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Ambiguity in a Real Option Game*

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Abstract

In this paper we study a two-player investment game with a first mover advantage in continuous time with stochastic payoffs, driven by a geometric Brownian motion. One of the players is assumed to be ambiguous with maxmin preferences over a strongly rectangular set of priors. We develop a strategy and equilibrium concept allowing for ambiguity and show that equilibira can be preemptive (a player invests at a point where investment is Pareto dominated by waiting) or sequential (one player invests *as if* she were the exogenously appointed leader). Following the standard literature, the worst case prior for the ambiguous player if she is the second mover is obtained by setting the lowest possible trend in the set of priors. However, if the ambiguous player is the first mover, then the worst case prior can be given by either the lowest or the highest trend in the set of priors. This novel result shows that "worst case prior" in a setting with geometric Brownian motion and κ-ambiguity does not equate to "lowest trend".

Keywords: Real Options, Knightian Uncertainty, Worst Case Prior, Optimal Stopping, Timing Game
JEL classification: C61, C73, D81, L13

1 Introduction

Since the seminal contribution of Chen and Epstein (2002), there has been a solid framework for dealing with Gilboa and Schmeidler (1989) maxmin preferences in a continuous time multiple prior model of ambiguity. This model has been applied to several problems in economics and finance to gain valuable insights in the consequences of a form of Knightian uncertainty, as opposed to risk, on economic decisions. The main insight of Chen and Epstein (2002) is that in order to find the maxmin value of a payoff stream under a

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particular kind of ambiguity (called *strongly rectangular*) we need to identify the *upper-rim generator* of the set of multiple priors, and value the payoff stream *as if* this is the true process governing the payoffs.

In the literature this process has become known as the *worst case prior*, because it identifies the prior that at any given time t gives the lowest expected discounted payoff from time t. In the literature on investment under uncertainty (so-called "real options") the approach has been used to value investment projects when the decision maker is not sure about the stochastic process governing the cash-flows resulting from the project. Typically, this literature models cash-flows as geometric Brownian motions and ambiguity takes the form of κ *ambiguity* over the true trend of the diffusion. In that case it has been shown by Nishimura and Ozaki (2007) that the worst case *at any time* t corresponds to the *lowest* possible trend that is considered under κ-ambiguity.

In this paper we extend the Nishimura and Ozaki model to a timing game between two firms, which both have the option to invest in a project, where one firm is ambiguous about the process governing cash-flows, and the other firm (potentially) has a cost disadvantage. In such timing games, players typically have to balance the expected future payoffs of being the first or second firm to invest; the *leader* and *follower* roles, respectively.

The purpose of our paper is threefold. Firstly, we want to explore the effects of ambiguity on the leader and follower payoffs to players. Secondly, we wish to extend the equilibrium concepts for stochastic timing games[1] to include ambiguous players. Thirdly, we want to investigate the interaction of ambiguity and cost (dis-) advantages on equilibrium investment scenarios.

Our main conclusions are as follows. First, contrary to all of the literature on ambiguity in the real options literature, the worst case prior is *not* always the lowest possible trend under κ ambiguity. As in any timing game, an ambiguous player has to consider the payoffs of the leader and follower roles. The payoffs of the latter role follow along very similar lines as in Nishimura and Ozaki (2007), i.e. the worst case payoff corresponds to valuing the follower's payoff stream *as if* the payoffs are driven by the diffusion with the lowest admissible trend under κ ambiguity. For the leader's payoff, however, the situation is different, because of the interplay between two opposing forces. On the one hand, the leader's payoff consists of current payoffs of being the leader. The worst case for these payoffs is represented by the lowest admissible trend, like in the follower payoff. There is, however, another force at work: the risk that the other player invests as well, which reduces the firm's monopoly payoff to a duopoly payoff. This event has a downward

[1]Since the seminal contribution of Fudenberg and Tirole (1985) for deterministic timing games, many attempts to defining equilibria in stochastic timing games has been made such as Thijssen (2010), Thijssen et al. (2012), de Villemeur et al. (2014), Boyarchenko and Levendorski (2014), Azevedo and Paxson (2014), Huisman and Kort (2014).

effect on the leader's payoff and is discounted using the expected time it takes until the other firm enters the market. This expected time is reached faster for higher values of the trend of the stochastic process, so that the worst case for this part of the leader's payoffs is represented by the *highest* admissible trend. We use an analysis based on backward stochastic differential equations and "g-expectations", as introduced by Peng (1997), to study which effect dominates. It turns out that for small values of the stochastic process, the worst case *always* corresponds to the lowest admissible trend, whereas for higher values the highest admissible trend *may* represent the worst case, depending on the underlying parameters.

Secondly, we show that equilibria can be of two types. First, there may be *preemptive* equilibria in which one of the firms invests at a time where it is not optimal for either firm to do so. This type of equilibrium is familiar from the literature (e.g. Fudenberg and Tirole (1985), Weeds (2002), Pawlina and Kort (2006)) but we use a technique recently developed by Riedel and Steg (2014) to rigorously prove existence of this type of equilibrium rather than relying on fairly *ad hoc* arguments that are often used in the existing literature. It should be pointed out here that in a preemptive equilibrium it is known a.s. *ex ante* which firm is going to invest first. This firm will invest at a point in time where its leader value exceeds its follower value, but where its competitor is indifferent between the two roles. A second type of equilibrium that can exist is a *sequential* equilibrium, in which one firm invests at a time where it is optimal for them to do so. By that we mean that the firm would choose the same time to invest even if it knew that the other firm could not preempt. Each game always has at least an equilibrium of one of these two types, which can not co-exist. These two types of equilibrium each lead to a clear prediction, a.s., as to which firm invests first. The role of first mover depends crucially on the levels of ambiguity and cost (dis-) advantage, as we show in a numerical analysis.

As mentioned above we obtain our equilibrium results by using techniques developed by Riedel and Steg (2014). It should be pointed out that we cannot simply adopt their strategies to our setting due to the presence of an ambiguous player. In fact, the notion of *extended mixed strategy* as introduced in Riedel and Steg (2014) presents a conceptual problem here. An extended mixed strategy consists, in essence, of a distribution over stopping times as well as a coordination device that allows players to coordinate in cases where equilibrium considerations require one and only one firm to invest and it is not clear *a priori* which firm this should be. In our model we need this coordination device as well, but we do not want ambiguity to extend to the uncertainty created by this coordination mechanism, i.e. ambiguity is over payoffs *exclusively*. This presents problems if we want to define payoffs to the ambiguous firm if it plays a mixture over stopping times. For equilibrium existence, however, such mixtures are not needed, so we choose to restrict attention to what we call *extended pure strategies*, which consist of a stopping time and an element related to the coordination mechanism mentioned above. By making this simplifying assumption, together with strong rectangularity of the set of priors, we can write the worst case payoff of a pair of extended pure strategies as a sum of worst cases of leader and follower payoffs.

2 The Model

We follow Pawlina and Kort (2006) in considering two firms who are competing to implement a new technology. Uncertainty in the market is modeled on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathsf{P})$ using a geometric Brownian motion

$$\frac{dX}{X} = \mu dt + \sigma dB,$$

where $(B_t)_{t\geq 0}$ is a Wiener process. The sunk costs of investment are $I > 0$ for Firm 1 and αI, $\alpha \geq 1$, for Firm 2. So, Firm 1 has a cost advantage.

The payoff streams are given by processes $(D_{k\ell}X_t)_{t\geq 0}$, where $D_{k\ell}$, $k, \ell = 0, 1$, denotes the scaling factor if the firm's investment status is k ($k = 0$ if the firm has not invested and $k = 1$ if the firm has invested) and the investment status of the competitor is ℓ. It is assumed that $D_{00} = 0$ (wlog), that $D_{10} > D_{11} \geq 0 \geq D_{01}$, and that there is a first mover advantage, i.e. $D_{10} > D_{11} - D_{01}$.

We assume that Firm 1 may have a cost advantage, but also that it is ambiguous about the trend μ. Following the recent literature on ambiguity in continuous time models we model the set of priors that the firm considers using a set of density generators. The set of measures that is considered by the firm is denoted by $\mathscr{P}^{\Theta}$, where Θ is a set of density generators. A process $(\theta_t)_{t\geq 0}$ is a density generator if it is such that the process $\left(M_t^\theta\right)_{t\geq 0}$, where

$$\frac{dM_t^\theta}{M_t^\theta} = -\theta_t dB_t, \quad M_0^\theta = 1, \tag{1}$$

is a P-martingale. Such a process $(\theta_t)_{t\geq 0}$ generates a new measure P^θ via the Radon-Nikodym derivative $d\mathsf{P}^\theta/d\mathsf{P} = M_\infty^\theta$.

In order to use density generators as a model for ambiguity the set Θ needs some more structure. Following Chen and Epstein (2002), the set of density generators, Θ, is chosen as follows. Let $(\Theta_t)_{t\geq 0}$ be a collection of correspondences $\Theta_t : \Omega \twoheadrightarrow \mathbb{R}$, such that

1. There is a compact subset $K \subset \mathbb{R}$, such that $\Theta_t(\omega) \subseteq K$, for all $\omega \in \Omega$ and all $t \in [0, T]$;

2. For all $t \in [0, T]$, Θ_t is compact-valued and convex-valued;

3. For all $t \in (0, T]$, the mapping $(s, \omega) \mapsto \Theta_s(\omega)$, restricted to $[0, t] \times \Omega$, is $\mathscr{B}[0, t] \times \mathscr{F}_t$-measurable;

4. $0 \in \Theta_t(\omega)$, $dt \otimes d\mathsf{P}$-a.e.

