Infrastructure investments and real options

Kjell Arne Brekke

We consider investment in energy transmission capacity between two region with uncertain demand in both regions. We show that even though an investor may learn about future price differences by waiting to invest, increased uncertainty also increases the value of the investment, and dominates the learning effect. As a consequence it is optimal to invest earlier the higher the uncertainty.

1 Introduction

Real option models demonstrate that uncertainty and learning often makes it optimal to delay irreversible investments. Early theory demonstrated that even when an investment has positive expected net present value it may be optimal to delay investment, as the stochastic variables may develop more favourably in the future.

But uncertainty may affect different types of investment very differently. Some investments are simply undertaken to enhance the ability to meet uncertainty. An example is people investing in a fire extinguisher in their home in case of fire. If the time of the fire was known, the optimal solution would be to rent one for that day and stop the fire at the outset. Or if we only knew in which homes fires would start this year, we could limit the investment to those homes. When we don't have this information it is optimal to install the equipment everywhere in case fire. In this case it is the uncertainty that justifies the investment, uncertainty is not an argument to dealy.

In this paper we consider investment in infrastructure, and transmission capacity in particular. We argue that this shares some which shares some similar features. If energy is produced by a known technology so the unit cost is the same in all regions, and we knew the exact demand in advance, we could build to demand. Daily variation in demand also tend to be similar across regions. But there are also fluctuations that are asymmetric and these are often due to unpredicted events. in demand also tend to be similar o But energy also has to be transported; Electricity require a transmission line from the producer to the consumer. Gas is often transported in pipelines. While transmission is needed also without uncertainty, it also has an element of the fire extinguisher. When it is cloudy and little sun power generated, or on cold days with high demand for heating, the electricity can be transported from places with excess supply.

In the following we will first discuss some simple theoretical model highlighting this element of infrastructure investment, that is, the benefits of infrastructure as a mean to meet uncertainty. The model is set up to focus on this effect, and leaves open the question as to how important this will be in reality. To address that question we present results from a general equilibrium model of the energy market in Western Europe, where we have both investment in capacity and infrastructure, and where there is uncertainty.

2. A simple model of transmission

We first consider the simplest possible model of transmission, simple enough to allow an analytical solution.

Let

$$X_t = x + \sigma B_t$$

denote the price difference between two regions, and consider the investment in a unit of transmission. We assume that the investment is sufficiently small that the price difference can be considered exogenous. (Or the X may be thought of as the price difference when the transmission is in place.) The transmission line can charge a price equal to the absolute value of the price difference, then the value of a unit trasmission line will be.

$$f(x) = E\int_0^{\infty} |X_t| e^{-rt} dt$$

The deterministic case is straightforward, as the difference will be x forever, thus

$$f_{\sigma=0}(x) = \frac{x}{r}$$

With a unit investment cost C it is optimal to invest if and only if

$$C < \frac{x}{r} \text{ that is } \quad x > rC$$

With uncertainty the value of an investment is less straightforward.

Theorem

$$f(x) = E\int_0^{\infty} |X_t| e^{-rt} dt = \frac{\sigma}{r^{3/2}}\left[|x| \frac{\sqrt{2r}}{\sigma} + exp(\frac{|x|\sqrt{2\pi}}{\sigma}) \right]$$

$$= \sqrt{2}\frac{|x|}{r} + \frac{\sigma}{r^{3/2}} exp(\frac{|x|\sqrt{2\pi}}{\sigma}) > f_{\sigma=0}(x) = \frac{|x|}{r}$$

Proof: See appendix.

To find the optimal investment criterion we next need to solve the optimal stopping problem:

$$\max_{\tau} E\left(\left(f(X_\tau) - C\right)e^{-r\tau}\right)$$

Let D denote the continuation area for this optimal stopping problem, then

Theorem $D \subset [0, rC)$

Proof: See appendix.

The result implies that the fact that $f(x) > f_{\sigma=0}(x)$ dominates the waiting to learn. That is, although there still is an argument for delaying investment due to learning when uncertainty increases, uncertainty also increases the value of the investment, and this effect dominates.

3 Three period information in two period model.

Representing the value of transmission as an exogenous price difference is admittedly a considerable simplification, still the model above is challenging to solve. To be able to solve more realistic models, I revert to a numerical model. I will here present a model that I will interpret as essentially a three period model with uncertainty, although it is formally a two period model. To explain the idea, consider first how a static equilibrium model may be given a two period interpretation. Essential to this construction is the presence of uncertainty, and we represent uncertainty in the simplest way possible, as two possible states of the world. And we also consider a very simple economy, with firms investing in production capacity to produce a homogenous good consumed by consumers.

We further assume that different decision are based on different types of information; When firms invest they do not know the state of the world, but when they choose how much to produce given their capital and when consumers decide on their demand, the state of the world is known. One reason why the actors use different information is that their decisions have to be taken at different points in time. Firms invest in capacity ahead of time, and thus do not yet know the state of the world. When the goods are traded the state of the world is known. Firms have perfect foresight and know the prices that will materialize for each possible state of the world.

