Strategic Acquisitions and Investments in a Duopoly Patent Race under Uncertainty

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Abstract

This paper uses a continuous-time real-options patent-race model to study a patent-race game in which a firm with larger research bandwidth competes with a firm with smaller bandwidth. The large firm can make strategic acquisitions or investments in the small firm subject to transactions costs. Acquisitions occur when the small firm is about to make pre-emptive investments or the large firm has set-backs and the two firms are about to enter head-to-head competition. Strategic investments tend to occur when the smaller leader has technological setbacks. Our continuous-time approach shows that some rich strategic play can occur.

This paper uses a continuous-time real-options patent-race model to study a patent-race game in which a firm with larger research bandwidth competes with a firm with smaller bandwidth in an environment with multiple sources of uncertainty. The large firm makes take-it-or-leave-it offers to the small firm's owner for all (acquisitions) or part (strategic investments) of the small firm's equity, but exogenously imposed transactions costs make such transactions costly. Each firm chooses an investment rate at which it pursues the

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patent, but the investment rates are constrained by capacity limits, with the large firm having a larger capacity (or research bandwidth) than the small firm. Each of the two firms has a research technology in which its expected cost-to-completion follows a diffusion process which is a special case of the technology described by Pindyck (1993). The value of the patent fluctuates randomly, following geometric Brownian. Duopoly competition for the patent is modeled as a game in which there are four publicly observed state variables: the value of the patent, the expected cost-to-completion of each of the two firms, and the size of the large firm's stake in the small firm. Using numerical approximations to the continuous time game, we develop a discrete-time implementation on a 201 x 61 x 61 x 4 grid, with 600 monthly decisions in which we describe the strategic investments by the large firm in the small firm and the investment strategies of the two firms. We also briefly examine some financial properties of the equilibrium and provide a welfare analysis of the effects of transactions costs on strategic investments and acquisitions.

The equilibrium decisions reflect a tradeoff between the costs of exogenously assumed transactions costs and the benefits of reducing rent dissipation resulting from "over-investment" (relative to joint monopoly). When the large firm is behind in the patent race, a strategic investment tends to occur when the small firm is investing actively, the large firm is not yet investing but is about to invest, and the small firm subsequently has technological setbacks. Another type of strategic investment occurs when the small firm is inactive, and the small firm is about to make pre-emptive investment if the strategic investment is not made.

Acquisitions can occur when only the large firm is active or neither firm is active in the patent race. When the large firm is investing, acquisitions occur when the large firm has technological set-backs, the patent value is increasing, and the small firm is about to enter into head-to-head competition against the large firm. Therefore, this type of acquisition is to prevent the small firm from investing immediately. After the acquisition, both firms invest according to the investment rule of joint monopoly. If the large firm continues having bad luck, the small firm will eventually resume investing. So the acquisition also serves to protect the large firm against bad luck in its own technology. When neither firm is investing in the race, acquisitions can occur to prevent the small firm from making pre-emptive investments.

Acquisitions and strategic investments tend to occur after the value of the patent has been increasing. Patent races without acquisitions or strategic investments are also possible in equilibrium.

Our continuous-time approach shows that some intuitively rich strategic play can occur. For example, if the small firm is ahead in the race and investing actively, but steadily has bad luck with its technology despite a rapidly growing patent value, the large firm may first buy a small stake which induces it not to invest, then buy a larger stake which continues to induce it not to invest, then compete head-to-head by investing (undermining the value of its own strategic investment), then acquire a 100% stake by making a cheap offer, and finally stop investing itself while continuing to operate the acquired small firm's technology to achieve a joint monopoly outcome. We believe that such complex behavior would be difficult to capture in a simple two- or three- period

model, and thus justifies our use of numerical approximations to a dynamic game in continuous time.

This paper generalizes Meng (2008) by allowing firms to have different research bandwidths and by allowing strategic investments and acquisitions to occur, subject to transactions costs. Thus, the model in Meng (2008) can be thought of as the version of this model with identical research bandwidths and infinite transactions costs. This research is also related to several other strands of literature.

Horizontal mergers in output markets have been used to study competitive effects (Salant, Switzer, and Reynolds (1983), Perry and Porter (1985), and Farrell and Shapiro (1990)), a potential entrant's "build or buy" decisions and market structure (Gilbert and Newbery (1992), and McCardle and Viswanathan (1994)), and the role of an equity stake in softening (potential) competition on product markets (Reynolds and Snapp (1986)) or eliminating (potential) competition on product markets (Mathews (2002b)). In our model, strategic investments occur when it is mutually valuable to alter the incentives of the two firms, especially the incentive of the large firm to invest. In our model, acquisitions tend to occur when both firms would otherwise invest, but it is more profitable from the perspective of joint monopoly for only one of the two firms to invest; typically, the firm which invests after the acquisition is the same as the firm which would invest in the scenario of joint monopoly.

Hellmann (2002) derives a model in which corporate strategic investors compete with independent venture capitalists to finance a new entrepreneurial firm, whose success affects the value of the strategic investor's assets. Smith and Triantis (1995) uses real-options techniques to value strategic synergies of growth options, flexibility options and

divestiture options associated with strategic acquisitions. They aim to demonstrate that the real-options methodology is a more appropriate vehicle for analysis than the net present value method. The use of real-options techniques implicitly underlies the valuations in this paper. For example, a firm which is well ahead of its competitor does not begin to make investments as soon as a naïve NPV rule says that the expected present value of winning the patent equals the present value of the expected costs, even in the risk neutral measure. Instead, the firm exercises an "option to wait" by choosing to invest when the patent value is higher.

This paper is also related to a growing literature investigating organizational design and the financing of R&D activities, including the work of Ambec and Poitevin (2001), Fulghieri and Sevilir (2001), and Gromb and Scharfstein (2002). In these papers, an R&D project can be undertaken by an entrepreneurial firm or developed in-house by an established firm. The authors are interested in deriving the equilibrium organizational form. The potential competition between the entrepreneurial firm and the established firm is not their main concern. This paper models such competition, although in the context of a patent race with exogenous transactions costs imposed on securities transactions which create cross-ownership between the two firms.

Lambrecht (2001) uses a continuous-time real-options model with complete information to investigate the timing and terms of takeovers. Unlike ours, his paper does not model the operational interactions between the bidder and the target. Morellec and Zhdanov (2003) extend that model to multiple bidders and incomplete information. In our model, both firms always have complete information. Thus, our model does not have the flavor of an auction model.

This paper is organized as follows. Section 2 describes the model set-up, including a self-contained discussion of the model in Meng (2008). Section 3 describes the numerical implementation. Section 4 discusses the equilibrium strategic investment decisions and acquisitions, as well as the equilibrium investment strategies in the underlying patent race. Section 5 discusses how transactions costs affect the amount of rent dissipation, as well as firm value and betas. Section 6 describes a rudimentary welfare analysis. Section 7 concludes with a defense of the continuous-time approach used in this paper.

I. THE MODEL

Patent Value. A firm receives no positive cash flows unless it is the first to obtain the patent successfully. If the winning firm obtains the patent at time t, it immediately begins to receive an infinitely-lived stochastic stream of cash flows, denoted $D(t)$, which is valued by the familiar Gordon growth model formula, using the Capital Asset Pricing Model (CAPM). The value of the winning firm is the value of this stream of cash flows. The losing firm receives nothing and becomes worthless. The stochastic process $D(t)$ is assumed to follow geometric Brownian. Let μ denote the drift, let β denote the CAPM beta, let σ_m denote the market volatility, and let σ_I denote idiosyncratic volatility. Then we can write

$$\frac{dD}{D} = \mu dt + \beta\sigma_m dz_m + \sigma_I dz_x . \qquad (1)$$

Let λ_m denote the risk premium on the market portfolio, let r_f denote the risk-free rate and let $P_m(t)$ denote the value of a share in the market portfolio with dividends reinvested. Then we can write

$$\frac{dP_m}{P_m} = \left(r_f + \lambda_m\right)dt + \sigma_m dz_m . \tag{2}$$

Using Gordon's growth formula, the present value of the stream of the patent cash flows, denoted $x(t)$, is given by

$$x(t) = \frac{D(t)}{r_f + \beta\lambda_m - \mu}. \tag{3}$$

The "patent value" $x(t)$ is one of the state variables affecting the value of both firms. The patent value $x(t)$ follows the process

$$\frac{dx}{x} = \mu dt + \beta\sigma_m dz_m + \sigma_I dz_x . \tag{4}$$

Let δ denote the "dividend yield," the denominator in Gordon's growth formula given by $\frac{D(t)}{x(t)} = r_f + \beta\lambda_m - \mu$. Intuitively, the dividend yield δ represents the rate at which the present value of the patent decays through time as a result of no patent being granted; it represents the opportunity cost of delaying winning the patent. Intuitively, if the dividend yield δ is smaller (holding the patent value x(t) constant), then the "option to delay" is more valuable and firms will have an incentive to pursue the patent less aggressively.

Investment Rates and Technological Uncertainty. The position of the two firms in the patent race is described by two state variables, $k^S(t)$ and $k^L(t)$, denoting the

small and large firms' expected "cost-to-completion", respectively. We assume both $k^S(t) > 0$ and $k^L(t) > 0$ at the beginning of the game. Let $I^S(t)$ and $I^L(t)$ denote the small and large firms' investment rates, respectively. A positive investment rate $I^S(t)$ tends to reduce $k^S(t)$, and similarly a positive investment rate $I^L(t)$ tends to reduce $k^L(t)$. A firm wins the patent race at the time that it becomes the first firm whose cost-to-completion $k^S(t)$ or $k^L(t)$ becomes zero.

A special case of Pindyck's (1993) random technology is used. Expected cost-to-completion $k^S(t)$ and $k^L(t)$ are assumed to follow a diffusion process of the following form, where θ^S and θ^L are cost technological volatility parameters: [1]

$$dk^S = -I^S dt + \theta^S \left(I^S k^S\right)^{1/2} dz^S. \tag{5}$$

$$dk^L = -I^L dt + \theta^L \left(I^L k^L\right)^{1/2} dz^L. \tag{6}$$

Here, dz^S and dz^L denote idiosyncratic white noise processes, also distributed independently from the financial risks dz_m and dz_x. Our CAPM assumption implies that any risk associated with the expected cost-to-completion is diversifiable, i.e. it is priced in a risk neutral manner. This specification has several attractive features. First, the terms $-I^S dt$ and $-I^L dt$ indicate that a dollar of investment reduces expected cost-to-completion by a dollar. Second, the terms $\theta^S \left(I^S k^S\right)^{1/2} dz^S$ and $\theta^L \left(I^L k^L\right)^{1/2} dz^L$ indicate that expected cost-to-completion changes randomly, with both many small unexpected improvements (reductions in $k^S(t)$ and $k^L(t)$) and many small unexpected setbacks (increases in $k^S(t)$ and $k^L(t)$). Third, there is a "learning by doing" feature: a firm's expected cost-to-completion do not change if it does not invest, i.e., $dk^S = 0$ if $I^S = 0$

and $dk^L = 0$ if $I^L = 0$. Investing not only tends to push the firms towards victory by reducing expected cost-to-completion but also tends to reduce uncertainty about how much investment will eventually be required to drive $k^S(t)$ or $k^L(t)$ to zero. Let $I^S_{\max}$ and $I^L_{\max}$ denote fixed maximum investment rates of the small firm and large firm, respectively. The large firm is "large" precisely in the sense that $I^L_{\max} > I^S_{\max}$, i.e., the large firm has larger "research bandwidth" than the small firm. The large and small firms choose $I^S(t)$ and $I^L(t)$ from the closed interval $\left[0, I^S_{\max}\right]$ and $\left[0, I^L_{\max}\right]$, respectively. As we shall see below, in the continuous-time formulation of the model, the square root applied to $I^S(t)$ and $I^L(t)$ in the diffusion terms (together with the assumption that there are no adjustment costs on capital) leads to corner solutions ("bang-bang") in which a firm invests at the maximum rate or not at all. Thus, the small or large firm faces essentially a binary choice between investing at its maximum rate $I^S_{\max}$ or $I^L_{\max}$, respectively, or not investing at all.