The set of density generators is then taken to be,

$$\Theta = \{(\theta_t)_{t\geq 0}\,|\theta_t(\omega) \in \Theta_t(\omega), d\mathsf{P} - \text{a.e., all } t \geq 0\},$$

and the resulting set of measures $\mathscr{P}^\Theta$ is called *strongly-rectangular*. For sets of strongly rectangular priors the following has been obtained by Chen and Epstein (2002):

1. $\mathsf{P} \in \mathscr{P}^\Theta$;

2. All measures in $\mathscr{P}^\Theta$ are uniformly absolutely continuous with respect to P and are equivalent to P;

3. For every $X \in \mathscr{L}^2(\Omega, \mathscr{F}, \mathsf{P})$, there exists $\mathsf{P}^* \in \mathscr{P}^\Theta$ such that for all $t \geq 0$,

$$\mathsf{E}^{\mathsf{P}^*}[X|\mathscr{F}_t] = \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q[X|\mathscr{F}_t]. \tag{2}$$

Finally, for further reference, define the *upper-rim generator* $(\theta_t^*)_{t\geq 0}$, where

$$\theta_t^* = \arg\max\{\sigma_w(t)\theta_t|\theta_t \in \Theta_t\}. \tag{3}$$

Note that $(\theta_t^*)_{t\geq 0} \in \Theta$.

From Girsanov's theorem it immediately follows that under $\mathsf{P}^\theta \in \mathscr{P}^\Theta$, the process $\left(B_t^\theta\right)_{t\geq 0}$, defined by

$$B_t^\theta = B_t + \int_0^t \theta_s ds,$$

is a P^θ-Brownian motion and that, under P^θ, the process $(X_t)_{t\geq 0}$ follows the diffusion

$$\frac{dX_t}{X_t} = \mu^\theta(t)dt + \sigma dB_t^\theta,$$

Furthermore,

$$\mu^\theta(t) = \mu - \sigma\theta_t.$$

In the remainder we will assume that $\Theta_t = [-\kappa, \kappa]$, for all $t > 0$, for some $\kappa > 0$. Denote $\Delta = [\underline{\mu}, \overline{\mu}] = [\mu - \sigma\kappa, \mu + \sigma\kappa]$. This form of ambiguity is called κ*-ignorance* (cf. Chen and Epstein (2002)). The advantages of using this definition of ambiguity are that (i) Θ is strongly rectangular so that the results stated above apply and (ii) the upper-rim generator takes a convenient form, namely $\theta_t^* = \kappa$, for all $t \geq 0$. In addition, it can easily be shown that $\left(B_t^\theta\right)_{t\geq 0}$ is a P-martingale for every $(\theta_t)_{t\geq 0} \in \Theta$.

Notice, Cheng and Riedel (2013) show that $\kappa-$ignorance can be applied in an infinite time-horizon. In particular they show that value functions taken under drift ambiguity in the infinite time horizon are nothing but the limits of value functions of finite time horizons T if $T \to \infty$.

In our model, we assume Firm 1 to be ambiguity averse.

Finally, the discount rate is assumed to be $r > \overline{\mu}$.

3 Value Functions: Leader Value and Follower Value

Assume Firm 1 becomes the leader at t. Then the non-ambiguous Firm 2 solves the optimal stopping problem

$$F_2(x_t) = \sup_{\tau_2^F \geq t} \mathsf{E}^{\mathsf{P}}\Big[\int_t^{\tau_2^F} e^{-r(s-t)} D_{01} X_s ds + \int_{\tau_2^F}^{\infty} e^{-r(s-t)} D_{11} X_s - e^{-r(\tau_2^F - t)} \alpha I | \mathscr{F}_t\Big]. \tag{4}$$

Thus, τ_2^F is the optimal time Firm 2 invests as a follower.

On the other hand, if the non-ambiguous firm becomes the leader at a certain point in time t, its value function is

$$L_2(x_t) = \mathsf{E}^{\mathsf{P}}\Big[\int_t^{\tau_1^F} e^{-r(s-t)} D_{10} X_s ds + \int_{\tau_1^F}^{\infty} e^{-r(s-t)} D_{11} X_s ds - \alpha I | \mathscr{F}_t\Big], \tag{5}$$

where τ_1^F denotes the optimal time at which the ambiguous firm invests as a follower. From the standard literature on real option games (cf. Pawlina and Kort (2006)) we know that the former value function can be written as

$$F_2(x_t) = \begin{cases} \frac{x_t D_{01}}{r-\mu} + \left(\frac{x_2^F (D_{11} - D_{01})}{r-\mu} - \alpha I\right)\left(\frac{x_t}{x_2^F}\right)^{\beta(\mu)}, & \text{if} \quad x_t \leq x_2^F, \\ \frac{x_t D_{11}}{r-\mu} - \alpha I & \text{if} \quad x_t > x_2^F, \end{cases} \tag{6}$$

where τ_2^F is the first hitting time of x_2^F, i.e

$$\tau_2^F = \inf\{s \geq t | X_s \geq x_2^F\}.$$

The standard procedure of dynamic programming yields that the threshold x_2^F is given by

$$x_2^F = \frac{\beta(\mu)}{\beta(\mu) - 1} \frac{\alpha I (r - \mu)}{D_{11} - D_{01}},$$

where $\beta(\mu)$ is the positive root of the fundamental quadratic $1/2\sigma^2 \beta(\mu)(\beta(\mu) - 1) + \mu\beta(\mu) - r = 0$, which is

$$\beta(\mu) = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{s^2}} > 1.$$

Applying the standard techniques of backward induction and dynamic programming, one can show that the leader value (5) turns out to be

$$L_2(x_t) = \begin{cases} \frac{x_t D_{10}}{r-\mu} - \alpha I + \frac{x_F^1 (D_{11} - D_{10})}{r-\mu}\left(\frac{x_t}{x_1^F}\right)^{\beta(\mu)}, & \text{if} \quad x_t \leq x_1^F, \\ \frac{x_t D_{11}}{r-m} - I_1, & \text{if} \quad x_t > x_1^F. \end{cases}$$

Accordingly, the real value x_1^F describes the optimal time for the ambiguous firm to become the follower, i.e.

$$\tau_1^F = \inf\{s \geq t | X_s \geq x_1^F\}.$$

If ambiguity is introduced, the standard techniques for computing the value functions are not applicable any longer. In our case, where ambiguity is modeled by a strongly rectangular set of density generators, one needs, in contrast to the standard case, to allow for changing priors over time.

The value functions of the ambiguous Firm 1 are given by

$$F_1(x) := \sup_{\tau_1^F \geq t} \inf_{Q \in \mathscr{P}^\Theta} \mathsf{E}^Q \Big[\int_t^{\tau_1^F} e^{-r(s-t)} D_{01} X_s ds + \int_{\tau_1^F}^{\infty} e^{-r(s-t)} D_{11} X_s - e^{-r(\tau_1^F - t)} I | \mathscr{F}_t \Big] \tag{7}$$

and

$$L_1(x_t) = \inf_{Q \in \mathscr{P}^\Theta} \mathsf{E}^Q \Big[\int_t^{\tau_2^F} e^{-r(s-t)} D_{10} X_s ds + \int_{\tau_2^F}^{\infty} e^{-r(s-t)} D_{11} X_s ds \Big| \mathscr{F}_t \Big] - I, \tag{8}$$

respectively.

If the set of priors $\mathscr{P}^\Theta$ is strongly rectangular, it turns out that problem (7) can be reduced to a standard optimal stopping problem and, hence, can be solved by using standard techniques. This reduction is possible due to the following lemma, the proof of which is standard and is, thus, omitted.

Lemma 1. *Let $\mathscr{P}^\Theta$ be strongly-rectangular. Then*

$$F_1(x_t) = \sup_{\tau_1^F \geq t} \mathsf{E}^{P^{\theta^*}} \Big[\int_t^{\tau_1^F} e^{-r(s-t)} D_{01} X_s ds + \int_{\tau_1^F}^{\infty} e^{-r(s-t)} D_{11} X_s ds - e^{-r(\tau_1^F - t)} I | \mathscr{F}_t \Big], \tag{9}$$

where $(\theta_t^*)_{t \geq 0}$ is the upper-rim generator (3)

Hence, for the follower problem of the ambiguous firm, the worst case is always induced by the worst possible drift $\underline{\mu}$. This observation indeed makes sense; the actions of the opponent have, essentially, no influence of the decision as a follower. The problem therefore reduces to one of a "monopolistic" decision-maker. Nishimura and Ozaki (2007) already showed that for such decisions, the worst case is always given by the worst trend $\underline{\mu}$.

In other words, we find that the follower value of the ambiguous firm can be expressed by

$$F_1(x_t) = \begin{cases} \frac{x_t D_{01}}{r - \underline{\mu}} + \left(\frac{x_1^F (D_{11} - D_{01})}{r - \underline{\mu}} - I \right) \left(\frac{x_t}{x_1^F} \right)^{\beta(\underline{\mu})}, & \text{if} \quad x_t \leq x_1^F, \\ \frac{x_t D_{11}}{r - \underline{\mu}} - I & \text{if} \quad x_t > x_1^F, \end{cases} \tag{10}$$

where

$$x_1^F = \frac{\beta(\underline{\mu})}{\beta(\underline{\mu}) - 1} \frac{I(r - \underline{\mu})}{D_{11} - D_{01}}.$$

Determining the leader value function of the ambiguous firm, however, is a different issue. The action of the opponent (in this case the decision when to invest as a follower) is crucial for the computation of the leader function which might lead, as we will see, to a non-trivial behaviour of the worst case prior.

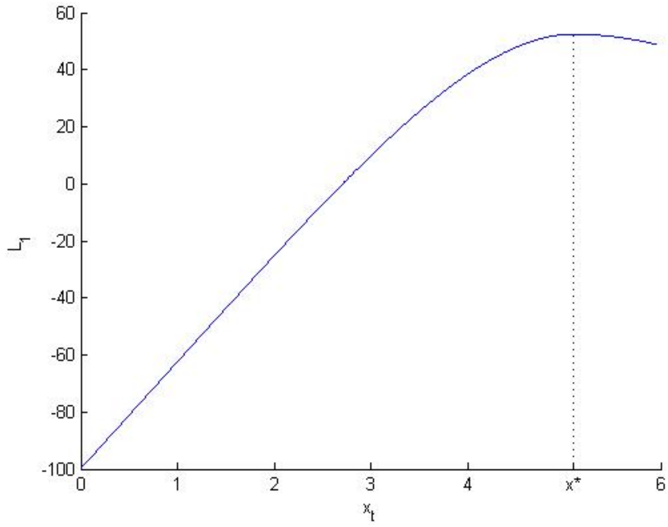

Figure 1: The Leader value function $L_1(x_t)$ for the values $D_{10} = 1.5$, $D_{11} = 1$, $D_{01} = -0.01$, $r = 0.05$, $\sigma = 0.1$, $I = 100$, $\mu = 0.02$, $\overline{\mu} = 0.3$, $\underline{\mu} = 0.01$ and $x_2^F = 5.95$.