The example with two scenarios is illustrated in Figure 1, note that while there are two decision times, there is only one market clearing:

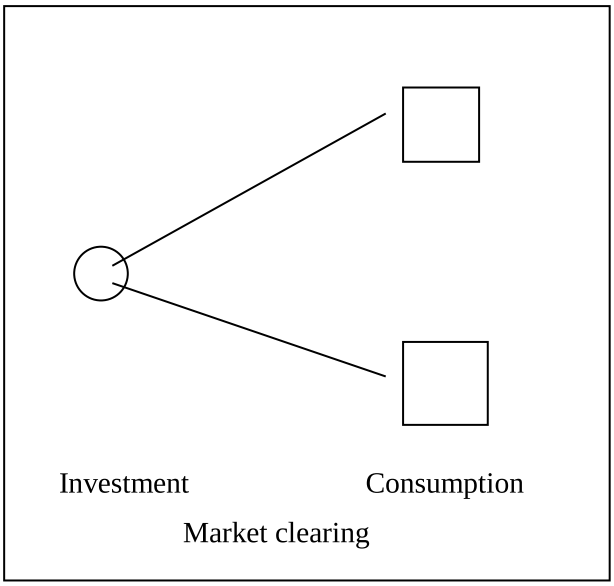

Figure 1: The information structure within one period. Firms decide investment before uncertainty is resolved while consumers decide on demand after observing the scenario. There is market clearing within the period.

Now, to make a model with three time periods we need markets to clear twice. First firms invest, then consumers make their demand based on more information than what was available at the time of investment. The investment and consumption decisions are tied together in one market clearing condition. While consumers demand goods, the firms also demand goods for investment. This is for the capital that will be available when consumers demand product in period 2, and are tied together in a second market clearing. This is illustrated in figure 2 below.

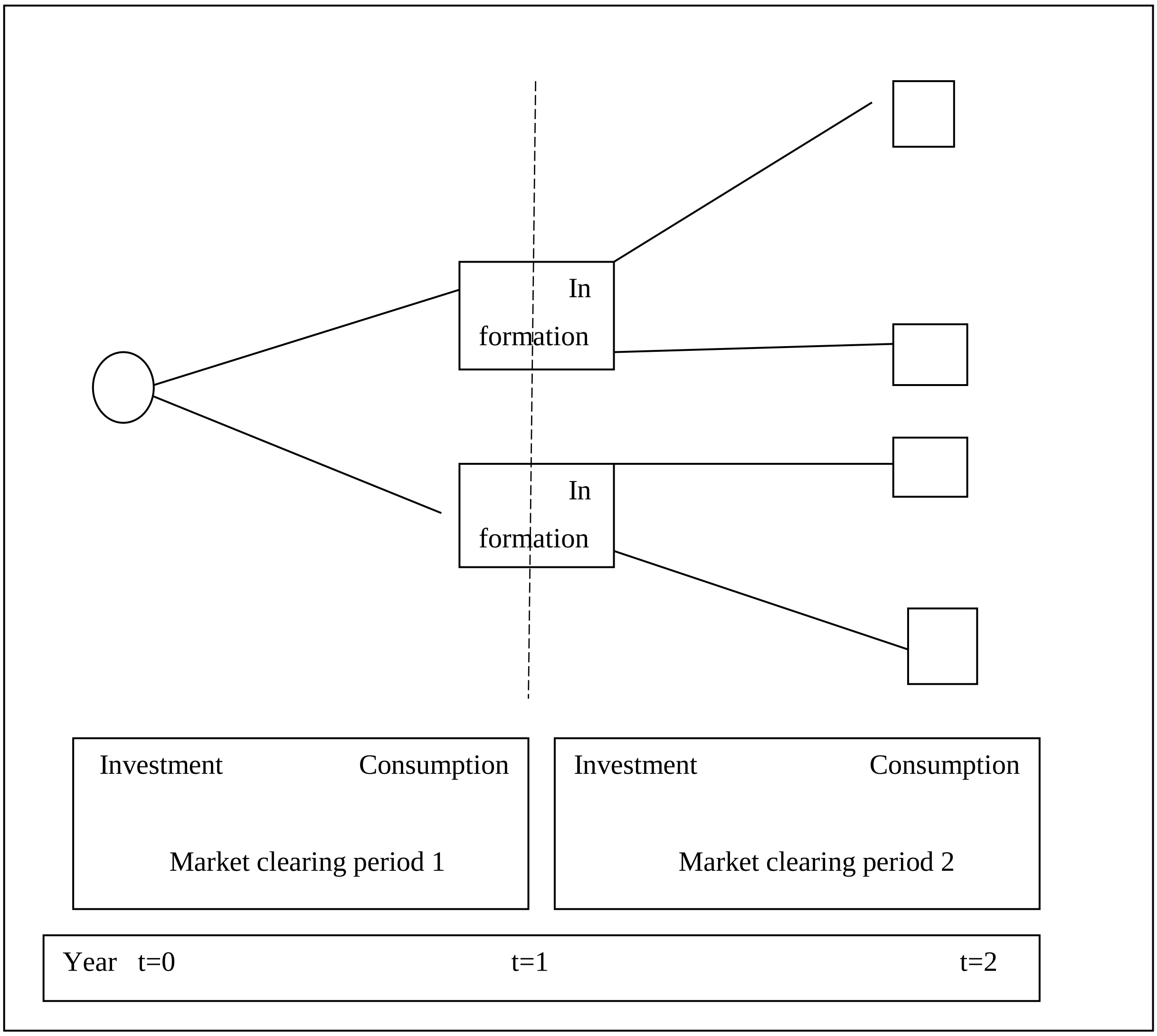

Figure 2: With two market clearing periods. Note that consumption in period 1 and investment for period 2 is made at the same time, but still belong to two different market clearing periods. The investment at t=1 will only affect capital t=2 and hence not influence market clearing for t=1.