Firm Value, Portfolio Value, Strategic Investments and Acquisitions. There are three state variables which describe the state of competition between the large firm and the small firm: the expected cost-to-completion of the large firm $k^L(t)$, the expected cost-to-completion of the small firm $k^S(t)$, and the present value of the patent $x(t)$. We assume that both the manager of the small firm and the manager of the large firm observe all three state variables at all times; there is no asymmetry of information in our model. We also assume that the managers of the small and large firms represent distinct groups of investors. Each manager maximizes the value of its firm's shares without taking into account the impact of its decision on its shareholders' other portfolio holdings.

If the large firm owns less than 100% of the small firm, i.e., the large firm makes a "strategic investment" in the small firm, then the small firm maximizes the value of the remaining shares, not taking into account the impact on the value of the large firm; if the small firm chooses to invest, the large firm shares in the costs of the investment proportionally to its ownership stake in the small firm. The large firm maximizes its value, including its stake in the small firm. If the large firm owns 100% of the small firm – i.e., it has made an "acquisition" – then the large firm assumes control of the small firm, and the large firm maximizes the combined "joint-monopoly" value of both firms. When a strategic investment is made, we assume that the large firm pays an exogenously specified constant proportional transactions cost at rate c on the value of the transaction where c is set to 20% in the numerical implementation.

We assume that the small firm and the large firm are held by investors with diversified portfolios. We also assume that investors can hedge market risk by trading the market portfolio. The holdings in the large and small firms themselves are not assumed to be tradable. For example, the small firm might be owned by venture capitalists, whose investors are well-diversified. This makes it possible for us to assume that the idiosyncratic risks are priced in a risk neutral manner, while market risk can be fully hedged, e.g. by investors trading market index funds in the tradable part of their portfolios.

If a market price for the large or small firm exists, we do not assume that it reflects all information. In fact, we do not even assume that market prices for the two firms exist. Our model is consistent with an interpretation that both firms are privately held, perhaps in the portfolios of venture capitalists who represent the interests of their

investors. To apply the theoretical hedging argument used to justify the "risk neutral" pricing approach, we do, however, need to assume that the ultimate owners (e.g., the limited partners investing in the venture capitalists' funds) can observe enough about the value of the underlying portfolio firms to hedge out systematic risk appropriately.

Equilibrium**.** We employ the solution concept of Markov Perfect Equilibrium (see Maskin and Tirole (1988)). Let α denote the fraction of the small firm's shares owned by the large firm. For a given date t and for every state $\left(x, k^S, k^L, \alpha\right)$ there is a sub-game starting from that state. The Markovian strategies imply that firms cannot engage in history-dependent punishment strategies which would support a more collusive equilibrium. It tends to support an equilibrium in which there is more rent dissipation than under alternative equilibrium concepts where firms behave more co-operatively. For example, we assume that firms cannot commit to invest in the future; they cannot make threats to invest in certain future states, when these threats are not credible. How much this issue of equilibrium concept matters is an issue for future research.

Although our notation is based on continuous time, we implement the solution concept in discrete time, with decisions made at N+1 dates $0 = t_0, t_1, \ldots, t_N = \mathrm{T}$ (where $t_{n+1} - t_n$ is one month in the numerical implementation below). At time t_n, there are two rounds of decision making, a "bargaining round" followed by an "investment round."

In the bargaining round, the large firm is assumed to make a take-it-or-leave-it offer for all or part of the small firm, and the small firm immediately accepts or rejects the offer. This is immediately followed by a Nash investment round, in which each firm simultaneously chooses an investment rate, which is held constant over the interval from t_n to t_{n+1}. At time t_{n+1}, there is another bargaining round and another investment round,

and the process is repeated over and over until the large firm buys a 100% stake in the small firm, one firm wins the race, or time T is reached. If the large firm buys a 100% stake in the small firm, the firms stop competing but the research bandwidth of the two firms cannot be combined. The large firm makes the "joint monopoly" investment decision for each of the combined firm's two "divisions." The large firm is allowed to sell shares in the small firm in the future bargaining rounds (but we believe the large firm never does so in equilibrium). If one firm wins the race, then the winner gets the patent value $x(t)$ and the loser gets zero. If neither firm has won by time T, then both firms get zero.

Since the manager of the large firm maximizes the combined value of the large firm and its holdings in the small firm, some notation is needed to distinguish between the value of the large firm with- and without the holdings in the small firm. Let $W^L\left(t,x,k^S,k^L,\alpha\right)$, the value function of the large firm, denote the value of the large firm, including the shares it owns in the small firm. Similarly, let $W^S\left(t,x,k^S,k^L,\alpha\right)$, the value function of the small firm, denote the value of the shares of the small firm not held by the large firm. The large firm maximizes $W^L(...)$ and the small firm maximizes $W^S(...)$. Note that in addition to the three state variables describing the state of competition between the small firm and the large firm, the arguments to the functions include time t and the strategic investment α of the large firm. Thus, there are five arguments: four state variables and time.

For our discrete-time implementation, we need notation which distinguishes between the value functions before and after the bargaining round. Let

$W^L\left(t,x,k^S,k^L,\alpha\right)$ and $W^S\left(t,x,k^S,k^L,\alpha\right)$ refer to the value functions before the bargaining round, and let $W^{L+}\left(t,x,k^S,k^L,\alpha\right)$ and $W^{S+}\left(t,x,k^S,k^L,\alpha\right)$ refer to the value functions after the bargaining round but before the investment round.

It is also useful to have notation for the value of the large firm, excluding its holdings in the small firm, and the value of the small firm, including the fraction of it held by the large firm. Let $F^L\left(t,x,k^S,k^L,\alpha\right)$ and $F^S\left(t,x,k^S,k^L,\alpha\right)$ denote these implied values, respectively. These functions are related by the equations

$$W^S\left(t,x,k^S,k^L,\alpha\right)=\left(1-\alpha\right)F^S\left(t,x,k^S,k^L,\alpha\right) \tag{7}$$

and

$$W^L\left(t,x,k^S,k^L,\alpha\right)=F^L\left(t,x,k^S,k^L,\alpha\right)+\alpha F^S\left(t,x,k^S,k^L,\alpha\right). \tag{8}$$

The maximization problems of the two firms are based on the functions $W^L\left(t,x,k^S,k^L,\alpha\right)$ and $W^S\left(t,x,k^S,k^L,\alpha\right)$. The implied share prices $F^L\left(t,x,k^S,k^L,\alpha\right)$ and $F^S\left(t,x,k^S,k^L,\alpha\right)$ can be obtained by inverting the above functions to obtain

$$F^S=W^S/(1-\alpha) \tag{9}$$

and

$$F^L=W^L-\alpha W^S/(1-\alpha). \tag{10}$$

The Bargaining Round. In the bargaining round, the large firm makes a take-it-or-leave-it offer in the form of price-quantity pair $\left\langle p,\Delta\alpha\right\rangle$, where p is the price offered per share and $\Delta\alpha$ is the ownership percentage which the large firm offers to purchase. The small firm must accept the offer for the entire quantity $\Delta\alpha$ or reject the offer. We

assume that the small firm accepts the offer in the case of a tie. Thus, the bargaining round is described by a maximization problem

$$W^L\left(t,x,k^S,k^L,\alpha\right)=\max_{p,\Delta\alpha}\left\{W^{L+}\left(t,x,k^S,k^L,\alpha+\Delta\alpha\right)-p\Delta\alpha-c\left|p\Delta\alpha\right|\right\} \tag{11}$$

s.t.

$$W^S\left(t,x,k^S,k^L,\alpha+\Delta\alpha\right)+p\Delta\alpha\geq W^S\left(t,x,k^S,k^L,\alpha\right), \tag{12}$$

$$\alpha+\Delta\alpha\leq 1. \tag{13}$$

The constraints (12) and (13) state, respectively, that the small firm cannot be made worse off (participation constraint) and the large firm cannot buy more than 100% of the small firm. Note that the large firm can buy or sell shares in the small firm (but we believe the large firm never sells shares in equilibrium). For the small firm, we have

$$W^S\left(t,x,k^S,k^L,\alpha\right)=W^S\left(t,x,k^S,k^L,\alpha+\Delta\alpha^*\right)+p^*\Delta\alpha^*, \tag{14}$$

where $\left\langle p^*,\Delta\alpha^*\right\rangle$ is the optimal price-quantity pair offered by the large firm. Note that in the bargaining round, the optimal price offered p^* is not necessarily equal to the implied value of the shares of the small firm $F^S\left(t,x,k^S,k^L,\alpha\right)$. Since the large firm has bargaining power, we might conjecture that the large firm makes successful offers at a lower price (and this conjecture is shown to be true in the numerical implementation below).

For numerical implementation, we assume that the α must be chosen from a finite set (assumed to be {0,20%,40%,100%} in the numerical implementation below). This makes it possible to calculate the optimal strategy for the large firm by checking all the cases allowed by the finite number of possibilities. In the event of a tie, we assume

the large firm chooses $\Delta\alpha = 0$ (but in our numerical implementation this only happens when the small firm's value is very tiny).

The Investment Round. In the investment round, the large firm and the small firm simultaneously (Nash) choose investment rates I^L and I^S to maximize $W^{L+}\left(t, x, k^S, k^L, \alpha^*\right)$ and $W^{S+}\left(t, x, k^S, k^L, \alpha^*\right)$. The notation α^* emphasizes that the ownership share for the investment round was determined in the bargaining round, i.e. $\alpha^* = \alpha + \Delta\alpha^*$. We thus have

$$W^{L+}\left(t, x, k^S, k^L, \alpha^*\right) = \max_{I^L}\left\{-\alpha^* I^S \Delta t - I^L \Delta t + e^{-r\Delta t} E^*\left\{W^L\left(t+\Delta t, x+x, k^S+\Delta k^S, k^L+\Delta k^L, \alpha^*\right)\right\}\right\}, \tag{15}$$

$$W^{S+}\left(t, x, k^S, k^L, \alpha^*\right) = \max_{I^S}\left\{-(1-\alpha^*) I^S \Delta t + e^{-r\Delta t} E^*\left\{W^S\left(t+\Delta t, x+x, k^S+\Delta k^S, k^L+\Delta k^L, \alpha^*\right)\right\}\right\} \tag{16}$$

$$s.t.\begin{cases} \Delta x = \mu^* x \Delta t + \sigma x \Delta w \\ \Delta k^S = -I^S \Delta t + \theta^S (I^S k^S)^{1/2} \Delta z^S \\ \Delta k^L = -I^L \Delta t + \theta^L (I^L k^L)^{1/2} \Delta z^L \end{cases} . \tag{17}$$

In the above notation for the Nash equilibrium for an investment stage, t refers to some decision date t_n and $t + \Delta t$ refers to the next date t_{n+1}. The standard deviation of innovations in x is defined by $\sigma = (\beta^2 \sigma_m^2 + \sigma_I^2)^{1/2}$, i.e. the standard deviation includes both market volatility and idiosyncratic volatility. The notation for the evolution of the state variables suggests an Euler approximation to the continuous-time version of the model (but we use a different approximation – Euler in transformed variables – in the numerical implementation).

The expectation operator notation $E^*\{...\}$ refers to the expectation with respect to the risk-neutral probabilities. The risk neutral probabilities are obtained by changing μ to $\mu^* = \mu - \beta\lambda_m$, i.e. by deflating the true growth rate on the patent's cash flows by the risk premium on the patent's cash flows, so that Gordon's growth formula gives the same value for winning the patent in a risk neutral model as in the CAPM model, i.e.

$$x(t) = D(t)/(r_f + \beta\lambda_m - \mu) = D(t)/(r_f - \mu^*). \tag{18}$$

Note that the dividend yield δ remains the same and is given by

$$\delta = r_f + \beta\lambda_m - \mu = r_f - \mu^*. \tag{19}$$

The intuition behind this result is that the investors can hedge out market risk associated with investments in the two firms; this leaves them with idiosyncratic risk, with respect to which they are risk neutral; thus, the value of the firm is the same as it would be in a risk neutral model with zero risk premium. A proof of this point (omitted here) is analogous to Merton's derivation of the Black-Scholes model.

