Discounting, value and dollar beta matching in flexible cashflow systems.

Steinar Ekern, Mark B. Shackleton and Sigbjørn Sødal

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Abstract

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The paper applies a real options framework to a value a firm with flexibility to switch between different operational modes. Developing conditions that are equivalent to smooth pasting, it uses discount factors to capture the value and beta of each options that is present.

When betas and investment thresholds are known, it shows how to solve for option values and investment costs. For multiple modes and thresholds, the paper proposes a matrix based solution that significantly improves computations and intuition.

C61, G31. Key Words; Value matching, smooth pasting, dollar beta and rate of return equalisation, cashflow networks and bipartite directed graphs.

1 Introduction

When a *financial* option is exercised, tradeable quantities are exchanged; for instance upon exercise a call option's value coupled with the strike price is swapped for a stock. This is a simple example of the benefit of flexibility which allows the present value of one cash flow (the certain interest on a strike price) to be converted into another (the uncertain dividend flow on a stock).[1]

This article follows those in the literature on management of real assets, where there is flexibility to switch between *operational* cashflow types. Many articles have made the analogy between operational and financial options, where capital investment including running costs, is viewed as a strike price and operational value a stock price.[2] These techniques are employed to evaluate capacity and other investment decisions.[3]

Although less has been written on the acquisition or creation[4] of such option flexibility, their worth is also based on an exercise condition at investment. Managers maximise flexibility value by ensuring optimal timing for these investment decisions. After an investment, if further flexibility remains (i.e. the use of one option creates another) optimal timing should be driven by maximisation of the joint value of both initial and subsequent options.

When an option is exercised and an investment transition made, we therefore recognise that flexibility is unlikely to end and this article accounts for the simultaneous application and creation of operational options. Furthermore these follow-on options influence the timing of prior exercise; a decision to proceed can not be taken without anticipating follow on flexibility and its optimal decision too. Our valuation procedure tracks sequential flexibility through operating modes, requiring the values at optimal investment

[1] See Black and Scholes [4], Merton [17]; also Cox, Ross and Rubinstein [7] for the risk neutral valuation of options.

[2] See Myers [18], Brennan and Schwartz [5], McDonald and Siegel [16] and the texts of Dixit and Pindyck [10] and Trigeorgis [24]. Baldwin and Clark [3] with Gamba and Fusari [14] motivate and value the real options of project design using six principles of modularity (splitting, substitution, augmenting, excluding, inversion and porting). Latterly these are valued using multidimensional stochastic processes and methods.

[3] Including capacity (Pindyck [19]), land (Capozza and Li [6]), costly reversibility (Abel and Eberly [2] and Eberly and van Mieghem [12]), marginal cost of capital (Abel, Dixit, Eberly and Pindyck [1]), information (Lambrecht and Perraudin [15]), capital structure and debt valuation (Sarkar and Zapatero [20]).

[4] Simple calls (and puts where the exchange is reversed) in financial markets are sold by option traders to clients for a premium or fee which is designed to cover the hedging strategy and some profit. In return the client enjoys the financial flexibility described. Operational or real options are not explicitly purchased in this manner but are embedded in ownership of any physical asset.

or divestment times, i.e. when the options are applied and created, to be tracked.

For an option's value to be maximised by choice of its decision threshold and time, so called smooth pasting (or tangency) must also hold. At an optimal threshold, the value functions of the option and that of the net payoff must converge (value matching) and at that transition point, do so tangentially (smooth pasting). That is the two value functions must meet and join with equal first derivatives there.

When flexibility is exercised at a decision threshold, we make use of the intuition embedded in the first order condition. Take for example an American style call option on a stock (i.e. one that can be exercised early). At the critical threshold that represents the optimal exercise point, the derivatives (slope of functions with respect to the stock price) for both the option and its payoff must match. The first derivative of an option price with respect to its stock is known as *delta*, so this condition says that upon exercise a call option should have a delta of one (unity being the slope of a call).

Options also have an *elasticity*, defined as this delta (or slope) multiplied by the ratio of the stock price to the option price. Although similar to a profit mark-up (as shown by Dixit, Pindyck and Sødal [11]), we interpret the option elasticity as a (CAPM) *beta* relative to its underlying investment. For instance a call option that is exercised in the money at a critical price to strike ratio of two, will have a relative beta of two.[5] Because it contains a 2:1 levered position, this means that the local rate of return (or drift) is double that of the underlying. Thus the option beta (or local rate of return in Shackleton and Sødal [21]) can be used as a metric in optimal exercise timing.

When the use of one option generates the creation of another, we apply value and weighted beta matching to account for their interactions at exercise. In addition to equations that track the value before and after each transition, this is done by representing first order conditions including the beta of options weighted by the dollar value of the options.

The other element required to complete our analysis, is a discount factor representation of option values.[6] This provides both the means to determine betas and also to link option values that are separated by threshold and time.

Section two gives examples of how dollar beta matching is useful in the early exercise of American style put and call options, both individually and when they interact in an operational setting. Section three incorporates

[5] In the next section a $4 stock price threshold and an option value of $2 commands $4 of stock with $2 of borrowing implicit in the option hedge; this has a relative beta of 2.

[6] See Dixit, Pindyck and Sødal [11] and Sødal [23] for the discount factor approach.

interactions with a third threshold and mode, using geometric Brownian to illustrate and matrix algebra to solve the system. Section four introduces a fourth threshold and solves a more advanced problem with two way switching. Section five concludes.

2 Beta matching and the discount function

2.1 Definition of beta matching with a simple application

For an option $V(P)$, its sensitivity to underlying state variable P is known as its delta, $\Delta(P)$. All assets, not just stocks with respect to the market, have a beta; for *options* this quantity may be dynamic like delta. This beta, or elasticity $\beta(P)$, tracks the sensitivity of $\partial V(P)/V(P)$ an option's percentage change to the return $\partial P/P$ of its stochastic underlying driver P (assumed to have unit beta). The beta is closely linked to the delta and a third quantity shown in (1).

$$\Delta(P) = \frac{\partial V(P)}{\partial P} \qquad \beta(P) = \frac{P}{V(P)}\frac{\partial V(P)}{\partial P} \qquad \beta(P)V(P) = P\frac{\partial V(P)}{\partial P} \quad (1)$$

The last quantity in (1) is the dollar beta, i.e. the regular beta scaled by the dollar value $V(P)$ of the option itself; it measures the dollar impact of the beta, i.e. weighted by its value and this is equivalent to the delta scaled by the underlying P.

Since the delta at early exercise of an option has a particular value ($\Delta(P) = 1, -1$), knowledge of the critical threshold in P implies knowledge of the dollar beta. Furthermore knowledge of $\beta(P)$ at that point would then also imply the value of the option value there $V(P) = P\Delta(P)/\beta(P)$.

There may be empirical situations where $V(P)$ and optimal exercise decisions are observable; for example in the market for exchange traded American call or put options. Given synchronous high frequency observations of options and underlying, close to exercise a local regression of the returns on $V(P)$ against P would recover this beta.

If data is not available and $V(P)$ needs modelling, we will show how the assumed process for P determines $\beta(P)$. Typically such modelling also determines optimal exercise thresholds but in this paper we assume that these are given first and see what other option characteristics are then determined.

There are two motivations for this assumption; firstly at least in some empirical contexts, thresholds may be observable. Secondly, along with the dynamics for P and therefore $\beta(P)$, from the structures in this paper we show that starting with given thresholds allows values and costs to be derived using matrix algebra.

Although the objective of stochastic modelling is usually to determine thresholds from option strikes, the set of thresholds in our system provides an adequate information set to determine other variables explicitly. The disadvantage is that whilst option and strike values can be expressed as a matrix product of payoff information at thresholds, due to the mixed and non-linear dependency of payoffs on their threshold and discount rates between thresholds, despite using a matrix system, these thresholds cannot be expressed as linear algebra of strikes etc.

To partially illustrate this approach, here is an example that shows how option prices are calculated when thresholds and betas are known ahead of values and strikes.

To emphasize the prime role of fixed thresholds as a framework they are numbered e.g. $P_1 ... P_4$ and set numerically to 1...4 in examples. Whilst it may seem more attractive to label these with a subscript that denotes their nature, two points are noteworthy; firstly their ordering is important and secondly, since a matrix row will come to depend on each threshold, numbering the thresholds according to their row in that matrix or vector aides their tracking. Initially only $P_4 = 4$ and $P_1 = 1$ are used; later $P_3 = 3$ and $P_2 = 2$ are added.

To quantify and demonstrate, take an American style call with value[7] $V_c(P)$ on a stock P. The delta of this option is $\Delta_c(P) = \partial V_c(P)/\partial P$; multiplying this by the underlying price gives the dollar beta, then dividing this by the option value yields its beta $\beta_c(P) = P\Delta_c(P)/V_c(P)$. Note that an option's *scale* plays a role in its dollar beta but not in its beta.

Suppose upon the stochastic price P reaching the fixed value $P_4 = \$4$ we see this option being exercised; since the delta there is unity $\Delta_c(P_4) = 1$, the dollar beta at this point is \$4. We will assume that the location of thresholds such as P_4 are known.

Suppose we also know that the beta $\beta_c(P_4)$ at this threshold is 2; if not measure by regression from market data, then it needs derivation from an uncertainty and stochastic model (in due course). From these facts we can infer $V_c(P_4) = \$2$.

$$\beta_c(P_4) = 2 = \frac{P}{V_c(P)} \frac{\partial V_c(P)}{\partial P}\bigg|_{P=P_4} = \frac{P_4}{V_c(P_4)} = \frac{4}{2} \tag{2}$$

[7]Option values and betas are labelled $V(P)$ and $\beta(P)$ but have subscripts that distinguish their type, c for calls, p for puts and later i for idle, f for full. Apart from this labelling, in the GBM case constants c, p are also used for their particular beta values.

Typically it is the strike price that is a given and the exercise threshold $P_4(X_4)$ that is determined as a function of X_4, but here treating the threshold $P_4 = \$4$ and option beta $\beta_c(P_4) = 2$ as given we can infer the implied strike price $X_4(P_4)$. At this critical underlying price P_4, the call option value converts into a payoff. With a strike labelled X_4 (because it is applied at P_4) and from $P_4 = \$4$ we also can infer $X_4(P_4) = \$2$.

$$\begin{aligned} V_c(P_4) &= P_4 - X_4 = 2 \\ \beta_c(P_4)V_c(P_4) &= P_4 = 4 \end{aligned} \tag{3}$$

At this threshold, smooth pasting implies that the slope or delta of the option and its payoff match so that the differential of the first line in (3) at P_4 confirms this. However the second line of (3) is a *dollar beta* representation (from (2)), where the beta $\beta_c(P_4)$ has been applied on the left to $V_c(P)$ and the unit beta (of P) to P on the right. At the point of exercise, these equations say that the beta of $V_c(P)$ is double that of the underlying and that the option (\$4 of stock net of \$2 borrowing) has twice the local rate of return (or drift) of P in the CAPM world.