The next theorem describes the leader value function of the ambiguous firm. Two cases are distinguished there. If the difference $D_{10} - D_{11}$ is sufficiently small, we find that the worst case is, as before, always induced by $\underline{\mu}$. In case this condition is not satisfied, the worst case is given by $\underline{\mu}$ for values x_t up to a certain threshold x^*, where it jumps to $\overline{\mu}$. The intuition for this fact can already be derived from equation (8); the lowest trend $\underline{\mu}$ gives the minimal values for the payoff stream $(D_{kl}X_t)$. However, the higher the trend μ the sooner the stopping time τ_2^F is expected to be reached. The higher payoff stream $(D_{10}X_t)$ is then sooner replaced by the lower one $(D_{11}X_t)$. If the drop of the payoffs becomes sufficiently small, the former effect always dominates the latter. In this case the worst case is given by $\underline{\mu}$ for each x_t.

Theorem 1. *The worst case for the leader function of the ambiguous firm is always given by the worst possible drift $\underline{\mu}$ if and only if the following condition holds*

$$\frac{D_{10} - D_{11}}{D_{10}} \leq \frac{1}{\beta(\underline{\mu})}. \tag{11}$$

In this case, the leader function becomes

$$L_1(x_t) = \begin{cases} \frac{D_{10}x_t}{r-\underline{\mu}} + \left(\frac{x_t}{x_2^F}\right)^{\beta(\underline{\mu})} \frac{D_{11}-D_{10}}{r-\underline{\mu}} x_2^F - I & \text{if } x_t < x_2^F \\ \frac{D_{11}x_t}{r-\underline{\mu}} - I & \text{if } x_t \geq x_2^F. \end{cases} \tag{12}$$

On the other hand, if $\frac{D_{10}-D_{11}}{D_{10}} > \frac{1}{\beta(\underline{\mu})}$*, then there exists a unique threshold*

$$x^* = \left(- \frac{\frac{D_{10}}{r-\overline{\mu}}}{\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}} \frac{1}{\beta(\overline{\mu})} \right)^{\frac{1}{\beta(\overline{\mu})-1}} x_2^F,$$

such that $\underline{\mu}$ *is the worst case on the set* $\{X_t < x^*\}$ *and* $\overline{\mu}$ *is the worst case on* $\{x^* \leq X_t < x_2^F\}$*. Furthermore, in this case the leader value function is given by*

$$L_1(x_t) = \begin{cases} \frac{D_{10}x_t}{r-\underline{\mu}} + A_1 {x_t}^{\beta(\underline{\mu})} - I & \text{if } x_t < x^* \\ \frac{D_{10}x_t}{r-\overline{\mu}} + \left(\frac{x_t}{x_2^F}\right)^{\beta(\overline{\mu})} \left(\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right) x_2^F - I & \text{if } x^* \leq x_t < x_2^F \\ \frac{D_{11}x_t}{r-\underline{\mu}} - I & \text{if } x_t \geq x_2^F, \end{cases} \tag{13}$$

where

$$A_1 = \left(\frac{D_{10}x^*}{r-\overline{\mu}} - \frac{D_{10}x^*}{r-\underline{\mu}}\right) \frac{1}{(x^*)^{\beta(\underline{\mu})}} + \left(\frac{(x^*)}{x_2^F}\right)^{\beta(\overline{\mu})} \left(\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right) x_2^F \frac{1}{(x^*)^{\beta(\underline{\mu})}}.$$

Figure (1) shows a typical run of the leader value function for the case that the worst case changes. The critical value x^* determines the maximal value of this function in the interval $[0, x_2^F]$. The worst case is given by the lowest possible trend in the region where L_1 is increasing, whereas the highest possible trend depicts the worst case whenever the leader function is decreasing.

For the proof of Theorem (1), we need a completely different approach compared to the standard literature on real option games. We use backward stochastic differential equations and $g-$expectations introduced by Peng (1997). The advantage of this approach lies in the fact that we know the value of our problem at the entry point of the follower. This value yields the starting point for a backward stochastic differential equation. The non-linear Feynman-Kac formula reduces the problem to solving a certain non-linear partial differential equation. From this non-linear PDE we are eventually able to derive the worst case prior.

Proof.

Denote

$$Y_t := \inf_{Q \in \mathscr{P}^\Theta} \mathsf{E}^Q \left[\int_t^{\tau_2^F} e^{-r(s-t)} D_{10} X_s ds + \int_{\tau_2^F}^{\infty} e^{-r(s-t)} D_{11} X_s ds \Big| \mathcal{F}_t \right]$$

Applying the time consistency property of a rectangular set of density generators gives

$$
\begin{aligned}
Y_t &= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[\int_t^{\tau_2^F} e^{-r(s-t)}D_{10}X_s ds + \int_{\tau_2^F}^{\infty} e^{-r(s-t)}D_{11}X_s ds \Big| \mathcal{F}_t\right] \\
&= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[\inf_{Q'\in\mathscr{P}^\Theta} \mathsf{E}^{Q'}\left[\int_t^{\tau_2^F} e^{-r(s-t)}D_{10}X_s ds + \int_{\tau_2^F}^{\infty} e^{-r(s-t)}D_{11}X_s ds \Big| \mathcal{F}_{\tau_2^F}\right]\Big|\mathcal{F}_t\right] \\
&= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[\int_t^{\tau_2^F} e^{-r(s-t)}D_{10}X_s ds + e^{-r(\tau_2^F-t)}\inf_{Q'\in\mathscr{P}^\Theta} \mathsf{E}^{Q'}\left[\int_{\tau_2^F}^{\infty} e^{-r(s-\tau_2^F)}D_{11}X_s ds \Big| \mathcal{F}_{\tau_2^F}\right]\Big|\mathcal{F}_t\right] \\
&= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[\int_t^{\tau_2^F} e^{-r(s-t)}D_{10}X_s ds + e^{-r(\tau_2^F-t)}\Phi(x_{\tau_2^F})\Big|\mathcal{F}_t\right],
\end{aligned}
$$

where

$$\Phi(x_t) := \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Big[\int_t^{\infty} e^{-r(s-t)}D_{11}X_s ds\Big|\mathcal{F}_t\Big] = \frac{D_{11}x_t}{r-\underline{\mu}}. \tag{14}$$

Chen and Epstein (2002) show that Y_t solves the BSDE

$$-dY_t = g(Z_t)dt - Z_t dB_t,$$

for the *generator*

$$g(z) = -\kappa|z| - rY_t + X_t D_{10}.$$

The boundary condition is given by

$$Y_{\tau_2^F} = \Phi(x_2^F),$$

In the terminology of Peng (2013) we say that the leader value is the g-expectation of the random variable $\Phi(x_2^F)$, and denote it by

$$Y_t = \mathsf{E}_g[\Phi(x_2^F)|\mathcal{F}_t].$$

Denote the present value of the leader payoff by L, i.e.

$$L(x_t) = Y_t.$$

The non-linear Feynman-Kac formula[2] (Peng, 2013, Theorem 3) implies that L solves the non-linear PDE

$$\mathscr{L}_X L(x) + g(\sigma x L'(x)) = 0.$$

Hence, L solves

$$\frac{1}{2}\sigma^2 x^2 L''(x) + \mu x L'(x) - \kappa\sigma x\left|L'(x)\right| - rL(x) + D_{10}x = 0. \tag{15}$$

[2]Note that Peng (1991) shows that the non-linear Feynman-Kac formula not only holds for deterministic times but also first exit times like τ_2^F.

Expression (15) implies that $\underline{\mu}$ is the worst case on the set $\{x \leq x_2^F | L'(x) > 0\}$ and $\overline{\mu}$ is the worst case on $\{x \leq x_2^F | L'(x) < 0\}$.

The general increasing solution to the PDE (15) is

$$L(\mu, x) = \frac{D_{10} x}{r - \mu} + A x^{\beta(\mu)},$$

where μ equals either $\underline{\mu}$ or $\overline{\mu}$. The constant A is determined by the boundary condition $L(\mu, x_2^F) = \Phi(x_2^F)$. Notice that L is a concave function in x. Indeed, L consists of a linear function plus a term $Ax^{\beta(\mu)}$, in which A denotes a constant and $x^{\beta(\mu)}$ a convex function (since $\beta(\mu) > 1$ for all μ). Furthermore, we have $A < 0$, which follows directly from the boundary condition (14) and the fact that $D_{10} > D_{11}$. Hence, L is concave.

In order to determine the constant A, we suppose for the moment that $\underline{\mu}$ is the worst case in the region close to x_2^F. The value matching condition at x_2^F then gives

$$L(\underline{\mu}, x_2^F) = \frac{D_{10} x_2^F}{r - \underline{\mu}} + A_1 {x_2^F}^{\beta(\underline{\mu})} = \frac{D_{11} x_2^F}{r - \underline{\mu}}.$$

This implies

$$A_1 = \frac{D_{10} - D_{11}}{r - \underline{\mu}} {x_2^F}^{1 - \beta(\underline{\mu})},$$

and therefore

$$L(x_t) = \frac{D_{10} x_t}{r - \underline{\mu}} + \left(\frac{x_t}{x_2^F}\right)^{\beta(\underline{\mu})} \frac{D_{11} - D_{10}}{r - \underline{\mu}} x_2^F. \tag{16}$$

On the other hand, if we suppose that $\overline{\mu}$ is the worst case close to x_2^F, the value matching condition gives

$$L(\overline{\mu}, x_2^F) = \frac{D_{10} x_2^F}{r - \overline{\mu}} + A_2 {x_2^F}^{\beta(\overline{\mu})} = \frac{D_{11} x_2^F}{r - \underline{\mu}}.$$

This implies

$$A_2 = \frac{D_{11}}{r - \underline{\mu}} - \frac{D_{10}}{r - \overline{\mu}} {x_2^F}^{1 - \beta(\overline{\mu})}.$$

Hence, in that region, we have

$$L(x_t) = \frac{D_{10} x_t}{r - \overline{\mu}} + \left(\frac{x_t}{x_2^F}\right)^{\beta(\overline{\mu})} \left(\frac{D_{11}}{r - \underline{\mu}} - \frac{D_{10}}{r - \overline{\mu}}\right) x_2^F \tag{17}$$

Now two cases are possible. Either $\underline{\mu}$ is always the worst case or there exists a critical point $x^* < x_2^F$ such that $\underline{\mu}$ is the worst case on $\{X_t < x^*\}$ and $\overline{\mu}$ is the worst case on $\{x^* < X_t \leq x_2^F\}$.