For simplicity we will use period to denote the time *t*, and the subscript *t* will denote the market clearing period, thus the capital that is installed at *t-1* using the information available at that time, will still be denoted K_t as it determines the supply in year t and hence is set to clear markets in year t.

4. Energy demand in two regions, an example

We now consider a simple example with energy demand in two regions. The main purpose of this section is to show how the general idea in Rockafellar and Wets (1991) to find the solution of a stochastic problem for a *single* actor can be used to transform a deterministic *equilibrium* model to a stochastic equilibrium model. To this end we first set up a simple deterministic equilibrium model for a two-regions electricity market. We then discuss how to extend this to add uncertainty and then two market clearing periods with an information structure and Bayesian learning. While the model is simple enough to be transparent, the basic formulation is similar to the numerical LIBEMOD model, see Section 3, or more generally to a CGE model.

We consider a market where there is only one energy good and only one technology, and there is uncertainty about the demand in each region in each period. To highlight the importance of real options as well as the value of flexibility under uncertainty, it turns out to be convenient to make some rather unrealistic assumption. These assumptions make the deterministic case very simple and the impact of uncertainty very transparent. The key assumption is that there is constant unit capital cost in energy production and in transmission, and neither production nor transmission has any variable cost. In addition we assume that the interest rate is zero and there is no depreciation.

The simplicity of the model makes the impact of uncertainty more transparent. In the model there is one demand parameter in each region. We show that in the case of no uncertainty, there will be no investment in transmission between the two regions, and hence no trade in electricity. We demonstrate that this result does *not* depend on the values of the demand parameters. In contrast, with stochastic demand parameters there will be investment in transmission in the stochastic equilibrium. This capacity will be utilized if the realizations of the two stochastic demand parameters differ, which means that one region has higher demand for electricity than the other.

We consider a very simple model of electricity production and consumption in two regions. In each region i, $i=1,2$, there is a representative producer i. Initially a producer has no production capacity, but he can invest in capacity at a constant unit cost c. There is no cost of operating the capacity, so production (x_i) will equal capacity (K_i). There is also a transmission company which may invest in a transmission line between the two regions. Initially, there is no transmission capacity, but the transmission company can invest in capacity (κ) at a constant unit cost (η).

In each region there is one representative consumer. His gross utility of consuming electricity (x_i) is $\theta_i \ln x_i$ where θ_i is the utility parameter of the representative consumer in market *i*. The consumers in region *i* maximize his net utility $\theta_i \ln x_i - p_i x_i$.

We will consider a stochastic model where markets clear in two periods. But to introduce one idea at the time, we start first with the static model, just to describe the standard complementary slackness conditions for such a problem. Then we add uncertainty and discuss how this will change the complementary slackness conditions. Finally, we extend the model to two periods with learning.

The model has two periods (but we neglect discounting between the periods). In period 1 the actors may invest in capacity. In the beginning of period 2, the new capacities are available, and there is production and consumption like in any standard deterministic model. We assume that the electricity producer in market *i* can sell electricity in this market only, whereas the transmission company can buy electricity in one market and sell this electricity in the other market.

In period 1 the electricity producer knows that in the next period the price of electricity will be p_i. The electricity producer in region *i* will therefore maximize $(p_i - c)K_i$. The first-order condition of this problem is

$$p_i \leq c \perp K_i \geq 0,$$

that is, $p_i = c$ if it is optimal to invest in production capacity $(K_i > 0)$.

In period 1 the transmission company determines its investment in transmission capacity. Let z_1 be electricity bought in market 1 by the transmission company. This quantity is exported to market 2 and then sold in market 2 by the transmission company. Correspondingly, let z_2 be electricity bought in market 2 by the transmission company and then exported to market to 1. Profits of the transmission company are then $(p_1 - p_2)z_2 + (p_2 - p_1)z_1 - c_T K_T$. Of course, exports cannot exceed the transmission capacity, and hence in period 2 the transmission company faces the following two restrictions:

$$z_1 - z_2 \leq K_T \perp \mu_1 \geq 0$$

$$z_2 - z_1 \leq K_T \perp \mu_2 \geq 0$$

where μ_i is the shadow price associated with the constraint on the amount of imports to market i. Maximizing profits with respect to transmission capacity and export quantities, the first-order conditions are:

$$\begin{aligned} &\mu_1 + \mu_2 \leq c_T \perp K_T \geq 0 \\ &p_2 - p_1 \leq \mu_1 - \mu_2 \perp z_1 \geq 0 \\ &p_1 - p_2 \leq \mu_2 - \mu_1 \perp z_2 \geq 0. \end{aligned}$$

The last two conditions imply that $p_1 - p_2 = \mu_2 - \mu_1$.

In period 2, the electricity producers will use their entire production capacity because there are no costs of production, whereas consumers will buy electricity. The first-order condition for the consumers is:

$$x_i = \frac{\theta_i}{p_i}$$

Finally, the market clearing conditions are:

$$\begin{aligned} K_1 + z_2 &= x_1 \\ K_2 + z_1 &= x_2. \end{aligned}$$

The market equilibrium.

