A Stochastic Supply/Demand Model for Storable Commodity Prices

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ABSTRACT

We develop a two-factor mean reverting stochastic model for forecasting storable commodity prices and valuing commodity derivatives. We define a variable called “normalized excess demand” based on the observable production rate, consumption rate, and inventory levels of the commodity. Moreover, we formulate and quantify the impact of this factor on the commodity spot and futures prices. We apply this model to crude oil prices from 1995 to 2016 via a Kalman filter. Our analysis indicates a strong correlation between normalized excess demand and crude oil spot and futures prices. We analyze the term structure of futures prices under the calibrated parameters as well as the implications of changes in the underlying factors.

I. Introduction

Commodities are an integral part of the global economy. Economies of both producing and consuming nations are vastly affected by commodities' prices. For instance, Hamilton (2008) and Morana (2013) demonstrate that oil price increase has preceded nine out of ten most recent US recessions. Hamilton (2011) and Cuñado & Gracia (2003) show the detrimental effects of oil price shocks on US and European Economies. The importance of commodities to national security has prompted governments and financial markets to pay close attention to commodities' price levels. Commodity pricing plays an important role in real options valuation. Kobari et al. (2014) demonstrate the importance of oil prices in the real options valuation, operational decisions, and expansion rate of Alberta oil sands projects. Bastian-Pinto et al. (2009) demonstrate the impact of sugar and ethanol price processes on the real option valuations of ethanol production from sugarcane. Brandão et al. (2013) show the significance of the soybean and castor bean price processes for real options valuation of managerial flexibility embedded in a biodiesel plant.

Moreover, in recent decades, commodities have become an investment asset in financial portfolios providing protection against inflation and as a diversifying factor, boosting portfolios' risk-reward profiles (Geman 2005; Gorton & Rouwenhorst 2006). Juvenal & Petrella (2015) estimated that assets allocated to commodity index trading had increased from \$13 billion in 2004 to \$260 billion in 2008. Importance and widespread use of commodities has led to growth of variety and trading volume of financial products related to commodities. According to statistics published by the Bank for International Settlements (*Semiannual OTC derivatives statistics*), notional value of over-the-counter commodities’ derivatives increased from \$415 billion in 1998 to over \$2 trillion in 2015. The increase in trading volume and importance of commodities in real project / option valuation has made the pricing of commodity products a priority.

Commodity prices exhibit stochastic behavior. Many factors contribute to the pricing of commodity related financial products. The majority of previous literature has either focused on modeling stochasticity of the prices or finding causal links between commodities' prices and a range of commodity specific and economic factors.

Gibson & Schwartz (1990), Schwartz (1997), Casassus & Collin-Dufresne (2005), Hikspoors & Jaimungal (2008), Trolle & Schwartz (2009), Liu & Tang (2011), Chen & Insley (2012), Mirantes, Población, & Serna (2013), and Lai & Mellios (2016) are some recent examples of literature focused on modeling stochasticity of commodity prices, convenience yield, interest rates, and volatility. Schwartz (1997) is a seminal work on the modeling mean reversion and stochasticity of commodity prices. Routledge et al. (2000) is an influential work on modeling term structure of forward prices in equilibrium. In this work, the impacts of changes in inventory, net demand shocks, and convenience yield on forward prices are discussed. They show convenience yield is a results of interaction between supply, demand, and inventory.

Another category of literature is concerned with analyzing the effects of commodity prices on commodity-specific and economic factors. Commodity specific factors are factors directly affecting the production and consumption of commodities, such as supply, demand, inventory levels, and even geopolitical tensions in the case of energy products. Güntner (2014) analyzes the impact of oil prices and demand on oil production within OPEC and non-OPEC producers. Bu (2014) shows that weekly inventory data published by the U.S. Energy Information Agency (EIA) has a significant impact on oil prices.

Alternatively, economic factors indirectly impact commodity prices by changing the expected supply and demand for commodities. For instance, global economic activity is expected to have an impact on global energy demand. Kaminski (2014) classifies the North American energy market’s determinants as physical, financial, and socioeconomic layers, and measures their impacts on the market. Stefanski (2014) analyzes the impact of structural changes in the industrialization level of developing countries on the energy markets.

Employing an econometric approach, Kilian’s work demonstrates the importance of supply, demand, and inventory levels on commodity prices within a vector autoregressive model. Kilian & Lee (2014) analyze the impact of speculative demand shocks on oil prices. Kilian (2014) demonstrates that supply and demand levels are integral parts of oil price dynamics. Baumeister & Kilian (2016) create a vintage dataset consisting of global oil production, US and OECD oil inventory levels and other factors, and demonstrate improved forecasting capabilities compared to similar vector autoregressive models.

The literature demonstrates strong evidence of stochasticity of the commodities' prices, convenience yield, and volatility. Moreover, literature also suggests strong explanatory power of production, consumption, and inventory levels. In this work, we propose a two-factor stochastic model incorporating commodities’ prices as well as supply, demand, and inventory data.

In the equilibrium setting, commodity prices should follow a mean reverting process. With increasing price, supply of the commodity will increase as higher marginal cost producers enter the market. This increase in supply will in turn reduce the demand pressure on the price and slow or reverse the price increase. Conversely, with a decrease in price, higher marginal cost producers would halt production. The lower supply of the commodity would reduce the downward pressure on the price and slow the fall in production rates, resulting in an upward price pressure. Thus, in equilibrium, commodity prices are expected to follow a mean reverting process. Schwartz (1997) models commodity prices via an Ornstein-Uhlenbeck process reflecting this mean reverting dynamic.