Note first that the value function L is increasing at 0. That means $\underline{\mu}$ is the worst case close to 0.

So, $\underline{\mu}$ is always the worst case if and only if $L'(x) \geq 0$ for all $x \leq x_2^F$. Due to the continuity and concavity of the value function, this is equivalent to the condition

$$L'(x_2^F) \geq 0.$$

Therefore,

$$
\begin{aligned}
L'(x_2^F) = \frac{D_{10}}{r-\underline{\mu}} + \left(\frac{D_{11}-D_{10}}{r-\underline{\mu}}\right)\beta(\underline{\mu})\left(\frac{x_2^F}{x_2^F}\right)^{\beta(\underline{\mu})-1} &\geq 0 \\
D_{11} - D_{10} &\geq -\frac{D_{10}}{\beta(\underline{\mu})} \\
\frac{D_{10}-D_{11}}{D_{10}} &\leq \frac{1}{\beta(\underline{\mu})}.
\end{aligned}
$$

If the last condition is not satisfied, the worst case changes at some point $x^* \leq x_2^F$ from $\underline{\mu}$ to $\overline{\mu}$. Using again the continuity and concavity of the leader value function, we can easily verify that x^* is uniquely determined by $L'(x^*) = 0$. Hence, x^* is the solution to

$$
\begin{aligned}
\frac{D_{10}}{r-\overline{\mu}} + \left(\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)\beta(\overline{\mu})\left(\frac{x^*}{x_2^F}\right)^{\beta(\overline{\mu})-1} &= 0 \\
\left(\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)\beta(\overline{\mu})\left(\frac{x^*}{x_2^F}\right)^{\beta(\overline{\mu})-1} &= -\frac{D_{10}}{r-\overline{\mu}} \\
\left(\frac{x^*}{x_2^F}\right)^{\beta(\overline{\mu})-1} &= -\frac{\frac{D_{10}}{r-\overline{\mu}}}{\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}}\frac{1}{\beta(\overline{\mu})} \\
x^* &= \left(-\frac{\frac{D_{10}}{r-\overline{\mu}}}{\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}}\frac{1}{\beta(\overline{\mu})}\right)^{\frac{1}{\beta(\overline{\mu})-1}} x_2^F.
\end{aligned}
$$

At x^* we get a further value matching condition firstly observed by Cheng and Riedel (2013), namely that $L(\underline{\mu}, x^*) = L(\overline{\mu}, x^*)$. This implies

$$
\frac{D_{10}x^*}{r-\underline{\mu}} + A_1(x^*)^{\beta(\underline{\mu})} = \frac{D_{10}x^*}{r-\overline{\mu}} + \left(\frac{x^*}{x_2^F}\right)^{\beta(\overline{\mu})}\left(\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)x_2^F.
$$

Thus,

$$
A_1 = \left(\frac{D_{10}x^*}{r-\overline{\mu}} - \frac{D_{10}x^*}{r-\underline{\mu}}\right)\frac{1}{(x^*)^{\beta(\underline{\mu})}} + \left(\frac{x^*}{x_2^F}\right)^{\beta(\overline{\mu})}\left(\frac{D_{11}}{r-\underline{\mu}} - \frac{D_{10}}{r-\overline{\mu}}\right)x_2^F\frac{1}{(x^*)^{\beta(\underline{\mu})}},
$$

and the result follows. ■

3.1 Optimal Leader Threshold

Next we want to determine the optimal time to invest as a leader. Suppose Firm 2 knows it becomes the leader and searches for the optimal time to invest. It then faces at time t the following optimal stopping problem

$$
L^*(x_t) = \sup_{\tau_{L,2}^t \geq t} \mathsf{E}^P\Big[\int_{\tau_{L,2}^t}^{\tau_1^F} e^{-r(s-t)}D_{10}X_s ds + \int_{\tau_1^F}^{\infty} e^{-r(s-t)}D_{11}X_s ds - e^{-r(\tau_{L,2}^t - t)}\alpha I\Big|\mathcal{F}_t\Big]. \tag{18}
$$

The solution can be found by applying the standard techniques and is given by

$$\tau_{L,2}^t = \inf\{s \geq t | X_s \geq x_2^L\},$$

where

$$x_2^L = \frac{\beta}{\beta - 1}\frac{\alpha I(r-\mu)}{D_{10} - D_{00}}.$$

The ambiguous firm solves the following optimal stopping problem

$$L^*(x_t) = \sup_{\tau_{L,1}^t \geq t} \inf_{Q \in \mathscr{P}^\Theta} \mathsf{E}^Q\Big[\int_{\tau_{L,1}^t}^{\tau_2^F} e^{-r(s-t)} D_{10} X_s ds + \int_{\tau_2^F}^{\infty} e^{-r(s-t)} D_{11} X_s ds - e^{-r(\tau_{L,1}^t - t)} I \Big| \mathcal{F}_t\Big]. \quad (19)$$

Again, in order to determine this stopping time for the ambiguous firm, we cannot apply the standard procedure. Nevertheless, the stopping time does not differ from the one of a non-ambiguous firm given a drift $\underline{\mu}$.

Proposition 1. *The optimal time to invest as a leader for the ambiguous firm is*

$$\tau_{L,1}^t = \inf\{s \geq t | X_s \geq x_1^L\},$$

where

$$x_1^L = \frac{\beta(\underline{\mu})}{\beta(\underline{\mu}) - 1}\frac{I(r-\underline{\mu})}{D_{10} - D_{00}}.$$

For the proof we refer to the appendix.

4 Equilibrium Analysis

The appropriate equilibrium concept for a game with ambiguity as described here is not immediately clear. In this paper we consider two types of equilibria: *preemptive equilibria* in which firms try to preempt each other at some times where it is sub-optimal to invest, and *sequential equilibria*, where one firm invests at its optimal time.

4.1 Strategies and Payoffs

The appropriate notion of subgame perfect equilibrium for our game is developed in Riedel and Steg (2014). Let $\mathscr{T}$ denote the set of stopping times with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$. The set $\mathscr{T}$ will act as the set of (pure) strategies. Given the definitions of the leader and follower payoffs above, the timing game is

$$\Gamma = \left\langle (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathsf{P}), \mathscr{P}^\Theta, \mathscr{T} \times \mathscr{T}, (L_i, F_i, M_i)_{i=1,2}, (\pi_i)_{i=1,2} \right\rangle,$$

where, for $(\tau_1, \tau_2) \in \mathscr{T} \times \mathscr{T}$,

$$\pi_1 = \inf_{Q \in \mathscr{P}^\Theta} \mathsf{E}^Q[L_1 1_{\tau_1 < \tau_2} + F_1 1_{\tau_1 > \tau_2} + M_1 1_{\tau_1 = \tau_2}], \quad \text{and}$$
$$\pi_2 = \mathsf{E}^{\mathsf{P}}[L_2 1_{\tau_1 > \tau_2} + F_2 1_{\tau_1 < \tau_2} + M_2 1_{\tau_1 = \tau_2}].$$

The *subgame starting at stopping time* $\vartheta \in \mathscr{T}$ is the tuple

$$\Gamma = \left\langle (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathsf{P}), \mathscr{P}^\Theta, \mathscr{T}_\vartheta \times \mathscr{T}_\vartheta, (L_i, F_i, M_i)_{i=1,2}, (\pi_i^\vartheta)_{i=1,2} \right\rangle,$$

where $\mathscr{T}_\vartheta$ is the set of stopping times no smaller than ϑ a.s.,

$$\mathscr{T}_\vartheta := \{\tau \in \mathscr{T} | \tau \geq \vartheta, \mathsf{P} - a.s.\},$$

and, for $(\tau_1, \tau_2) \in \mathscr{T}_\vartheta \times \mathscr{T}_\vartheta$,

$$\pi_1^\vartheta = \inf_{Q \in \mathscr{P}^\Theta} \mathsf{E}^Q[L_1 1_{\tau_1 < \tau_2} + F_1 1_{\tau_1 > \tau_2} + M_1 1_{\tau_1 = \tau_2} | \mathscr{F}_\vartheta], \quad \text{and}$$
$$\pi_2^\vartheta = \mathsf{E}^{\mathsf{P}}[L_2 1_{\tau_1 > \tau_2} + F_2 1_{\tau_1 < \tau_2} + M_2 1_{\tau_1 = \tau_2} | \mathscr{F}_\vartheta].$$

As it is argued in Riedel and Steg (2014) careful consideration has to be given to the appropriate notion of strategy. They show that the notion of *extended mixed strategy* is versatile and intuitively appealing. For the subgame Γ^ϑ this is a pair of processes $(G^\vartheta, \alpha^\vartheta)$, both taking values in $[0, 1]$, with the following properties.