The market equilibrium in this case is obvious. For prices approaching zero, demand is infinite. Hence, there will be production of electricity, which requires investment in production capacity in period 1; $K_i > 0$. With an interior solution for production capacity, we have $p_i = c$. Therefore, prices are equal between the two markets, and it will not be profitable to invest in transmission capacity to export electricity between the two regions. Hence, $K_T = 0$. Technically, $p_1 = p_2$ and $\mu_1 = \mu_2 = 0$. Thus the equilibrium is characterized as follows:

$$K_i = x_i = \frac{\theta_i}{c}$$
$$p_i = c$$
$$K_T = 0.$$

Note that no matter the value of (θ_1, θ_2), the optimal solution is always $K_T = 0$. While this may seem like a very robust result, as demonstrated below $K_T = 0$ is not the equilibrium in the stochastic model.

4.2 Modelling uncertainty

We now transform the deterministic model to a stochastic model by letting the demand parameters be random. Suppose that there are two possible values of $\theta_i \in \{\theta_L, \theta_H\}$ for each market. This makes four possible combinations:

$$(\theta_1, \theta_2) \in \{(\theta_L, \theta_L), (\theta_L, \theta_H), (\theta_H, \theta_L), (\theta_H, \theta_H)\}.$$

We denote each of the four outcomes as a scenario *s*, $s \in \{1,2,3,4\} = S$. The probability that scenario *s* materializes is q_s where $\sum_{s=1}^{4} q_s = 1$.

With uncertainty we need to specify the information available to the decision maker at the time of making the decision. We will assume that actors learn the true scenario in the beginning of period 2. Hence, investment decisions are taken under uncertainty (in period 1) whereas trade, consumption and production decisions are taken after the uncertainty has been resolved (in period 2).

Let us now consider the maximization problem of the electricity producer *i*. The straight forward formulation would be to maximize

$$\sum_{s=1}^{4} q_s (p_{is} - c) K_i = \left(Ep_i - c\right) K_i.$$

This would give the first-order condition

$$Ep_i \leq c \perp K_i \geq 0.$$

While this would of course work, we want to use an alternative formulation that makes the changes in the model as small as possible when we move from the deterministic case to the stochastic. To this end we employ a model formulation from Rockafellar and Wets (1991). To explain this approach, suppose we simultaneously solve the deterministic model for each of the four scenarios. This would simply amount to specify the first-order conditions four times, once of each scenario. We could do this by adding an index *s* for the scenarios to each variable. Thus the first-order condition for the electricity producers would be

$$p_{is} \leq c \perp K_{is} \geq 0.$$

This condition has to be satisfied for each electricity producer *i* and each scenario *s*. Such a simultaneous solution would only require an extra scenario index on the variables relative to the deterministic case. However, this will *not* be the solution to the stochastic problem: with no link between the scenarios, the solution would be as above. In particular, production capacity will be $K_{is} = \theta_{is} / c$, and a producer will have a different capital stock for each scenario. But this does not make sense: because capital has to be chosen *before* the scenario is revealed, the capital stock must be the *same* in all scenarios. Or to put it differently: a solution with $K_{i1} \neq K_{i4}$ is impossible to implement if capital is chosen before the firm knows the scenario. We therefore have to impose the condition that $K_{is} = K_{is'}$ for all $s, s' \in S$. Below this restriction is specified as $K_{is} = K_{i4}$ for $s = 1,2,3,$ and it is referred to as the implementability constraint.

The discussion above implies that under uncertainty the investor cannot maximize profit for each individual scenario separately. With uncertainty, the aim of the electricity producer is to find the production capacity in each scenario (K_{is}) that solves the following problem:

$$\max \sum_{s=1}^{4} q_s (p_{is} - c) K_{is} \quad \text{subject to}$$
$$K_{is} = K_{i4} \text{ for } s = 1, 2, 3.$$

The first-order condition is

$$q_s p_{is} = q_s c + \mu_{is}'$$
$$\sum \mu_{is}' = 0$$

where μ' is the shadow price of the implementability constraint. The second condition follows from the fact that K_{i4} enters all side constraints. Now, defining $\mu_{is} = \frac{\mu_{is}'}{q_s}$ - the probability adjusted shadow prices - the first-order conditions can be rewritten as

$$p_{is} = c + \mu_{is}$$
$$E\mu_i = \sum q_s \mu_{is} = 0.$$

Compared to the first-order condition in the deterministic case ($p_i = c$), we have only added the (probability adjusted) shadow price of the implementability constraint (μ_{is}) and indexed all variables by *s*. In addition, we have a condition for the shadow price; its expected value should be zero.

The first-order conditions for investment in transmission capacity are changed in the same way as the conditions for investment in electricity production capacity; the first-order condition in the deterministic model is extended by an additive term ν_s, which is the (probability adjusted) shadow price of the impelemtability constraint $K_{Ts} = K_{T4}$, $s = 1,2,3$, and all variables are indexed by s. In addition, the expected value of the shadow price ν is zero:

$$\mu_{1s} + \mu_{2s} \le c_T + \nu_s \perp K_T \ge 0$$
$$E\nu = 0.$$

Actual transmission (trade) and consumption is decided in period 2, that is, after the scenario is known. Thus, to characterize these decisions no implementability constraint is needed; the conditions are therefore similar to the ones for the deterministic case, except that all variables are indexed by *s*. The first-order conditions for trade are thus

$$p_{2s} - p_{1s} \le \mu_{1s} - \mu_{2s} \perp z_{1s} \ge 0$$
$$p_{1s} - p_{2s} \le \mu_{2s} - \mu_{1s} \perp z_{2s} \ge 0.$$

whereas the first-order condition for the consumers is

$$x_{is} = \left(\frac{\theta_{is}}{p_{is}} \right)^2 .$$

Finally, market clearing requires

$$K_{1s} + z_{2s} = x_{1s}$$
$$K_{2s} + z_{1s} = x_{2s}.$$

Again, the only difference to the deterministic case is that all variables have been indexed by *s*.