In this work, we propose a two factor stochastic model. The first factor is the commodity's spot price and the second factor is the normalized excess demand for the commodity. This model is unique in that it proposes a mean reverting factor related to supply, demand, and inventory levels and measures the impact of changes in these variables on the commodity prices. Moreover, it is calibrated not only on the commodity prices, but also on the observable supply, demand, and inventory levels. This model is then applied to oil markets. Following the Schwartz (1997) framework, since the spot price, supply/demand, and inventory data of commodities are uncertain and unobservable, we use oil futures prices for model calibration. Specifically, the model is put in a state-space form and a Kalman filter is applied to estimate the model parameters and values of the state parameters. We apply the model to oil market data from 1996 to present time. We use the West Texas Intermediate (WTI) futures data on a monthly basis as well as quarterly data on supply, demand, and inventory published by the International Energy Agency (IEA).

The remainder of this work is organized as follows. Section 2 explains the economic rationale for the model. In Section 3, the forecasting model is explained and reviewed. In Section 4, the data is presented and described. Section 5 explains the calibration methodology. Section 6 presents results of the model calibration and forecasting. Finally, Section 7 presents the concluding remarks.

II. Economic Rationale

In this section, we will discuss the economic rationale behind the proposed model. As mentioned in the introduction, commodity spot prices are often modeled as mean reverting processes. With increasing demand, commodity prices increase as well. The increase in price leads to an increase in supply as producers with higher production costs enter the market. The increase in supply, in turn, balances the excess demand and leading to increasing price levels. In reverse, with increasing excess supply, the prices fall to reflect the abundance of the commodity. Falling prices then push out the producers with higher production cost and reduce the excess demand and balance the price. In this version of mean reversion, every price increase is followed by a price fall and every price decrease is followed by a price rise due to a balancing process between supply and demand.

In this work, we propose a different mean reverting model for commodities. We postulate that commodity spot prices grow exponentially at a rate dependent on various factors such GDP growth rate, inflation, and other market factors. For instance, as GDP increases, price levels in both financial and physical markets have to grow to accommodate both producers and consumers. Chiang et al. (2015) conclude that oil prices have a statistically significant and economical relationship with real GDP. Another factor is inflation. Inflation leads to devaluation of currencies. A unit of a given commodity and costs associated with production of that unit do not decrease with the unit of currency. This leads to an increase in the nominal value of a unit of commodity as inflation devalues the currency. Szymanowska et al. (2014) find evidence of sizeable impact of inflation as well other factors on spot and term risk premia for a high-minus-low portfolio of seven commonly used commodities ranging from energy to agricultural products and industrial metals. Moreover, commodities returns are shown to be positively correlated with inflation (Greer 2000; Erb & Harvey 2006; Gorton & Rouwenhorst 2006).

Finally, we believe commodities are impacted by supply, demand, and inventory levels observable by the market. A positive demand shock (negative supply shock) would increase the price of the commodity while a negative demand shock (positive supply shock) would decrease the price. However, it is not the absolute demand or supply shock that would impact the price but the relative value of the shock. For instance, if the positive shock in demand is followed by a positive shock in supply, the value of the commodity should not be impacted by the shocks as increasing supply would offset increasing demand. Moreover, inventories act as a buffer for the markets by absorbing excess supply and offsetting excess demand. However, it should be noted that inventories only provide buffering capacity for storable commodities (see Routledge et al. 2000; Carlson et al. 2007; Sockin & Xiong 2015 for discussion on impacts of inventory on storable commodities). Moreover, inventories cease to provide buffering protection for positive supply shocks at near maximum capacity. This is due to limited capacity for storage and not being able to mix different qualities of a commodity such as crude oil in the same container (Kaminski 2014). We propose a ratio called "normalized excess demand" and defined as

$$q = \frac{D - S - I}{D}, \tag{1}$$

to represent the impact of supply, demand, and inventory on commodity spot prices. Here D, S and I represent observable demand, supply and inventory of the commodity. We postulate that this ratio follows a mean reverting process. In the long term, there should be a balance between supply, demand, and inventory levels. Supply and inventory levels should always be in line with demand. Therefore, the ratio of excess total supply to demand should be mean reverting in the long term. Deviation from this long term mean level would have an impact on prices. With increasing total supply, this ratio would decrease and in turn, prices would decrease. We formulate the model such that this ratio impacts the growth rate of spot prices through a convenience yield factor defined as

$$\delta = aq + b, \tag{2}$$

where a and b are constants that can be fitted to data. For instance, with increasing supply, the normalized excess demand factor would decrease. Producers and market participants are expected to store the commodity at this point. It is then expected that the convenience yield would increase providing incentive for participants to store the commodity. The increase in convenience would also reduce the growth rate of spot prices and even reduce the prices if the magnitude of the supply increase is significant. In the next section, details of the proposed model are presented.