As well as a call option to capture revenue once potential value reaches P_4, there is an analogous put. The operating argument for puts rests on the flexibility to suspend operations from running to the idle condition. For a stand alone American style put $V_p(P)$, we consider exercise at $P_1 = \$1$. Smooth pasting requires a delta at P_1 of -1 and (from an uncertainty model) if we also know the put's beta is $\beta_p(P_1) = -1$, we can infer the put value itself $V_p(P_1) = \$1$ at exercise.

$$\beta_p(P_1) = -1 = \frac{P}{V_p(P)} \frac{\partial V_p(P)}{\partial P}\bigg|_{P=P_1} = -\frac{P_1}{V_p(P_1)} = -\frac{1}{1} \tag{4}$$

Due to the payoff and dollar beta (-1) conditions in the two lines of (5), this is consistent with an exercise price $X_1 = \$2$ for the put (labelled X_1 because it occurs at P_1; it is only the same as X_4 because $\beta_p(P_1)$ was chosen as -1).

$$\begin{aligned} V_p(P_1) &= X_1 - P_1 = 1 \\ \beta_p(P_1)\, V_p(P_1) &= -P_1 = -1 \end{aligned} \tag{5}$$

The first line of (5) smooth pastes with a delta of $\Delta_p(P_1) = -1$, i.e. the put payoff has unit negative slope. The negative beta indicates a put return or local drift rate of the risk free less the market risk premium (\$2 on deposit with a \$1 liability on the short hedge). The second line of (5) is the dollar beta matching condition, and given $\beta_p(P_1), P_1$ this is written beneath the value matching condition to give it equal status to the value condition (first line of 5) in jointly determining $X_1, V_c(P_1)$ via linear algebra.

Thus if the betas and thresholds of options are known, they can be used to infer option prices and strikes at exercise. So far we have not yet utilised a specific diffusion and β_c, β_p were chosen arbitrarily; these can be determined for a stochastic flow π and value P that follows a Brownian motion B geometrically.

$$\frac{d\pi}{\pi} = \frac{dP}{P} = (r - \delta)\, dt + \sigma dB \tag{6}$$

Under this risk neutral diffusion, if the revenue flow π has drift $r - \delta$ (where r is the risk free rate and δ the yield) $P = \pi/\delta$ follows the same diffusion.[8]

Claims such as discount factors $V(P)$ that are contingent on reaching a certain level must satisfy a Bellman equation. This condition is derived from the risk neutral expectation of a local change generating a risk free rate of return over an interval dt. Applying expectations to the Ito's lemma expansion[9], we require $E^{RN}[dV(P,.)] = rV(P,.)dt$.

For an option claim $V(P)$ that depends on the proximity of P to a threshold, an asset pricing equation generates option solutions $V_c(P) \propto P^{\beta_c}$ and $V_p(P) \propto P^{\beta_p}$. Beta constants $\beta_c = c$ (for the call) or $\beta_p = p$ (for the put) are given by a quadratic in (7).

$$\begin{aligned} 0 &= \frac{1}{2}\sigma^2 P^2 \frac{\partial^2 V(P)}{\partial P^2} + (r-\delta) P \frac{\partial V(P)}{\partial P} - rV(P) \\ c, p &= \frac{1}{2} - \frac{r-\delta}{\sigma^2} \pm \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \end{aligned} \tag{7}$$

2.2 A two mode case and the definition of discount factors

As suggested, the put may not be financial but operational in nature and therefore not stand alone. Now we represent the present value P of revenues π and costs X, whilst option values V_c, V_p represent the flexibility value of launch or closure decisions on π.

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[8]

$$P = \int_0^\infty E^{RN}[\pi(t)]\, e^{-rt} dt = \int_0^\infty \pi e^{-\delta t} dt = \left[-\frac{\pi}{\delta} e^{-\delta t}\right]_0^\infty = \frac{\pi}{\delta}.$$

[9] Unlike risk neutral, using physical or real world (CAPM) expectations $E^{RW}[.]$ to determine local expected returns of dV generates a beta dependent risk premium, i.e. $E^{RW}[dV(P,.)]/V(P,.) = (r + \beta(P) * \text{risk premium})\, dt$. However, for the isoelastic GBM, discounting can occur using CAPM expecations (see Shackleton and Wojakowski [22]).

Upon its exercise into an idle state (when the operational value P and revenue π are sacrificed in return for retrieving, from a prior investment, a fixed value X_1), it is possible that the call option to relaunch may be acquired. The condition at the put exercise point must then embrace the acquisition of $V_c(P)$.

On the left hand side of the next equation (8), the put value $V_p(P)$ is added to the present value of perpetual revenue P less a present value cost[10] X_1. This sum is foregone upon suspension but a call option to relaunch $V_c(P)$ is acquired; this goes on the right of equation (8) which matches value when $P = P_1$.

$$V_p(P_1) + P_1 - X_1 = V_c(P_1) \tag{8}$$

This differs from (5) because now more flexibility is present. Until the optimal point $P = P_1$, values are dynamic in P and equality only holds at P_1. However up to and including that time, the slopes and betas with respect to P can be calculated to portray proximity to the first order condition through betas and dollar betas.

Smooth pasting is akin to examining slopes on either side of such value equations. Differentiating equation (8) by P (in the limit as it reaches P_1) and multiplying by P represents $P\partial V(P)/\partial P = \beta(P)V(P)$ the dollar beta of $V(P)$ on either side of (8).[11] Therefore as well as (8), if smooth pasting holds, *dollar* (or value weighted) betas must match at $P = P_1$ in (9).

$$\beta_p\,(P_1)\,V_p(P_1) + 1.P_1 = \beta_c(P_1)V_c(P_1) \tag{9}$$

Now compared to (8), the put value $V_p(P)$ has been weighted by its beta $\beta_p\,(P)$, the value of operations P (foregone) by a beta of 1 and the call value $V_c(P)$ weighted by its beta $\beta_c(P)$. Since its beta is zero, X_1 has disappeared.

For decisions involving follow on flexibility, smooth pasting demands rate of return equalisation across value, including acquired options. Optimal exercise of the put V_p requires that the sum of dollar betas must match at P_1 including options acquired V_c, so this call has an influence through $\beta_c(P)V_c(P)$ at P_1.

[10] This strike or costs might have composition $k/r - K_1$, where k is a perpetual cost rate of operations saved on suspension and K_1 is a one time and irrecoverable cost that diminishes these savings. Whilst this paper solves for X_1, it is not directly concerned with the breakdown $k/r - K_1$ and these quantities are shown to motivate X_1 only (similarly for $X_4 = k/r + K_4$).

[11] Taking the beta of the each side of (8), would make sense for the right ($V_c(P)$) but not for the left hand side ($V_p(P)+P-X$). The dollar beta operator however produces the sum of dollar components directly (i.e. $\beta_p\,(P_1)\,V_p(P_1) + 1.P_1$ on the left of 8). This is also important if an option subject to knock out can attain a zero value (due to complementarity, in section 4). The dollar beta operator $P\partial V_p(P)/\partial P$ is robust to this situation.

Now suppose that this system is closed by the two thresholds P_1, P_4; this standard hysteresis has been studied before (Dixit [9]). We apply the same logic at P_4 where the call is exercised (not as in (3)) but including recovery of the put we have motivated for use at P_1. Value matching is easy to monitor by inclusion of the put on the right hand side of (3), and smooth pasting via the means of dollar beta matching just described (for both options and the payoffs at the boundary, differentiation and P scaling).

$$\begin{aligned} V_c(P_4) &= V_p(P_4) + P_4 - X_4 \\ \beta_c(P_4)V_c(P_4) &= \beta_p(P_4)\,V_p(P_4) + 1.P_4 \end{aligned} \tag{10}$$

Now equations (8-10) give four conditions for the six unknowns $V_c(P_4)$, $V_c(P_1)$, $V_p(P_4)$, $V_p(P_1)$, X_4, X_1 as a function of known thresholds P_4, P_1 and assumed betas $\beta_c(P_4)$, $\beta_p(P_4)$, $\beta_c(P_1)$, $\beta_p(P_1)$. Two more conditions are required, i.e. six in total.

For the threshold pair P_1, P_4 the aim is to include the betas $\beta(P)$ there in boundary conditions (such as 8) for use in simultaneous linear algebra alongside those of value (jointly in 10).

The four conditions presented so far apply at investment or divestment; the two extra required equations link between these transitions. They involve a relationship between $V_c(P_1)$ & $V_c(P_4)$ and another between $V_p(P_4)$ & $V_p(P_1)$.

For a given separation, the means to link the relative scale of option values at two different transition times (i.e. thresholds) is to use discount functions. Discount functions are option solutions with a standardised unit payoff. They also provide the means to determine $\beta_{p,c}(P)$.

The exercise time and life of each option depends on an unknown time taken for its state variable P to travel from a current value to a given threshold. For time homogeneous problems, valuation can occur using stochastic stopping time methods. Although the time taken for P to evolve from P_1 to P_4 is unknown, expectations can be formed with respect to τ and its random continuous discount $e^{-r\tau}$ where[12] $P(\tau) = P_4$.

In common with Dixit, Pindyck and Sødal [11], we consider the present value of \$1 paid at the boundary P_4. From the perspective of $P = P_1 \ll P_4$, this threshold is far away in both terms of P and time τ. Thus the

[12] To separate them for subscripts indicating fixed thresholds, indices for stopping times $P(\tau)$ indicate a value of the uncertain variable P at time τ. These are usually omitted for the initial (known) level $P(0)$ which is abbreviated to P (similarly for π).

For stochastic processes other than GBM, the probability distribution of the stopping time τ can be evaluated and an expected discount factor can be found analytically (see Appendix); numerical methods are also possible in other cases.

present value of the future sum is less than \$1 but increases[13] toward \$1 as P approaches P_4.

Excluding changes from cash flows (with present values $P_4 - X_4$), the payoff to timing flexibility at a threshold is taken as the option value. Values at the option's thresholds (e.g. $V_c(P_4)$ or $V_p(P_1)$) assume the role of scaling constants and the functional form of the flexibility value is carried in a multiplier which is labelled as a discount $D_c(P, P_4)$ for the call or $D_p(P, P_1)$ for the put.

Having previously labelled the call $V_c(P)$, we now label it $V_i(P)$ to indicate the flexibility present when operations are in *idle* mode i. The put $V_p(P)$ is relabelled[14] $V_f(P)$ since it is active when operations are *full*, hence f. Discount functions D_p, D_c are labelled with p, c notation to associate put or call with full or idle states.

Also more than one call or put may present; this allows labels f, i (rather than p, c) to identify the state in which the option is active. Furthermore a third state $V_g(P)$ is introduced later that contains an option that is neither uniquely a put nor a call but a combination of $D_p(P, .)$ and $D_c(P, .)$ and which therefore cannot be labelled by c or p.

For a unit payoff at the random time τ, conditional on current value P the valuation of discount $0 < D_c(P, P_4) \leqslant 1$ is given by the risk neutral expected[15], risk free discounted time τ when $P(\tau) = P_4$.

$$D_c(P, P_4) = E_P^{RN}\left[e^{-r\tau}\middle|\, P(\tau) = P_4\right] \tag{11}$$

The discount factor becomes 1 at the time when P reaches P_4 and is stochastic up to that point. Applied to a dollar payoff at the stopping time, from a prior point P it gives the present value of a one off future cashflow at an uncertain time in the future τ.

From the perspective of a call $V_i(P)$ created at P_1, the discount function has fixed value $D_c(P_1, P_4)$; the first argument represents the initial point and the second the final one when the option is used. Until P_4 is reached the discount factor is dynamic in P, furthermore the required beta at P_4 rests upon the sensitivity of the discount factor with respect to this dynamic variable. Thus the discount factors in (12) perform two option roles in this

[13] Note that if the boundary is above P then the discount factor increases with P (call) but it decreases with P if the boundary to be hit is below (put). This is relevant for the put's time τ to traverse from P_4 down to P_1.

[14] Their betas $\beta_{i,f}$ too are labelled with i, f but for GBM $\beta_i = c$ and $\beta_f = p$.