1. For every $Q \in \mathscr{P}^\Theta$, G^ϑ is adapted, has right-continuous and non-decreasing sample paths, with $G^\vartheta(s) = 0$ for all $s < \vartheta$, $Q - a.s.$
2. For every $Q \in \mathscr{P}^\Theta$, α^ϑ is progressively measurable with right-continuous sample paths whenever its value is in $(0, 1)$, $Q - a.s.$
3. For every $Q \in \mathscr{P}^\Theta$, on $\{t \geq \vartheta\}$, it holds that

$$\alpha^\vartheta(t) > 0 \Rightarrow G^\vartheta(t) = 1, \quad Q - a.s.$$

We use the convention that

$$G^\vartheta(0-) \equiv 0, \quad G^\vartheta(\infty) \equiv 1, \quad \text{and} \quad \alpha^\vartheta(\infty) \equiv 1.$$

For our purposes extended mixed strategies are, in fact, more general than necessary. Therefore, we will restrict attention to what we will call *extended pure strategies*. For the subgame Γ^ϑ this is a pair of extended mixed strategies $(G_i^\vartheta, \alpha_i^\vartheta)_{i=1,2}$, where G_i^ϑ is restricted to take values in $\{0, 1\}$. In other words, in an extended pure strategy a firm does not mix over stopping times, but potentially mixes over its "investment intensity" α^ϑ.

An *extended pure strategy* for the game Γ is then a collection $(G^\vartheta, \alpha^\vartheta)_{\vartheta \in \mathscr{T}}$ of extended pure strategies in subgames Γ^ϑ, $\vartheta \in \mathscr{T}$ satisfying the time consistency conditions that for all $\vartheta, \nu \in \mathscr{T}$ it holds that

1. $\nu \leq t \in \mathbb{R}_+ \Rightarrow G^\vartheta(t) = G^\vartheta(\nu-) + (1 - G^\vartheta(\nu-))G^\nu(t)$, $\mathscr{P}^\Theta$-q.s. on $\{\vartheta \leq \nu\}$, and
2. $\alpha^\vartheta(\tau) = \alpha^\nu(\tau)$, $\mathscr{P}^\Theta$-q.s., for all $\tau \in \mathscr{T}$.

The importance of the α component in the definition of extended pure strategy becomes obvious in the definition of payoffs. Essentially α allows both for immediate investment and coordination between firms. It leads to investment probabilities that can be thought of as the limits of conditional stage investment probabilities of discrete-time behavioural strategies with vanishing period length. In the remainder, let $\hat{\tau}_i^\vartheta$ be the first time that α_i^ϑ is strictly positive, and let $\hat{\tau}^\vartheta$ be the first time that at least one α^ϑ is non-zero in the subgame Γ^ϑ, i.e.

$$\hat{\tau}_i^\vartheta = \inf\left\{ t \geq \vartheta \;\middle|\; \alpha_i^\vartheta(t) > 0 \right\}, \quad \text{and} \quad \hat{\tau}^\vartheta = \inf\left\{ t \geq \vartheta \;\middle|\; \alpha_1^\vartheta(t) + \alpha_2^\vartheta(t) > 0 \right\},$$

respectively. At time $\hat{\tau}^\vartheta$ the extended pure strategies induce a probability measure on the state state space

$$\Lambda = \{ \{\text{Firm 1 becomes the leader}\}, \{\text{Firm 2 becomes the leader}\}, \{\text{Both firms invest simultaneously}\} \},$$

for which we will use the shorthand notation

$$\Lambda = \{ (L, 1), (L, 2), M \}.$$

Riedel and Steg (2014) show that the probability measure on Λ, induced by the pair $(\alpha_1^\vartheta, \alpha_1^\vartheta)$, is given by

$$\lambda_{L,i}^\vartheta(\hat{\tau}^\vartheta) = \begin{cases} \frac{\alpha_i^\vartheta(\hat{\tau}^\vartheta)(1-\alpha_j^\vartheta(\hat{\tau}^\vartheta))}{\alpha_i^\vartheta(\hat{\tau}^\vartheta)+\alpha_j^\vartheta(\hat{\tau}^\vartheta)-\alpha_i^\vartheta(\hat{\tau}^\vartheta)\alpha_j^\vartheta(\hat{\tau}^\vartheta)} & \text{if } \hat{\tau}_i^\vartheta = \hat{\tau}_j^\vartheta \text{ and } \alpha_i^\vartheta(\hat{\tau}_i^\vartheta), \alpha_j^\vartheta(\hat{\tau}_i^\vartheta) > 0 \\ 1 & \text{if } \hat{\tau}_i^\vartheta < \hat{\tau}_j^\vartheta, \text{ or } \hat{\tau}_i^\vartheta = \hat{\tau}_j^\vartheta \text{ and } \alpha_j^\vartheta(\hat{\tau}_j^\vartheta) = 0 \\ 0 & \text{if } \hat{\tau}_i^\vartheta > \hat{\tau}_j^\vartheta, \text{ or } \hat{\tau}_i^\vartheta = \hat{\tau}_j^\vartheta \text{ and } \alpha_j^\vartheta(\hat{\tau}_j^\vartheta) = 0 \\ \frac{1}{2}\big(\liminf_{t \downarrow \hat{\tau}_i^\vartheta} \frac{\alpha_i^\vartheta(t)(1-\alpha_j^\vartheta(t))}{\alpha_i^\vartheta(t)+\alpha_j^\vartheta(t)-\alpha_i^\vartheta(t)\alpha_j^\vartheta(t)} & \text{if } \hat{\tau}_i^\vartheta = \hat{\tau}_j^\vartheta, \alpha_i^\vartheta(\hat{\tau}_i^\vartheta) = \alpha_j^\vartheta(\hat{\tau}_j^\vartheta) = 0, \\ +\limsup_{t \downarrow \hat{\tau}_i^\vartheta} \frac{\alpha_i^\vartheta(t)(1-\alpha_j^\vartheta(t))}{\alpha_i^\vartheta(t)+\alpha_j^\vartheta(t)-\alpha_i^\vartheta(t)\alpha_j^\vartheta(t)}\big) & \text{and } \alpha_i^\vartheta(\hat{\tau}_i^\vartheta+), \alpha_j^\vartheta(\hat{\tau}_j^\vartheta+) > 0, \end{cases}$$

and

$$\lambda_M^\vartheta(\hat{\tau}^\vartheta) = \begin{cases} 0 & \text{if } \hat{\tau}_i^\vartheta = \hat{\tau}_j^\vartheta, \alpha_i^\vartheta(\hat{\tau}_i^\vartheta) = \alpha_j^\vartheta(\hat{\tau}_i^\vartheta) = 0, \text{ and } \alpha_i^\vartheta(\hat{\tau}_i^\vartheta+), \alpha_j^\vartheta(\hat{\tau}_i^\vartheta+) > 0 \\ \frac{\alpha_i^\vartheta(\hat{\tau}^\vartheta)\alpha_j^\vartheta(\hat{\tau}^\vartheta)}{\alpha_i^\vartheta(\hat{\tau}^\vartheta)+\alpha_j^\vartheta(\hat{\tau}^\vartheta)-\alpha_i^\vartheta(\hat{\tau}^\vartheta)\alpha_j^\vartheta(\hat{\tau}^\vartheta)} & \text{otherwise.} \end{cases}$$

Note the following:

1. if $\hat{\tau}_i^\vartheta < \hat{\tau}_j^\vartheta$ there is no coordination problem: Firm i becomes the leader a.s.;

2. if $\hat{\tau}_i^\vartheta = \hat{\tau}_j^\vartheta$, but $\alpha_j^\vartheta(\hat{\tau}_j^\vartheta) = 0$, there is no coordination problem: Firm i becomes the leader a.s.;

3. in the degenerate case where $\alpha_i^\vartheta(\hat{\tau}_i^\vartheta) = \alpha_j^\vartheta(\hat{\tau}_j^\vartheta) = 0$, and $\alpha_i^\vartheta(\hat{\tau}_i^\vartheta+), \alpha_j^\vartheta(\hat{\tau}_j^\vartheta+) > 0$, the leader role is effectively assigned on the basis of the flip of a fair coin;

4. there is no ambiguity (for Firm 1) over the measure λ.

In order to derive the payoffs to firms, let $\tau_{G,i}^\vartheta$ denote the first time that G_i^ϑ jumps to one, i.e.

$$\tau_{G,i}^\vartheta = \inf\left\{ t \geq \vartheta \,\middle|\, G_i^\vartheta(t) > 0 \right\}.$$

The payoff to the ambiguous firm of a pair of extended pure strategies $((G_1, \alpha_1), (G_2, \alpha_2))$ in the subgame Γ^ϑ is given by

$$
\begin{aligned}
&V_1^\vartheta(G_1^\vartheta, \alpha_1^\vartheta, G_2^\vartheta, \alpha_2^\vartheta) := \\
&\inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[1_{\tau_{G,1}^\vartheta<\min\{\tau_{G,2}^\vartheta,\hat{\tau}^\vartheta\}}\left(\int_{\tau_{G,1}^\vartheta}^{\tau_2^F} e^{-r(s-\vartheta)}D_{10}X_s ds + \int_{\tau_2^F}^{\infty} e^{-r(s-\vartheta)}D_{11}X_s ds - e^{-r(\tau_{G,1}^\vartheta-\vartheta)}I\right)\middle|\mathscr{F}_\vartheta\right] \\
&+ \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[1_{\tau_{G,2}^\vartheta<\min\{\tau_{G,1}^\vartheta,\hat{\tau}^\vartheta\}}\left(\int_{\tau_{G,2}^\vartheta}^{\tau_1^F} e^{-r(s-\vartheta)}D_{01}X_s ds + \int_{\tau_1^F}^{\infty} e^{-r(s-\vartheta)}D_{11}X_s - e^{-r(\tau_1^F-\vartheta)}I\right)\middle|\mathscr{F}_\vartheta\right] \\
&+ \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[1_{\tau_{G,1}^\vartheta=\tau_{G,2}^\vartheta<\hat{\tau}^\vartheta}\left(\int_{\tau_{G,1}^\vartheta}^{\infty} e^{-r(s-\vartheta)}D_{11}X_s ds\right)\middle|\mathscr{F}_\vartheta\right] \\
&+ \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Bigg[1_{\hat{\tau}^\vartheta\leq\min\{\tau_{G,1}^\vartheta,\tau_{G,2}^\vartheta\}}\lambda_{L,1}^\vartheta(\hat{\tau}^\vartheta)\Bigg(\int_{\hat{\tau}^\vartheta}^{\tau_2^F} e^{-r(s-\vartheta)}D_{10}X_s ds \\
&\qquad\qquad + \int_{\tau_2^F}^{\infty} e^{-r(s-\vartheta)}D_{11}X_s ds - e^{-r(\tau_{G,1}^\vartheta-\vartheta)}I\Bigg)\Bigg|\mathscr{F}_\vartheta\Bigg] \\
&+ \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Bigg[1_{\hat{\tau}^\vartheta\leq\min\{\tau_{G,1}^\vartheta,\tau_{G,2}^\vartheta\}}\lambda_{L,2}^\vartheta(\hat{\tau}^\vartheta)\Bigg(\int_{\hat{\tau}^\vartheta}^{\tau_1^F} e^{-r(s-\vartheta)}D_{01}X_s ds \\
&\qquad\qquad + \int_{\tau_1^F}^{\infty} e^{-r(s-\vartheta)}D_{11}X_s - e^{-r(\tau_1^F-\vartheta)}I\Bigg)\Bigg|\mathscr{F}_\vartheta\Bigg] \\
&+ \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[1_{\hat{\tau}^\vartheta\leq\min\{\tau_{G,1}^\vartheta,\tau_{G,2}^\vartheta\}}\lambda_M^\vartheta(\hat{\tau}^\vartheta)\left(\int_{\hat{\tau}^\vartheta}^{\infty} e^{-r(s-\vartheta)}D_{11}X_s ds\right)\middle|\mathscr{F}_\vartheta\right]
\end{aligned}
$$