III. Model

This section outlines the proposed model and derivatives' valuations. The two factors of this model are the spot price, S, and the normalized excess demand of the commodity, q. The two factors are modeled as the following joint stochastic processes:

$$\frac{dS}{S} = (\mu - \delta)dt + \sigma_s dz^s, \tag{3}$$

$$\delta = aq + b, \tag{4}$$

$$dq = \kappa(\theta - q)dt + \sigma_q dz^q , \tag{5}$$

where,

- μ is the drift of the prices based on macroeconomic factors such as GDP and Inflation
- δ is the convenience yield,
- σ_s is the volatility of the oil prices,
- Z^s is a standard Brownian motion,
- a and b are the constants relating normalized excess demand to convenience yield,

- κ is the rate of mean reversion,
- θ is the level of mean reversion,
- σ_q is volatility of the normalized excess demand, and
- Z^q is a standard Brownian motion,

and the two standard Brownian motions are correlated as

$$dz^s dz^q = \rho dt, \tag{6}$$

where ρ is the correlation between the two motions.

Equation (3) models the spot process of the commodity as a geometric Brownian motion (GBM) including a convenience yield term incorporating the benefits of storage to commodity holders in favor of consuming the commodity. Equation (4) describes the convenience yield process related to the normalized excess demand factor. Equation (5) describes the normalized excess demand process. Normalized excess demand (q) is modeled as an Ornstein-Uhlenbeck mean reverting stochastic process. As already mentioned, the normalized excess demand is defined as

$$q = \frac{D - S - I}{D}, \tag{7}$$

where D is market demand, S is supply, and I is the inventory level per day.

Following standard transformation and applying Ito's Lemma, the spot process can be defined in logarithm form as

$$dX = \left(\mu - \delta - \frac{{\sigma_s}^2}{2}\right) dt + \sigma_s dz^s, \tag{8}$$

where $X = \log(S)$.

Under the risk-neutral framework, applying the Girsanov's theorem, the model can be presented as

$$dX = \left(r - \delta - \frac{{\sigma_s}^2}{2}\right) dt + \sigma_s dz^{s^*}, \tag{9}$$

$$\delta = aq + b, \tag{10}$$

$$dq = [\kappa(\theta - q) - \lambda]dt + \sigma_q dz^{q^*}, \tag{11}$$

$$dz^{s^*} dz^{q^*} = \rho dt, \tag{12}$$

where,

- Z^{s^*} and Z^{q^*} are standard Brownian motions under the equivalent risk-neutral measure, and
- λ is the market price of risk.

This joint distribution for log-spot price and normalized excess demand can be then presented as

$$\begin{pmatrix} X(t) \\ q(t) \end{pmatrix} \sim N\left(\begin{bmatrix} \mu_X \\ \mu_q \end{bmatrix} , \begin{bmatrix} \underline{\sigma_X^2} & \sigma_{Xq} \\ \sigma_{Xq} & \underline{\sigma_q^2} \end{bmatrix} \right), \tag{13}$$

where $N(\cdot)$ represents a bivariate normal distribution, the terms μ_X, μ_q, $\underline{\sigma_X^2}$, $\underline{\sigma_q^2}$, and σ_{Xq} are derived in Appendix A and defined as

$$\mu_X = X_0 + \left(\mu - \frac{1}{2}\sigma_s^2 - a\theta - b\right) t - \frac{a}{\kappa}(q_0 - \theta)(1 - e^{-\kappa t}), \tag{14}$$

$$\mu_q = q_0 e^{-\kappa t} + \theta(1- e^{-\kappa t}), \tag{15}$$

$$\underline{\sigma_X^2} = \sigma_s^2 t + \frac{a^2\sigma_q^2}{\kappa^2}\left[t + \frac{(1-e^{-2\kappa t})}{2\kappa} - \frac{2(1-e^{-\kappa t})}{\kappa}\right] - \frac{2a\sigma_s\sigma_q\rho}{\kappa}\left[t - \frac{(1-e^{-\kappa t})}{\kappa}\right], \tag{16}$$

$$\underline{\sigma_q^2} = \frac{\sigma_q^2}{2\kappa}(1-e^{-2\kappa t}), \tag{17}$$

$$\underline{\sigma_q^2} = \frac{\sigma_q^2}{2\kappa}(1-e^{-2\kappa t}). \tag{18}$$

Applying the Feynman-Kac theorem, any derivative under this model should satisfy the following partial differential equation for undiscounted derivatives (g)

$$g_t + (r-\delta)Sg_S + [\kappa(\theta-q)-\lambda]g_q + \frac{1}{2}\sigma_1^2 S^2 g_{SS} + \sigma_1\sigma_2\rho S g_{Sq} + \frac{1}{2}\sigma_2^2 g_{qq} = 0, \tag{19}$$

and the following PDE for discounted derivatives (f)

$$f_t + (r-\delta)Sf_S + [\kappa(\theta-q)-\lambda]f_q + \frac{1}{2}\sigma_1^2 S^2 g_{SS} + \sigma_1\sigma_2\rho S f_{Sq} + \frac{1}{2}\sigma_2^2 f_{qq} = rf. \tag{20}$$