[15] For a discrete outcome case, if there were risk neutral chances of 1/3 each for the process stopping at $\tau = 1$, $\tau = 10$ or never stopping ($\tau \to \infty$), with risk free $r = 5\%$ the expected discount factor is $(e^{-.05} + e^{-0.5} + 0)/3 = 0.519$. Discount factors can adapt to structures where stopping is never certain or where $E^{RN}[\tau]$ is unbounded.

paper.

$$V_i(P) = D_c\left(P, P_4\right) V_i\left(P_4\right) \qquad\qquad V_i(P_1) = D_c\left(P_1, P_4\right) V_i\left(P_4\right). \tag{12}$$

In (12), the first interpretation gives the means to separate an option's value at the threshold $V_i\left(P_4\right)$ from its dynamics $D_c\left(P, P_4\right)$ whilst the second shows the fixed discount factor used to find the initial value of the option as the expected present value of itself at the next threshold.

Rather than one scaling constant for each option, for a linear algebra approach two are needed; however one is determined from the other using a fixed discount (e.g. on the right in 12). The interpretation on the left is used for calculating option betas.

With option values portrayed via discounting, the end value $V_i\left(P_4\right)$ can be treated as a scaling constant. Therefore differentiating the option $V_i(P)$ or discount function $D_c(P, P_4)$ (in P) is equivalent and gives the same beta function. Since boundary conditions include both value matching and smooth pasting, from (1) the beta at P_1 that is important to rate of return matching can be expressed in $D_c(P, P_4)$ (first line of 13) and evaluated at certain thresholds.

$$\begin{aligned} \frac{P}{V_i\left(P\right)} \frac{\partial V_i\left(P\right)}{\partial P} &= \frac{P}{D_c\left(P, P_4\right)} \frac{\partial D_c\left(P, P_4\right)}{\partial P} = \beta_i\left(P\right) \\ \beta_i\left(P_1\right) &= \left. \frac{P}{D_c\left(P, P_4\right)} \frac{\partial D_c\left(P, P_4\right)}{\partial P} \right|_{P=P_1} = c \end{aligned} \tag{13}$$

The put depends on a discount factor that increases if P falls toward a lower threshold $P_1 \leqslant P$. Its discount factor $D_p(P, P_1)$ applies to \$1 paid at a random time occurring when P falls to P_1

$$D_p\left(P, P_1\right) = E_P^{RN}\left[e^{-r\tau} \right| P\left(\tau\right) = P_1 \left.\right]. \tag{14}$$

From this representation the put value function $V_f(P)$ is given as a dynamic fraction of its final value or for $V_f(P_4)$ a static fraction of $V_f\left(P_1\right)$

$$V_f(P) = D_p\left(P, P_1\right) V_f\left(P_1\right) \qquad\qquad V_f(P_4) = D_p\left(P_4, P_1\right) V_f\left(P_1\right). \tag{15}$$

The put value and discount functions have negative slope and beta in P. Since $\partial D_p\left(P, P_1\right)/\partial P < 0$, differentiating $V_f(P)$ by P and scaling by P gives a negative beta; the beta at P_1 is necessary for rate of return matching at the point of put usage P_1.

$$\begin{aligned} \frac{P}{V_f\left(P\right)} \frac{\partial V_f\left(P\right)}{\partial P} &= \frac{P}{D_p\left(P, P_1\right)} \frac{\partial D_p\left(P, P_1\right)}{\partial P} = \beta_f\left(P\right) \\ \beta_f\left(P_1\right) &= \left. \frac{P}{D_p\left(P, P_1\right)} \frac{\partial D_p\left(P, P_1\right)}{\partial P} \right|_{P=P_1} = p \end{aligned} \tag{16}$$

The asset pricing equation in (7) says that the discount factors $D(P,.) \propto P^{\beta}$ are iso–elastic claims, either depending on a positive or negative power (beta) $\beta = c > 1$ or $\beta = p < 0$. Their dollar beta is a constant multiple of their value i.e. $P\partial D(P,.)/\partial P = \beta D(P,.)$.

Option claims must satisfy no–bubble conditions $D_c(P,.) \to 0$ as $P \to 0$ and $D_p(P,.) \to 0$ as $P \to \infty$; these match c to the idle state i (the call) and p (the put) to f. Both discount factors reach a unit value, but at their own boundary; $D_c(P_4, P_4) = 1$ and $D_p(P_1, P_1) = 1$.

These complete the specification of standard unit discount factor claims with their constant betas.

$$\begin{array}{lll} i \text{ idle (call) discount} & P \leqslant P_4 & D_c(P, P_4) = (P/P_4)^c \\ f \text{ full (put) discount} & P \geqslant P_1 & D_p(P, P_1) = (P/P_1)^p \end{array} \tag{17}$$

These forms appear in real options paper with GBM processes; they can be combined to form other discount factors that satisfy boundary conditions at both of $P_{4,1}$ (in section 4, terminology D_b, D_t labels these discounts that combine D_c, D_p). The discount factors for other processes are shown in the appendix.

For the GBM process (6), not only are these betas known at both thresholds, but there are constant inbetween, i.e. for all P the $\beta_i(P) = c$. The only restriction is that this elasticity should be greater than unity, here we choose r, δ, σ so that $c = 2$. Also the put betas under GBM $\beta_f(P) = p$ are constant and it is also possible to choose r, δ, σ to also match $p = -1$ which satisfies its constraint of being negative.

To reiterate, discount functions fulfil two useful roles; first they capture the dynamics of values and their betas and second they fix the ratio of options at the time of their creation compared to their use. Even though discount factors do not appear in the value matching condition, their first property is used at that instant when smooth pasting or dollar beta matching in the second line of (10).

Although defined by P_4, P_1, the flexibility system described so far could not have been solved without the discount factors; these provided both the means to determine the betas and also to link beginning and end options.

In Table 1, the extra two conditions are in the first panel along side the two value, two beta matched equations (in the later panels) required to solve the closed hysteresis system; the solutions (to 3 dp) are on their right (knowns are in plain text and solutions in bold).

We present these results in this section without demonstrating the numerical method used to calculate them. Before presenting the matrix solution, we include another possible operating state and intervention point P_2. In

Type	Condition	Values (sols in bold)
Discounting	$V_f(P_4) = D_p(P_4, P_1)\, V_f(P_1)$ $V_i(P_1) = D_c(P_1, P_4)\, V_i(P_4)$	$\mathbf{0.190} = 0.250 \times \mathbf{0.762}$ $\mathbf{0.119} = 0.063 \times \mathbf{1.905}$
Value matching	$V_i(P_4) = V_f(P_4) + P_4 - X_4$ $V_f(P_1) + P_1 - X_1 = V_i(P_1)$	$\mathbf{1.905} = \mathbf{0.190} + 4 - \mathbf{2.286}$ $\mathbf{0.762} + 1 - \mathbf{1.643} = \mathbf{0.119}$
Dollar β match	$cV_i(P_4) = pV_f(P_4) + P_4$ $pV_f(P_1) + P_1 = c(P_1)V_i(P_1)$	$2 \times \mathbf{1.905} = -1 \times \mathbf{0.190} + 4$ $-1 \times \mathbf{0.762} + 1 = 2 \times \mathbf{0.119}$

Table 1: $V_f(P_4), V_f(P_1), V_i(P_1), V_i(P_4), X_4, X_1$ and 6 equations

order to do this, a third cashflow mode is included. Whilst this is easy to incorporate using a Geometric Brownian motion, the results of this paper also pertain for other stochastic processes. The advantage of GBM is that the claims $V_f, V_i(P)$ are iso-elastic, i.e. the betas are constant for all P. Even if betas are not constant but depend on P, the diffusion dynamics and the solution to its asset pricing equation provide the means to calculate particular $\beta_f(P_1)$ and $\beta_i(P_1)$ etc.

3 A three mode case with power flow and matrix solutions

For a different function of the cashflow driver π (which still follows (6)), we now motivate the present value of a third cashflow mode. Before the full state with cashflow π and value P is launched, suppose that there is operational flexibility to engage with π^γ, the γ power of underlying cashflow π. This might arise for instance when a test market is possible during which time the economies to scale γ are different to those in full operations f.

For a GBM flow, an expectation of $\pi(t)^\gamma$ (a uncertain cashflow t years in the future raised to power γ $(0 < \gamma < 1)$) is also proportional to the power of the current value π; i.e. $E^{RN}[\pi(t)^\gamma] \propto \pi^\gamma$.

Like that of the full flow ($P = \pi/\delta$), the perpetuity value for a power of cashflows (proportional to P^{γ}) has a fixed dividend yield[16] with respect to its flow π^{γ}. Before going to full value P, we thus compare a mode of operations engaging in a cashflow with value P^{γ}. Since this occurs with return to scale γ, we label this power state *gamma* with subscript g. Options whilst in this mode are labelled $V_g(P)$.

This mode is beneficial for $P \ll 1$ but detrimental for $P \gg 1$ so the new cashflow pattern π^{γ} is scaled so values of $P = \$1 = P^{\gamma}$ align there (i.e. when $\pi = \delta.\$1$ the present value of the power cashflow and full cashflow are equal $P^{\gamma} = P$ at $P = \$1$). Thus $P = \$1$ is also a rough (without option value) indicator of the desired switching level (however $P_1 = 1$ is maintained as the lowest threshold where idling commences). The actual switching level is designated P_2 so that the call from idle[17] operates from P_1 to P_2 and the call within the new gamma state from P_2 to P_4. In line with the idea of treating thresholds in a clearly labelled hierarchy, we set $P_2 = \$2$ and test the consequences of this policy.

If P_1 has already been encountered and switching out of the full state has occurred, we assess the implication of switching from idle (zero cashflow but with non–zero call/idling value $V_i(P)$) to cashflow π^{γ} with value P^{γ}. This occurs when a lump sum PV cost of X_2 is sunk; this is a strike price to be solved (it may compose a perpetual cost rate k_g/r and a one time cost K_2 but this paper does not treat its decomposition).

We label option values in the power state $V_g(P)$, operations start in this mode after P reaches P_2 and continue until it reaches P_4, there switching to full occurs and $V_f(P)$ is acquired. Thus the separation for $V_g(P)$ is between P_2, P_4 and compared to section 2, the original call from idle has been split into two separate calls, $V_i(P)$ and $V_g(P)$ (hence the need for notation other than $V_c(P)$).

Although we label the value of the second call option in the gamma

[16] The yield δ' on a perpetuity of the power cashflow arises through

$$\begin{aligned}\int_0^{\infty} E^{RN}\left[\pi(t)^{\gamma}\right] e^{-rt}dt &= \pi^{\gamma}\int_0^{\infty} e^{\left(\frac{1}{2}\sigma^2\gamma(\gamma-1)+\gamma(r-\delta)-r\right)t}dt = \pi^{\gamma}\left[\frac{-e^{-\delta' t}}{\delta'}\right]_0^{\infty} = \\ \pi^{\gamma}/\delta' &= P^{\gamma}\delta^{\gamma}/\delta' \text{ where } \delta' = r-\gamma\left(r-\delta\right)-\frac{1}{2}\sigma^2\gamma(\gamma-1).\end{aligned}$$

Note that $0 < \gamma < 1$ implies that δ' lies between r and δ, all are positive. The PV of power flows $\delta^{\gamma}P^{\gamma}/\delta'$ would revert to P for $\gamma = 1$ or to a constant $1/r$ for $\gamma = 0$.

Finally, note that the beta of the present values of the power of these flows with respect to the driver P, is $\frac{P}{P^{\gamma}}\frac{\partial P^{\gamma}}{\partial P} = \gamma$. Although this does not affect the beta of the options themselves, due to the presence of a cashflow with value P^{γ}, it is needed in smooth pasting and dollar beta matching.

