of this paper.

6 Conclusion

In this paper we studied the financing of firms with high growth uncertainty. We modelled growth uncertainty with a simple probability distribution for the arrival of the growth option at a deterministic time and we derived the values for a whole equity and a debt-equity financed firm. We found that a whole equity financed firm is a linear combination of a firm without growth options and of a firm with growth options. We found that for specific parameters the firm that chooses to finance itself with debt and equity could find multiple optimal capital structures and form financing habits. The economic forces are finely tuned to provide ambivalent incentives that trade-off default and high potential growth.

This result is in line with Miller's assumption of neutral mutations in capital structure theory. Moreover we found that the credit spreads of firms with growth uncertainty but high potential are significantly larger that those of firms with existing growth options. As a testable hypothesis, we predict that are firms partly financed with debt with uncertain but high potential growth, should have a larger variation of their debt ratios compared to firms with lower growth potential even if growth is uncertain.

For future research examining the conflicts between existing debtholders and new debtholders if the exercise of the growth option is partly financed with debt are of interest. In this case it will be interesting to study the conflicts between existing and new debtholders and the debt overhang problem for firms with high growth uncertainty. From a technical point of view, it would be interesting to solve for the more general case of a stochastic arrival of the growth option or the arrival of multiple options.

7 Appendix. Proofs and solutions

7.1 Proof of proposition 1

The proof breaks in four parts. First we calculate the value of the firm before the uncertainty regarding the existence of the growth option resolves. Second, we calculate the value of the firm if the real option does not show up. Third, we calculate the value of the firm if the real option shows up. In the fourth step we combine the following values to extract the value of the firm at $t = 0$.

Up to $t = T$

In this case the firm has only assets in place and its value is given by discounting cash flows up to $t = T$.

$$(1-\tau)\int_0^T \mathbb{E}\left[e^{-rs}X_s|X_0 = X\right] ds = (1-\tau)\left(1-e^{-T(r-\mu)}\right)\frac{X}{r-\mu} \tag{31}$$

After $t = T$ without the real option

After time $t = T$ and since the firm knows that a real option does not exist, its value comprises of only assets in place and it is given by:

$$(1-\tau)\int_T^\infty \mathbb{E}\left[e^{-r(s-T)}X_s|X_T\right] ds = (1-\tau)\frac{X_T}{r-\mu} \tag{32}$$

After $t = T$ with the real option

Now the case becomes more complicated since there is optionality pertaining to the expansion option. Simple integrals as those above cannot longer describe the value of the firm and we need PDEs. This is a very known problem in the real option literature. In fact it is a simplification of the seminal McDonald and Siegel (1986) paper where the exercise cost is non-stochastic. The value of the firm $\overline{V}$, is conditioned on $\mathcal{F}_T$ and we denote $X_T = X$. Firm value satisfies the following ODE:

$$\frac{1}{2}\sigma^2 X^2 \overline{V}_{xx} + \mu X \overline{V}_x + (1-\tau) X = r\overline{V} \tag{33}$$

and two boundary conditions. A value matching condition for the payoff at exercise and a smooth pasting one, stating the optimality of the investment threshold as to maximize firm value. These conditions are:

$$\overline{V}(X_I) = (1-\tau)\frac{KX_I}{r-\mu} - I \tag{34}$$

$$\overline{V}_x(X_I) = \frac{1-\tau}{r-\mu}K \tag{35}$$

This is a simple 2×2 linear system and can be easily solved analytically. The optimal investment threshold and the firm value are:

$$X_I = \frac{r-\mu}{1-\tau}\frac{b_1}{b_1-1}\frac{I}{K-1} \tag{36}$$

$$\overline{V}(X) = (1-\tau)\frac{X}{r-\mu} + \frac{I}{b_1-1}\left(\frac{X}{X_I}\right)^{b_1} \tag{37}$$

Value of the firm

Naively someone would just add all the values for the value of the firm. However, since the firm values are conditioned on the event that either the real option may or may not show up. These events are $\mathcal{F}_T$-measurable and not $\mathcal{F}_0$. For an arbitrary profit function $f(x)$, $\mathcal{F}_T$ -measurable its present value at $t=0$ is:

$$\int_T^{\infty} e^{-rs}\mathbb{E}\left[f(X_s)|\mathcal{F}_0\right]ds = e^{-rT}\int_T^{\infty} e^{-r(s-T)}\mathbb{E}\left[f(X_s)|\mathcal{F}_0\right]ds = e^{-rT}E\left[\int_T^{\infty} e^{-r(s-T)}\mathbb{E}\left[f(X_s)|\mathcal{F}_T\right]ds|\mathcal{F}_0\right] \tag{38}$$