As previously mentioned, we use futures data for calibration of this model. For a futures contract, the PDE in equation (19) is subject to terminal condition $F_T = S$. A futures contract's value at time t with maturity $\tau = T - t$ is obtained as the expectation of the spot price at maturity. Given the log-normal distribution of spot price, we have

$$F(S_t, q_t, \tau) = \mathrm{E}^{\mathrm{Q}}[S_T] = \mathrm{e}^{\mu_X^Q + 0.5\underline{\sigma_X^2}}, \tag{21}$$

where μ_X^Q is the mean of log-spot price under the risk-neutral measure derived as

$$\mu_X^Q = X_0 + \left(r - \frac{1}{2}\sigma_s^2 - a\theta - b + \frac{a\lambda}{\kappa}\right)t - \frac{a}{\kappa}\left(q_0 - \theta + \frac{\lambda}{\kappa}\right)(1-e^{-\kappa t}). \tag{22}$$

Substituting for μ_X^Q and $\underline{\sigma_X^2}$ in equation (21) and reorganizing the equation, we can obtain

$$F(S_t, q_t, \tau) = S_t e^{A(\tau)+B(\tau)q_t}, \tag{23}$$

where,

$$A(\tau) = \left(r - a\hat{\theta} - b - \frac{a\sigma_s\sigma_q\rho}{\kappa} + \frac{a^2\sigma_q^2}{2\kappa^2}\right)\tau + \left(a\hat{\theta} + \frac{a\sigma_s\sigma_q\rho}{\kappa} - \frac{a^2\sigma_q^2}{\kappa^2}\right)\frac{(1-e^{-\kappa\tau})}{\kappa} + \frac{a^2\sigma_q^2}{4\kappa^3}(1-e^{-2\kappa\tau}), \tag{24}$$

$$B(\tau) = -\frac{a(1-e^{-\kappa\tau})}{\kappa}, \tag{25}$$

$$\hat{\theta} = \theta - \frac{\lambda}{\kappa}. \tag{26}$$

It should be noted that log-futures prices follow a normal distribution $N\left(\mu_{F(t.\tau)}, \sigma_{F(t,\tau)}^2\right)$ with

$$\mu_{F(t.\tau)} = \mu_X^Q(t) + A(\tau) + B(\tau)\mu_q^Q(t), \tag{27}$$

$$\sigma^2_{F(t,\tau)} = \underline{\sigma^2_X} + B^2(\tau)\underline{\sigma^2_q} + 2B(\tau)\sigma_{X,q}. \tag{28}$$

In the next section, we present the data used for calibration.

IV. Data

In this section, we apply the proposed model to monthly oil market data from December 1995 to February 2016. We use WTI futures for oil prices. WTI futures are available every month of the year trading on the NYMEX. We select five futures contracts for calibration of the model, the 1st, 3rd, 6th, 9th, and 12th nearby contracts. These specific contracts are selected such that data corresponding to next 4 quarters are included in the model as well as representing some of the most highly liquid contracts. Figure 1 represents the WTI futures data used for calibration.

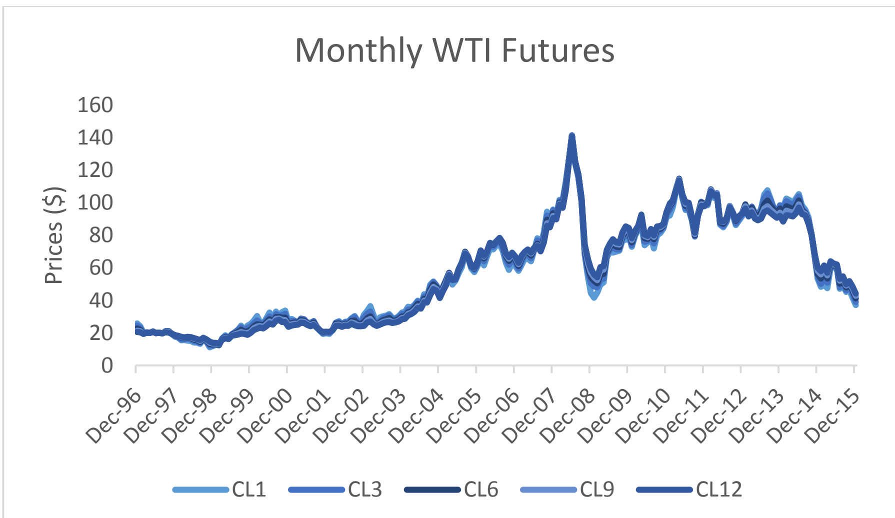

Figure 1 Monthly WTI Futures data for 1996-2016

Moreover, for supply, demand, and inventory values, we use data published by the International Energy Agency (IEA). IEA publishes data on global supply and demand on a quarterly basis. The inventory data is only available for the Organization for Economic Cooperation and Development (OECD) countries, which causes a discrepancy as there are other storage facilities outside of the domain of the OECD, mainly China, and non-OECD inventory data is not readily available. However, as demonstrated by Kilian & Murphy (2014), OECD inventory levels provide a good proxy for global oil inventory levels. Furthermore, as the data on supply and demand are rates of production and consumption per day, we adjust the inventory data by dividing them by 365, assuming 365 days per year. Figure 2 represents the historical values of normalized excess demand (q). It should be noted that the IEA data is published by a 2 week delay at the end of each quarter.