[17] In this section, (12) needs revising to $V_i(P_1) = D_c(P_1, P_2)V_i(P_2)$.

cashflow mode $V_g(P)$, this does not mean that $V_g(P)$ has dependency on γ ($V_g(P)$ is not proportional to P^γ). If the uncertainty driver is the same dB and P, π process (6) throughout then $V_g(P)$ will be another straight call $D_c(P, .)$ with beta also given by c from (7). That is to say under GBM a call on P^γ still has the same beta as a call on P.[18]

Using the discount framework, since this extra call also has beta c the same dynamic relationship between initial and final values hold for $V_g(P)$ as $V_i(P)$

$$\beta_g = c \qquad V_g(P) = D_c\left(P, P_4\right) V_g(P_4) \qquad V_g(P_2) = D_c\left(P_2, P_4\right) V_g(P_4). \quad (18)$$

In addition to the four option values in standard hysteresis, note that another two option values and another strike X_2 have been introduced.

When the first call from i idle $V_i(P)$ is used to get into power mode at P_2, the present value of the power cashflow is acquired along with the second call $V_g(P)$ (to go from power to full mode). If this operational transition requires a present value cost X_2 (a total cost) then the value matching condition at P_2 reflects use of the call from idle into an operational value net of costs plus the option in the next (power) state g.

$$\begin{aligned} V_i\left(P_2\right) &= V_g\left(P_2\right) + P_2^\gamma - X_2 \\ cV_i\left(P_2\right) &= cV_g\left(P_2\right) + \gamma P_2^\gamma. \end{aligned} \quad (19)$$

The second condition in (19) is the dollar beta matching condition at P_2, this says that not only must the values either side of the top line of (19) match but to do so smoothly i.e. with dollar betas $(Pd()/dP)$ matching too (note that the beta of the gamma payoff is γ and dollar beta γP^γ). Betas of $V_i(P)$ and $V_g(P)$ are both known because the option function takes the same form in this region g as idle i; however the size of the options there $V_i\left(P_2\right), V_g\left(P_2\right)$ remain unknown.

Now we close the cycle in this section; allowing transfer from power g mode to full f i.e. from present value of revenue P^γ to P by incurring a net cost X_4 at P_4. Thus the total present value cost is X_4 (which allows for the addition of k_f/r, the saving of k_g/r net of another one time friction K_4, the net cost being $X_4 = K_4 + k_f/r - k_g/r$; again this decomposition is not considered). The strike X_4 that is consistent with $P_{4,2,1}$ will be calculated.

[18] That is to say that claims $V_g(.)$ can either be formulated as functions of P or (say H where) $H = P^\gamma$. Under GBM, $dV(P)$ and $dV(H\left(P\right))$ generate asset pricing equations with betas β, β' that are different, but scaled by γ. Their solutions $V(P) \propto P^\beta$ and $V(H) \propto H^{\beta'}$ are equivalent because $\beta = \beta'\gamma$ ensures that $H^{\beta'} = P^{\beta'\gamma} = P^\beta$. This is true for calls or puts. The beta elasticity of P^γ is γ.

When the power call is used to exit the g mode and enter the f mode, this generates a value condition (20). This happens at $P = P_4$ when the g call exercises into full mode, upon which P_4^γ (the present value of the power flow) is lost (left hand side) along with $V_g(P_4)$ but P_4, the full present value, is gained (right hand side). In equation (20), items on the left represent the power state, including the option V_g and net present values of the cashflows there; on the right the full state contains the next option $V_f(P)$ a put, net present values of cashflows there less the one strike X_4.

$$\begin{aligned} V_g\left(P_4\right) + P_4^\gamma &= V_f\left(P_4\right) + P_4 - X_4 \\ cV_g\left(P_4\right) + \gamma P_4^\gamma &= pV_f\left(P_4\right) + 1.P_4. \end{aligned} \tag{20}$$

In the first line of (20), all fixed present values have been consolidated on the right into one sum X_4. This equation too must match by dollar beta (X_4 disappears) and since $\beta_g = c$ and $\beta_f = p$ we get the second line of (20).

The option $V_f(P_4)$ is then valued as a function of its discounted payoff via a discount factor which also produces betas at $P_{4,1}$

$$V_f\left(P_4\right) = D_p(P_4, P_1)V_f(P_1) \qquad \beta_f = p. \tag{21}$$

Now we have three value matching equations, three discountings, and three weighted beta matchings. Given thresholds P_1, P_2, P_4, present value cashflows P_4^γ etc. and discount factors $D_c(.)$ (used twice), $D_p(.)$ and betas, the three value equations, three discounting equations and three weighted beta equations can all be solved for the following variables $V_i(P_1)$, $V_i(P_2)$, $V_g(P_2)$, $V_g(P_4)$, $V_f(P_1)$, $V_f(P_4)$ and X_1, X_2, X_4. These are most easily summarised after describing them in a network or graph using its associated matrix.

3.1 The three mode problem and the resulting system of equations

These option value can be placed in a graph or network as in Table 2. The underlying value of full operations P is mapped onto the horizontal axis, whilst the vertical axis increases with flows P, P^γ, values V_i, V_g, V_f and with jumps up or down indicating investment or divestment sums. In particular it is important to track the passage of P toward each of the three thresholds $P_{4,2,1}$ which are marked on the horizontal. At these thresholds, investment jumps by $X_{4,2,1}$ upward (for a call, incurring a cost but accumulating investment) and downward (for a put which realises a saving but decumulates investment). The present value of the cashflow in that mode is shown in each line and is quantified at the threshold at which it starts or stops.

From the full state f, (in the top row of the network because this mode has highest cumulated investment) when P reaches P_1 from above i.e. moving from the right to the left, full operations are suspended with loss of present value revenue $P_1 = \pi_1/\delta$ but also with a saving of present value of operating and transition costs X_1 ($k_f/r - K_1$ to be thought of as the present value of full running costs less a friction).

In the idle state i (on the bottom row because this mode has lowest accumulated capital and cashflow), no operational cashflow is present but there is a call option $V_i(P)$. This is gained at P_1 with value $V_i(P_1)$ and is used at P_2 where its value is $V_i(P_2)$.

Due to required investment X_2, exercise of the call option from idle, lifts value upward. This X_2 represents the present value of running costs (k_g in g the gamma state as well as transition costs K_2, i.e. $X_2 = k_g/r + K_2$). In return for sacrificing $V_i(P_2)$ and committing this strike, the call in the power state $V_g(P_2)$ is gained along with the present value of operations in the power mode P_2^{γ}.

This cycle completes with the power state (middle row) but at P_4 the second call option is used with value $V_g(P_4)$. This is combined with a third transition cost X_4 (this might comprise three items; the cost saving of stopping power operations k_g/r, the present value cost of engaging in full operations $-k_f/r$ and finally another one off transaction cost K_4 so that $X_4 = k_f/r + K_4 - k_g/r$).[19] Along with the acquisition of the put option in the full f state, $V_f(P_4)$, this completes the augmented hysteresis cycle in this section (Ekern [13] presents another extension to switching hysteresis, one with a finite number of transitions that would also admit a matrix solution).

The total value at each stage is given by taking the value of these options and adding the present value of operations in all states, i.e. $V_i(P)$, $V_g(P) + P^{\gamma}$, or $V_f(P) + P$ in idle, gamma and full modes. The vertical separations are determined by $X_{4,2,1}$ the total investment and divestment sums (this is apparent in the Figure in Section 4 which displays solved and interim values for a four level system); this is where cash is injected into or withdrawn from the system.[20]

The nine equations that link the nine variables are summarised in Table 3 and Figure 1 by discounting, value matching and dollar beta matching; again for the same input parameters $c, p, \gamma = 2, -1, 0.5$ and $P_{4,2,1} = 4, 2, 1$

[19] Although the strikes are decomposed with components, the method in this paper only solves for $X_{4,2,1}$

[20] Note that horizontal arrows indicate diffusions; these take time to transit and involve possible movement (left or right) in P. Vertical arrows indicate either investment or divestment and occur instantaneously at a threshold; these are not reversible other than by further forward passage through the network and its graph.

f, full flow	$V_f(P_1)+P_1$		$\leftarrow$		$V_f(P_4)+P_4$
					$\uparrow -X_4$
g, power flow	$\downarrow +X_1$		$V_g(P_2)+P_2^\gamma$	$\rightarrow$	$V_g(P_4)+P_4^\gamma$
			$\uparrow -X_2$		
i, idle no flow	$V_i(P_1)$	$\rightarrow$	$V_i(P_2)$		
	P_1 threshold		P_2 threshold		P_4 threshold

Table 2: Investment graph at three thresholds (horizontal) for three state system (vertical, idle power full). Value matching at investment occurs vertically, diffusion and discounting occurs horizontally.

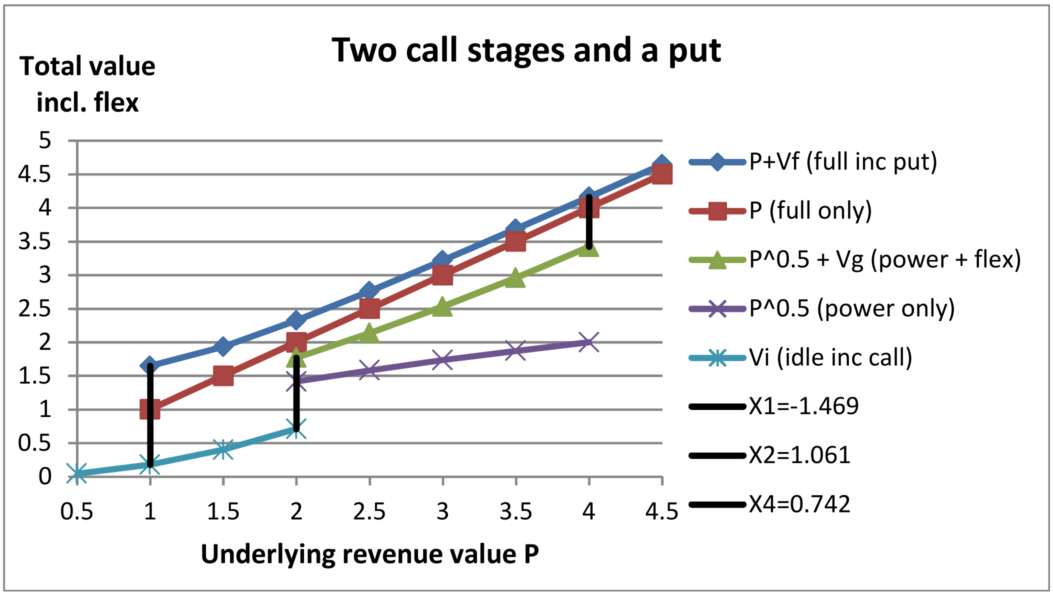

Figure 1: Investments across three mode example with $P_{4,2,1} = 4, 2, 1$ for GBM and power values $c, p, \gamma = 2, -1, 0.5$. Table 3 shows the investment graph whilst this plot shows the idle, power and full flex values with their investment quantities $X_{4,2,1}$ (value matching occurs vertically at $P = 4, 2, 1$).