Note that the equations state that the value of a firm is the present value of its risk-adjusted cash flows, discounted at the risk-free rate. The positive cash flows are the cash flows on the patent, received if and when the firm wins. The negative cash flows are the investment costs (e.g., in R&D), paid in pursuit of winning the patent; we think of this as new equity being injected into the firms by venture capitalists. In addition, there are cash flows (including transactions costs) associated with the large firm buying shares in the small firm. We can think of these cash flows as new equity injected into the large firm (to finance the purchase), but they are portfolio transactions by the owners of the small firm, i.e. these payments go directly to the shareholders of the small firm (who sell shares) and not into the firm itself.

In the continuous-time limit, the Bellman equations for the small firm's and the large firm's strategies (with notation described below) are given, respectively, by

$$\underset{I^S\in\left[0,I^S_{\max}\right]}{Max}\left\{\begin{array}{r}\left(\frac{1}{2}\sigma^2x^2W^{S+}_{xx}+\left(r_f-\delta\right)xW^{S+}_{x}\right)+I^S\left(\frac{1}{2}(\theta^S)^2k^SW^{S+}_{k^Sk^S}-W^{S+}_{k^S}-\left(1-\alpha^*\right)\right)\\ +I^L\left(\frac{1}{2}(\theta^L)^2k^LW^{S+}_{k^Lk^L}-W^{S+}_{k^L}\right)+\left(-\left(r_f+\eta\right)W^{S+}\right)\end{array}\right\}=0 \qquad (20)$$

$$\underset{I^L\in\left[0,I^L_{\max}\right]}{Max}\left\{\begin{array}{r}\left(\frac{1}{2}\sigma^2x^2W^{L+}_{xx}+\left(r_f-\delta\right)xW^{L+}_{x}\right)+I^S\left(\frac{1}{2}(\theta^S)^2k^SW^{L+}_{k^Sk^S}-W^{L+}_{k^S}-\alpha^*\right)\\ +I^L\left(\frac{1}{2}(\theta^L)^2k^LW^{L+}_{k^Lk^L}-W^{L+}_{k^L}-1\right)+\left(-r_fW^{L+}\right)\end{array}\right\}=0. \qquad (21)$$

The notation η represents a potentially higher cost-of-capital for the small firm than the large firm (possibly due to fees charged by the venture capitalists). In the current version of this paper, the value of η is set to zero.

Note that both of the Bellman equations (20) and (21) are linear in each of the investment strategies I^S and I^L. In the investment sub-stage, each firm takes as given the investment choice of the opponent, but each firm takes into account how its choice of investment rate affects its expected cost-to-completion. The linearity of the Bellman equations implies that in the continuous-time limit, the investment strategy of the small firm depends on the coefficient of I^S in its Bellman equation, and the investment strategy of the large firm depends on the coefficient of I^L in its Bellmen equation. If the coefficient is positive, the firm has a dominant strategy to invest at the maximium rate $I^S_{\max}$ or $I^L_{\max}$. If the coefficient is negative, the firm has a dominant strategy to choose a zero investment rate. If the coefficient is zero, the firm is indifferent as to its investment rate.

In the numerical implementation, the firms are assumed to invest at the maximum rate or not at all (an assumption which we believe has little effect on results). If the firms have dominant strategies (which happens at more than 99% of the decision nodes in the numerical implementation below), then each firm plays its dominant strategy. Otherwise, there are possibly multiple solutions and the solutions possibly involve mixed strategies. When there are multiple equilibria, we assume that the firms play the equilibrium strategies which maximize the large firm's value (an assumption which we believe has little effect on the results). When equilibrium involves mixed strategies, we assume that the firms play mixed strategies although this is not reflected in the above notation.

Except for the small region of mixed strategies, the strategies of the firms follow can be described by four sets of points, where both firms invest, only the small firm invests, only the large firm invests, and neither firm invests.

To complete a description of the equilibrium, it is also necessary to describe the solution of the joint monopoly problem, which is implemented after the large firm buys a 100% stake in the small firm. The Bellman equation now has the large firm maximizing value by choosing both investment rates and has the ownership share set to one:

$$\underset{I^L\in\left[0,I^L_{\max}\right],I^S\in\left[0,I^S_{\max}\right]}{Max}\left\{\begin{array}{c}\left(\frac{1}{2}\sigma^2x^2W^{L+}_{xx}+\left(r_f-\delta\right)xW^{L+}_{x}\right)+I^S\left(\frac{1}{2}(\theta^S)^2k^SW^{L+}_{k^Sk^S}-W^{L+}_{k^S}-1\right)\\ +I^L\left(\frac{1}{2}(\theta^L)^2k^LW^{L+}_{k^Lk^L}-W^{L+}_{k^L}-1\right)+\left(-r_fW^{L+}\right)\end{array}\right\}=0. \qquad (22)$$

The Bellman equation remains linear in both I^S and I^L. For numerical implementation, we assume that that each division invests at the maximum rate or not at all. Theoretically, the large firm can continue to run a "race" between the two divisions by having both divisions invest at the maximum rate.

II. NUMERICAL IMPLEMENTATION

Since we believe that it is hopeless to calculate a closed-form solution for the game, we provide a numerical implementation of the game and discuss the properties of the solution of this specific example.

Assumptions. The assumed values of the exogenous parameters are as follows:

$\delta = 3\%,\ r_f = 5\%,\ \beta = 1.00,\ \sigma_m = 20\%,\ \sigma = 30\%, \theta^S = \theta^L = 0.60,\ c = 20\%,\ I^S_{\max} = 5,\ I^L_{\max} = 10.$ These choices imply $\mu^* = r_f - \delta = 2\%,\ \sigma_I = (\sigma^2 - \beta^2\sigma_m^2)^{1/2}$. Time is discretized so that $\Delta t = t_{n+1} - t_n = one\,month$. The horizon T is chosen as 50 years. Thus, we have N=600. Notice that the large firm is assumed to invest at twice the rate of the small firm. Notice also that the winner of the race receives a patent value which fluctuates like a "typical" stock, with a beta of 1.00 and a volatility of 30%, with market volatility of 20%.

For the purpose of numerical implementation, we also discretize the state space of the four state variables. Instead of discretizing the three diffusion state variables themselves, we discretize the three transformed variables $\ln(x(t)), \left(k^S(t)\right)^{1/2}, \left(k^L(t)\right)^{1/2}$. Note that the dynamics of the transformed variables are given (from Ito's lemma) by

$$d(\ln(x)) = (r_f - \delta - \tfrac{1}{2}\sigma^2)dt + \sigma dz, \tag{23}$$

$$d((k^S)^{1/2}) = -\tfrac{1}{2} I^S (1 + \left(\theta^S\right)^2 / 4)(k^S)^{-1/2} dt + \tfrac{1}{2}(I^S)^{1/2}\theta^S dz^S, \tag{24}$$

$$d((k^L)^{1/2}) = -\tfrac{1}{2} I^L (1 + \left(\theta^L\right)^2 / 4)(k^L)^{-1/2} dt + \tfrac{1}{2}(I^L)^{1/2}\theta^L dz^L. \tag{25}$$

The chosen transformations have the property that the instantaneous variance in the transformed variables is constant. For the value of the patent $x(t)$, we choose 201 points, equally spaced in the transformed variable ln(x), such that the smallest point corresponds

to x=1 and the largest point corresponds to x = 10,000. The midpoint (101st point) corresponds to x = 100, the value used in the figures discussed below. For each of the expected cost-to-completion variables k^S and k^L, we choose 61 points, equally spaced in the transformed variables $(k^S)^{1/2}$ and $(k^L)^{1/2}$, such that the smallest value corresponds to $k^S = k^L = 0$ and the largest value corresponds to $k^S = k^L = 225$. The ownership percentage α is chosen from the set of four values {0%,20%,40%,100%}.

The game is solved by using the "brute force" lattice method of backward induction from the terminal date $t_N = 50\,years$, corresponding to N=600. We use an Euler approximation for all three transformed state variables. We approximate the normally distributed transition probabilities in the Euler approximation by mapping the true outcome to the closest transformed outcome in the space of transformed state variables within two standard deviations (rounded up) of the expected outcome. When a cost-to-completion variable hits its minimum value, we assume that the respective firm wins the patent at that point. If both firms win the patent at the same time, we split the patent equally between the two firms. When the state variables otherwise hit their maximum or minimum values, we do not assume that they are absorbed; random fluctuations are allowed to push them back into the interior of the space of state variables at a later date. Note that our numerical implementation allows the state to jump more than one grid point from one month to the next. A typical large jump is two or three grid points. In the continuous model, by contrast, a firm cannot move from one state to another discontinuously; it must pass through intervening states.[2]

Since the state space is four dimensional, for the purpose of describing the equilibrium, we choose a patent value of x=100 (e.g. $100 million) and we choose an

initial investment stake of $\alpha = 0$, then describe the investment strategies of the small firm and the large firm for various values of their costs-to-completion k^S and k^L. Note that since the small firm's maximum investment rate is 5 per year, it would take the small firm 20 years to achieve the patent if its expected cost-to-completion k^S were also 100. In general, we would not expect the small firm to invest in this situation, so in general we expect the small firm to invest when its expected time-to-completion is less than 20 years. Similarly, we expect the large firm, with a maximum investment rate of 10, to begin investing at a point where the expected time to completion is less than 10 years.

Note that the expected cost-to-completion variables k^S and k^L are undiscounted costs. The present values of the expected costs to completion are less because of discounting. Similarly, the value of the patent x is present value of the cash flows on the patent if a firm wins immediately. If a firms wins at some point in the future, the present value of the patent is reduced by the foregone dividends, which accrue at rate δ.

III. STRATEGIC INVESTMENTS, ACQUISITIONS, AND INVESTMENT STRATEGIES IN THE NUMERICAL EXAMPLE

Properties of Investment Strategies with Joint Monopoly. In solving for the equilibrium, it is also necessary to solve for the joint monopoly equilibrium. Joint monopoly also provides a useful benchmark for making comparisons with duopoly competition with transactions costs. Joint monopoly by definition implements the collusive outcome which maximizes the joint profits of the two firms. If transactions costs were zero, it would be an equilibrium for the large firm to offer the joint monopoly outcome to the small firm as soon as possible, with a price such that the small firm would

accept the offer. The investment behavior of the two firms is illustrated in Figure 1, where the value of the patent is fixed at x=100, the horizontal axis is the expected cost-to-completion of the small firm k^S and the vertical axis is the expected cost-to-completion of the large firm k^L. In this figure, the dark (blue) area shows where the large firm invests and the small firm does not invest; the light gray (orange) area shows where the small firm invests and the large firm does not; the unshaded white area shows where neither firm invests. The key properties to note in this figure are the following:

1. Although it is theoretically possible that the joint monopolist would choose to have both firms invest at the same time, this does not happen in this example. Evidently, the rent dissipation which results from both firms' investing simultaneously is always greater than the benefit associated with finishing sooner. This result is not going to hold in all examples. If the patent value is much higher for the set of parameter values examined here, both firms invest along the 45-degree line close to the origin (see Figure 5 below). Moreover, if the dividend yield were much greater than 3%, the joint monopoly would be in a great hurry to start the patent cash flows, and this would lead to the joint monopolist having both firms invest simultaneously if the patent value were large enough.

2. The point where each firm's expected cost-to-completion is equal to the value of the patent is well inside the area where neither firm invests. The real option value of waiting induces the joint monopolist to delay investment when a naïve NPV rule would suggest that the project has a positive NPV.

3. An approximate description of the optimal investment policy is that one of the two firms will invest if either of the two firms has an expected cost-to-completion less than about 50. In this case, the firm which invests is typically the firm with the lower

expected cost-to-completion. This can be seen in Figure 1, where the investment area of the large firm corresponds to an area below the 45-degree line, and the investment area of the small firm is above the 45-degree line.