The above formula is a consequence of the tower property of the conditional expectation. Now from the total probability theorem we have:

$$V(X) = (1-\tau)\left(1-e^{-T(r-\mu)}\right)\frac{X}{r-\mu} + e^{-rT}\left((1-\pi)\mathbb{E}\left[\frac{(1-\tau)X_T}{r-\mu}|\mathcal{F}_0\right] + \pi\mathbb{E}\left[\overline{V}(X_T)|\mathcal{F}_0\right]\right) = \tag{39}$$

$$\left(1-e^{-T(r-\mu)}\right)\frac{(1-\tau)X}{r-\mu} + e^{-rT}\left[(1-\pi)\frac{(1-\tau)\mathbb{E}\left[X_T|\mathcal{F}_0\right]}{r-\mu} + \pi\left(\frac{(1-\tau)\mathbb{E}\left[X_T|\mathcal{F}_0\right]}{r-\mu} + \frac{I\mathbb{E}\left[X_T^{b_1}|\mathcal{F}_0\right]}{(b_1-1)X_I^{b_1}}\right)\right] = \tag{40}$$

$$\left(1-e^{-T(r-\mu)}\right)\frac{(1-\tau)X}{r-\mu} + e^{-T(r-\mu)}\frac{(1-\tau)X}{r-\mu} + \pi\frac{Ie^{-rT}\mathbb{E}\left[X_T^{b_1}|\mathcal{F}_0\right]}{(b_1-1)X_I^{b_1}} = \tag{41}$$

$$\frac{(1-\tau)X}{r-\mu} + \pi\frac{Ie^{-T(r-\lambda)}}{b_1-1}\left(\frac{X}{X_I}\right)^{b_1} \text{ and } \lambda = \mu b_1 + \frac{1}{2}b_1(b_1-1)\sigma^2 \tag{42}$$

To show where this extra term λ comes from we prove the following lemma.

Lemma 4 *The expectation of* $X_t^{b_1}$ *is* $\mathbb{E}\left[X_t^{b_1}|X_0 = X\right] = X^{b_1}e^{\lambda t}$

Proof. Consider the function $f(x) = x^{b_1}$ and X_t is a Geometric Wiener process. From Ito's lemma we have:

$$\begin{aligned} dX_t^{b_1} &= b_1 X_t^{b_1-1} dX_t + \frac{1}{2} b_1 (b_1 - 1) X_t^{b_1-2} \left(dX_t^{b_1}\right)^2 & (43) \\ &= b_1 X_t^{b_1-1} (\mu X_t dt + \sigma X_t dW_t) + \frac{1}{2} b_1 (b_1 - 1) \sigma^2 X_t^{b_1-2} X_t^2 dt & (44) \\ &= \underbrace{\left(\mu b_1 + \frac{1}{2} b_1 (b_1 - 1) \sigma^2\right)}_{\lambda} X_t^{b_1} dt + \sigma X_t^{b_1} dW_t & (45) \end{aligned}$$

Now we integrate the above relationship and we have that:

$$X_t^{b_1} = X_0^{b_1} + \int_0^t \lambda X_s^{b_1} ds + \int_0^t \sigma X_s^{b_1} dW_s \Longleftrightarrow \tag{46}$$

$$\mathbb{E}\left[X_t^{b_1}|X_0 = X\right] = X_0^{b_1} + \int_0^t \lambda \mathbb{E}\left[X_s^{b_1}\right] ds \tag{47}$$

Stating $g(t) = \mathbb{E}\left[X_t^{b_1}|X_0 = X\right]$ we have a simple differential equation to solve with $g(0) = X^{b_1}$ and solution

$$g(t) = X^{b_1}e^{\lambda t} \tag{48}$$

■

7.2 Proof of proposition 2

Equity value $E_2(X)$ after the option has been exercised satisfies the following ODE:

$$\frac{1}{2}\sigma^2 X^2 E_{2xx} + \mu X E_{2x} + K(1-\tau) X - (1-\tau) c_0 = r E_2 \tag{49}$$

with general solution:

$$E_2(X) = K(1-\tau)\frac{X}{r-\mu} - (1-\tau)\frac{c_0}{r} + AX^{b_1} + BX^{b_2} \tag{50}$$