Type	Condition	Values (sols in bold)
Disc	$V_f(P_4) = D_p(P_4, P_1)V_f(P_1)$	$\mathbf{0.161} = 0.250 \times \mathbf{0.646}$
	$V_g(P_2) = D_c(P_2, P_4)V_g(P_4)$	$\mathbf{0.355} = 0.250 \times \mathbf{1.419}$
	$V_i(P_1) = D_c(P_1, P_2)V_i(P_2)$	$\mathbf{0.177} = 0.250 \times \mathbf{0.708}$
Value	$V_g(P_4) = V_f(P_4) + P_4 - P_4^\gamma - X_4$	$\mathbf{1.419} = \mathbf{0.161} + 4 - 2 - \mathbf{0.742}$
	$V_i(P_2) = V_g(P_2) + P_2^\gamma - X_2$	$\mathbf{0.708} = \mathbf{0.355} + 1.414 - \mathbf{1.061}$
	$V_f(P_1) = V_i(P_1) - P_1 + X_1$	$\mathbf{0.646} = \mathbf{0.177} - 1.000 + \mathbf{1.469}$
Dol β	$cV_g(P_4) = pV_f(P_4) + P_4 - \gamma P_4^\gamma$	$2 \times \mathbf{1.419} = -1 \times \mathbf{0.161} + 4 - 0.5 \times 2$
	$cV_i(P_2) = cV_g(P_2) + \gamma P_2^\gamma$	$2 \times \mathbf{0.708} = 2 \times \mathbf{0.355} + 0.5 \times 1.414$
	$pV_f(P_1) = cV_i(P_1) - P_1$	$-1 \times \mathbf{0.646} = 2 \times \mathbf{0.177} - 1.000$

Table 3: $V_f(P_4), V_f(P_1), V_g(P_2), V_g(P_4), V_i(P_1), V_i(P_2), X_{4,2,1}$ and 9 equations

(solutions to option and investment quantities are shown in bold). We next shown the matrix method that was used to retrieve these solutions.

3.2 A matrix based solution

With many option constants and conditions to consider (this method is generalised to a fourth threshold in Section 4 but can be larger), it is useful to collect similar items into vectors and then use matrix and vector equations to represent their linkages.

Given the three discount equations, rather than using one common vector of values (say $\mathbf{V}$) a more sensible grouping is to separate option values into one of two vectors $\mathbf{U}$ or $\mathbf{W}$. Option values at their beginning threshold (on the right hand side of a value equation) are put into $\mathbf{U} = [V_f(P_4), V_g(P_2), V_i(P_1)]^\intercal$ and[21] those at their end threshold (left hand side of a value transition equation) $\mathbf{W} = [V_g(P_4), V_i(P_2), V_f(P_1)]^\intercal$.

This allows the value of options at their birth to be expressed succinctly as a matrix multiplication $\mathbf{D}$ of values at their use, i.e. the relationship between beginning (left $\mathbf{U}$ in 22) compared to the end (right $\mathbf{W}$ in 22) threshold

$$\begin{array}{ccccc} \mathbf{U} & = & \mathbf{D} & & \mathbf{W} \\ \begin{bmatrix} V_f(P_4) \\ V_g(P_2) \\ V_i(P_1) \end{bmatrix} & = & \begin{bmatrix} 0 & 0 & D_p(P_4, P_1) \\ D_c(P_2, P_4) & 0 & 0 \\ 0 & D_c(P_1, P_2) & 0 \end{bmatrix} & & \begin{bmatrix} V_g(P_4) \\ V_i(P_2) \\ V_f(P_1) \end{bmatrix} \end{array}. \tag{22}$$

Note that the call discount factor $D_c(.)$ has been used twice, between intervals P_1, P_2 (idle i) and P_2, P_4 (gamma g) whilst with only one put present (in the full state f), $D_p(.)$ appears once (between P_4 and P_1).

[21]The transpose of a vector is indicated by $^\intercal$.

Two other vectors capture the present values of operational cash flows before option exercise $\mathbf{Z} = [P_4^\gamma, 0, P_1]^\intercal$ and after exercise at each level $\mathbf{Y} = [P_4, P_2^\gamma, 0]^\intercal$. These flows are treated as potentially perpetual in nature but their duration will depend upon the potential exercise and timing of the next option (when terminated the loss of the flow is accounted by deduction at the next exercise threshold). Their net effect on the right hand side of () is captured by $\mathbf{Y} - \mathbf{Z}$.

Finally the present values of cost rates and frictions in the option strikes are also grouped into a vector $\mathbf{X} = [X_4, X_2, -X_1]^\intercal$, collectively in (23), these vectors represent all three value matching equations, one line for each threshold.

$$\begin{array}{ccccccc} \mathbf{W} & = & \mathbf{U} & + & \mathbf{Y}-\mathbf{Z} & - & \mathbf{X} \\ \begin{bmatrix} V_g(P_4) \\ V_i(P_2) \\ V_f(P_1) \end{bmatrix} & = & \begin{bmatrix} V_f(P_4) \\ V_g(P_2) \\ V_i(P_1) \end{bmatrix} & + & \begin{bmatrix} P_4 - P_4^\gamma \\ P_2^\gamma \\ -P_1 \end{bmatrix} & - & \begin{bmatrix} X_4 \\ X_2 \\ -X_1 \end{bmatrix} \end{array} \tag{23}$$

Assuming $P_{4,2,1}$ represent an optimal policy set of thresholds, with their first order conditions, the aim is to calculate $\mathbf{W}, \mathbf{U}, \mathbf{X}$. This is possible since vectors $\mathbf{Z}, \mathbf{Y}$ and matrix $\mathbf{D}$ are all given as functions of $P_{4,2,1}$.

To represent the first order conditions, the dollar betas of operational flows are also required; these are given by $\boldsymbol{\beta}_Z\mathbf{Z} = [\gamma P_4^\gamma, 0, P_1]^\intercal$ and $\boldsymbol{\beta}_Y\mathbf{Y} = [P_4, \gamma P_2^\gamma, 0]^\intercal$ which are combined into $\boldsymbol{\beta}_Y\mathbf{Y} - \boldsymbol{\beta}_Z\mathbf{Z} = [P_4 - \gamma P_4^\gamma, \gamma P_2^\gamma, -P_1]^\intercal$ and used in (24) for dollar beta matching.[22]

Differentiating the items of (23) line by line in P and then multiplying by P gives the dollar betas both sides of a value transition. This is succinctly expressed using two diagonal beta matrices $\boldsymbol{\beta}_W, \boldsymbol{\beta}_U$ defined in (24).

$$\begin{array}{cccccc} \boldsymbol{\beta}_W & \mathbf{W} & = & \boldsymbol{\beta}_U & \mathbf{U} & + \quad \boldsymbol{\beta}_Y\mathbf{Y} - \boldsymbol{\beta}_Z\mathbf{Z} \\ \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & p \end{bmatrix} & \begin{bmatrix} V_g(P_4) \\ V_i(P_2) \\ V_f(P_1) \end{bmatrix} & = & \begin{bmatrix} p & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} & \begin{bmatrix} V_f(P_4) \\ V_g(P_2) \\ V_i(P_1) \end{bmatrix} & + \begin{bmatrix} P_4 - \gamma P_4^\gamma \\ \gamma P_2^\gamma \\ -P_1 \end{bmatrix} \end{array} \tag{24}$$

In order to separate the option constants in $\mathbf{W}, \mathbf{U}, \mathbf{X}$ from their (known) betas, matrices $\boldsymbol{\beta}_W, \boldsymbol{\beta}_U$ were used. These diagonal matrices operate on the vectors of option values to weight them with their betas. They are evaluated

[22] Although only the vector $\boldsymbol{\beta}_Y\mathbf{Y} - \boldsymbol{\beta}_Z\mathbf{Z}$ is required, matrices $\boldsymbol{\beta}_Y, \boldsymbol{\beta}_Z$ are given by

$$\boldsymbol{\beta}_Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{\beta}_Z = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

using the option beta in that mode; for GBM the flexibility options are iso-elastic and have the same beta at both thresholds. Consequently $\boldsymbol{\beta}_W, \boldsymbol{\beta}_U$ contain either c or p (V_i and V_g are both calls with the same function D_c and power $c > 1$ and V_f is a put with power $p < 0$ with factor on D_p).

In particular, note that the beta matching and discounting equations do not involve the transition costs $\mathbf{X}$. This is another reason why it is easier to solve for those at the end. Direct elimination from (22 and 24) gives $\mathbf{W}, \mathbf{U}$ that are consistent with smooth pasting i.e. dollar beta matching, then (23) gives $\mathbf{X}$.

$$\begin{aligned} \mathbf{W} &= [\boldsymbol{\beta}_\mathbf{W} - \boldsymbol{\beta}_\mathbf{U}\mathbf{D}]^{-1} (\boldsymbol{\beta}_Y \mathbf{Y} - \boldsymbol{\beta}_Z \mathbf{Z}) \\ \mathbf{U} &= \mathbf{D} [\boldsymbol{\beta}_\mathbf{W} - \boldsymbol{\beta}_\mathbf{U}\mathbf{D}]^{-1} (\boldsymbol{\beta}_Y \mathbf{Y} - \boldsymbol{\beta}_Z \mathbf{Z}) \\ \mathbf{X} &= \mathbf{U} - \mathbf{W} + \mathbf{Y} - \mathbf{Z} \end{aligned} \tag{25}$$

In the first two lines of (25), an inverse of a matrix combination is applied to a beta weighted difference (in brackets). The expression $\mathbf{Y} - \mathbf{Z}$ is a difference of present values of revenues before and after each transition (one in each row), however when weighted by their betas $\boldsymbol{\beta}_Y \mathbf{Y} - \boldsymbol{\beta}_Z \mathbf{Z}$ it is akin to a rate of return on this difference so acts like a flow. The matrix $\boldsymbol{\beta}_W - \boldsymbol{\beta}_U \mathbf{D}$ reflects the net beta of the change in flexibility value, $\boldsymbol{\beta}_W$ for the immediate $\mathbf{W}$ and $\boldsymbol{\beta}_U \mathbf{D}$ in anticipation of the demise of $\mathbf{U}$ at the end of the next stopping time. Thus option solutions $\mathbf{W}, \mathbf{U}$ can be thought of as *aperiodic perpetuities* on beta weighted cashflows $\boldsymbol{\beta}_Y \mathbf{Y} - \boldsymbol{\beta}_Z \mathbf{Z}$ with annuity factor $[\boldsymbol{\beta}_\mathbf{W} - \boldsymbol{\beta}_\mathbf{U}\mathbf{D}]^{-1}$. That is to say that their benefit is linked to the cashflow gain carried by $\boldsymbol{\beta}_Y \mathbf{Y} - \boldsymbol{\beta}_Z \mathbf{Z}$ which comes along with random future timing reflected in a perpetuity factor (an inverse matrix).

As the system cycles through i, g, f episodes, the expected timing is carried though this inverse $[\boldsymbol{\beta}_W - \boldsymbol{\beta}_U \mathbf{D}]^{-1}$. Although idle, power and full states last for different lengths of time, $\mathbf{D}$ carries the expected timings and present value of each through the discount factors embedded. It is this equation (25) that was used to produce the values in Table 3.

This method can now be extended to encompass four thresholds with two way flexibility in the power state.

4 Four thresholds & two way flexibility

In this section we illustrate a more complex investment scenario involving the same interim (power) state g. Now as well as the chance to invest further if things go well there is also the chance to recoup and if the situation worsens partially divest from full f by rejoining g (rather than i as before). This splits

f					$V_f(P_3) + P_3$	$\leftarrow$	$V_f(P_4) + P_4$
					$\downarrow +X_3$		$\uparrow -X_4$
g	$V_g(P_1) + P_1^\gamma$	$\leftarrow$	$V_g(P_2) + P_2^\gamma$	$\leftrightarrow$	$V_g(P_3) + P_3^\gamma$	$\rightarrow$	$V_g(P_4) + P_4^\gamma$
	$\downarrow +X_1$		$\uparrow -X_2$				
i	$V_i(P_1)$	$\rightarrow$	$V_i(P_2)$				
	P_1 thresh.		P_2 thresh.		P_3 thresh.		P_4 thresh.