4.4 The two period extension

Adding another time period, we assume that the demand for the first period is as above; two possible values of $\theta_{i1} \in \{\theta_l, \theta_h\}$ for each market. In the second period there are also two possible values $\theta_{i2} \in \{\theta_L, \theta_H\}$. This makes four possible combinations for each period, and hence 16 possible paths through the two periods. E.g. if market A start out low and becomes High in the second period, while B stay high throughout, the path would be: $(\theta_{A1}, \theta_{B1}), (\theta_{A2}, \theta_{B2}) = (\theta_l, \theta_h), (\theta_H, \theta_H)$. Each path is one possible scenario. We denote each of the four outcomes as a scenario *s*, $s \in \{1, 2, 3, ..., 16\} = S$. The probability that scenario *s* materializes is q_s where $\sum_{s=1}^{16} q_s = 1$.

Note that in period 1, even if we observe that demand (θ_l, θ_h) materializes, we cannot yet know if the scenario is $s' = (\theta_l, \theta_h), (\theta_H, \theta_H)$ or some of the other three possible realizations of demand in the second period. Let $S_{(\theta_l, \theta_h)}$ denote the set of all scenarios starting with (θ_l, θ_h). We denote the set $S_{(\theta_l, \theta_h)}$ an information set, as the agents know that the true scenario is in this set once (θ_l, θ_h) is observed. Note that there are four such information set.

Let *s'* be a scenario starting with (θ_l, θ_h). The probability that *s'* materializes given that we have observed (θ_l, θ_h) is then

$$\hat{q}_{s'} = \frac{q_{s'}}{\sum_{s \in S_{(\theta_l, \theta_h)}} q_s}$$

As each scenario is in only one information set, the information set in the denominator is well-defined for any scenario.

Note that the investment in the first period last both periods, and hence prices in both period enters the first order condition. In the first period there is also no new information, thus

$$p_{1is} + p_{2is} = c + \mu_{1is}$$
$$E\mu_i = \sum q_s \mu_{is} = 0.$$

For the second period, the capacity will be used to produce electricity only one period. Moreover we only require that investments must be constant over the information set, and not over all scenarios, as investors now can tell the different information sets apart. Thus

$$p_{2is} = c + \mu_{2is}$$
$$E\mu_i = \sum_{s \in S_{(\theta_{1A}, \theta_{1B})}} q_s \mu_{is} \geq 0 \perp I_{is} \geq 0 \text{ for all information sets.}$$

For the transmission company, it is similarly the case that the transmission line built the first year will last both period thus the first order condition is

$$\mu_{1As} + \mu_{1Bs} + \mu_{2As} + \mu_{2Bs} \le \eta + \nu_s \perp \kappa \ge 0$$
$$E\nu = 0.$$

While for the second period, the implementability constraint only apply within each information set:In addition, the expected value of the shadow price ν is zero:

$$\mu_{2As} + \mu_{2Bs} \le \eta + \nu_{2s} \perp \kappa \ge 0$$
$$E\nu = 0. \text{ (Within each information set)}$$

5 Results

We consider a market where there is only one energy good and only one technology, and there is uncertainty about the demand in each region in each period. The above assumptions allow us to highlight the importance of real options as well as the value of flexibility under uncertainty. These assumptions make the deterministic case very simple and the impact of uncertainty very transparent. The key assumption is that there is constant unit capital cost in energy production and in transmission, and neither production nor transmission has any variable cost. In addition we assume that the interest rate is zero and there is no depreciation.

Note that with zero interest rate and no depreciation there will – under certainty – be no reason to invest in period 2, as we noted above. It is costless to move the investment to period 1, as there is no interest rate, and get the profits from increased capacity one additional period. Thus –in the absence of uncertainty – all investments are made immediately and such that marginal investment cost equals the return over the two next periods.

The next thing to note is that with a constant unit cost in production, it will always be less costly to produce the energy in the region where it is to be consumed. Paying for transmission will only add to the cost. With full certainty there is no surprises in where the demand is, so capacity will be built to demand and the optimal solution is zero transmission capacity. Again any non-zero transmission investment is solely due to uncertainty.

Note finally that the conclusion above is true for any parameter value of demand in the two regions. Thus if we do many simulation, drawing the uncertain parameter randomly, the deterministic model will find that firms don't invest in capacity in period two and never in transmission. The average of all simulations will also be zero. Hence, a Monte Carlo or robustness analysis will not spot the error of this conclusion.

Consider a stochastic version of this model where the uncertainty of demand is as follows. In either region the demand is either high (h) or low (l) at t=1, and either High (H) or Low (L) at t=2. The two regions are independent. An if the state is h at t=1, it will be H with 80% probability at t=2. Similarly state l imply 80% probability of L next period. For t=1, high demand means demand *0.2/p* while low is *0.1/p* while in the last period High/Low is *2/p* or *1/2p* respectively. Unit cost of production capacity is 1 and unit cost of transmission capacity is 0.3. A more formalized version is given below.