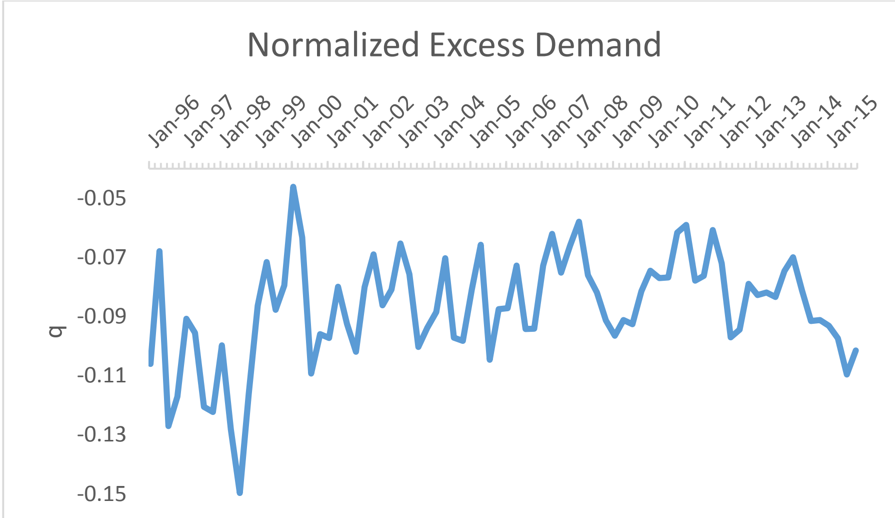

Figure 2 Normalized Excess Demand for 1996-2016

We test the data for mean reversion using the Generalized Hurst Exponent test (Hurst 1951; Mandelbrot 2003) to gauge the suitability of the mean reversion assumption.. We obtain a value of 0.16 for the test, indicating the time series is stationary and mean reverting.

Moreover, we obtain U.S. Treasury Rates with maturities ranging from 1 month to 5 years published by U.S. Federal Reserve. Then, appropriate interest rates for each specific derivative's maturity is calculated via cubic splines. The short term interest rate of less than 1 month is assumed constant at the 1 month maturity rate. Figure 3 demonstrates the yield rates for the period under study.

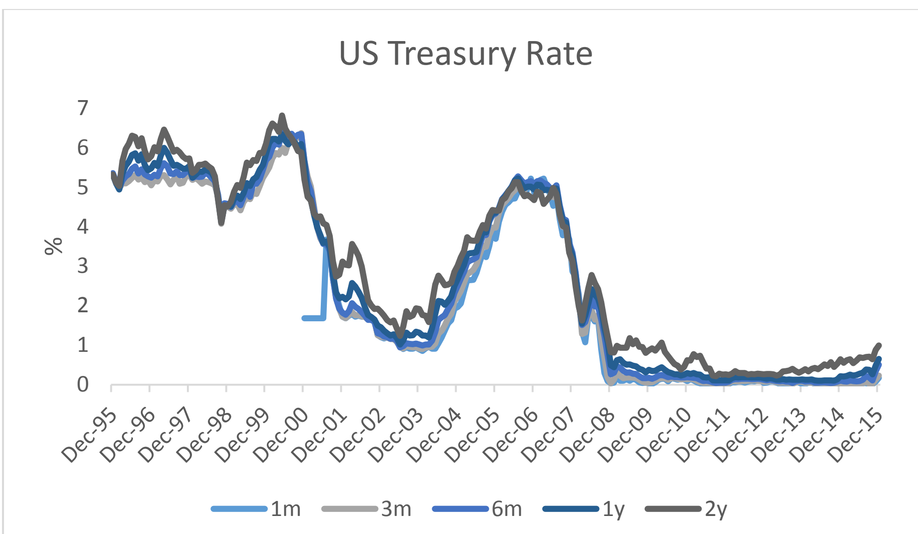

Figure 3 US Treasury Rate for 1996-2016

V. Calibration

For the purpose of calibration, a Kalman filter process similar to that of Schwartz (1997) is utilized.

The two underlying stochastic factors in the proposed model are spot prices and normalized excess demand. Spot prices are unobservable for oil prices. Often, 1st nearby futures' prices are used as a proxy for the spot prices. Moreover, the published supply, demand, and inventory data are only estimates of the true unobservable variables. This makes straight forward calibration of the processes inaccurate. Hence, the model is put in a state-space form to account for unknown true values of the state variables. The two state variables are the spot price and the normalized excess demand. The Kalman filter is then applied to estimate the true value of the state variables' time series. The Kalman filter is an iterative prediction-correction algorithm. In each time step t, based on the current estimate of parameters and state variables, the time $t+1$ value of state variables are estimated through transition equation. The measurement variables are then estimated based on the predicted state variables through the measurement equation. Finally, the predicted state variables are corrected based on the differences between actual and predicted observed parameters. The transition equation relating state parameters at time $t+1$ to state parameters at time t is set up as

$$\overline{x}_t^- = C_t + D_t \overline{x}_{t-\Delta t} + G_t, \tag{29}$$

where

$$\overline{x}_t^- = \begin{bmatrix} X_t^- \\ q_t^- \end{bmatrix}, C_t = \begin{bmatrix} (\mu - \frac{1}{2}\sigma_s^2 - a\theta - b) & \frac{a\theta}{\kappa} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta t \\ \theta(1-e^{-\kappa t}) \end{bmatrix}, D_t = \begin{bmatrix} 1 & -\frac{a}{\kappa}\theta(1-e^{-\kappa t}) \\ 0 & e^{-\kappa t} \end{bmatrix},$$

and G_t is defined as transition noise with

$$E[G_t] = 0\,, Var[G_t] = \begin{bmatrix} \sigma_X^2 & \sigma_{Xq} \\ \sigma_{Xq} & \sigma_q^2 \end{bmatrix}. \tag{30}$$