The first line of of the payoff function can be rewritten to

$$\inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[1_{\tau^\vartheta_{G,1}<\min\{\tau^\vartheta_{G,2},\hat\tau^\vartheta\}}\left(\int_{\tau^\vartheta_{G,1}}^{\tau_2^F} e^{-r(s-\vartheta)}D_{10}X_s ds+\int_{\tau_2^F}^{\infty} e^{-r(s-\vartheta)}D_{11}X_s ds - e^{-r(\tau^\vartheta_{G,1}-\vartheta)}I\right)\Big|\mathscr{F}_\vartheta\right]$$

$$\begin{aligned}
&= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Bigg[1_{\tau^\vartheta_{G,1}<\min\{\tau^\vartheta_{G,2},\hat\tau^\vartheta\}}e^{-r(\tau^\vartheta_{G,1}-\vartheta)}\Bigg(\int_{\tau^\vartheta_{G,1}}^{\tau_2^F} e^{-r(s-\tau^\vartheta_{G,1})}D_{10}X_s ds \\
&\qquad\qquad + \int_{\tau_2^F}^{\infty} e^{-r(s-\tau^\vartheta_{G,1})}D_{11}X_s ds - I\Bigg)\Big|\mathscr{F}_\vartheta\Bigg] \\
&= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Bigg[1_{\tau^\vartheta_{G,1}<\min\{\tau^\vartheta_{G,2},\hat\tau^\vartheta\}}\inf_{Q'\in\mathscr{P}^\Theta} \mathsf{E}^{Q'}\Bigg[e^{-r(\tau^\vartheta_{G,1}-\vartheta)}\Bigg(\int_{\tau^\vartheta_{G,1}}^{\tau_2^F} e^{-r(s-\tau^\vartheta_{G,1})}D_{10}X_s ds \\
&\qquad\qquad + \int_{\tau_2^F}^{\infty} e^{-r(s-\tau^\vartheta_{G,1})}D_{11}X_s ds - I\Bigg)\Big|\mathscr{F}_{\tau^\vartheta_{G,1}}\Bigg]\Big|\mathscr{F}_\vartheta\Bigg] \\
&= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Bigg[1_{\tau^\vartheta_{G,1}<\min\{\tau^\vartheta_{G,2},\hat\tau^\vartheta\}}e^{-r(\tau^\vartheta_{G,1}-\vartheta)}\inf_{Q'\in\mathscr{P}^\Theta} \mathsf{E}^{Q'}\Bigg[\int_{\tau^\vartheta_{G,1}}^{\tau_2^F} e^{-r(s-\tau^\vartheta_{G,1})}D_{10}X_s ds \\
&\qquad\qquad + \int_{\tau_2^F}^{\infty} e^{-r(s-\tau^\vartheta_{G,1})}D_{11}X_s ds - I\Big|\mathscr{F}_{\tau^\vartheta_{G,1}}\Bigg]\Big|\mathscr{F}_\vartheta\Bigg] \\
&= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[1_{\tau^\vartheta_{G,1}<\min\{\tau^\vartheta_{G,2},\hat\tau^\vartheta\}}e^{-r(\tau^\vartheta_{G,1}-\vartheta)}L_1(x_{\tau^\vartheta_{G,1}})\Big|\mathscr{F}_\vartheta\right]
\end{aligned}$$

Using the same arguments, the other lines can be rewritten[3] and the payoff of the ambiguous firm eventually becomes

$$\begin{aligned}
V_1^\vartheta(G_1^\vartheta,\alpha_1^\vartheta,G_2^\vartheta,\alpha_2^\vartheta) := &\inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[1_{\tau^\vartheta_{G,1}<\min\{\tau^\vartheta_{G,2},\hat\tau^\vartheta\}}e^{-r(\tau^\vartheta_{G,1}-\vartheta)}L_1(x_{\tau^\vartheta_{G,1}})\Big|\mathscr{F}_\vartheta\right] \\
&+ \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[1_{\tau^\vartheta_{G,2}<\min\{\tau^\vartheta_{G,1},\hat\tau^\vartheta\}}e^{-r(\tau^\vartheta_{G,2}-\vartheta)}F_1(x_{\tau^\vartheta_{G,2}})\Big|\mathscr{F}_\vartheta\right] \\
&+ \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[1_{\tau^\vartheta_{G,1}=\tau^\vartheta_{G,2}<\hat\tau^\vartheta}e^{-r(\tau^\vartheta_{G,1}-\vartheta)}M_1(x_{\tau^\vartheta_{G,1}})\Big|\mathscr{F}_\vartheta\right] \\
&+ \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[1_{\hat\tau^\vartheta\le\min\{\tau^\vartheta_{G,1},\tau^\vartheta_{G,2}\}}\lambda^\vartheta_{L,1}(\hat\tau^\vartheta)e^{-r(\hat\tau^\vartheta-\vartheta)}L_1(x_{\hat\tau^\vartheta})\Big|\mathscr{F}_\vartheta\right] \\
&+ \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[1_{\hat\tau^\vartheta\le\min\{\tau^\vartheta_{G,1},\tau^\vartheta_{G,2}\}}\lambda^\vartheta_{L,2}(\hat\tau^\vartheta)e^{-r(\hat\tau^\vartheta-\vartheta)}F_1(x_{\hat\tau^\vartheta})\Big|\mathscr{F}_\vartheta\right] \\
&+ \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\left[1_{\hat\tau^\vartheta\le\min\{\tau^\vartheta_{G,1},\tau^\vartheta_{G,2}\}}\lambda^\vartheta_{M}(\hat\tau^\vartheta)e^{-r(\hat\tau^\vartheta-\vartheta)}M_1(x_{\hat\tau^\vartheta})\Big|\mathscr{F}_\vartheta\right].
\end{aligned}$$

[3]For the last three lines it is important to note that Firm 1 is not ambiguous over the measure λ.

In the same way, the payoff for the unambiguous firm can be written as

$$
\begin{aligned}
V_2^\vartheta(G_2^\vartheta, \alpha_2^\vartheta, G_1^\vartheta, \alpha_1^\vartheta) := & \mathsf{E}^{\mathsf{P}}\Big[1_{\tau_{G,2}^\vartheta < \min\{\tau_{G,2}^\vartheta, \hat{\tau}^\vartheta\}} e^{-r(\tau_{G,1}^\vartheta - \vartheta)} L_2(x_{\tau_{G,2}^\vartheta}) \Big| \mathscr{F}_\vartheta\Big] \\
& + \mathsf{E}^{\mathsf{P}}\Big[1_{\tau_{G,1}^\vartheta < \min\{\tau_{G,2}^\vartheta, \hat{\tau}^\vartheta\}} e^{-r(\tau_{G,1}^\vartheta - \vartheta)} F_2(x_{\tau_{G,1}^\vartheta}) \Big| \mathscr{F}_\vartheta\Big] \\
& + \mathsf{E}^{\mathsf{P}}\Big[1_{\tau_{G,1}^\vartheta = \tau_{G,2}^\vartheta < \hat{\tau}^\vartheta} e^{-r(\tau_{G,2}^\vartheta - \vartheta)} M_2(x_{\tau_{G,2}^\vartheta}) \Big| \mathscr{F}_\vartheta\Big] \\
& + \mathsf{E}^{\mathsf{P}}\Big[1_{\hat{\tau}^\vartheta \le \min\{\tau_{G,1}^\vartheta, \tau_{G,2}^\vartheta\}} \lambda_{L,2}^\vartheta(\hat{\tau}^\vartheta) e^{-r(\hat{\tau}^\vartheta - \vartheta)} L_2(x_{\hat{\tau}^\vartheta}) \Big| \mathscr{F}_\vartheta\Big] \\
& + \mathsf{E}^{\mathsf{P}}\Big[1_{\hat{\tau}^\vartheta \le \min\{\tau_{G,1}^\vartheta, \tau_{G,2}^\vartheta\}} \lambda_{L,1}^\vartheta(\hat{\tau}^\vartheta) e^{-r(\hat{\tau}^\vartheta - \vartheta)} F_2(x_{\hat{\tau}^\vartheta}) \Big| \mathscr{F}_\vartheta\Big] \\
& + \mathsf{E}^{\mathsf{P}}\Big[1_{\hat{\tau}^\vartheta \le \min\{\tau_{G,1}^\vartheta, \tau_{G,2}^\vartheta\}} \lambda_{M}^\vartheta(\hat{\tau}^\vartheta) e^{-r(\hat{\tau}^\vartheta - \vartheta)} M_2(x_{\hat{\tau}^\vartheta}) \Big| \mathscr{F}_\vartheta\Big],
\end{aligned}
$$