where

$$b_1 = \frac{1}{\sigma^2}\left[-\left(\mu-\frac{\sigma^2}{2}\right)+\sqrt{\left(\mu-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}\right] \tag{51}$$

$$b_2 = -\frac{1}{\sigma^2}\left[\left(\mu-\frac{\sigma^2}{2}\right)+\sqrt{\left(\mu-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}\right] \tag{52}$$

with the following boundary conditions:

$$E_2\left(X_{D2}\right)=0 \tag{53}$$

$$E_{2x}\left(X_{D2}\right)=0 \tag{54}$$

$$A=0 \tag{55}$$

Value matching condition 53 states that equity holders at default get nothing from the firm. Smooth pasting condition 54 states that the default threshold will be chosen optimally by equityholders as to maximize equity value. Condition 55 is a no-bubbles requirement. After the algebra is carried out, we have that the default threshold X_{D1} is given by:

$$X_{D2}=\frac{r-\mu}{rK}\frac{b_2}{b_2-1}c_0 \tag{56}$$

Finally, equity value $E_2\left(X\right)$ is given by:

$$E_2\left(X\right)=K\left(1-\tau\right)\frac{X}{r-\mu}-\left(1-\tau\right)\frac{c_0}{r}-\frac{1-\tau}{b_2-1}\frac{c_0}{r}\left(\frac{X}{X_{D2}}\right)^{b_2} \tag{57}$$

In a similar manner, debt value $D_2\left(X\right)$ satisfies the following ODE:

$$\frac{1}{2}\sigma^2X^2D_{2xx}+\mu XD_{2x}+c_0=rD_2 \tag{58}$$

with general solution:

$$D_2\left(X\right)=\frac{c_0}{r}+\overline{A}X^{b_1}+\overline{B}X^{b_2} \tag{59}$$

and boundary conditions:

$$D_2\left(X_{D2}\right)=\left(1-\alpha\right)\left(1-\tau\right)\frac{X_{D2}}{r-\mu} \tag{60}$$

$$\overline{A}=0 \tag{61}$$

Once more, condition 60 is a value matching stating the payoff of debtholders at default, while condition 61 is a no-bubbles requirement. Solving the system of equations for debt value $D_2(X)$ we have:

$$D_2(X) = \frac{c_0}{r} - \left(\frac{c_0}{r} - D_2(X_{D2})\right)\left(\frac{X}{X_{D2}}\right)^{b2} \tag{62}$$

7.3 Proof of proposition 3

Equity value $E_1(X)$ satisfies the following ODE:

$$\frac{1}{2}\sigma^2 X^2 E_{1xx} + \mu X E_{1x} + (1-\tau)X - (1-\tau)c_0 = rE_1 \tag{63}$$

with the following boundary conditions:

$$E_1(X_{D0}) = 0 \tag{64}$$
$$E_{1x}(X_{D0}) = 0 \tag{65}$$
$$E_1(X_I) = E_2(X_I) - I \tag{66}$$
$$E_{1x}(X_I) = E_{2x}(X_I) \tag{67}$$

and general solution:

$$E_1(X) = (1-\tau)\frac{X}{r-\mu} - (1-\tau)\frac{c_0}{r} + \widetilde{A}X^{b_1} + \widetilde{B}X^{b_2} \tag{68}$$

Condition 64 states that at default equity holders don't get anything since they are junior in priority to debtholders. Smooth pasting condition 65 states that equityholders optimize their default choice. The value matching condition 66 is for the payoff of equityholders when the firm expands by paying an irreversible cost I. The final smooth pasting 67 condition is an optimality condition for equityholders, as to how they should choose the investment barrier in order to maximize the equity value. Rewriting those conditions we have:

$$(1-\tau)\frac{X_{D1}}{r-\mu} - (1-\tau)\frac{c_0}{r} + \widetilde{A}(X_{D1})^{b_1} + \widetilde{B}(X_{D1})^{b_2} = 0 \tag{69}$$
$$\frac{1-\tau}{r-\mu} + b_1\widetilde{A}(X_{D1})^{b_1-1} + b_2\widetilde{B}(X_{D1})^{b_2-1} = 0 \tag{70}$$
$$(1-\tau)\frac{X_I}{r-\mu} - (1-\tau)\frac{c_0}{r} + \widetilde{A}(X_I)^{b_1} + \widetilde{B}(X_I)^{b_2} = E_2(X_I) - I \tag{71}$$
$$\frac{1-\tau}{r-\mu} + b_1\widetilde{A}(X_I)^{b_1-1} + b_2\widetilde{B}(X_I)^{b_2-1} = E_{2x}(X_I) \tag{72}$$