With the given parameters the deterministic solution is to invest 1.4 units of production capacity in both regions in period 1. As pointed out above there will be no investment in transmission and no investment in period 2. In the stochastic solution however, investment depends on the information set at the outset of period 2. There are 4 information sets. (hh,hl,lh,ll) whith the state in region A stated first and region B second. The model is simple enough to be easily solved in an Excel spreadsheet, and the results are given below.

Full certainty and uncertainty with no learning is presented in the table below. In both cases there is no investment in period 2, but for period 1 there are some differences. The main difference is that with uncertainty, but no learning, we get a positive investment in transmission capacity. The investment in production capacity is also slightly lower than with full certainty.

	Period 1			Period 2				
	No L	Cert.	Uncert.	Cert/No L	hh	hl	lh	ll
Prod. Cap_A	1,34	1.40	**1.16**	0	**0.52**	**0.26**	**0**	**0**
Prod. Cap. B	1.34	1.40	**1.16**	0	**0.52**	**0**	**0.26**	**0**
Transmission	0.42	0	**0.22**	0	**0**	**0.19**	**0.19**	**0**

Note that while investment in production capacity amounts to 1.40 under certainty the investment drops to 1.16 in the first period. This is consistent with a real option argument. Investors delay investment waiting for more information, and only invest in the second period when the received information is favourable. The total investment over the two periods is less affected. With an expected investment the second period of 0,195, the total is 1.359, only a slight reduction from 1.40.

For transmission the picture is very different. Since transmission is useful in the face of uncertainty the investment increases in the first period to 0.22 while it would have been 0 under full certainty. It is still interesting to note that even with a further expected investment of 0.095 in period two, the total investment expected investment is 0.315, much lower than the 0.42 without learning. The real option takes a larger toll here.

6 The general case.

Above we presented a framework extending to two market clearing periods. We argued that the basic difference to the deterministic model is that variables will be indexed by scenario, we add a implementability constraint that the variable has to be the same within a information set and that a shadow price corresponding to this constraint is added to the first order condition. In this section we will consider the adjustment of the first order condition in the general case with T market clearing periods.

To extend the method to dynamic models and learning, new information may be represented as a gradual refinement of the partition of the set of scenarios. At time t the investor knows that the true scenario is in some subset S_{ti} of the set S, where $\{S_{ti}\}$ is a partition of the set S. By partition we mean that sets S_{ti} do not overlap and their union is the total set of scenarios *S*. To illustrate, suppose that there are four scenarios, and two periods. In the first period it would be natural to assume that the agents do not know which scenario that will materialize, all scenarios are still possible. But in the second period some information is revealed, but not everything, e.g., an agent may know in period 2 that the true scenario is in one of two sets, e.g., $\{1,2\}$ or $\{3,4\}$. Let $x_{t,s}$ denote the vector of decision variables in period t and scenario *s*. With the given information structure the constraint for this agent becomes $x_{1,1}=x_{1,2}=x_{1,3}=x_{1,4}$ for $t=1$, but only $x_{2,1}=x_{2,2}$ and $x_{2,3}=x_{2,4}$ for $t=2$. Note that in the last period we do *not* require that $x_{2,2}=x_{2,3}$ since the agent knows that the true scenario is either (1 or 2) or (3 or 4) and we may use this information to pursue different policies in the different cases. As agents learn, the information gets gradually more precise, thus for all t and all S_{ti} there is a $S_{t-1,j}$ such that $S_{ti} \subset S_{t-1,j}$.

For concreteness we consider the first order condition for a firm investing in a capacity

K_{ts}, where t is (market clearing) period and s is scenario. We assume as above that there is no variable cost so actual production will equal capacity and the product is sold a price p_{ts}. The maximization problem is then

$$\max \sum_{t=0}^{T} \sum_{s \in S} q_s \left(p_{ts} K_{ts} - c(K_{ts} - (1-\delta) K_{t-1,s}) \right) (1+r)^{-t}$$

with side constraints

$$K_{ts} \geq (1-\delta) K_{t-1,s}$$
$$K_{ts} = K_{ti} \text{ for } s \in S_{ti}$$

Let I_t denote the set of information sets at time t such that

$$\bigcup_{i=1}^{I_t} S_{ti} = S \text{ and } S_{ti} \cap S_{tj} = \varnothing \text{ for all } i, j \leq I_t \text{ and } i \neq j$$

It is further assumed that the information structure gets gradually finer, such that for any S_{ti} and any $t' < t$ there will be an $S_{t'j}$ such that $S_{ti} \subseteq S_{t'j}$.

Now, the Lagrangian for the problem is

$$\begin{aligned} L = & \sum_{t=0}^{T} \sum_{s \in S} q_s \left(p_{ts} K_{ts} - c(K_{ts} - (1-\delta) K_{t-1,s}) \right) (1+r)^{-t} \\ & + \sum_{t=0}^{T} \left(\sum_{i=1}^{I_t} \sum_{\substack{s \in S_{ti} \\ s \neq i}} \hat{\lambda}_{ts} \left(K_{ts} - K_{ti} \right) \right) \\ & + \sum_{t=0}^{T} \left(\sum_{i=1}^{I_t} \sum_{\substack{s \in S_{ti} \\ s \neq i}} \mu_{ti} \left(K_{ts} - (1-\delta) K_{t-1,s} \right) \right) \end{aligned}$$

With first order conditions

$$q_s (p_{ts} - c(1 - \frac{1-\delta}{1+r})) - \hat{\lambda}_{ts} - \mu_{ti} = 0$$