Table 4: Investment graph with four thresholds (horizontal) and three states (vertical). Value matching at investment occurs vertically, diffusion and discounting occurs horizontally.

the put into two, creating a high level put joining g in addition to the one that returns to the lowest mode; the gamma state g becomes more complex as a result. The network of investment transitions, is given in Table 4 which is similar to Table 2 other than retreat (like advance) has two stages. The middle stage g has reversion on the left (low $P = P_1$ rather than the full) and the full state f has reversion to the g state at an additional level P_3 (upward moves occur at even thresholds $P_{2,4}$ and downward odd $P_{1,3}$).[23]

Since the power state no longer has a unique direction of propagation (up, i.e. a call D_c as in section 3), first we need to construct new discounts, ones that can accommodate two different diffusion outcomes. This is done with D_t, D_b discount factors that are subject to knockout features (i.e. where a claim value of zero is attainable as well as unity). These are combinations of the simple one way factors already used D_c, D_p (each of which remains strictly positive). Their betas also require growth factors which we describe next.

It is more convenient from now on, to use compact notation for elements that are static and non–dynamic, i.e. to use D_{c12} rather than $D_c(P_1, P_2)$, D_{p43} rather than $D_p(P_4, P_3)$ and V_{f4} rather than $V_f(P_4)$ etc. However $D_c(P, P_4)$ and $V_f(P)$ will be maintained for quantities that are dynamic in P and not fixed at a threshold.

4.1 Growth factors and their betas

The two simplest options in Table 3 are the straight call (bottom row) and put (top row) that can be discounted easily and whose dollar betas are already known. (These form part of the overall $\mathbf{D}$ and beta matrices required in this section); for the top and bottom thresholds, the value and beta relationships

[23]Now diffusion from the centre of V_g is possible to the left or right (double arrow).

are laid out in (26).

$$\begin{bmatrix} V_{f4} \\ V_{i1} \end{bmatrix} = \begin{bmatrix} D_{p43} & 0 \\ 0 & D_{c12} \end{bmatrix} \begin{bmatrix} V_{f3} \\ V_{i2} \end{bmatrix} \qquad \begin{bmatrix} \beta_{f4}V_{f4} \\ \beta_{i1}V_{i1} \end{bmatrix} = \begin{bmatrix} pD_{p43} & 0 \\ 0 & cD_{c12} \end{bmatrix} \begin{bmatrix} V_{f3} \\ V_{i2} \end{bmatrix} \tag{26}$$

The first (discount) matrix can be inverted to form a *growth matrix* which expresses end values V_{f3}, V_{i2} as a function of V_{f4}, V_{i1} at the beginning (this forms part of the overall growth matrix $\mathbf{G}$ defined in 38). This has used inverse discounts in relationships such as $V_{f3} = (D_{p43})^{-1} V_{f4} = D_{p34}V_{f4}$ where it is understood that the *put growth factor* $D_{p34} > 1$ from P_4 to P_3 is the inverse of the discount factor from P_4 to P_3 (similarly for call). Thus the subscripts fixing the thresholds, not only determine which thresholds are under scrutiny but also whether $D_{c,p}$ represents a discount or a growth.[24] Due to the definition of beta, growth factors have the same elasticity as their discounts; both growth and its dollar beta relationships are shown in (27) for the simple, non–compound options.

$$\begin{bmatrix} V_{f3} \\ V_{i2} \end{bmatrix} = \begin{bmatrix} D_{p34} & 0 \\ 0 & D_{c21} \end{bmatrix} \begin{bmatrix} V_{f4} \\ V_{i1} \end{bmatrix} \qquad \begin{bmatrix} \beta_{f3}V_{f3} \\ \beta_{i2}V_{i2} \end{bmatrix} = \begin{bmatrix} pD_{p34} & 0 \\ 0 & cD_{c21} \end{bmatrix} \begin{bmatrix} V_{f4} \\ V_{i1} \end{bmatrix} \tag{27}$$

These simple relationships were used directly in section 2, but for the compound options, growth factors are integral to establishing their betas. This is because the discount and growth option betas rest upon two possible payoffs; although only one affects the value at a boundary, both affect its slope there.

4.2 Two way discount functions and their betas

For the other values V_g in Table 4, we require discount factors which have a unit payment at one threshold but are subject to a knockout possibility, becoming strictly worthless at a second attainable threshold (e.g. when $D_t = 1$ at a threshold $D_b = 0$ or vice versa).

In this section, starting from $P = P_3$ there are two conversion thresholds of interest $P_4 > P_3$ and $P_1 < P_3$. For conversion at the top i.e. P_4, further

[24]For GBM the call growth factor is $D_{c21} = (P_2/P_1)^c > 1$. Not only do we assume that $(D_{c12})^{-1} = D_{c21}$, i.e. that inverse discount factors are growth factors with the same functional form applied to a growth ratio P_2/P_1, but also implied are relationships which chain discounts (or growths) together across intermediate thresholds i.e. $D_{c12}D_{c23} = D_{c13}$ etc. This is possible due to concatenation of two stopping times; one from 1 to 2 and the other from 2 to 3.

Note that D_p, D_c have different elasticities, $D_{p12}D_{c23}$ which represents a put at P_2 from P_1 followed by a call at P_3 from P_2, has a mixed beta.

investment is completed but its complementary divestment option (at the bottom P_1) becomes worthless; the top discount factor D_t attains 1 at P_4 but that at the bottom D_b attains zero there. At P_1 the complementary divestment activity occurs; it involves the discount D_b attaining unit value at the bottom and D_t becoming zero.

Since D_t is used at the *top* of the power state, we use notation t, also three arguments are required $D_t(P, P_4, P_1)$; at the *bottom* $D_b(P, P_1, P_4)$ is used. Normalised for unit and zero payoffs, in static notation these claims are subject to payoff or boundary conditions in (28).

$$\begin{bmatrix} D_{t441} & D_{b414} \\ D_{t141} & D_{b114} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{28}$$

The first argument in $D_t(P, P_4, P_1)$ is the current dynamic level of the state variable P, the second a static level (above or below P) at which the discount factor becomes one and the third an attainable static level at which the discount becomes zero. Formally, conditional on current P, these claims are risk neutral expectations but each subject to an extra condition $\min P$ or $\max P$ for their acceptable path.

$$\begin{aligned} & [D_t(P, P_4, P_1);\ D_b(P, P_1, P_4)] \qquad (29) \\ = & \left[E_P^{RN}\left[e^{-r\tau}\middle| P(\tau) = P_4, \min P > P_1\right];\ E_P^{RN}\left[e^{-r\tau}\middle| P(\tau) = P_1, \max P < P_4\right]\right] \\ = & [D_c(P, P_4);\ D_p(P, P_1)] \begin{bmatrix} 1 & -D_{p41} \\ -D_{c14} & 1 \end{bmatrix} (1 - D_{c14}D_{p41})^{-1} \end{aligned}$$

We already have two claims $D_c(P, P_4)$ and $D_p(P, P_1)$ (straight call and put discount factors without knockouts) that satisfy single conditions. For $P_4 > P > P_1$ the claims $D_t(P, .)$, $D_b(P, .)$ are linear combinations with coefficients fixed by the matrix on the right of (29).

For D_t, this values the direct (successful) path via a call from P to P_4 less an indirect (and negating) path from P to P_1 via the put multiplied by the call path from P_1 to P_4 (see Darling and Siegert [8] for complementary stopping times using a reflection principle).

For D_b the direct put path is from P to P_1 however an indirect call path from P via P_4 then a put from P_1 to P_4 must be subtracted.

In the valuation of both D_t, D_b these must be scaled by the determinant of a weighting matrix to account for the discounted round trip, i.e. call from P_1 to P_4 times put from P_4 to P_1.[25]

---

[25]

$$\det \begin{bmatrix} 1 & -D_{p41} \\ -D_{c14} & 1 \end{bmatrix} = 1 - D_{c14}D_{p41} = 1 - \left(\frac{P_1}{P_4}\right)^{c-p}$$

The effect of (29) is that a discount factor D_t that pays off \$1 at P_4 but knocks out and becomes worthless at P_1 is a long position (with magnitude $(1-D_{c14}D_{p41})^{-1}$) in the call $D_c(P,.)$ and a short position ($D_{c14}(1-D_{c14}D_{p41})^{-1}$) in the put $D_p(P,.)$ (the bottom discount D_b is a weighted put less a call).

For D_t as P increases, the value of the straight call increases and that of the straight put decreases so $D_t(P,.)$ increases in value until it reaches 1 when $P=P_4$. Conversely as P decreases, the call decreases and the put increases its short contribution. As P reaches the knockout level P_1, even though the straight call D_c still has some residual value (D_{c14}) this is eliminated by the negative element from the straight put and the whole claim D_t becomes worthless. For similar reasons, $D_b(P,.)$ is decreasing in P and achieves 0,1 at $P_{4,1}$.

Within the power state, these knock out options D_t, D_b are applied to two possible payoffs; at P_1 a value of $V_g(P_1)$ will be realised but at P_4 a different value $V_g(P_4)$ will result. These components have expected value $D_t(P,P_4,P_1)\,V_{g4}$ and $D_b(P,P_1,P_4)\,V_{g1}$ and from the perspective of a dynamic point P from within this range, in (30) the current expected value of flexibility is a probabilistic and discount weighted sum of two options (only one of which can be attained).

$$\begin{aligned} V_g(P) &= [D_c(P,P_4);\ D_p(P,P_1)] \begin{bmatrix} 1 & -D_{p41} \\ -D_{c14} & 1 \end{bmatrix} \begin{bmatrix} V_{g4} \\ V_{g1} \end{bmatrix} (1-D_{c14}D_{p41})^{-1} \qquad (30) \\ \beta_g(P)\,V_g(P) &= [cD_c(P,P_4);\ pD_p(P,P_1)] \begin{bmatrix} 1 & -D_{p41} \\ -D_{c14} & 1 \end{bmatrix} \begin{bmatrix} V_{g4} \\ V_{g1} \end{bmatrix} (1-D_{c14}D_{p41})^{-1} \end{aligned}$$

These decompositions of D_t, D_b via D_c, D_p makes the dollar betas in the second line of (30) particular easy to evaluate; since the new discount factors are a linear combinations of the straight put and call, their slopes at any point P are too. Rescaling the slopes by P gives (30, line two) $\beta_g(P)\,V_g(P)$ the dollar weighted beta for $V_g(P)$.

Although under GBM the component betas c, p of D_c, D_b are fixed, the weights $D_c(P,P_4)$ and $D_p(P,P_1)$ change with P so in (30) the dollar beta of claims are not constant weights of c and p (furthermore, the top call with knock out D_t is more levered than the straight call D_c and has a higher slope and dollar beta; D_b has a more negative beta than D_p). Note that the payoffs V_{g4}, V_{g1} at $P_{4,1}$ are static as are the discounts D_{c14}, D_{p41} in the weighting matrix and its determinant.

Having valued the option in the power state and its dollar beta through end outcomes, we can focus on the option values at two particular points of their creation e.g. V_{g3}, V_{g2}. Since these are static we also annotate non–

dynamically as items occurring at fixed thresholds, e.g. $D_t(P_2, P_4, P_1) = D_{t241}$ etc.