4. The large firm invests when its cost-to-completion is slightly higher than the small firm when the cost-to-completion is relatively high (in the investment region). Intuitively, this probably reflects the fact that the large firm can win the patent faster. Although the discounted cost of winning faster is greater than winning more slowly, this extra expected cost is more than offset by the additional dividends collected from winning the patent faster.[3]

Properties of Investment Strategies with Infinite Transactions Costs. Another useful benchmark for making comparisons with the equilibrium with transactions costs is the case where transactions costs are infinite. This case corresponds to that of Meng (2008) if firms are allowed to have asymmetric research bandwidths. Equilibrium investment strategies for this case are illustrated in Figure 2. The layout for this figure is the same as in Figure 1, with the patent value fixed at 100, the large firm's cost-to-completion plotted on the vertical axis and the small firm's cost-to-completion plotted on the horizontal axis. Similarly, the dark (blue) area shows where the large firm invests and the small firm does not, and the light gray (yellow) area shows where the small firm invests but the large firm does not. In this figure, there is also a medium gray (green) area, where both firms invest. In addition to the 45-degree line, which represents "equal-cost-to-completion" line, there is a steeper "equal-time-to-completion" line with a slope of two (since the large firm's maximum rate of 10 is twice as large as the small firm's investment rate of 5). Important properties of this graph are the following:

1. There is an extensive area where both firms invest, "inefficiently" in comparison with joint monopoly. The area where both firms invest tends to lie between the equal-cost-to-completion line (45-degree line) and the equal-time-to-completion line (slope of two). It covers almost all of the area between the two lines for costs-to-completion less than about 50.

2. The small firm does not invest when its cost-to-completion is greater than that of the large firm, but it invests much more aggressively (pre-emptively) than in the joint monopoly case when its cost-to-completion is less than that of the large firm.

3. The large firm also invests much more aggressively than in the joint monopoly case, especially when the small firm is investing pre-emptively but the large firm is ahead in terms of time-to-completion. There is a significant region where both firms invest simultaneously, although neither firm would invest with joint monopoly.

Investment Strategies and Bargaining Strategy with 20% Transactions Costs. With 20% transactions costs, we expect the investment strategies to lie somewhere between the zero transactions cost case (joint monopoly) and the infinite transactions cost case.

The investment strategies for the case of 20% transactions costs are illustrated in Figure 3A. The layout of the axes and the value of the patent are the same as in Figures 1 and 2. The color scheme for the investment strategies is also the same as in Figure 2, with light gray (yellow) indicating investment by the small firm, dark (blue) indicating investment by the large firm, and medium gray (green) indicating investment by both firms. It is assumed that the initial holdings by the large firm in the small firm are zero. Figure 3A uses the symbols "X," "O," and "Δ" to indicate bargaining outcomes. An "O"

represents a successful offer for 20% of the small firm when the initial holdings of the large firm are zero; a "Δ" represents a successful offer for 40% of the small firm when the initial holdings of the large firm are 0%; and an "X" represents a successful offer for 100% of the small firm when the initial holdings are zero.

Key features of the investment strategies are the following:

1. The area where both firms invest is approximately 50% smaller than the corresponding area when transactions costs are infinite. Areas where investment strategies are different are in the vast majority of cases associated with successful strategic investments or takeovers. Although not shown in Figure 3A, in all cases where there are strategic investments (20% "O" or 40% "Δ") or takeovers (100% "X"), if the large firm were to deviate from its optimal strategy and not made an offer at that point, the subsequent investment strategies would be different, looking more like the case of infinite transactions costs. As a result of strategic investments and takeovers, the investment strategies look more like those in joint monopoly.

2. Small 20% strategic investments tend to be made along the northwest side of the region where both firms would have been investing with infinite transactions costs. If a strategic investment were not made at these points, both the large firm and the small firm would simultaneously invest. Intuitively, the large firm is credibly threatening to invest (and dissipate rents) if the small firm does not accept its offer of strategic investment. The strategic investment alters the large firm's incentives so that it does not invest; the small firm knows this, and accepts the offer. The offer price can be much lower than the value of the shares to the small firm. This results from our assumption that the large firm can make a take-it-or-leave-it offer, thus internalizing all gains.

3. Successful 40% strategic bids tend to be made at points just south of points where successful 20% bids are made. Intuitively, in these cases a 20% stake in the small firm does not provide enough incentive for the large firm not to compete with the small firm. The larger stake provides the necessary incentive.

4. Along the southeast side of the region where both firms would be investing with 20% transactions costs, successful 100% offers occur. In this region, the large firm always prevents the small firm from investing, and—if its own costs are low enough—it invests itself. Intuitively, these 100% offers are different from the strategic investments for 20% or 40% of the small firms equity in that their purpose is to make the small firm stop investing, not to make the large firm stop investing. To make the small firm stop investing, our rules require 100% ownership of the small firm by the large firm.

A Comparison of 20% Transactions Costs with Joint Monopoly. Figure 4 provides a comparison of the investment strategies with 20% transactions costs and joint monopoly. In this figure, the medium gray (green) region where both firms invest in Figure 3A is divided into two sub-regions: a lighter (yellow-green) sub-region where only the small firm would be investing in joint monopoly and a darker (blue-green) region where only the large firm would be investing in joint monopoly. The light gray (yellow) region where only the small firm invests is also divided into two sub-regions: a darker (orange) region where the small firm also invests in joint monopoly and a lighter (yellow) region where the small firm would not be investing with joint monopoly. Notice that strategic investments and acquisitions tend to be followed by the investment strategies consistent with joint monopoly, i.e. strategic investments and acquisitions tend to occur in the orange, blue, and white regions. The only major exception is the 20% and 40%

strategic investments tend to occur in the yellow region where the small firm invests with 20% transactions costs but not with joint monopoly. These strategic investments avoid an outcome where both firms would otherwise invest with 20% transactions costs but neither firm would invest with joint monopoly.

The Effect of Changing Patent Value on Investment Dynamics. The effect of an increase in the patent value from 100 to 151 is shown in Figure 5. It has the same scale and color scheme as Figure 4. The main difference is that since the patent is more profitable, the investment regions are larger. Intuitively, the investment regions shift outward along rays from the origin. The regions where strategic investments and acquisitions occur are also larger as the size of the market increases.[4]

When the large firm invests, the resolution of technological uncertainty results in movements north and south on figures 1-5, but not in movements east and west. Success pushes the point on the graph south, and the large firm wins the patent when the point hits the horizontal axis. When the small firm invests, the resolution of technological uncertainty results in movements east and west, but not north and south. Success pushes the point west, and the small firm wins when the point hits the vertical axis. This intuition that the overall shape of the investment regions does not change, but rather shifts out from the origin, as the patent value increases, makes it possible to describe the dynamics of the patent race. Since increases in patent values shift the entire investment region along a ray away from the origin, we can also think of the shift as having a similar relative effect to staying on the same Figure but shifting the point $\left(k^S, k^L\right)$ towards the origin. In other words, from a graphical perspective, an increase in patent value is like a reduction in both firms' cost-to-completion, with a change in scale. Using this intuition,

we can think of an increase in patent value as a shift along a ray towards the origin and a decrease in patent value as a shift along a ray away from the origin.

Suppose that we begin on Figure 4 in the northeastern white area of the graph, where neither firm is investing. The only source of uncertainty operating is changes in the size of the market. For one of the two firms to start investing it is necessary for the patent's value to grow enough so that the investment area reaches the initial point. Using the change of scale intuition, we think of this as a movement of the point along a ray towards the origin.

Acquisitions When Only the Large Firm is Active. If the initial point $\left(k^S, k^L\right)$ is below the 45-degree line, i.e., the large firm has a lower expected cost-to-completion than the small firm, then (assuming the shape of the investment region remains more-or-less constant as the patent value increases), an increase in the size of the market (i.e. shift along a ray towards the origin) will result in the large firm investing before the small firm invests. If the large firm has good luck with its technology, the point will move south and the large firm will win the patent race when the point hits the horizontal axis. If the large firm has bad luck, the point will move north and the large firm will eventually stop investing. It will start investing again if the patent value increases. If the patent value is increasing while the large firm is investing and having bad luck, the point can move horizontally to the west. When the point gets close to the 45-degree line, it will hit an "X," indicating that an acquisition occurs. These acquisitions occur at precisely the point where the small firm would begin to invest also.

Thus, the acquisitions which occur when only the large firm is investing tend to occur when the large firm is having bad luck, the patent value is increasing, and the small

firm is about to enter into competition by investing head-to-head against the large firm. After the acquisition, the point shifts to the joint monopoly graph. The large firm continues to invest for a while, but if it has bad luck, the point will move north towards the 45-degree line and eventually the large firm will stop investing and the small firm will begin to invest. Thus, although the purpose of the acquisition is to prevent the small firm from investing immediately, the acquisition protects the large firm against bad luck in its own technology, since the technology of the small firm, acquired cheaply, is also in the large firm's portfolio.

Acquisitions when Neither Firm is Active. Suppose that the initial point is above the 45-degree line in the white area where neither firm is investing. Suppose further that the patent value increases, which we visualize as a movement along a ray towards the origin. If the point is close to the 45-degree line, the movement along the ray towards the origin eventually hits an "X" in the white area, indicating that an acquisition occurs even though neither firm is investing. Such acquisitions prevent the small firm from making pre-emptive investments. Immediately after the acquisition, neither firm invests. If the market grows even more, either the large or small firm will eventually begin investing according to the joint monopoly strategy.

Strategic Investments when the Small Firm is Active. Suppose that the initial point is in the non-investment (white) area, not only above the equal-cost-to-completion line (45-degree) line but also above the equal-time-to-completion (slope of two) line. In this case the small firm is ahead of the large firm in both cost- and time-to-completion. If the market grows, we visualize the point $\left(k^S, k^L\right)$ moving along a ray towards the origin, and it eventually hits the light gray (yellow) area where the small firm begins to invest. If

the small firm's investments are successful, the point moves west, and the small firms wins the patent race when the point hits the vertical axis. If the small firm's investments are not particularly successful, but the value of the patent increases, then the point moves southward, crossing the line with a slope of two and eventually hitting a point where a strategic investment occurs ("O"). If a strategic investment does not occur at this point (because one of the firms behaves sub-optimally in the bargaining round), then the large firm immediately begins investing, in competition with the small firm. Intuitively, the strategic investment alters the large firm's incentives to compete against the small firm; without the strategic investment, it would have competed by investing also, but with the strategic investment it does not compete because this would cannibalize the value of its investment. After the strategic investment, the small firm continues to invest, but the strategic investment induces the large firm not to compete immediately.

Now suppose that the small firm continues to have bad luck, but the value of the patent continues to increase, i.e. the point continues to move south from a strategic investment point (identified by an "O"). Since the large firm now has a 20% stake in the small firm, both firms' investments strategies are perhaps different from what they would be if the large firm held a zero stake. The acquisition and investment strategies with the large firm owning a 20% and a 40% stake in the small firm are shown in Figures 3B and 3C, respectively. These graphs look very similar to Figure 3A, with the exception that the "O" symbols are missing from Figures 3B and 3C (because the large firm has already acquired a 20% stake in the small firm), and the "Δ" symbol is missing from Figure 3C, because the large firm already has a 40% stake in the small firm. If the strategic investment point is far enough from the origin in Figure 3B, it moves into the area where

the large firm purchases a 100% stake in the small firm ("X") and neither firm invests (white), i.e., the large firm acquires the small firm and implements the joint monopoly strategy of mothballing the project. If the strategic investment point is close enough to the origin, the large firm may initially increase its stake to 40%. If the point continues to move southward, the large firm chooses to compete with the small firm (see both Figures 3B and 3C). This occurs in the finger-shaped green area of Figures 3B and 3C. We thus have both firms investing head-to-head, even though the large firm has a 20% or 40% stake in the small firm.

Now suppose that the point continues to move south from the green finger-shaped region where both firms are competing head-to-head, in Figures 3A, 3B, or 3C. This can occur because the large firm is successful, or both the small firm is unsuccessful and the value of the patent increases. Eventually, the large firm acquires 100% of the small firm. After the acquisition, the joint monopoly strategy is played, in which the small firm continues investing and the large firm stops investing.