Note further that for the generic element $i = \overline{s}_{ti} \in S_{ti}$, there is no shadow price, but K_{ti} enters in all the other constraints, we can define a shadow price

$$\hat{\lambda}_{t\overline{s}_i} = -\sum_{\substack{s \in S_{ti} \\ s \neq i}} \hat{\lambda}_{ts}$$

to make the above equation apply to all scenarios. Note also that as a consequence

$$\sum_{s \in S_{ti}} \hat{\lambda}_{ts} = 0$$

As in Brekke et al (2014) we further divide by q_s and define $\lambda_{ts} = (\hat{\lambda}_{ts} + \mu_{ti}) / q_s$ then the equation simplifies to

$$q_s(p_{ts} - c(1 - \frac{1-\delta}{1+r})) - \lambda_{ts} = q_s(p_{ts} - c\frac{\delta}{1+r}) - \lambda_{ts} = 0$$

with

$$E\lambda_{ts} = \mu_{ti} \text{ with } \mu_{ti} = 0 \text{ if } K_{ts} - (1-\delta)K_{t-1,s} \text{ for } s \in S_{ti}$$

Thus the traditional complementary slackness condition is replaced by a scenario-specific multiplier where the standard condition applies in expectation across the relevant information sets.

Note here that the expectation in $E\lambda_{ts} = \mu_{ti}$ is derived using the original probabilities, summing over all $s \in S_{ti}$. The Bayesian updated probabilities are

$$\Pr(s \mid S_{it}) = \frac{\Pr(S_{it} \mid s)\Pr(s)}{\Pr(S_{it})} = \frac{q_s}{\sum_{s' \in S_{it}} q_{s'}}$$

Here the denominator is the same for all $s \in S_{ti}$. Thus $E(\lambda_{ts} \mid S_{ti}) = \frac{E\lambda_{ts}}{\sum_{s' \in S_{it}} q_{s'}}$, and it follows that the last condition can be restated as

$$E\lambda_{ts} = \mu_{ti} \text{ with } \mu_{ti} = 0 \text{ if } K_{ts} - (1-\delta)K_{t-1,s} \text{ for } s \in S_{ti}$$

With only an implicit re-normalization of μ_{ti}.

6.2 Furhter extensions: Risk averse firms

In the model in section 2 we assumed that the firms are risk neutral. A problem with risk adjustment in this setting is that energy prices are endogenous, and thus we cannot know before we run the model how returns to energy investments will correlate with consumption. A standard CAPM, consumption based or not, is thus hard to implement. State-price models on the other hand, are easy to implement but may be harder to calibrate. This model assumes that firms choose investments to maximize the utility of a well diversified owner. It may be argued that the manager will have an other attitude to risk, but we will return to this below.

Each scenario will represent a state of the world. In a complete Arrow-Debreu equilibrium model where consumers trade state contingent commodities, the equilibrium will reach a first best where all agents chooses optimally given their preferences which also reflect their risk aversion. (Debreu, 1959) The same real allocation emerges in general equilibrium if consumers only trade products after the true state is revealed, but are allowed to trade in state contingent claims also called Arrow securities (Arrow, 1964), or equivalently that the financial markets are complete. For an introduction se MasCollell et al (1995, Chapter 19), or Duffie (1996).

Reconsider the investment problem of the simple model above, and for the moment we ignore discounting as above. The producer maximizes expected profits subject to the latter restriction:

$$\max \sum_{s \in S} q_s \left[p_s K_s - cK_s \right]$$
$$\text{s.t. } K_s = K \text{ for all } s \in S.$$

Assuming risk neutrality, the return from a unit increase in capital is thus $\sum_{s \in S} q_s p_s$ and the cost is c. Now, suppose that there is a market for state contingent claims. A contingent claim on scenario s ensures a payment of 1$ in period 1 if scenario s materializes. The price of the contingent claim in period 0 is on a payment in scenario s is denoted Q_s, and we will refer to them as state prices. Note also that with no discounting, a contingent claim on all scenarios must be worth exactly 1, thus $\sum_{s \in S} Q_s = 1$. If an investor buy p_s claims contingent on scenario s, and does this for each scenario, that will yield a payoff of p_s in each scenario, exactly as the investment in an extra unit of capacity. The value, or market price, of such a portfolio would be $\sum_{s \in S} Q_s p_s$.

For any consumer the state prices Q_s are proportional to the probability times the consumers marginal utility of income in the relevant scenario, $Q_s = au'(c_s)q_s$ for some constant a. In scenarios with low consumption marginal utility is high, and in these scenarios $Q_s > q_s$, while in scenarios with high consumption $Q_s < q_s$. Low consumption scenarios are thus overweighed, and the high consumption scenarios are underweighted relative to probability weighing. If consumers are risk neutral, marginal utility will be the same in all scenarios and the state prices will equal the probabilities. The value $\sum_{s \in S} Q_s p_s$ would thus take the risk aversion of the owners into account. The optimal investment, that is, the one maximizing the value of the firm on the market for contingent claims is thus the one maximizing.

$$\max \sum_{s \in S} Q_s \left[p_s K_s - c K_s \right]$$
$$\text{s.t. } K_s = K \text{ for all } s \in S.$$

Note that the only difference is that the probabilities q_s have been replaced with state prices Q_s, which looks like probabilities in the sense that the prices are non-negative and $\sum_{s \in S} Q_s = \sum_{s \in S} q_s = 1$. There is thus no need to change the model; we only need to recalibrate the probabilities.