Acquisition of $V_g(P)$ at either of the thresholds P_2 or P_3 is represented in (31) at those two particular values of P.

$$\begin{bmatrix} V_{g3} \\ V_{g2} \end{bmatrix} = \begin{bmatrix} D_{t341} & D_{b314} \\ D_{t241} & D_{b214} \end{bmatrix} \begin{bmatrix} V_{g4} \\ V_{g1} \end{bmatrix} \tag{31}$$

For D_{t341}, D_{t241} in the first column, the first threshold in the subscripts is either P_3 or P_2, the second P_4 the target or call level and the third e.g. P_1 the level at which the flexibility dies or is knocked out (the complementary options in the second column D_{b314}, D_{b214} does the opposite; put at 1 but knock out at 4). In order to work out how $V_{g4,1}$ depend on $V_{g3,2}$ this last equation can be inverted to produce growth factors (32) comparable to (26), this is easiest to do by decomposing the matrix in (31) using (29).

$$\begin{bmatrix} D_{t341} & D_{b314} \\ D_{t241} & D_{b214} \end{bmatrix} = \begin{bmatrix} D_{c34} & D_{p31} \\ D_{c24} & D_{p21} \end{bmatrix} \begin{bmatrix} 1 & -D_{p41} \\ -D_{c14} & 1 \end{bmatrix} (1 - D_{c14}D_{p41})^{-1} \tag{32}$$

The inverses of both matrices[26] in (32) are applied in reverse order, allowing V_{g4}, V_{g1} to be expressed from (32) in V_{g2}, V_{g3}

$$\begin{bmatrix} V_{g4} \\ V_{g1} \end{bmatrix} = \begin{bmatrix} 1 & D_{p41} \\ D_{c14} & 1 \end{bmatrix} \begin{bmatrix} D_{p21} & -D_{p31} \\ -D_{c24} & D_{c34} \end{bmatrix} \begin{bmatrix} V_{g3} \\ V_{g2} \end{bmatrix} (D_{c34}D_{p21} - D_{c24}D_{p31})^{-1} . \tag{33}$$

This can be simplified back to a similar form to the standard discounts subject to knock out. Multiplying the denominator in the determinant's fraction by a growth factor $D_{c43}D_{p12}$ (each row of the last equation must also be multiplied by $D_{c43}D_{p12}$ to compensate) and using discount and growth concatenation yields a round trip determinant between $P_{3,2}$ (resting on $1 - D_{c23}D_{p32} = (D_{c34}D_{p21} - D_{c24}D_{p31})\, D_{c43}D_{p12}$ and the production of a new determinant). Applying these and evaluating the matrix product yields the simplification (34)

$$\begin{bmatrix} V_{g4} \\ V_{g1} \end{bmatrix} = \begin{bmatrix} D_{c43} & D_{p42} \\ D_{c13} & D_{p12} \end{bmatrix} \begin{bmatrix} 1 & -D_{p32} \\ -D_{c23} & 1 \end{bmatrix} \begin{bmatrix} V_{g3} \\ V_{g2} \end{bmatrix} (1 - D_{c23}D_{p32})^{-1} . \tag{34}$$

[26]

$$\begin{bmatrix} 1 & -D_{p41} \\ -D_{c14} & 1 \end{bmatrix}^{-1} = \frac{1}{1 - D_{c14}D_{p41}} \begin{bmatrix} 1 & D_{p41} \\ D_{c14} & 1 \end{bmatrix}$$

$$\begin{bmatrix} D_{c34} & D_{p31} \\ D_{c24} & D_{p21} \end{bmatrix}^{-1} = \frac{1}{D_{c34}D_{p21} - D_{c24}D_{p31}} \begin{bmatrix} D_{p21} & -D_{p31} \\ -D_{c24} & D_{c34} \end{bmatrix}$$

Thus not only are two way discount factors $D_{t,b}$ easily expressed as a linear combination of standard put and call discounts, but their inverse or growth functions are too[27]. The growth elements reflect the value at extreme points $P_{4,1}$ as a function of starting from one or other of the interior points $P_{3,2}$, but for these end values the weights are fixed as a function of the internal points (and the reciprocal of their new round trip determinant $(1 - D_{c23}D_{p32})^{-1}$). The resultant growth matrix contains both positive and negative quantities; these result from the long and short positions present in the knock out discount factors.

This formulation also holds true for dynamic $V_g(P)$, i.e. any dynamic P within the region so another version of (30, line two) of the dynamic dollar beta $\beta_g(P)V_g(P)$ is easy to obtain as a function of $P_{3,4}$. This version (35) allows the dollar betas of end option values (in $\mathbf{W}$) to be expressed as a function of beginning ($\mathbf{U}$) values (while 30 expresses $\beta_g(P)V_g(P)$ as a function of $\mathbf{W}$).

$$\begin{bmatrix} \beta_{g4}V_{g4} \\ \beta_{g1}V_{g1} \end{bmatrix} = \begin{bmatrix} cD_{c43} & pD_{p42} \\ cD_{c13} & pD_{p12} \end{bmatrix} \begin{bmatrix} 1 & -D_{p32} \\ -D_{c23} & 1 \end{bmatrix} \begin{bmatrix} V_{g3} \\ V_{g2} \end{bmatrix} (1 - D_{c23}D_{p32})^{-1} \tag{35}$$

The dollar betas of all four V_{g4}, V_{g1} and V_{g3}, V_{g2} are necessary for smooth pasting dollar beta equivalency. The growth inversion allows the dollar beta at one threshold pair to be expressed as a function of values at the other pair (and vice versa). It is necessary for two way options when the beta matrices $\boldsymbol{\beta}_W, \boldsymbol{\beta}_U$ are not simple diagonal ones (as in Section 2 and 3).

4.3 Solution and a numerical example

We can now formalise the system for three investment levels with the flexibility to move up or down between idle i, full f and intermediate power state g with elasticity γ of an underlying flow.[28] We use the same linear algebra

[27] Eq (30) shows that

$$\begin{aligned} \begin{bmatrix} V_{g4} \\ V_{g1} \end{bmatrix} &= \begin{bmatrix} D_{c43} - D_{p42}D_{c23} & D_{p42} - D_{c43}D_{p32} \\ D_{c13} - D_{p12}D_{c23} & D_{p12} - D_{c13}D_{p32} \end{bmatrix} \begin{bmatrix} V_{g3} \\ V_{g2} \end{bmatrix} (1 - D_{c23}D_{p32})^{-1} \\ &= \begin{bmatrix} D_{t432} & D_{b423} \\ D_{t132} & D_{b123} \end{bmatrix} \begin{bmatrix} V_{g3} \\ V_{g2} \end{bmatrix}. \end{aligned}$$

In the bottom left corner, D_{t132} would normally imply (for a discount) achieving P_3 from a start point of P_1 without hitting P_2. Because P_1 cannot grow to P_3 without hitting P_2, a similar and literal interpretation is hampered for the growth factor by the relative levels and the growth matrix contains negative quantities.

[28] More modes regions can be considered using different γ_1, γ_2 and more regions etc. Note that unless two thresholds converge, the up (even) and down (odd) rungs of this

but vary the matrix and vector contents (from Table 2) in order to extend the flexibility setup to Table 4. Table 4 summarises no operational value in the idle state, value that varies with P^γ in the power state and full P (power 1) in the highest state (i.e. flow values varying with $0, P^\gamma$ and P).

Starting with the transition at the highest threshold, from power to full, one investment occurs at threshold P_4 with a payoff net of exercise cost of $P_4 - P_4^\gamma - X_4$. That is to say at the top threshold on going to full, P_4 is gained but P_4^γ is lost along with incremental investment cost X_4.

A reversion to the power state is possible at P_3 yielding $P_3^\gamma - P_3 + X_3$, a reverse payoff including a partial return of fixed investment cost $X_3 < X_4$ (this imposes a restriction on a quantity of recoverable capital). After this two outcomes are possible.

Another downward movement, from the power state to the idle, can occur at P_1; this foregoes flow value P_1^γ but recoups costs $X_{1.}$.

Finally from the idle state, recovery to the power state is possible at P_2 where P_2^γ is gained at the cost of $-X_1$.

The value matching conditions within this section and Table 3 are shown in (36).

$$\begin{array}{ccccccc} \mathbf{W} & = & \mathbf{U} & + & \mathbf{Y}-\mathbf{Z} & - & \mathbf{X} \\ \begin{bmatrix} V_{g4} \\ V_{f3} \\ V_{i2} \\ V_{g1} \end{bmatrix} & = & \begin{bmatrix} V_{f4} \\ V_{g3} \\ V_{g2} \\ V_{i1} \end{bmatrix} & + & \begin{bmatrix} P_4 - P_4^\gamma \\ P_3^\gamma - P_3 \\ P_2^\gamma \\ -P_1^\gamma \end{bmatrix} & - & \begin{bmatrix} X_4 \\ -X_3 \\ X_2 \\ -X_1 \end{bmatrix} \end{array} \tag{36}$$

Table 4 depicts the graph of the investment network.[29] Note that with two new possible outcomes from the power state $V_g(P) + P^\gamma$, branching of outcomes is now possible.

From the power state, since reversion to the idle state is possible (at P_1) as well as elevation to the full state (at P_4), the discount matrix is populated with six elements, and in particular two rows now contain complementary discount factors with mutual exclusivity, i.e. conditional upon each other not paying off. For example in row three (within the power state) from the point of view of threshold P_3, the element D_{t341} represents the discounted value of a dollar paid at P_4 conditional upon P_1 not being reached (where the top opportunity dies and has zero value).

ladder are different.

[29] This mathematical graph is bipartite and directed (see Wilson [25]); bipartite in that investment nodes are either beginning $\mathbf{U}$ or end $\mathbf{W}$ of state and directed in the sense that $\mathbf{W}$ maps forward (one to one) to $\mathbf{U}$ upon value matching and smooth pasting/dollar beta. Beginning of state values $\mathbf{U}$ are linked to $\mathbf{W}$ (one to two) by discounting but this also has a backward i.e. growth interpretation.

The second D_{b314} in row two is complementary to D_{t341} and pays off at P_1 assuming P_4 is not hit (this put is knocked out with zero value if P_4 is reached). Other discount factors are interpreted similarly but simple D_{p43}, D_{c12} are one way factors in full and idle states respectively (one way factors do not knock out for zero value). The detailed discounting equations $\mathbf{U} = \mathbf{DW}$ are given in (37) which contains equation (31) in rows two and three.

$$\begin{array}{ccc} \mathbf{U} & = & \mathbf{D} \qquad\qquad\qquad \mathbf{W} \end{array} \tag{37}$$

$$\begin{bmatrix} V_{f4} \\ V_{g3} \\ V_{g2} \\ V_{i1} \end{bmatrix} = \begin{bmatrix} 0 & D_{p43} & 0 & 0 \\ D_{t341} & 0 & 0 & D_{b314} \\ D_{t241} & 0 & 0 & D_{b214} \\ 0 & 0 & D_{c12} & 0 \end{bmatrix} \begin{bmatrix} V_{g4} \\ V_{f3} \\ V_{i2} \\ V_{g1} \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & D_{p43} & 0 & 0 \\ D_{c34} & 0 & 0 & D_{p31} \\ D_{c24} & 0 & 0 & D_{p21} \\ 0 & 0 & D_{c12} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{1-D_{c14}D_{p41}} & 0 & 0 & \frac{-D_{p41}}{1-D_{c14}D_{p41}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-D_{c14}}{1-D_{c14}D_{p41}} & 0 & 0 & \frac{1}{1-D_{c14}D_{p41}} \end{bmatrix}$$

We also collate elements of $\mathbf{G}$ the inverse of $\mathbf{D}$, this is the matrix that values $\mathbf{W} = \mathbf{GU}$

$$\begin{array}{ccc} \mathbf{W} & = & \mathbf{G} \qquad\qquad\qquad \mathbf{U} \end{array} \tag{38}$$

$$\begin{bmatrix} V_{g4} \\ V_{f3} \\ V_{i2} \\ V_{g1} \end{bmatrix} = \begin{bmatrix} 0 & D_{t432} & D_{b423} & 0 \\ D_{p34} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{c21} \\ 0 & D_{t132} & D_{b123} & 0 \end{bmatrix} \begin{bmatrix} V_{f4} \\ V_{g3} \\ V_{g2} \\ V_{i1} \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 0 & D_{c43} & D_{p42} & 0 \\ D_{p34} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{c21} \\ 0 & D_{c13} & D_{p12} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{1-D_{c23}D_{p32}} & \frac{-D_{p32}}{1-D_{c23}D_{p32}} & 0 \\ 0 & \frac{-D_{c23}}{1-D_{c23}D_{p32}} & \frac{1}{1-D_{c23}D_{p32}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we have assembled two thirds of the equations required to solve the four level system; the remaining set comes from (scaled) smooth pasting and construction of $\boldsymbol{\beta}_W, \boldsymbol{\beta}_U$.