4.2 Preemptive and Sequential Equilibria

An *equilibrium* for the subgame Γ^ϑ is a pair of extended pure strategies $\big((\bar{G}_1^\vartheta, \bar{\alpha}_1^\vartheta), (\bar{G}_2^\vartheta, \bar{\alpha}_2^\vartheta)\big)$, such that for each Firm $i = 1, 2$ and every extended pure strategy $(G_i^\vartheta, \alpha_i^\vartheta)$ it holds that

$$
V_i^\vartheta(\bar{G}_i^\vartheta, \bar{\alpha}_i^\vartheta, \bar{G}_j^\vartheta, \bar{\alpha}_j^\vartheta) \ge V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, \bar{G}_j^\vartheta, \bar{\alpha}_j^\vartheta),
$$

for $j \neq i$. A *subgame perfect* equilibrium is a pair of extended pure strategies $\big((\bar{G}_1, \bar{\alpha}_1), (\bar{G}_2, \bar{\alpha}_2)\big)$, such that for each $\vartheta \in \mathscr{T}$ the pair $\big((\bar{G}_1^\vartheta, \bar{\alpha}_1^\vartheta), (\bar{G}_2^\vartheta, \bar{\alpha}_2^\vartheta)\big)$ is an equilibrium in the subgame Γ^ϑ.

There are several types of equilibria of interest in this model. Fix $\vartheta \in \mathscr{T}$. For Firm i we denote the optimal time of investment, assuming that the other firm cannot preempt, in the subgame Γ^ϑ by $\tau_{L,i}^\vartheta$, i.e.

$$
\tau_{L,i}^\vartheta = \inf\left\{ t \ge \vartheta \;\middle|\; X_t \ge x_i^L \right\}.
$$

We also define the *preemption region* as the part of the state space where both firms prefer to be the leader rather than the follower, i.e.

$$
\mathcal{P} = \left\{ x \in \mathbb{R}_+ \mid (L_1(x) - F_1(x)) \wedge (L_2(x) - F_2(x)) > 0 \right\}.
$$

The first hitting time of $\mathcal{P}$ in the subgame Γ^ϑ is denoted by τ_P^ϑ.

We distinguish between two different equilibrium concepts. Lemma (2) determines a preemptive equilibrium.

Lemma 2. *(Riedel and Steg (2014)) Suppose $\vartheta \in \mathscr{T}$ satisfies $\vartheta = \tau_P^\vartheta$ a.s. Then $\big((G_1^\vartheta, \alpha_1^\vartheta), (G_2^\vartheta, \alpha_2^\vartheta)\big)$ given by*

$$
\alpha_i^\vartheta(t) = \begin{cases} 1 & \text{if } t = \tau_P^t,\ L_t^j = F_t^j, \text{ and } (L_t^i > F_t^i \text{ or } F_t^j = M_t^j) \\ 1_{L_t^1 > F_t^1} 1_{L_t^2 > F_t^2} \frac{L_t^j - F_t^j}{L_t^j - M_t^j} & \text{otherwise.} \end{cases}
$$

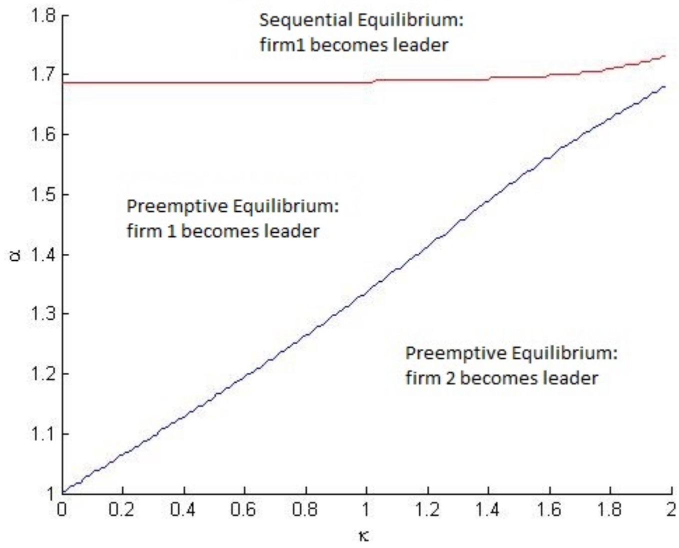

Figure 2: The resulting equilibria with respect to κ and α for the values $D_{10} = 1.5$, $D_{11} = 1$, $D_{01} = -0.01$, $r = 0.05$, $\sigma = 0.1$ and $\mu = 0.02$.

for any $t \in [\vartheta, \infty)$ *and* $G_i^\vartheta = 1_{t \geq \vartheta}$*,* $i = 1, 2$*,* $j \in \{1, 2\}$ i*, are an equilibrium in the subgame at* ϑ*.*

In a preemptive equilibrium both firms try to preempt each other. Investment takes place sooner than it optimally would, i.e. the time one firm would invest without the fear of being preempted. The resulting equilibrium in the latter case is called sequential equilibrium. For certain underlying parameters (see Figure (2)) the preemption time τ_P^ϑ is greater than the the optimal time to invest $\tau_{L,i}^\vartheta$ for firm i. A sequential equilibrium is given by the next lemma.

Lemma 3. *Suppose* $\vartheta = \tau_{L,i}^\vartheta < \tau_P^\vartheta$ *for one* $i \in \{1, 2\}$*. Then* $\left((G_1^\vartheta, \alpha_1^\vartheta), (G_2^\vartheta, \alpha_2^\vartheta)\right)$ *given by*

$$\alpha_i^\vartheta(\vartheta) = 1, G_i^\vartheta(t) = 0 \textit{ for all } t < \vartheta, G_j^\vartheta(t) = 0 \textit{ for all } t \leq \vartheta$$

are an equilibrium in the subgame at ϑ*.*

Proof. The stopping time $\tau_{L,i}^\vartheta$ is determined in Proposition (1) as the stopping time that maximizes the leader payoff. Hence, without the threat of being preempted by its opponent, i.e. $\tau_{L,i}^\vartheta < \tau_P^\vartheta$, it is not optimal to deviate for firm i. Firm j does not want to stop before $\tau_{L,i}^\vartheta$ as its payoff of becoming the leader is strictly smaller than becoming the follower up to τ_P^ϑ. ■

Now, we are finally able to formulate a subgame perfect equilibrium for our game.

Theorem 2. *There exists a subgame perfect equilibrium* $((G_1, \alpha_1), (G_2, \alpha_2))$ *with* α_i^ϑ *and* G_1^ϑ *given by* *(i) Lemma* (2) *if either* $\vartheta \geq \tau_P^\vartheta$ *or* $\tau_P^\vartheta \leq \tau_{L,i}^\vartheta$*.*

(ii) Lemma (3) *otherwise (i.e.* $\vartheta < \tau_P^\vartheta$ *and* $\tau_P^\vartheta > \tau_{L,i}^\vartheta$*).*

Proof. Optimality for case (ii) follows along the same lines as the proof of Lemma (3).

If $\vartheta \geq \tau_P^\vartheta$, then optimality for case (i) follows directly from Lemma (2). What remains to proof is that, in case $\vartheta < \tau_P^\vartheta$, neither of the firms wants to invest sooner than τ_P^ϑ.

We start with firm 1. Suppose that Firm 2 plays the preemption equilibrium strategy. Then if firm 1 plays the preemption strategy, its payoff is $V_1(x) = \mathsf{E}_x[e^{-r\tau_P} L_1(x_P)]$, for any $x < x_P$. [This is the case, because, either the other firm is indifferent between the leader and follower role at x_P, in which case firm 1 becomes the leader, or firm 1 is indifferent in which case $F_1(x_P) = L_1(x_P)$.] Note that V_1 is a strictly increasing function, with $V_1(x_P) = L_1(x_P)$ and $V_1(0) = 0 > L_1(0)$, so that $V_1(x) > L_1(x)$ for any $x < x_P$. The only deviations $\hat{\tau}$ that could potentially give a higher payoff have $\hat{\tau} < \tau_P$, P-a.s. Consider the first hitting time $\hat{\tau}$ of some $\hat{x} < x_P$. Let $\hat{V}_1$ denote the payoff to firm 1 of this strategy (while the other firm plays its preemption strategy). For $\hat{x} \leq x < x_P$, it holds that $\hat{V}_1(x) = L_1(x) < V_1(x)$.

For $x < \hat{x}$, note that $\hat{V}_1(x) = \left(\frac{x}{\hat{x}}\right)^\beta L_1(\hat{x}) = \frac{L_1(\hat{x})}{\hat{x}^\beta}$. Consider the mapping $x \mapsto \frac{L_1(x)}{x^\beta}$. This function attains its maximum at $x_1^L > x_P$. Therefore, its derivative is positive on $(0, x_P)$, implying that $V_1(x) > \hat{V}_1(x)$. Any stopping time τ can be written as a mixture of first hitting times. So, no stopping time $\hat{\tau}$ with $\hat{\tau} < \tau_P$, P-a.s. yields a higher payoff than τ_P. For firm 2, the argument is similar after realizing that $V_1(x) = \frac{L_2(x_P)}{x_P^{\beta(\underline{\mu})}} x^{\beta(\underline{\mu})}$ and $\hat{V}_1(x) = \frac{L_2(\hat{x})}{\hat{x}^{\beta(\underline{\mu})}} x^{\beta(\underline{\mu})}$. This holds because $x_P < x_2^L < x^*$, so that $\underline{\mu}$ is the trend under the worst-case measure for every $x \in (0, x_P]$.

■

Figure (2) shows the resulting equilibria with respect to different values of cost disadvantage and degree of ambiguity. In a world without ambiguity, the firm that has the lower investment cost always becomes the leader (cf. Pawlina and Kort (2006)). This result, however, might change if ambiguity is introduced. Although the non-ambiguous firm has a higher cost of investment, it might become the leader anyway. Ambiguity, therefore, might outbalance the cost advantage.