The approach readily extends to discounting, which is the standard case in the literature. Note that with discounting, a portfolio of one unit of each contingent claim will yield a certain return of 1, worth $1/(1+r)$ where r is the risk free rate of return. That is $\sum_{s \in S} Q_s = 1/(1+r)$. Normalizing the state prices to $\tilde{q}_s = Q_s(1+r)$ we get $\sum_{s \in S} \tilde{q}_s = 1$ and the value of an unit of capacity equals

$$\sum_{s \in S} Q_s p_s = \left(\sum_{s \in S} \tilde{q}_s p_s \right) / \left(1+r\right) \tag{0.1}$$

That is, after replacing the original probability with the normalized state prices, all investments can be discounted at a risk free rate of return.

Calibrating

While the approach only requires a shift in probabilities, the challenge is to calibrate the new probabilities. Consider first the case of GDP uncertainty, we may assume that aggregate consumption and GDP is perfectly correlated. In the simplest case $c_i = y_i$ and with and the state price should be proportional to marginal utility of income (or aggregate consumption). (This is sated as a Theorem in Duffie (1996), chapter 1.C). Assuming $u(c) = \frac{1}{1-\sigma} c^{1-\sigma}$, it follows that $u'(c) = c^{-\sigma}$ and the new probabilities are

$$\tilde{q}_s = \frac{q_s y_s^{-\sigma}}{\sum q_s y_s^{-\sigma}}.$$

With GDP different in the different regions of the model, it would be reasonable to assume that an investor is well diversified and hence use average GDP, weighted by population size.

Next in the case of oil price uncertainty, it may be argued that oil represent an alternative investment opportunity. Let Po_s denote the future price of oil in scenario *s* while Po_0 denotes the current price. An investor may buy oil today and keep it into the future which yields Po_s in the future. As it is equivalent[1] to buy a portfolio of Po_s units of the contingent claim for each state *s* it follows, using equation (1.4) that

$$Po_0 = \frac{\sum \tilde{q}_s Po_s}{1+r} .$$

This gives an additional constraint that can be used to calibrate the new probabilities.

Consider the latter calibration in the case of two scenarios reflecting high and low oil price. If keeping oil give a risk premium, the original return would be higher than the risk free rate, and to apply the equation above, we should increase the probability of the low price scenario, to make the expected return equal to the risk free rate. We thus put most weight on the low price scenario, as oil price is the only source of uncertainty and investors will have a low return on their oil investment in this scenario, hence also a low marginal utility of income. This takes the perspective of the owner.

Risk averse managers

A manager running a power company and deciding whether to extend capacity by investment in wind or thermal oil power, may think different from the owner. E.g. suppose managers' compensation is proportional to the firms profit. (See Murphy (1999) for more elaborate descriptions of executive compensation schemes.) Managers total wealth would be $W=\beta\pi+W_0$, where β is a proportionality factor. A risk averse manager would choose investments to maximize an increasing and concave function $\phi(\beta\pi + W_0)$ of profit. This is easily incorporated in our model. To simplify notation let $=\beta=1$ and $W_0=0$.

If firms maximize $\phi(\pi)$, their first-order condition would be:

$$\phi'((p_s - c)K_s)(p_s - c) = \tilde{\mu}_s ,$$

with the corresponding condition on the multipliers as in (1.3):

$$E\tilde{\mu} = 0 .$$

Due to the shadow price, we cannot eliminate the term $\phi'\left(\left(p_s - c\right)K_s\right)$.

[1] It may be argued that sitting on oil yields an additional "convenience yield" (see e.g. Gibson and Schwaartz, 1990) which is not earned by the equivalent portfolio. In this case the return to the equivalent portfolio should be reduced accordingly.

Note that we could recalibrate shadow prices, using $\hat{\mu}_s = \mu_s / \left(q_s \varphi' \left(\left(p_s - c \right) K_s \right) \right)$ rather than $\tilde{\mu}_s = \mu_s / q_s$. This would yield the simple first order condition $p_s = c + \hat{\mu}_s$ [2]. Apparently this corresponds to new probabilities $\hat{q}_s \propto q_s \varphi' \left(\left(p_s - c \right) K_s \right)$. But these probabilities would be firm specific, and not apply throughout the economy. In an equilibrium with different kinds of firms such a replacement of probabilities thus does not work.

7 Conclusions

We have shown how a deterministic dynamic model can be made into a stochastic model by adding a set of scenarios and with a dynamic information structure, and that this information structure can itself be given a timing interpretation.

We also solved a very simple model with equilibrium investment in production capacity and transmission capacity and with uncertain demand. The model is set up such that there will be no transmission without uncertainty, to highlight an aspect of infrastructure investment; the flexibility it generates. The value of transmission is comes from the price differences generated by different stochastic shocks in different regions. This also makes information more valuable, and we see a larger difference in first period investment in transmission than in production capacity when we assume that there is no learning in the second period, thus the real option is more valuable in this context.

Numerical results are very dependent on the parameter values used, and we have made no effort to make the model realistic. But the approach has already been used to make a static version of the model LIEBMOD stochastic, and in the next stage we will extend this to a dynamic version that allows us to study the tradeoff between the real option element of investments as well as the different kinds of flexibility otherwise generated by different types of investment.

[2] The condition on shadow prices would then become $\sum_s \left(q_s \varphi' \left(\left(p_s - c \right) K_s \right) \right) \hat{\mu}_s = 0$