Subsequently, with $y_t^{\tau_i} = \ln\big(F(S_t, q_t, \tau_i)\big)$, the measurement equation relating the observables to state variables is set up as

$$\overline{y}_t = \overline{A}_t + \overline{B}_t \overline{x}_t^- + H_t, \tag{31}$$

where

$$\overline{y}_t = \begin{bmatrix} y_t^{\tau_1} \\ \vdots \\ y_t^{\tau_{12}} \\ q_t \end{bmatrix}, \overline{A}_t = \begin{bmatrix} A(\tau_1) \\ \vdots \\ A(\tau_{12}) \\ 0 \end{bmatrix}, \overline{B}_t = \begin{bmatrix} 1 & B(\tau_1) \\ \vdots & \vdots \\ 1 & B(\tau_{12}) \\ 0 & 1 \end{bmatrix},$$

and H_t is defined as a vector of serially uncorrelated measurement noises with

$$E[H_t] = 0\,, Var[H_t] = \begin{bmatrix} h_{11} & & \\ & \ddots & \\ & & h_{66} \end{bmatrix}. \tag{32}$$

Equations 23-25 set out the prediction step of the Kalman filter. The correction steps are

$$P_t^- = D_t P_{t-1} D_t' + G_{t-1}, \tag{33}$$

$$K_t = P_t^- \overline{B}_t' \left(\overline{B}_t P_t^- \overline{B}_t' + H_t \right)^{-1}, \tag{34}$$

$$\overline{x}_t = \overline{x}_t^- + K_t\big(\overline{y}_t - \overline{A}_t - \overline{B}_t \overline{x}_t^-\big), \tag{35}$$

$$P_t = \left(I - K_t\overline{A}_t\right)P_t^-. \tag{36}$$

The negative log-likelihood function is then derived as

$$\frac{1}{2}\sum_{all\ t} \log(det(V_t)) + e_t^- V_t^{-1} e_t, \tag{37}$$

where

$$V_t = \overline{B}_t P_t^- \overline{B}_t' + H_t\,, e_t = \ \overline{y}_t - \overline{A}_t - \overline{B}_t \overline{x}_t^-. \tag{38}$$

Finally, this iterative system is passed to an optimization engine to maximize the likelihood function over the underlying parameters. Lastly, the maximum likelihood estimate of the parameters, the standard error of the parameters, and the optimal estimates of the state variables are obtained.

VI. Results

The calibration process is currently in progress and the results will be presented at the conference.

VII. Conclusion

Our preliminary analysis shows the importance of announcements regarding supply, demand, and changes in inventory for the oil market. The impact of normalized excess demand is directly observable on the market prices of crude oil. Observable normalized excess demand provides a promising channel for quantifying the convenience yield and calibrating the underlying mean reverting process. Final conclusions regarding the exact nature of relationship between normalized excess demand and crude oil prices will be presented at the conference.

VIII. Appendix A

This appendix represents the derivation for the join distribution of log-price and normalized excess demand. Log-price and normalized excess demand follow a bivariate normal distribution.

Joint stochastic process for log-price and normalized excess demand can be expressed as

$$dX = \left(\mu - aq_t - b - \frac{{\sigma_s}^2}{2}\right)dt + \sigma_s\sqrt{1-\rho^2}dz^s + \sigma_s \rho dz^q, \tag{39}$$

$$dq = [\kappa(\theta - q) - \lambda]dt + \sigma_q dz^q, \tag{40}$$

where processes z^s and z^q are independent standard Brownian motions.

Solution to normalized excess demand can be obtained as a general Ornstein-Uhlenbeck process

$$q_t = e^{-\kappa t}q_0 + \theta(1 - e^{-\kappa t}) + \sigma_q e^{-\kappa t}\int_0^t e^{\kappa u} dz_u^q. \tag{41}$$

Replacing q_t in equation (39) with the solution (41) we obtain

$$X_t = X_0 + \left(\mu - b - \frac{1}{2}\sigma_s^2\right)t - \int_0^t aq_u du + \int_0^t \sigma_s\sqrt{1-\rho^2}dz_u^s + \int_0^t \sigma_s \rho dz_u^q, \tag{42}$$

where

$$\int_0^t aq_u du = \int_0^t ae^{-\kappa u}q_0 du + \int_0^t a\theta(1-e^{-\kappa u})du + \int_0^t a\sigma_q e^{-\kappa u}\left(\int_0^u e^{\kappa w}dz_w^q\right)du. \tag{43}$$

Applying Fubini's theorem, the order of integration in last part of equation (21) can be changed as

$$\begin{aligned}\int_0^t a\sigma_q e^{-\kappa u}\left(\int_0^u e^{\kappa w}dz_w^q\right)du &= a\sigma_q\int_0^t\left(\int_w^t e^{-\kappa u}e^{\kappa w}du\right)dz_w^q \\ &= a\sigma_q\int_0^t \frac{1}{\kappa}\left(1-e^{-\kappa(t-w)}\right)dz_w^q.\end{aligned} \tag{44}$$

Simplifying equation (21) yields

$$\int_0^t aq_u du = \frac{aq_0}{\kappa}(1-e^{-\kappa t}) - \frac{a\theta}{\kappa}(1-\kappa t - e^{-\kappa t}) + a\sigma_q\int_0^t \frac{1}{\kappa}\left(1-e^{-\kappa(t-w)}\right)dz_w^q. \tag{45}$$