Solving this 4×4 non linear system defined by equations 69-72 is very difficult, however we can reduce its dimensionality. If we expand the right hand sides of 69-72 and do the algebra we have that:

$$(1-\tau)\frac{X_{D1}}{r-\mu}-(1-\tau)\frac{c_0}{r}+\widetilde{A}\left(X_{D1}\right)^{b_1}+\widetilde{B}\left(X_{D1}\right)^{b_2}=0 \tag{73}$$

$$\frac{1-\tau}{r-\mu}+b_1\widetilde{A}\left(X_{D1}\right)^{b_1-1}+b_2\widetilde{B}\left(X_{D1}\right)^{b_2-1}=0 \tag{74}$$

$$(1-\tau)\frac{X_I}{r-\mu}-(1-\tau)\frac{c_0}{r}+\widetilde{A}\left(X_I\right)^{b_1}+\widetilde{B}\left(X_I\right)^{b_2}=E_2\left(X_I\right)-I \tag{75}$$

$$\frac{1-\tau}{r-\mu}+b_1\widetilde{A}\left(X_I\right)^{b_1-1}+b_2\widetilde{B}\left(X_I\right)^{b_2-1}=E_{2x}\left(X_I\right) \tag{76}$$

$$\Leftrightarrow$$

$$\widetilde{A}\left(X_{D1}\right)^{b_1}+\widetilde{B}\left(X_{D1}\right)^{b_2}=-(1-\tau)\frac{X_{D1}}{r-\mu}+(1-\tau)\frac{c_0}{r} \tag{77}$$

$$b_1\widetilde{A}\left(X_{D1}\right)^{b_1}+b_2\widetilde{B}\left(X_{D1}\right)^{b_2}=-(1-\tau)\frac{X_{D1}}{r-\mu} \tag{78}$$

$$\widetilde{A}\left(X_I\right)^{b_1}+\widetilde{B}\left(X_I\right)^{b_2}=(1-\tau)(K-1)\frac{X_I}{r-\mu}-\Gamma\left(\frac{X_I}{X_{D2}}\right)-I \tag{79}$$

$$b_1\widetilde{A}\left(X_I\right)^{b_1}+b_2\widetilde{B}\left(X_I\right)^{b_2}=(1-\tau)(K-1)\frac{X_I}{r-\mu}-b_2\Gamma\left(\frac{X_I}{X_{D2}}\right) \tag{80}$$

With $\Gamma=\frac{1-\tau}{b_2-1}\frac{c_0}{r}$.Now we perform the algebraic transformations $79-\frac{1}{b_2}80, 79-\frac{1}{b_1}80$ and we get:

$$\widetilde{A} = \left(X_I\right)^{-b_1}\left((1-\tau)(K-1)\frac{X_I}{r-\mu}\frac{b_2-1}{b_2}-I\right)\frac{b_2}{b_2-b_1} \tag{81}$$

$$\widetilde{B} = \left(X_I\right)^{-b_2}\left((1-\tau)(K-1)\frac{X_I}{r-\mu}\frac{b_1-1}{b_1}-\Gamma\left(\frac{X_I}{X_{D2}}\right)^{b_2}\frac{b_1-b_2}{b_1}-I\right)\frac{b_1}{b_1-b_2} \tag{82}$$

At the same time we get from $77-\frac{1}{b_2}78, 77-\frac{1}{b_1}78$

$$\widetilde{A} = \left(X_{D1}\right)^{-b_1}(1-\tau)\left(\frac{c_0}{r}-\frac{X_{D1}}{r-\mu}\frac{b_2-1}{b_2}\right)\frac{b_2}{b_2-b_1} \tag{83}$$

$$\widetilde{B} = \left(X_{D1}\right)^{-b_2}(1-\tau)\left(\frac{c_0}{r}-\frac{X_{D1}}{r-\mu}\frac{b_1-1}{b_1}\right)\frac{b_1}{b_1-b_2} \tag{84}$$