Appendix

A Proof of Proposition (1)

The proof is very similar to the proof of Theorem (1). We take the same procedure, but now considering the value function in the continuation region, i.e. before any investment has taken place. Applying the BSDE approach and adding beside the value matching condition a smooth pasting condition eventually yield the desired stopping time.

Proof.

Denote

$$Y_t = \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Big[\int_t^{\tau_{L,1}^t} e^{-r(s-t)}D_{00}X_s ds + \int_{\tau_{L,1}^t}^{\tau_2^F} e^{-r(s-t)}D_{10}X_s ds + \int_{\tau_2^F}^{\infty} e^{-r(s-t)}D_{11}X_s ds\Big|\mathcal{F}_t\Big]$$

Using the time consistency property of a rectangular set of density generators yields

$$\begin{aligned}
Y_t &= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Big[\int_t^{\tau_{L,1}^t} e^{-r(s-t)}D_{00}X_s ds + \int_{\tau_{L,1}^t}^{\tau_2^F} e^{-r(s-t)}D_{10}X_s ds + \int_{\tau_2^F}^{\infty} e^{-r(s-t)}D_{11}X_s ds\Big|\mathcal{F}_t\Big] \\
&= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Bigg[\inf_{Q'\in\mathscr{P}^\Theta} \mathsf{E}^{Q'}\Bigg[\int_t^{\tau_{L,1}^t} e^{-r(s-t)}D_{00}X_s ds + \int_{\tau_{L,1}^t}^{\tau_2^F} e^{-r(s-t)}D_{10}X_s ds \\
&\qquad + \int_{\tau_2^F}^{\infty} e^{-r(s-t)}D_{11}X_s ds\Big|\mathcal{F}_{\tau_{L,1}^t}\Bigg]\Bigg|\mathcal{F}_t\Bigg] \\
&= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Bigg[\int_t^{\tau_{L,1}^t} e^{-r(s-t)}D_{00}X_s ds + e^{-r(\tau_{L,1}^t-t)}\inf_{Q'\in\mathscr{P}^\Theta} \mathsf{E}^{Q'}\Bigg[\int_{\tau_{L,1}^t}^{\tau_2^F} e^{-r(s-\tau_{L,1}^t)}D_{10}X_s ds \\
&\qquad + \int_{\tau_2^F}^{\infty} e^{-r(s-\tau_{L,1}^t)}D_{11}X_s ds\Big|\mathcal{F}_{\tau_{L,1}^t}\Bigg]\Bigg|\mathcal{F}_t\Bigg] \\
&= \inf_{Q\in\mathscr{P}^\Theta} \mathsf{E}^Q\Bigg[\int_t^{\tau_{L,1}^t} e^{-r(s-t)}D_{00}X_s ds + L_1(x_{\tau_{L,1}^t})\Big|\mathcal{F}_t\Bigg],
\end{aligned}$$

Chen and Epstein (2002) show that Y_t solves the BSDE

$$-dY_t = g(Z_t)dt - Z_t dB_t,$$

for the *generator*

$$g(z) = -\kappa|z| - rY_t + X_t D_{00}.$$

The boundary condition is given by

$$Y_{\tau_{L,1}^t} = L(x_1^L),$$

Denote the present value of the leader payoff by Λ, i.e.

$$\Lambda(x_t) = Y_t.$$

The non-linear Feynman-Kac formula implies that Λ solves the non-linear PDE

$$\mathscr{L}_X\Lambda(x) + g(\sigma x\Lambda'(x)) = 0.$$

Hence, Λ solves

$$\frac{1}{2}\sigma^2 x^2 \Lambda''(x) + \mu x \Lambda'(x) - \kappa\sigma x \left|\Lambda'(x)\right| - r\Lambda(x) + D_{00}x = 0. \tag{A.1}$$

In the continuation region the leader function has to be increasing, hence $\Lambda' > 0$. This implies that $\underline{\mu}$ is the worst case in the continuation region.

Therefore, equation (A.1) becomes

$$\frac{1}{2}\sigma^2 x^2 \Lambda''(x) + (\mu - \kappa\sigma)x\Lambda'(x) - r\Lambda(x) + D_{00}x = \frac{1}{2}\sigma^2 x^2 \Lambda''(x) + \underline{\mu} x \Lambda'(x) - r\Lambda(x) + D_{00}x = 0.$$

The general increasing solution to this PDE is

$$\Lambda(x) = \frac{D_{00}x}{r - \underline{\mu}} + A_2 x^{\beta(\underline{\mu})}.$$

We have to distinguish two cases here. Either the condition given in theorem (1) holds which means that the boundary condition takes the form (12) or the boundary condition becomes (13).

We will show that for both cases, the optimal threshold to invest becomes

$$x_1^L = \frac{\beta(\underline{\mu})}{\beta(\underline{\mu}) - 1} \frac{I_1(r - \underline{\mu})}{D_{10} - D_{00}}. \tag{A.2}$$

Let's first take a look at the former case. The value matching condition is given by

$$L_1(x_{\tau_{L,1}^t}) = \frac{D_{10}x_t}{r - \underline{\mu}} + \left(\frac{x_t}{x_2^F}\right)^{\beta(\underline{\mu})} \frac{D_{11} - D_{10}}{r - \underline{\mu}} x_2^F - I. \tag{A.3}$$

In addition to the value matching condition we apply another boundary condition called smooth pasting. Smooth pasting implies that the derivatives of the value function Λ and L coincide at $x_{\tau_{L,1}^t}$, i.e.

$$\Lambda'(x_{\tau_{L,1}^t}) = L_1'(x_{\tau_{L,1}^t}). \tag{A.4}$$

This condition ensures differentiability at the investment threshold.

Applying condition (A.4) yields

$$\frac{D_{00}}{r - \underline{\mu}} + \beta(\underline{\mu}) A_2 {x_1^L}^{\beta(\underline{\mu})-1} = \frac{D_{10}}{r - \underline{\mu}} + \left(\frac{x_1^L}{x_2^F}\right)^{\beta(\underline{\mu})-1} \frac{D_{11} - D_{10}}{r - \underline{\mu}}$$

$$A_2 = \frac{D_{10} - D_{00}}{r - \underline{\mu}} \frac{1}{\beta(\underline{\mu})} \frac{1}{{x_1^L}^{\beta(\underline{\mu})-1}} + \frac{D_{11} - D_{10}}{r - \underline{\mu}} \frac{1}{{x_2^F}^{\beta(\underline{\mu})-1}}.$$

The value matching condition gives

$$\frac{D_{00} - x_1^L}{r - \underline{\mu}} + \left(\frac{D_{10} - D_{00}}{r - \underline{\mu}} \frac{1}{\beta(\underline{\mu})} \frac{1}{x_1^{L\beta(\underline{\mu})-1}} + \frac{D_{11} - D_{10}}{r - \underline{\mu}} \frac{1}{x_2^{F\beta(\underline{\mu})-1}} \right) x_1^{L\beta(\underline{\mu})} = \frac{D_{10} - x_1^L}{r - \underline{\mu}} + \left(\frac{x_1^L}{x_2^F} \right)^{\beta(\underline{\mu})} \frac{D_{11} - D_{10}}{r - \underline{\mu}} x_2^F - I_1$$

$$\frac{D_{00} - x_1^L}{r - \underline{\mu}} + \frac{D_{10} - D_{00}}{r - \underline{\mu}} \frac{1}{\beta(\underline{\mu})} x_1^L = \frac{D_{10} x_1^L}{r - \underline{\mu}} - I_1$$

$$I = \frac{\beta(\underline{\mu}) - 1}{\beta(\underline{\mu})} \frac{D_{10} - D_{00}}{r - \underline{\mu}} x_1^L,$$

and therefore

$$x_1^L = \frac{\beta}{\beta - 1} \frac{I_1 (r - \underline{\mu})}{D_{10} - D_{00}}.$$

In case $\underline{\mu}$ is not always the worst case, the boundary value $L_1(x_1^L)$ differs. We have

$$L_1(x_1^L) = \frac{D_{10} x_1^L}{r - \underline{\mu}} + A_1 x_1^{L\beta(\underline{\mu})} - I_1,$$

where

$$A_1 = \left(\frac{D_{10} x^*}{r - \overline{\mu}} - \frac{D_{10} x^*}{r - \underline{\mu}} \right) \frac{1}{x^{*\beta(\underline{\mu})}} + \left(\frac{x^*}{x_2^F} \right)^{\beta(\overline{\mu})} \left(\frac{D_{11}}{r - \underline{\mu}} - \frac{D_{10}}{r - \overline{\mu}} \right) x_2^F \frac{1}{x^{*\beta(\underline{\mu})}}.$$

Here smooth pasting gives

$$\frac{D_{00}}{r - \underline{\mu}} + \beta(\underline{\mu}) A_2 x_1^{L\beta(\underline{\mu})-1} = \frac{D_{10}}{r - \underline{\mu}} + \beta(\underline{\mu}) A_1 x_1^{L\beta(\underline{\mu})-1}$$

$$A_2 = \frac{D_{10} - D_{00}}{r - \underline{\mu}} \frac{1}{\beta(\underline{\mu})} \frac{1}{x_1^{L\beta(\underline{\mu})-1}} + A_1.$$

Applying the value matching condition finally yields

$$\frac{D_{00} x_1^L}{r - \underline{\mu}} + \left(\frac{D_{10} - D_{00}}{r - \underline{\mu}} \frac{1}{\beta(\underline{\mu})} \frac{1}{x_1^{L\beta(\underline{\mu})-1}} + A_1 \right) x_1^{L\beta(\underline{\mu})} = \frac{D_{10} x_1^L}{r - \underline{\mu}} + A_1 x_1^{L\beta(\underline{\mu})}$$

$$I = \frac{\beta(\underline{\mu}) - 1}{\beta(\underline{\mu})} \frac{D_{10} - D_{00}}{r - \underline{\mu}} x_1^L,$$

and the result follows. ■

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