The spot process is then expressed as

$$\begin{aligned}X_t = X_0 &+ \left(\mu - b - \frac{1}{2}\sigma_s^2 - a\theta\right)t - \frac{a}{\kappa}(q_0-\theta)(1-e^{-\kappa t}) + \int_0^t \sigma_s\sqrt{1-\rho^2}dz_u^s \\ &+ \int_0^t \left(\sigma_s\rho - \frac{a\sigma_q}{\kappa}\left(1-e^{-\kappa(t-u)}\right)dz_u^q\right).\end{aligned} \tag{46}$$

The following moments are then obtained for the joint distribution of log-spot price and excess demand

$$E[X_t] = \mu_X = X_0 + \left(\mu - \frac{1}{2}\sigma_s^2 - a\theta - b\right)t - \frac{a}{\kappa}(q_0-\theta)(1-e^{-\kappa t}), \tag{47}$$

$$E[q_t] = \mu_q = q_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}), \tag{48}$$

$$\begin{aligned} var[X_t] = \underline{\sigma_X^2} = \sigma_s^2 t + \frac{a^2\sigma_q^2}{\kappa^2}\left[t + \frac{(1-e^{-2\kappa t})}{2\kappa} - \frac{2(1-e^{-\kappa t})}{\kappa}\right] \\ - \frac{2a\sigma_s\sigma_q\rho}{\kappa}\left[t - \frac{(1-e^{-\kappa t})}{\kappa}\right], \end{aligned} \tag{49}$$

$$var[q_t] = \underline{\sigma_q^2} = \frac{\sigma_q^2}{2\kappa}(1 - e^{-2\kappa t}), \tag{50}$$

$$cov(X_t, q_t) = \sigma_{Xq} = \left(\frac{\sigma_s\sigma_q\rho}{\kappa} - \frac{a\sigma_q^2}{\kappa^2}\right)(1 - e^{-\kappa t}) + \frac{a\sigma_q^2}{2\kappa}(1 - e^{-2\kappa t}). \tag{51}$$

Under the risk-neutral measure, only μ_x and $\mu_q$require to be updated. Under this measure

$$E^Q[X_t] = \mu_X^Q = X_0 + \left(r - \frac{1}{2}\sigma_s^2 - a\theta - b + \frac{a\lambda}{\kappa}\right)t - \frac{a}{\kappa}\left(q_0 - \theta + \frac{\lambda}{\kappa}\right)(1 - e^{-\kappa t}), \tag{52}$$

$$E^Q[q_t] = \mu_q^Q = q_0 e^{-\kappa t} + \left(\theta - \frac{\lambda}{\kappa}\right)(1 - e^{-\kappa t}). \tag{53}$$

References

Bastian-Pinto, C., Brandão, L. & Hahn, W.J., 2009. Flexibility as a source of value in the production of alternative fuels : The ethanol case. *Energy Economics*, 31(3), pp.411–422. Available at: http://dx.doi.org/10.1016/j.eneco.2009.02.004.

Baumeister, C. & Kilian, L., 2016. Real-Time Forecasts of the Real Price of Oil Real-Time Forecasts of the Real Price of Oil. *Journal of Business & Economic Statistics*, 30(2), pp.326–336.

BIS, 2015. *Semiannual OTC derivatives statistics*,

Brandão, L.E.T., Penedo, G.M. & Bastian-Pinto, C., 2013. The value of switching inputs in a biodiesel production plant. *The European Journal of Finance, 2013*, 19(7-8), pp.674–688.

Bu, H., 2014. Effect of inventory announcements on crude oil price volatility. *Energy Economics*, 46, pp.485–494. Available at: http://dx.doi.org/10.1016/j.eneco.2014.05.015.

Carlson, M., Khokher, Z. & Titman, S., 2007. Resource Price Dynamics. *The Journal of Finance*, LXII(4), pp.1663–1703.

Casassus, J. & Collin-Dufresne, P., 2005. Stochastic Convenience Yield Implied from Commodity Futures and Interest Rates. *The Journal of Finance*, 60(5), pp.2283–2331.

Chen, S. & Insley, M., 2012. Regime switching in stochastic models of commodity prices: An application to an optimal tree harvesting problem. *Journal of Economic Dynamics and Control*, 36(2), pp.201–219. Available at: http://dx.doi.org/10.1016/j.jedc.2011.08.010.

Chiang, I.E., Hughen, W.K. & Sagi, J.S., 2015. Estimating Oil Risk Factors Using Information from Equity and Derivatives Markets. *The Journal of Finance*, LXX(2), pp.769–804.

Cuñado, J. & Gracia, F.P. de, 2003. Do oil price shocks matter? Evidence for some European countries. *Energy Economics*, 25(2), pp.137–154. Available at: http://www.sciencedirect.com/science/article/pii/S0140988302000993.

Erb, C.B. & Harvey, C.R., 2006. The Strategic and Tactical Value of Commodity Futures. *Financial Analysts Journal*, 62(2), pp.69–97.

Geman, H., 2005. *Commodities and Commodity Derivatives: Modeling and Pricing for Agriculturals, Metals and Energy*, Chichester, West Sussex, England: Wiley & Sons, Ltd.