We have described one type of strategic investment: An active small firm has somewhat poor luck pursuing the patent despite the fact that the patent is becoming more valuable. The increasing value of the patent attracts the attention of a less efficient larger firm, which "threatens" to enter the market. The implicit threat to harm the small firm is sufficiently great that the large firm can buy a stake in the small firm for a cheap enough price that it is more profitable for the large firm to make a strategic investment than to compete. Immediately after the investment the small firm continues investing and the large firm does not compete. If the value of the patent continues to grow and the small firm has bad luck with its technology, the large firm will either compete with the small

firm (both firms close to the completion) or the large firm will acquire 100% of the small firm and shut it down (far from the origin). If it competes successfully against the small firm, the small firm becomes so weakened that it is acquired by the large firm (presumably cheaply).

Thus, in a rapidly growing market (increasing patent value) where only the small firm is active but having somewhat poor luck, the large firm first makes a strategic investment, then increases its investment, then competes, and finally (if successful) makes a 100% acquisition. This sequence of competitive behavior occurs when the small firm is ahead of the large firm in cost-to-completion, but the large firm is ahead of the small firm in time-to-completion.

Strategic Investments When the Small Firm is Inactive. Another type of strategic investment occurs when the small firm is inactive. To see this, suppose that the initial point in Figure 4 is in the non-investment (white) area, well above the 45-degree line but below the equal-time-to-completion line. If the patent value increases, which we visualize as a movement along a ray towards the origin, eventually an "O" or "Δ" is reached, indicating that a strategic investment occurs. If the point is close enough to the 45-degree line, neither the large firm nor the small firm is investing when the strategic investment is made, and neither the small firm nor the large firm invests after the strategic investment is made. If a strategic investment is not made (because one of the firms deviated from its optimal strategy in the bargaining stage), then the small firm invests but the large firm does not invest (not shown in Figure 4). Thus, the strategic investment somehow alters the incentive of the small firm to invest.

The intuition for this result is not as straightforward as in the previous example. Straightforward intuition suggests that the large firm makes a strategic investment in order to reduce its own incentives to compete with the small firm. In this case, the direct effect of reducing the incentive of the large firm to compete also has the indirect effect of reducing the incentive for the small firm to invest pre-emptively. In this manner, the strategic investment also reduces the incentive for the small firm to invest in such a way as to dissipate rents.

If the value of the patent continues to increase after the strategic investment is made, two things can happen. If the strategic investment is made at a point close enough to the 45-degree line, then the large firm buys a 100% stake in the small firm, immediately after which neither firm invests; the large firm exercises the option to wait in a manner optimal for joint monopoly. If the strategic investment is made at a point sufficiently far from the 45-degree line, there is a limited range of points where the small firm begins to invest, pre-emptively from the perspective of joint monopoly.

Note that both strategic investments when the small firm is active and strategic investments when the small firm is inactive occur when the patent is becoming more valuable. Immediately after these strategic investments, the large firm does not invest and the small firm continues to invest if and only if it was investing before the strategic investment.

Patent Races Without Strategic Investments or Acquisitions. With 20% transactions costs, how can head-to-head competition occur without either a strategic investment or acquisition? This can occur if the small firm is initially ahead in both time-to-completion and cost-to-completion, the patent value increases so that the small firm

becomes active, and the small firm initially has such good luck that the point moves close to the vertical axis. If at this point the small firm begins to have bad luck but the market grows dramatically, the point drops straight southward into the green region of head-to-head competition. We then have both firms competing head-to-head without a strategic investment of acquisition occurring.

IV. FINANCIAL PROPERTIES OF EQUILBRIUM

This section examines some financial properties of the equilibrium.

Firm Value. Intuitively, we expect the value of both the large firm and the small firm to increase when its own cost-to-completion falls and to decrease when the other firm's cost-to-completion falls. Figures 6A and 6B show iso-value lines for the large firm and the small firm, respectively, when the large firm has a zero stake in the small firm. The intervals between the iso-value lines represent factors of two in value. As expected, these iso-value lines are upward sloping. Furthermore, the figures show that dramatic changes in value occur in response to seemingly modest changes in the firm's expected costs-to-completion. South of the equal-cost-to-completion line (45-degree line), the large firm is worth more than 32 in the investment region, while the small firm is worth less than 2 (unless both firms are close to winning, in which case the small firm is worth more). Thus, the large firm's value typically exceeds the small firm's value by a factor of 16x. North of the equal-time-to completion line (slope of 2), the situation is reversed. The small firm is typically worth more than 64 and the large firm is worth less than 1. Thus, the value of the small firm exceeds the value of the large firm by a factor of at least 64x. Between the equal-cost-to-completion line and equal-time-to-completion

line, the values of both firms change dramatically as in response to technological shocks. Notice also that the iso-profit lines get closer together as either firm's value decreases. This indicates that a firm's value plunges increasingly rapidly towards zero as it gets further behind in the race, becoming economically meaningless. For example, if the value of the patent is $100 million, the value of either firm on its lowest graphed iso-value line is about 0.37 cents.

Rent Dissipation. Intuitively, we expect that lowering transactions costs allows the firms to lessen value dissipation (relative to joint monopoly), which results from head-to-head competition. Figure 7 provides level curves for the percentage value dissipation as a function of the two firm's costs-to-completion. Expected rent dissipation is measured as the percentage difference between the combined values of the two firms under joint monopoly and 20% transactions costs (with zero ownership of the small firm by the large firm). The graph shows that rent dissipation is greater than 2% only in the region between the equal-cost-to-completion line and the equal-time-to-completion line. In this region, it is not usually efficient (from the perspective of joint monopoly) for the large firm to invest, but the large firm often does invest because it expects to be able to beat the small firm in the patent race. The maximum amount of rent dissipation is about 10%. For comparison purposes, Figure 8 provides analogous level curves for rent dissipation comparing the case of infinite transactions costs with joint monopoly. This figure shows that rent dissipation can be in excess of 30% in the case of infinite transactions costs.

CAPM Betas. Figures 9 and 10 show CAPM betas for the large firm and the small firm, respectively, under the assumption that the beta of the patent value is 1.00.

The figures make it clear that the betas of both firms are complicated non-linear functions of their costs-to-completion. The figures show that the betas of the more efficient of the two firms tend to be somewhat greater than one. Intuitively, this reflects the fact that the firms incorporate real options on the value of the patents. Interestingly, the small firm's beta is unusually low (i.e., closer to one) in the area just above the region of head-to-head competition. Intuitively, this reflects the fact that as the patent value increases in this region, the large firm becomes more likely to invest, and this is bad news for the small firm.

V. WELFARE

A complete welfare analysis of the effects of transactions costs on strategic investments and acquisitions depends on several issues which are not modeled in this paper.[5] Our model is based on a reduced form for the profits of the two competing firms. For a complete welfare analysis, we also need to consider consumer surplus and producer surplus of other firms. If the winner of the patent is a perfectly discriminating monopolist and the product represented by the patent does not affect other product markets (so that producer surplus of other firms is not affected), the profits of the large and small firms internalize all social costs and benefits. In this case, joint monopoly is socially optimal and lower transactions costs which encourage strategic acquisitions and investments increase social welfare.

Let us assume that the winner of the patent is not a perfectly discriminating monopolist, but there are no other firms whose surplus is affected by the patent. Then the profits of the two firms understate the social gains of achieving the patent. Social welfare

is increased if the two firms invest more aggressively than they do with joint monopoly. In this case, as welfare analysis involves some interesting trade-offs. On the one hand, pre-emptive investments by the small firm may increase welfare since this represents earlier investment by the more efficient firm. On the other hand, head-to-head competition will in most cases lower welfare unless consumer surplus is huge relative to the profits of the two firms. Thus, to the extent that strategic investments prevent pre-emptive investments by the small firm, welfare may be harmed; but strategic investments and acquisitions which prevent head-to-head competition probably improve welfare by reducing rent dissipation. A policy rule which pushes investment strategies in the direction of the social optimum (but does not achieve it) is to allow strategic acquisitions and investments after which either the large firm or the small firm continues to invest, but to frown on strategic acquisitions or investments after which neither firm invests. Such a rule would encourage the small firm's pre-emptive investments (by disallowing strategic acquisitions and investments designed to prevent it) but would allow strategic acquisitions and investments which forestalled rent-dissipating head-to-head competition. With such a rule, lower transactions costs would tend to increase welfare, at least in the example studied in this paper.

If we suppose that there are other firms whose surplus is reduced by successful completion of the patent, then there is a welfare argument for discouraging investment. This favors joint monopoly.

VI. CONCLUSION

This paper has shown that a model of strategic acquisitions with transactions costs, in the context of a patent race, results in complex enough strategies that it is unlikely they could be adequately captured by a simple two- or three-period model. The approach taken here, numerical approximation to a discrete-time formulation close to the continuous model, can be used to show the complexity of the strategic interactions and provide intuition about how competition plays out in these models.

A topic for further research is to investigate a richer set of contracts between the large firm and the small firm. For example, the large firm may offer the small firm equity capital at a price which depends on other terms of the deal, such as anti-dilution protection. For example, strong anti-dilution protection for the large firm makes it more difficult for the small firm to raise capital in the future, and might thus promote the interests of joint monopoly, while incurring low transactions costs. Another example is based on the idea that a firm which receives venture capital financing will spend all of the money it receives. This might lead to an interesting model in which venture capitalists can pre-commit to have their portfolio firms follow aggressive investment strategies by giving them many months of financing in advance. Of course, this might also lead to unprofitable investments by the portfolio firms as well.

We leave these extensions for further research.

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Figure 1

Investment Strategies under Joint Monopoly

X(t) = 100

The horizontal axis is the small firm's cost-to-completion. The vertical axis is the large firm's cost-to-completion. The light gray (orange) area is where the small firm invests and the large firm does not (area SM). The dark (blue) area is where the large firm invests and the small firm does not (area LM). The un-shaded white area is where neither firm invests (area Φ). The dash-dot (green) line is the equal-cost-to-completion line (45-degree line) and the dotted (black) line is the equal-time-to-completion line with a slop of 2.

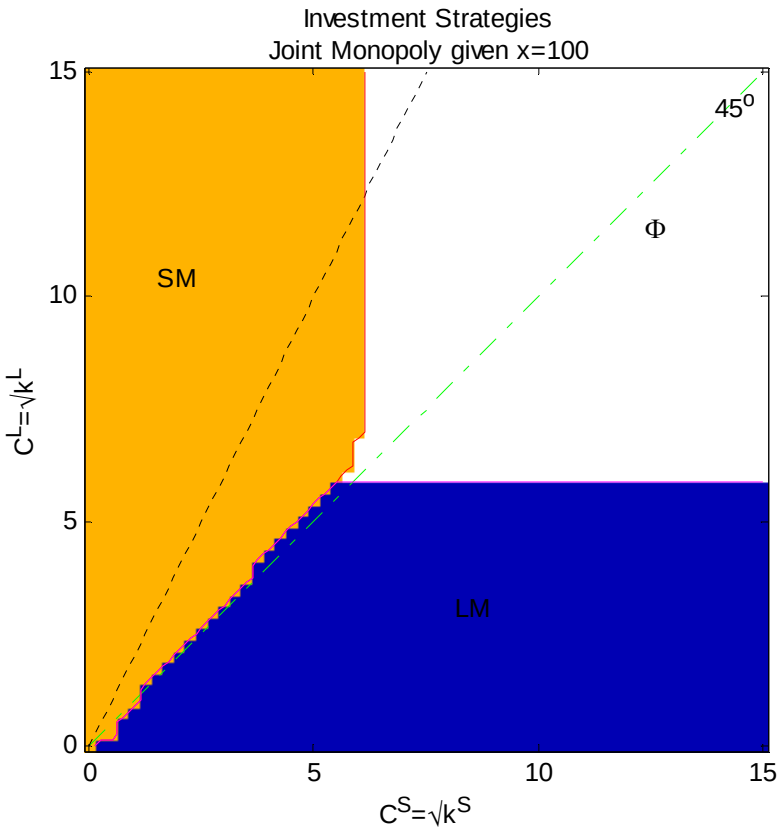


Figure 2

Investment Strategies with Infinite Transactions Costs

X(t) = 100

The horizontal axis is the small firm's cost-to-completion. The vertical axis is the large firm's cost-to-completion. The light gray (yellow) area is where the small firm invests and the large firm does not (area Sinf). The dark (blue) area is where the large firm invests and the small firm does not (area Linf). The medium gray (green) area shows where both firms invest (area Sinf & Linf). The un-shaded white area is where neither firm invests (area Φ). The dash-dot (green) line is the equal-cost-to-completion line (45-degree line) and the dotted (black) line is the equal-time-to-completion line with a slop of 2.