Now from these relationships it is evident that $81=83, 82=84$ and this is a 2×2 non linear system of equations with unknowns X_I, X_{D1}, which can be solved by a least squares solver. Given the optimal values of X_I, X_{D1} we can back out the optimal equity value. Note that for the optimal implementation of the non linear solver algorithm, the relationships $81 = 83, 82 = 84$ must be as linearized as possible. From the equations $81 = 83, 82 = 84$, we get two implicit equations for the investment and default boundary:

$$\begin{aligned}
X_I &= \frac{r-\mu}{b_1-1}\frac{b_1}{(1-\tau)(K-1)}\left[I+\Gamma\left(\frac{X_I}{X_{D2}}\right)^{b_2}\frac{b_1-b_2}{b_1}\right] \\
&\quad +\frac{r-\mu}{b_1-1}\frac{b_1}{(1-\tau)(K-1)}\left[(1-\tau)\left(\frac{c_0}{r}-\frac{X_{D0}}{r-\mu}\frac{b_1-1}{b_1}\right)\left(\frac{X_I}{X_{D1}}\right)^{b_2}\right] \qquad (85)\\
X_I &= \frac{r-\mu}{b_2-1}\frac{b_2}{(1-\tau)(K-1)}\left[I+(1-\tau)\left(\frac{c_0}{r}-\frac{X_{D0}}{r-\mu}\frac{b_2-1}{b_2}\right)\left(\frac{X_I}{X_{D1}}\right)^{b_1}\right] \qquad (86)
\end{aligned}$$

Solving the above system of equations requires an iterative algorithm and a non-linear solver. Non-linear solvers require a set of starting values. Since we are interested in the optimal capital structure we want to find optimal default and investment thresholds with respect to the coupon. We start with a very low coupon $c_{0,1} > 0$ and use the following starting values:

$$X_I = \frac{r-\mu}{1-\tau}\frac{b_1}{b_1-1}\frac{I}{K-1} \qquad (87)$$

$$X_{D1} = \varepsilon \qquad (88)$$

Notice that the initial value for the investment threshold corresponds to the value of the investment threshold when the firm is financed only with equity. The initial value $\varepsilon > 0$ is very small but different from zero. Using a non-linear solver and these starting values, we get a new pair of (X_I, X_{D1}). Assuming a new coupon $c_{0,2} > c_{0,1}$, we use for starting values the pair of values extracted for $c_{0,1}$. Continue for $c_{0,3} > c_{0,2}$ using the previous pair as the new pair of starting values and keep the process going up to the desired coupon level.

The debt $D_1(X)$ will satisfy the following PDE:

$$\frac{1}{2}\sigma^2 X^2 D_{1xx} + \mu X D_{1x} + c_0 = rD_1 \qquad (89)$$

with the following boundary conditions:

$$D_1(X_{D1}) = (1-\alpha)(1-\tau)\frac{X_{D1}}{r-\mu} \qquad (90)$$

$$D_1(X_I) = D_2(X_I) \qquad (91)$$

and general solution:

$$D_1(X) = \frac{c_0}{r} + \underline{A}X^{b_1} + \underline{B}X^{b_2} \qquad (92)$$

Condition 90 is a value matching condition stating that at default debtholders receive the value of unlevered assets reduced by the bankruptcy costs. At the same time when the firm invests, we have a regime change at an absorbing barrier. Condition 91 is a value matching, stating the value change of debt after investment has occurred. Note that for debt, there are no smooth pasting conditions and this is because debtholders do not optimize default and investment decisions. Having solved for X_I and X_{D1} we define the following matrices and we use Cramer's rule to solve for the debt value:

$$
\begin{aligned}
D &= \begin{bmatrix} (X_{D1})^{b_1} & (X_{D1})^{b_2} \\ (X_I)^{b_1} & (X_I)^{b_2} \end{bmatrix} \\
D_{\underline{A}} &= \begin{bmatrix} (1-\alpha)(1-\tau) X_{D1}/(r-\mu) - c_0/r & (X_{D1})^{b_2} \\ D_2(X_I) - c_0/r & (X_I)^{b_2} \end{bmatrix} \\
D_{\underline{B}} &= \begin{bmatrix} (X_{D1})^{b_1} & (1-\alpha)(1-\tau) X_{D1}/(r-\mu) - c_0/r \\ (X_I)^{b_1} & D_2(X_I) - c_0/r \end{bmatrix}
\end{aligned} \tag{93}
$$

Debt value is now given by:

$$
D_1(X) = \frac{c_0}{r} + \frac{|D_{\underline{A}}|}{|D|} X^{b_1} + \frac{|D_{\underline{B}}|}{|D|} X^{b_2} \tag{94}
$$

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