Gibson, R. & Schwartz, E.S., 1990. Stochastic Convenience Yield and the Pricing of Oil Contingent Claims. *The Journal of Finance*, 45(3), pp.959–976. Available at: http://onlinelibrary.wiley.com/doi/10.1111/j.1540-6261.1990.tb05114.x/abstract.

Gorton, G. & Rouwenhorst, K.G., 2006. Facts and Fantasies about Commodity Futures. *Financial Analysts Journal*, 62(2), pp.47–68. Available at: http://www.jstor.org/stable/4480744.

Greer, R.J., 2000. The Nature of Commodity Index Returns. *THE JOURNAL OF ALTERNATIVE INVESTMENTS*, 3, pp.45–53.

Güntner, J.H.F., 2014. How do oil producers respond to oil demand shocks ? *Energy Economics*, 44, pp.1–13. Available at: http://dx.doi.org/10.1016/j.eneco.2014.03.012.

Hamilton, J.D., 2011. Nonlinearities and the Macroeconomic Effects of Oil Prices. *Macroeconomic Dynamics*, 15(Supplement 3), pp.364–378.

Hamilton, J.D., 2008. "Oil and the Macroeconomy." In S. N. Durlauf & L. E. Blume, eds. *In The New Palgrave Dictionary of Economics*. Palgrave Macmillan.

Hikspoors, S. & Jaimungal, S., 2008. Asymptotic Pricing of Commodity Derivatives using Stochastic Volatility Spot Models. *Applied Mathematical Finance*, 15(5-6), pp.449–477.

Hurst, H.E., 1951. Long-term storage capacity of reservoirs. *Transactions of American Society of Civil Engineers*, 116, pp.770–808.

Juvenal, L. & Petrella, I., 2015. SPECULATION IN THE OIL MARKET. *Journal of Applied Econometrics*, 30(4), pp.621–649.

Kaminski, V., 2014. The microstructure of the North American oil market. *Energy Economics*, 46,

pp.S1–S10. Available at: http://dx.doi.org/10.1016/j.eneco.2014.10.017.
Kilian, L., 2014. Oil Price Shocks : Causes and Consequences. *The Annual Review of Resource Economics*, 6, pp.133–154.
Kilian, L. & Lee, T.K., 2014. Quantifying the speculative component in the real price of oil: The role of global oil inventories. *Journal of International Money and Finance*, 42, pp.71–87. Available at: http://dx.doi.org/10.1016/j.jimonfin.2013.08.005.
Kilian, L. & Murphy, D.P., 2014. THE ROLE OF INVENTORIES AND SPECULATIVE TRADING IN THE GLOBAL MARKET FOR CRUDE OIL. *Journal of Applied Econometrics*, 29(3), pp.454–478.
Kobari, L., Jaimungal, S. & Lawryshyn, Y., 2014. A real options model to evaluate the effect of environmental policies on the oil sands rate of expansion. *Energy Economics*, 45, pp.155–165. Available at: http://dx.doi.org/10.1016/j.eneco.2014.06.010.
Lai, A.N. & Mellios, C., 2016. Valuation of commodity derivatives with an unobservable convenience yield. *Computers and Operations Research*, 66, pp.402–414. Available at: http://dx.doi.org/10.1016/j.cor.2015.03.007.
Liu, P. & Tang, K., 2011. The stochastic behavior of commodity prices with heteroskedasticity in the convenience yield. *Journal of Empirical Finance*, 18(2), pp.211–224. Available at: http://dx.doi.org/10.1016/j.jempfin.2010.12.003.
Mandelbrot, B.B., 2003. HEAVY TAILS IN FINANCE FOR INDEPENDENT OR MULTIFRACTAL PRICE INCREMENTS. In S.T. Rachev, ed. *Handbook of Heavy Tailed Distributions in Finance*. Elsevier Science B.V.
Mirantes, A.G., Población, J. & Serna, G., 2013. The stochastic seasonal behavior of energy commodity convenience yields. *Energy Economics*, 40, pp.155–166. Available at: http://dx.doi.org/10.1016/j.eneco.2013.06.011.
Morana, C., 2013. The Oil Price-Macroeconomy Relationship Since the Mid-1980s: A Global Perspective. *The Energy Journal*, 34(3), pp.153–189.
Routledge, B.R., Seppi, D.J. & Spatt, C.S., 2000. Equilibrium Forward Curves for Commodities. *The Journal of Finance*, LV(3), pp.1297–1338.
Schwartz, E.S., 1997. The Stochastic Behavior of Commodity Prices : Implications for Valuation and Hedging. *The Journal of Finance*, LII(3), pp.923–973.
Sockin, M. & Xiong, W., 2015. Informational Frictions and Commodity Markets. *The Journal of Finance*, LXX(5), pp.2063–2098.
Stefanski, R., 2014. Structural transformation and the oil price. *Review of Economic Dynamics*, 17(3), pp.484–504. Available at: http://dx.doi.org/10.1016/j.red.2013.09.006.
Szymanowska, M. et al., 2014. An Anatomy of Commodity Futures Risk Premia. *The Journal of Finance*, LXIX(1), pp.453–482.
Trolle, A.B. & Schwartz, E.S., 2009. Unspanned Stochastic Volatility and the Pricing of Commodity Derivatives. *The Review of Financial Studies*, 22(11), pp.4423–4461.