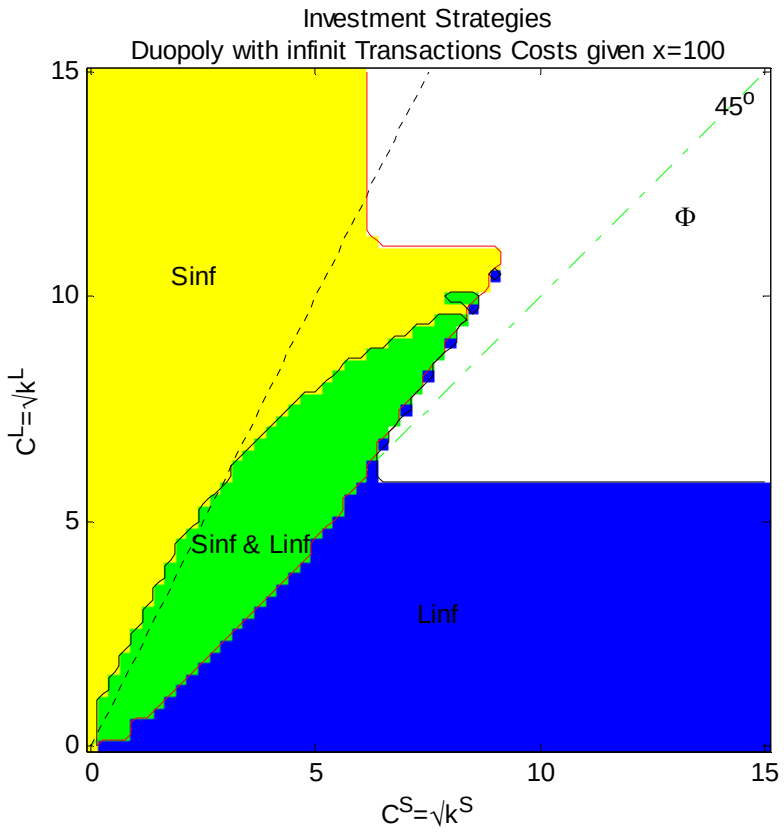


Figure 3A

Investment Strategies with 20% Transactions Costs

X(t) = 100, α = 0%

The horizontal axis is the small firm's cost-to-completion. The vertical axis is the large firm's cost-to-completion. The light gray (yellow) area is where the small firm invests and the large firm does not (area S20). The dark (blue) area is where the large firm invests and the small firm does not (area L20). The medium gray (green) area shows where both firms invest (area S20 & L20). The un-shaded white area is where neither firm invests (area Φ). The dash-dot (green) line is the equal-cost-to-completion line (45-degree line) and the dotted (black) line is the equal-time-to-completion line with a slop of 2. The symbols "X," "O," and "Δ" indicate bargaining outcomes. An "O" represents a successful offer for 20% of the small; a "Δ" represents a successful offer for 40% of the small firm; and an "X" represents a successful offer for 100% of the small firm.

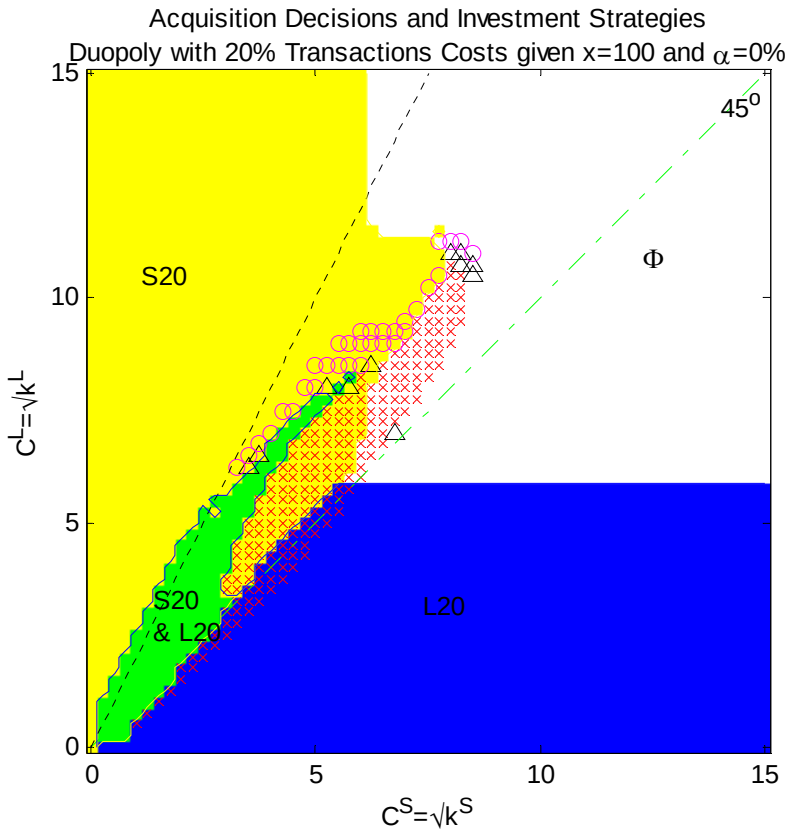


Figure 3B

Investment Strategies with 20% Transactions Costs

X(t) = 100, , α = 20%

The horizontal axis is the small firm's cost-to-completion. The vertical axis is the large firm's cost-to-completion. The light gray (yellow) area is where the small firm invests and the large firm does not (area S20). The dark (blue) area is where the large firm invests and the small firm does not (area L20). The medium gray (green) area shows where both firms invest (area S20 & L20). The un-shaded white area is where neither firm invests (area Φ). The dash-dot (green) line is the equal-cost-to-completion line (45-degree line) and the dotted (black) line is the equal-time-to-completion line with a slop of 2. The symbols "X" and "Δ" indicate bargaining outcomes. A "Δ" represents a successful offer for 40% of the small firm; and an "X" represents a successful offer for 100% of the small firm.

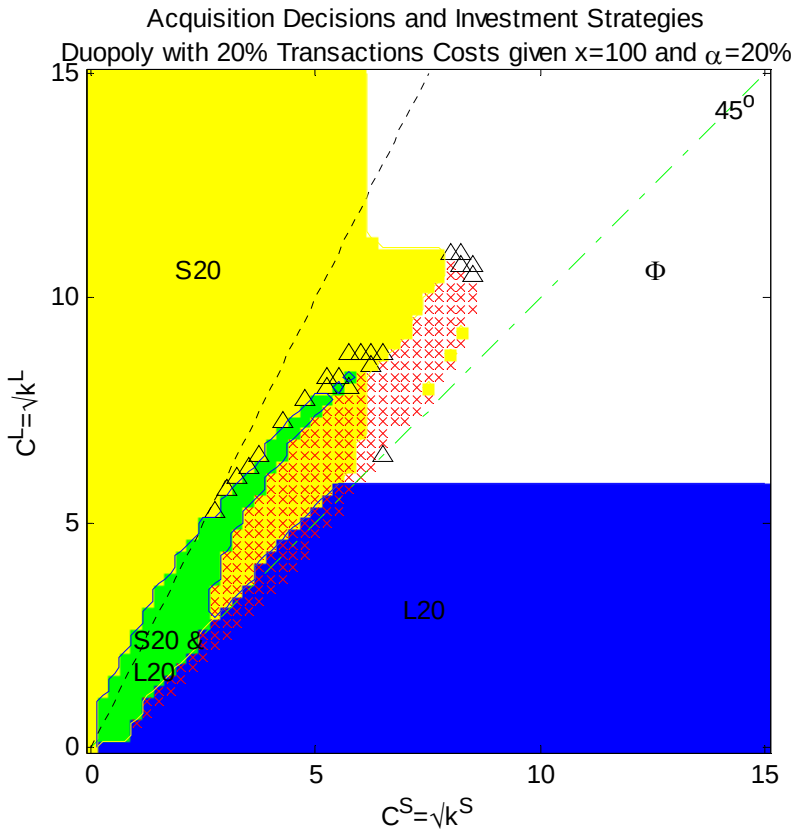


Figure 3C

Investment Strategies with 20% Transactions Costs

X(t) = 100, α = 40%

The horizontal axis is the small firm's cost-to-completion. The vertical axis is the large firm's cost-to-completion. The light gray (yellow) area is where the small firm invests and the large firm does not (area S20). The dark (blue) area is where the large firm invests and the small firm does not (area L20). The medium gray (green) area shows where both firms invest (area S20 & L20). The un-shaded white area is where neither firm invests (area Φ). The dash-dot (green) line is the equal-cost-to-completion line (45-degree line) and the dotted (black) line is the equal-time-to-completion line with a slop of 2. The symbols "X" represents a successful offer for 100% of the small firm.

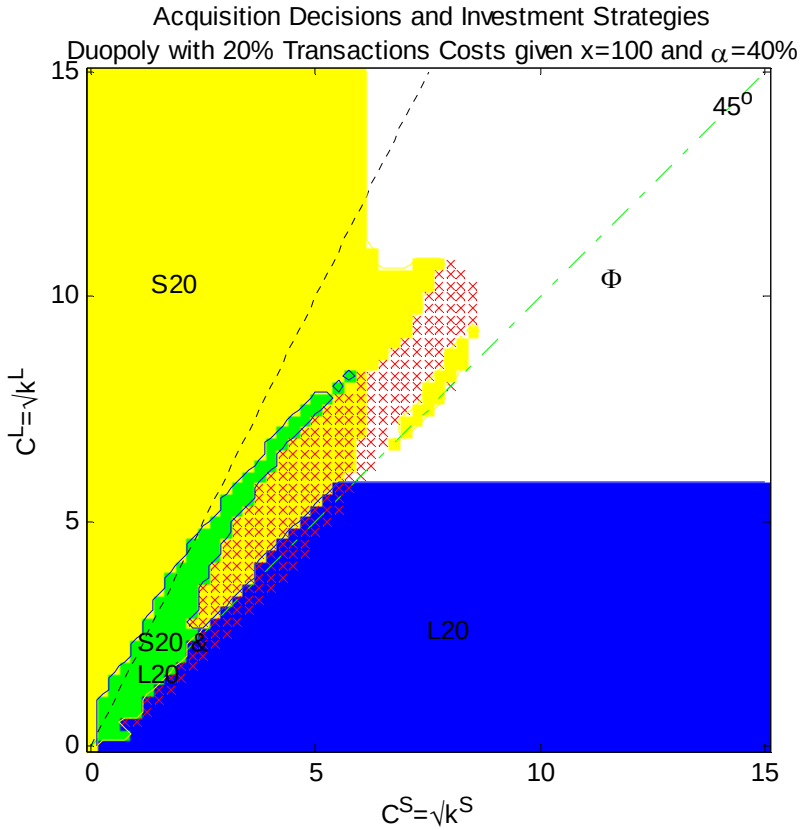


Figure 4

A Comparison of Investment Strategies:

20% Transactions Costs versus Joint Monopoly

X(t) = 100

The horizontal axis is the small firm's cost-to-completion. The vertical axis is the large firm's cost-to-completion. The light gray (orange) area is where the small firm invests in both joint monopoly and duopoly and the large firm does not invest (area SM & S20). The lighter gray (yellow) area is where the small firm invests only in duopoly but not in the joint monopoly and the large firm does not invest (area S20). The dark (blue) area is where the large firm invests in both joint monopoly and duopoly and the small firm does not invest (area LM & L20). The lighter medium gray (yellow-green) area shows where both firms invest in duopoly and only the small firm invests in joint monopoly (area SM & S20 & L20). The darker medium gray (blue-green) area shows where both firms invest in duopoly and only the large firm invests in joint monopoly (area LM & S20 & L20).The un-shaded white area is where neither firm invests (area Φ). The dash-dot (green) line is the equal-cost-to-completion line (45-degree line) and the dotted (black) line is the equal-time-to-completion line with a slop of 2. The symbols "X," "O," and "Δ" indicate bargaining outcomes. An "O" represents a successful offer for 20% of the small; a "Δ" represents a successful offer for 40% of the small firm; and an "X" represents a successful offer for 100% of the small firm.

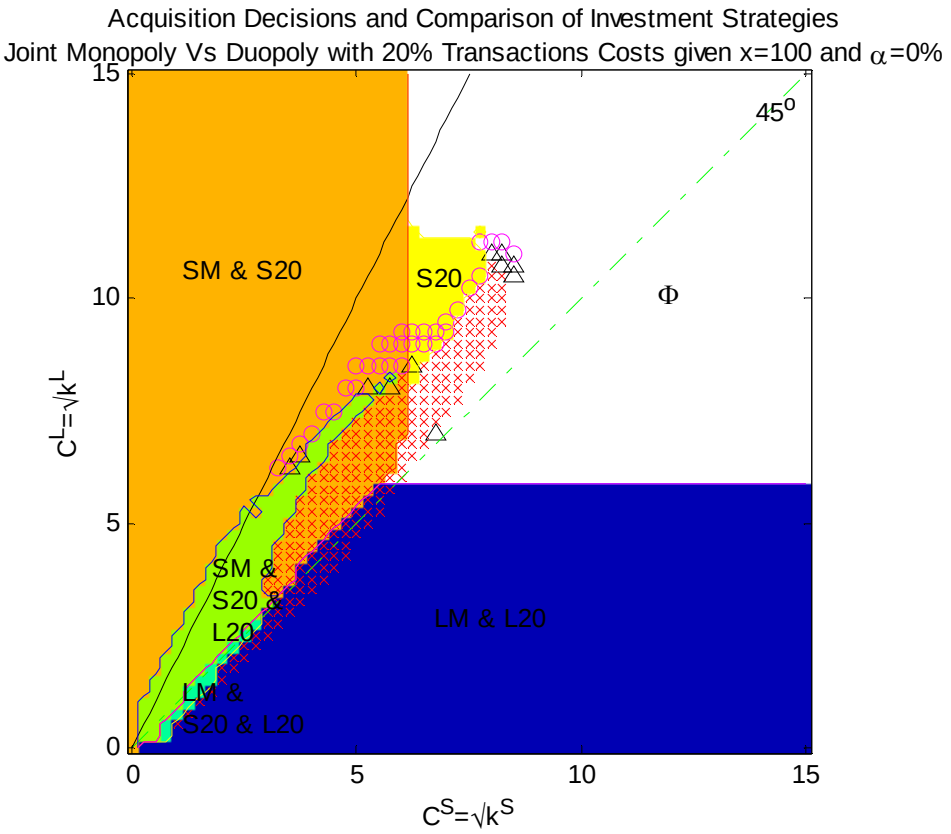


Figure 5

Investment Strategies with 20% Transactions Costs

And a Larger Value of X(t)

X(t) = 151

The horizontal axis is the small firm's cost-to-completion. The vertical axis is the large firm's cost-to-completion. The light gray (orange) area is where the small firm invests in both joint monopoly and duopoly and the large firm does not invest (area SM & S20). The lighter gray (yellow) area is where the small firm invests only in duopoly but not in the joint monopoly and the large firm does not invest (area S20). The dark (blue) area is where the large firm invests in both joint monopoly and duopoly and the small firm does not invest (area LM & L20). The lighter medium gray (yellow-green) area shows where both firms invest in both joint monopoly and duopoly (area SM & LM & S20 & L20). The medium gray (green) area shows where both firms invest in duopoly and only the small firm invests in joint monopoly (area SM & S20 & L20). The darker medium gray (blue-green) area shows where both firms invest in duopoly and only the large firm invests in joint monopoly (area LM & S20 & L20).The un-shaded white area is where neither firm invests (area Φ). The dash-dot (green) line is the equal-cost-to-completion line (45-degree line) and the dotted (black) line is the equal-time-to-completion line with a slop of 2. The symbols "X," "O," and "Δ" indicate bargaining outcomes. An "O" represents a successful offer for 20% of the small; a "Δ" represents a successful offer for 40% of the small firm; and an "X" represents a successful offer for 100% of the small firm.

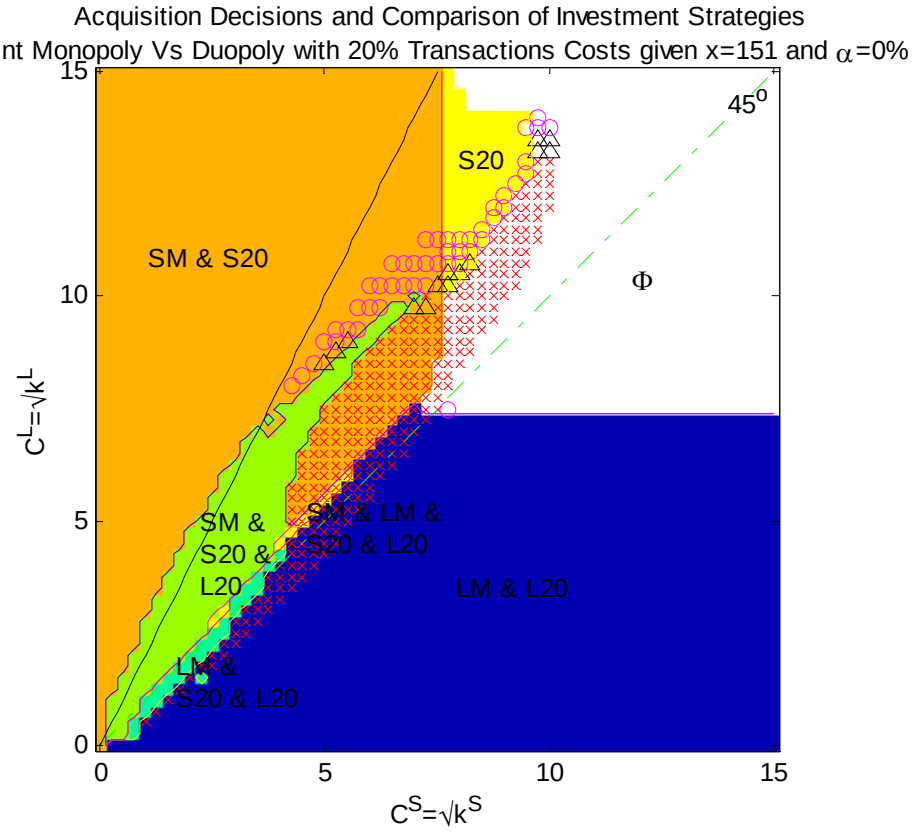


Figure 6A

Value of Large Firm

20% Transactions Costs

X(t) = 100, α = 0%

The horizontal axis is the small firm's cost-to-completion. The vertical axis is the large firm's cost-to-completion. The dash-dot (green) line is the equal-cost-to-completion line (45-degree line) and the dotted (black) line is the equal-time-to-completion line with a slop of 2. The intervals between the iso-value lines represent factors of 2 in value.

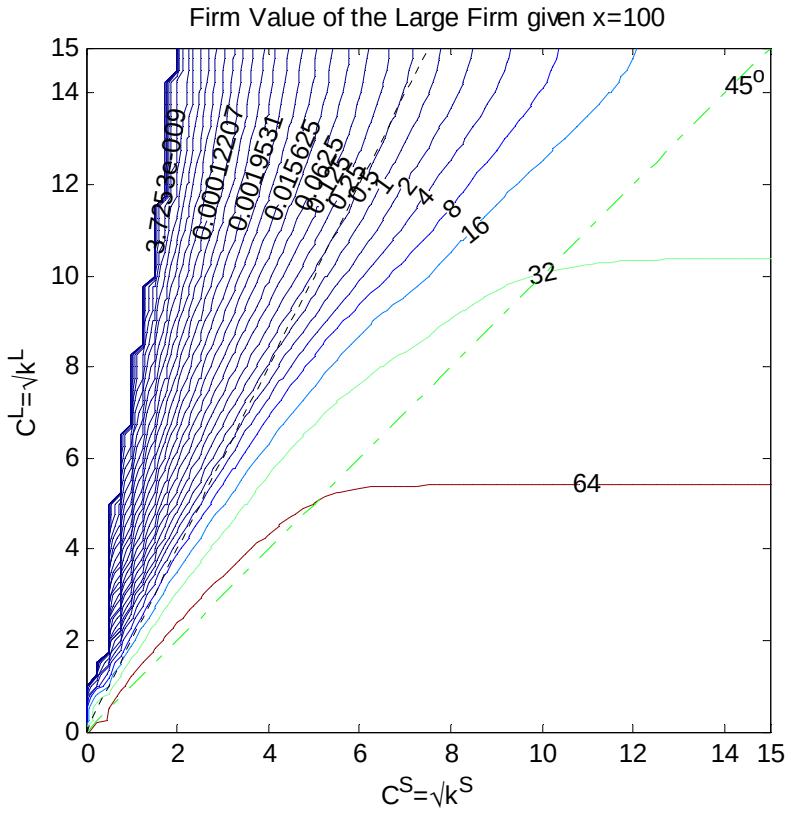


Figure 6B

Value of Small Firm

20% Transactions Costs

X(t) = 100, α = 0%

The horizontal axis is the small firm's cost-to-completion. The vertical axis is the large firm's cost-to-completion. The dash-dot (green) line is the equal-cost-to-completion line (45-degree line) and the dotted (black) line is the equal-time-to-completion line with a slop of 2. The intervals between the iso-value lines represent factors of 2 in value.

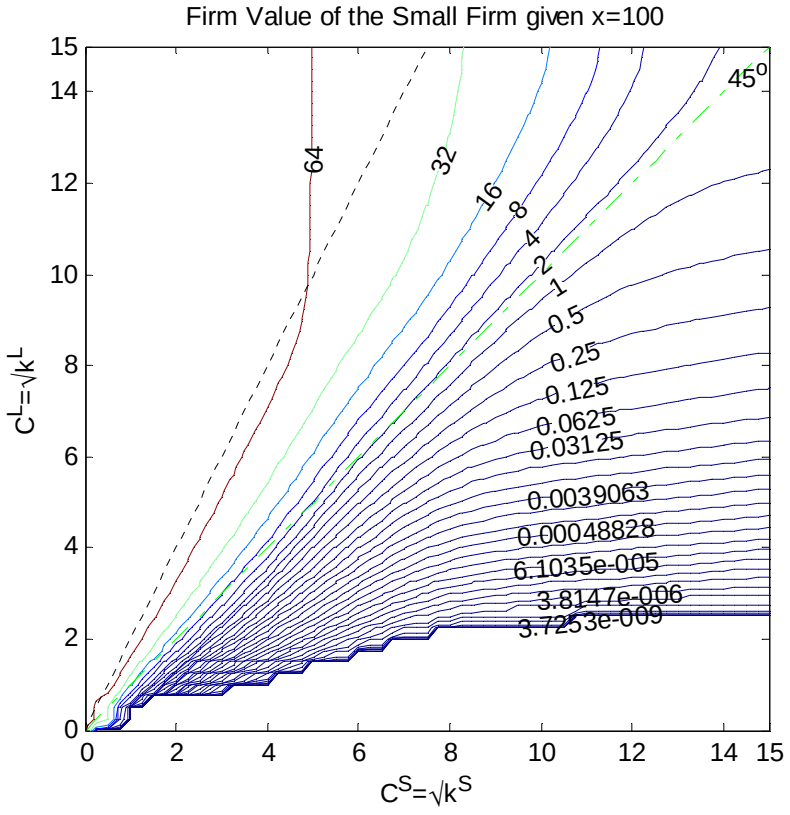


Figure 7

Value Dissipation:

20% Transactions Costs versus Joint Monopoly

X(t) = 100

Expected value dissipation is measured as the percentage difference between the combined values of the two firms under the joint monopoly and the duopoly patent race with 20% transactions costs.

The horizontal axis is the small firm's cost-to-completion. The vertical axis is the large firm's cost-to-completion. The dash-dot (green) line is the equal-cost-to-completion line (45-degree line) and the dotted (black) line is the equal-time-to-completion line with a slop of 2.

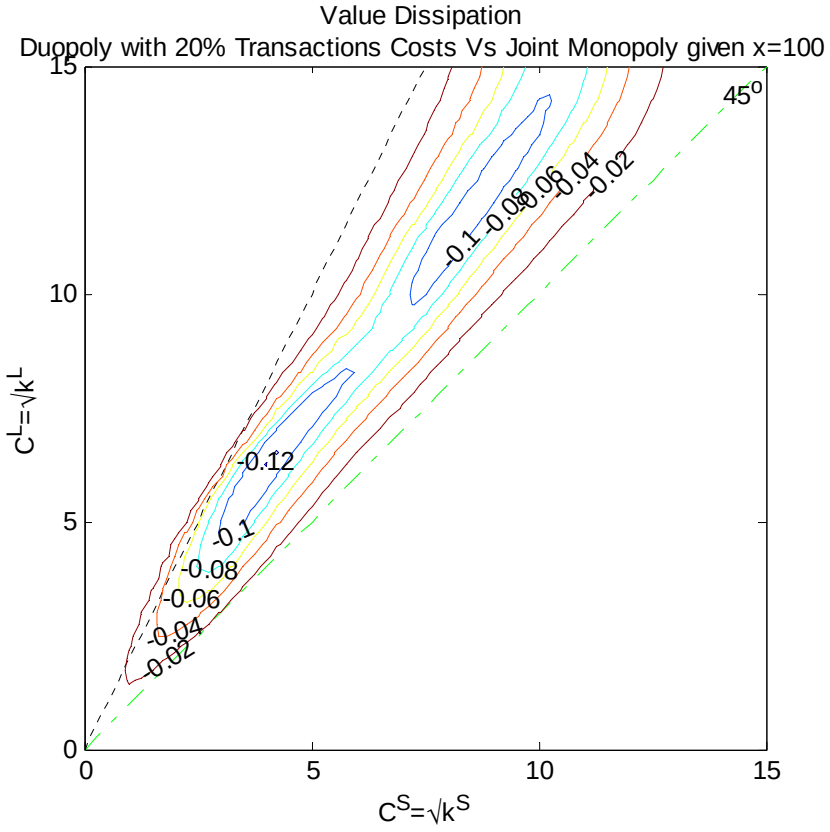


Figure 8

Value Dissipation:

Infinite Transactions Costs versus Joint Monopoly

X(t) = 100

Expected value dissipation is measured as the percentage difference between the combined values of the two firms under the joint monopoly and the duopoly patent race with infinite transactions costs.

The horizontal axis is the small firm's cost-to-completion. The vertical axis is the large firm's cost-to-completion. The dash-dot (green) line is the equal-cost-to-completion line (45-degree line) and the dotted (black) line is the equal-time-to-completion line with a slop of 2.

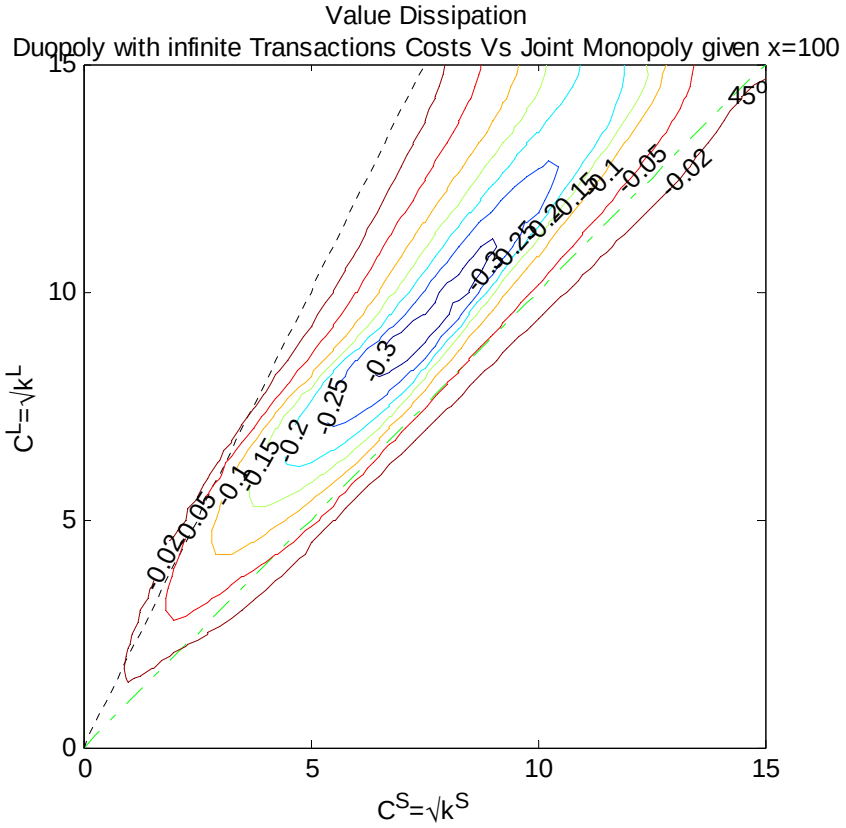


Figure 9

Large Firm's Beta

Assuming Patent Beta = 1

20% Transactions Costs

X(t) = 100

The horizontal axis is the small firm's cost-to-completion. The vertical axis is the large firm's cost-to-completion. The dash-dot (green) line is the equal-cost-to-completion line (45-degree line) and the dotted (black) line is the equal-time-to-completion line with a slop of 2.

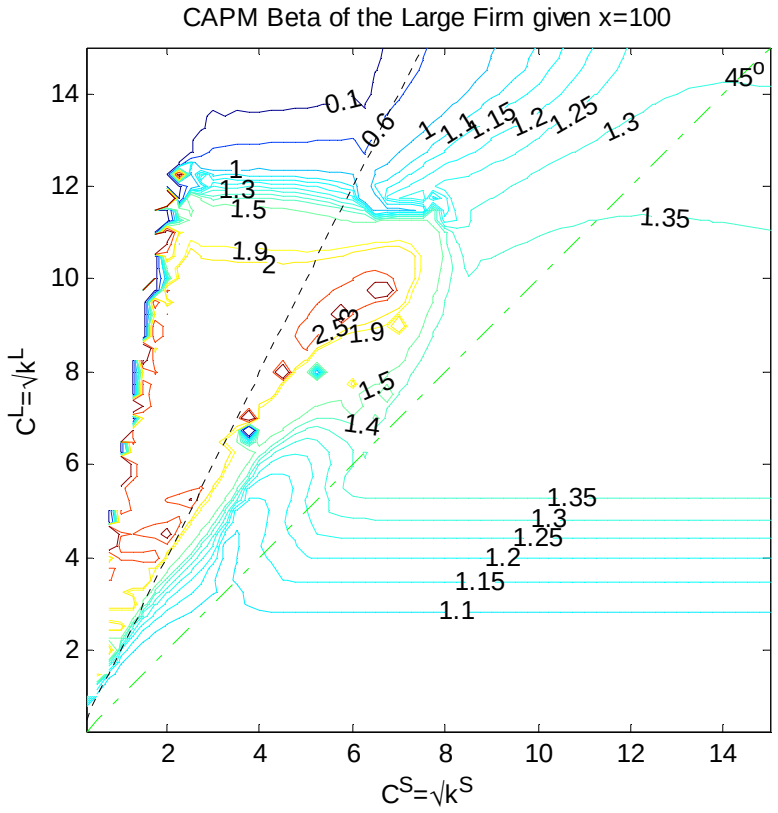


Figure 10

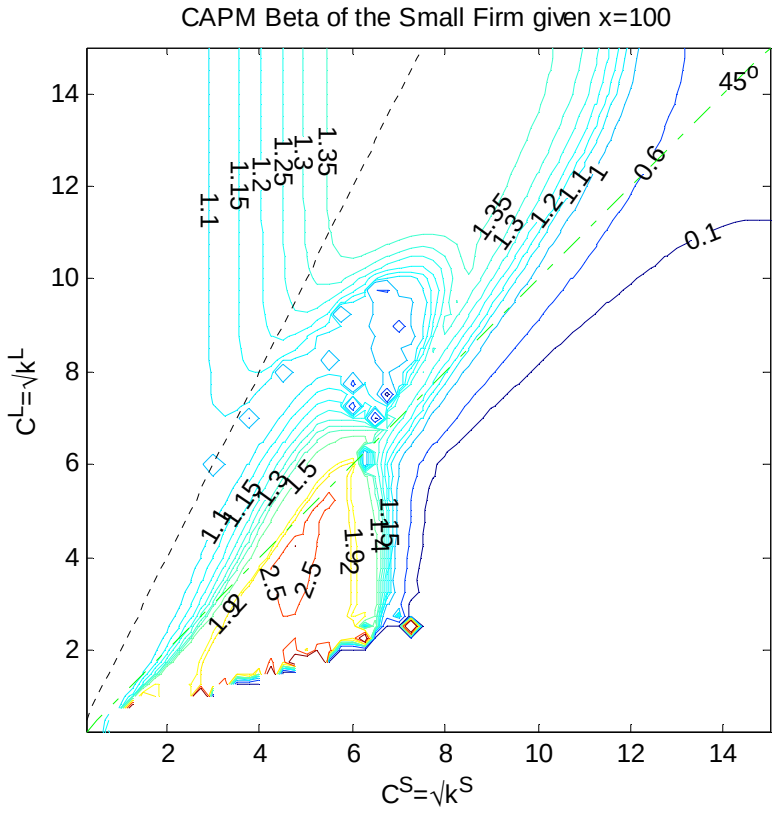
Small Firm's Beta

Assuming Patent Beta = 1

20% Transactions Costs

X(t) = 100

The horizontal axis is the small firm's cost-to-completion. The vertical axis is the large firm's cost-to-completion. The dash-dot (green) line is the equal-cost-to-completion line (45-degree line) and the dotted (black) line is the equal-time-to-completion line with a slop of 2.



[1] The general process of expected cost-to-completion in Pindyck (1993), allowing both technological uncertainty and input cost uncertainty, is $dk = -Idt + \theta k(\frac{I}{k})^{\phi} dz$, $0 \le \phi \le \frac{1}{2}$. The case of ϕ =0 corresponds to input cost uncertainty. The version we use corresponds to $\phi = \frac{1}{2}$.

[2] We also replaced the model's assumption that the payoffs are zero at time T with a set of liquidation values designed to approximate in an ad hoc manner what might happen if the game were to continue. Specifically, we first calculate the expected profits of both firms under the assumption that the firm is a monopolist and there is no more volatility in either the patent value or the cost-to-completion. If one of the firms has a positive NPV and the other a negative NPV, then we give the firm with the positive NPV this NPV value as it liquidation value and give the other firm a zero liquidation value. If both firms have a negative NPV value, both firms get zero. If both firms have a positive NPV value then we give each firm this NPV value multiplied by the ratio of the other firm's expected time to completion divided by the sum of both firms' expected time to completion, i.e. we make the "probability" of winning the NPV value inversely proportion to the expected time to completion. We do not believe that this approximation has an economically significant effect on the results.

[3] The small firm invests when its cost-to-completion is somewhat higher than the large firm's when both firm's cost-to-completion is very close to zero. Intuitively, this probably reflects the fact that when the both firms have a cost-to-completion of about 5, this represents a two-week investment for the large firm but a one-month investment for the small firm. If the large firm invested at its assumed maximum rate of 10 for the minimum period of one month, it would be inefficiently investing much more than necessary amount, which is expected to be 5. It saves resources to let the small firm invest only 5. Obviously, this result is an artifact of our assumption that the decision making interval Δt is one month.

[4] This is consistent with the empirical evidence found by Weston et al. (1990), Maksimovic and Philips (2001) and Jovanovic and Rousseau (2001).

[5] Miltersen and Schwarz (2003) and Moscarini and Squintani (2003) also provide a welfare analysis.