From the value matching condition $\mathbf{W} = \mathbf{U} + \mathbf{Y} - \mathbf{Z} - \mathbf{X}$ (36), on the right we use $\mathbf{U}$ as a discounted $\mathbf{DW}$ and on the left $\mathbf{W}$ as a discounted $\mathbf{GU}$ to produce the first line of (39). Then in the second line of (39), we apply a vector of dollar beta operator(s) $[.]' = \left[P\partial(.)/\partial P|_{P=P_4}; ...; P\partial(.)/\partial P|_{P=P_1}\right]^\intercal$ i.e. line by line differentiation and scaling by P at each threshold.

$$\begin{aligned} \mathbf{GU} &= \mathbf{DW} + \mathbf{Y} - \mathbf{Z} - \mathbf{X} \\ \mathbf{G'U} &= \mathbf{D'W} + [\mathbf{Y} - \mathbf{Z}]' \end{aligned} \tag{39}$$

The second equation in (39) holds because neither $\mathbf{G}$ nor $\mathbf{D}$ have diagonal elements and therefore $[\mathbf{GU}]' = \mathbf{G}'\mathbf{U}$ and $[\mathbf{DW}]' = \mathbf{D}'\mathbf{W}$ with the line elements of $\mathbf{G}', \mathbf{D}'$ operating on them alone. Secondly $[\mathbf{Y}-\mathbf{Z}]' = \boldsymbol{\beta}_Y\mathbf{Y} - \boldsymbol{\beta}_Z\mathbf{Z} = [P_4 - \gamma P_4^{\gamma}, \gamma P_3^{\gamma} - P_3, \gamma P_2^{\gamma}, -\gamma P_1^{\gamma}]^{\intercal}$ gives the dollar beta of the payoffs.[30]

To retrieve the beta matrices (previously in (24) this was done ad hoc), the forward and back projections for $\mathbf{U}, \mathbf{W}$ in the second line of (29) are unwound to refine the dollar beta matching set of equations into a form including $\boldsymbol{\beta}_W$ and $\boldsymbol{\beta}_U$ suitable for the solution method (25).

$$\begin{aligned} \mathbf{G}'\mathbf{D}\mathbf{W} &= \mathbf{D}'\mathbf{G}\mathbf{U} + [\mathbf{Y}-\mathbf{Z}]' \\ \boldsymbol{\beta}_W\mathbf{W} &= \boldsymbol{\beta}_U\mathbf{U} + \boldsymbol{\beta}_Y\mathbf{Y} - \boldsymbol{\beta}_Z\mathbf{Z} \end{aligned} \tag{40}$$

where in (40) $\boldsymbol{\beta}_W = \mathbf{G}'\mathbf{D}$ and $\boldsymbol{\beta}_U = \mathbf{D}'\mathbf{G}$ are derived from (37 & 38) in (41)[31]

$$\begin{aligned} \mathbf{D}' &= \begin{bmatrix} 0 & pD_{p43} & 0 & 0 \\ cD_{c34} & 0 & 0 & pD_{p31} \\ cD_{c24} & 0 & 0 & pD_{p21} \\ 0 & 0 & cD_{c12} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{1-D_{c14}D_{p41}} & 0 & 0 & \frac{-D_{p41}}{1-D_{c14}D_{p41}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-D_{c14}}{1-D_{c14}D_{p41}} & 0 & 0 & \frac{1}{1-D_{c14}D_{p41}} \end{bmatrix} \\ \mathbf{G}' &= \begin{bmatrix} 0 & cD_{c43} & pD_{p42} & 0 \\ pD_{p34} & 0 & 0 & 0 \\ 0 & 0 & 0 & cD_{c21} \\ 0 & cD_{c13} & pD_{p12} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{1-D_{c23}D_{p32}} & \frac{-D_{p32}}{1-D_{c23}D_{p32}} & 0 \\ 0 & \frac{-D_{c23}}{1-D_{c23}D_{p32}} & \frac{1}{1-D_{c23}D_{p32}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{41}$$

Results for this system under GBM were evaluated for $P_{4,3,2,1} = 4, 3, 2, 1$ with elasticity/beta parameters $c, p, \gamma = 2, -1, 0.5$. Figure 2 plots the relationship between the flexibility values (on the vertical) and underlying P value and Table 5 summaries the 12 equations and 12 solved quantities (in bold).

Using one way factors, the system accommodated uncertain stopping (e.g. in section 3, there was a chance that if P remained high the value of flexibility would become negligible because switching would cease). With two way

[30] See section three to form the comparable four by fours $\boldsymbol{\beta}_Y, \boldsymbol{\beta}_Z$.

[31] Rather than the diagonal matrices in (24), for the numerical example in this section, the beta matrices are:-

$$\boldsymbol{\beta}_U = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3.263 & -2.842 & 0 \\ 0 & 1.895 & -2.263 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \boldsymbol{\beta}_W = \begin{bmatrix} 2.048 & 0 & 0 & -0.762 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0.190 & 0 & 0 & -1.048 \end{bmatrix}.$$

Type	Condition	Values (sols in bold)
Disc	$V_{f4} = D_{p43}V_{f3}$ $V_{g3} = D_{t341}V_{g4} + D_{b314}V_{g1}$ $V_{g2} = D_{t241}V_{g4} + D_{b214}V_{g1}$ $V_{i1} = D_{c12}V_{i2}$	$\mathbf{0.672} = 0.750 \times \mathbf{0.895}$ $\mathbf{0.804} = 0.550 \times \mathbf{1.303} + 0.196 \times \mathbf{0.445}$ $\mathbf{0.487} = 0.222 \times \mathbf{1.303} + 0.444 \times \mathbf{0.445}$ $\mathbf{0.141} = 0.250 \times \mathbf{0.564}$
Value	$V_{g4} = V_{f4} + P_4 - P_4^\gamma - X_4$ $V_{f3} = V_{g3} + P_3^\gamma - P_3 + X_3$ $V_{i2} = V_{g2} + P_2^\gamma - X_2$ $V_{g1} = V_{i1} - P_1^\gamma + X_1$	$\mathbf{1.303} = \mathbf{0.672} + 2.000 - \mathbf{1.369}$ $\mathbf{0.895} = \mathbf{0.804} - 1.268 + \mathbf{1.359}$ $\mathbf{0.564} = \mathbf{0.487} + 1.414 - \mathbf{1.338}$ $\mathbf{0.445} = \mathbf{0.141} - 1.000 + \mathbf{1.304}$
DoI β	$\boldsymbol{\beta}_W \mathbf{W} =$ $\boldsymbol{\beta}_U \mathbf{U} + \boldsymbol{\beta}_Y \mathbf{Y} - \boldsymbol{\beta}_Z \mathbf{Z}$	$2.048 \times \mathbf{1.303} - 0.762 \times \mathbf{0.445} = -1 \times \mathbf{0.672} + 3$ $-1 \times \mathbf{0.895} = 3.263 \times \mathbf{0.804} - 2.842 \times \mathbf{0.487} - 2.134$ $2 \times \mathbf{0.564} = 1.895 \times \mathbf{0.804} - 2.263 \times \mathbf{0.487} + 0.707$ $0.190 \times \mathbf{1.303} - 1.048 \times \mathbf{0.445} = 2 \times \mathbf{0.141} - 0.500$

Table 5: $V_{g4}, V_{f3}, V_{i2}, V_{g1}, V_{f4}, V_{g3}, V_{g2}, V_{i1}, X_{4,3,2,1}$ and 12 equations (note that V_g terms have been expressed as combinations)

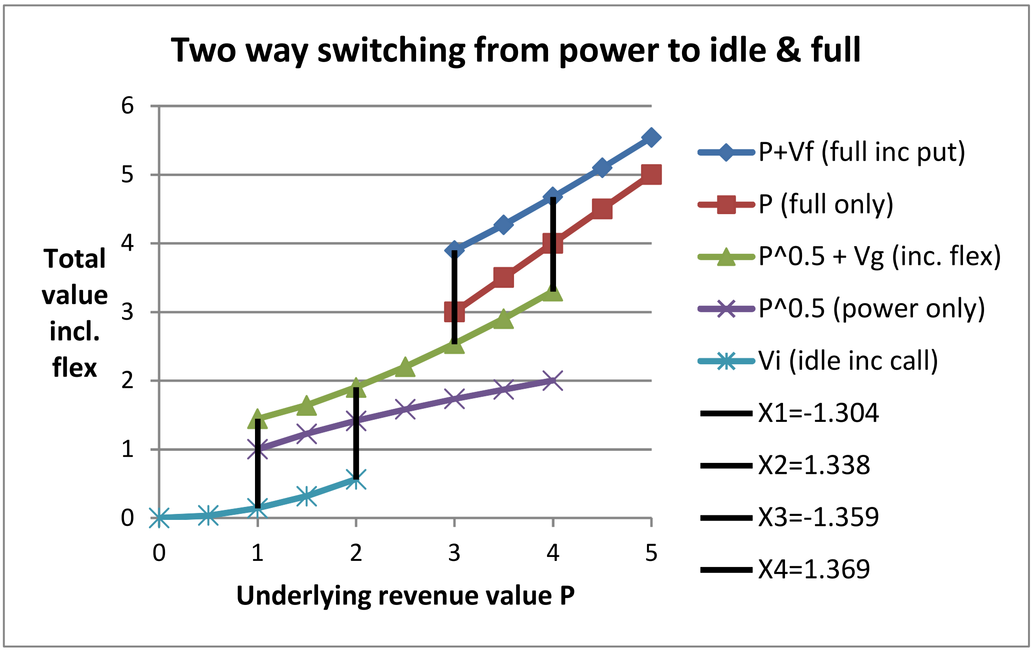

Figure 2: Investments across three mode example with $P_{4..1} = 4, 3, 2, 1$ for GBM and power values $c, p, \gamma = 2, -1, 0.5$. Table 3 shows the investment graph whilst this plot shows the idle, power and full flex values with their investment quantities $X_{4..1}$ (value matching occurs vertically at $P = 4, 3, 2, 1$).

discount factors, this system also embraces the potential for different scenarios and branching. Not only is switching uncertain, but it is also unclear in which direction switching from the power state will go; different sequences i.e. 2,4,3,1 or 2,1,2,1 etc. are possible.

The advantages of this solution system are apparent. It has building blocks in the sense that individual cashflow components can be placed individually within a system that governs their intertwined flexibilities where a common framework and solution method is available (these could then be analysed using modularity[32]). This is an important aspect of sequential investment. Secondly, although the example has used GBMs, other processes could be used by replacing the discount/growth components within $\mathbf{D}, \mathbf{G}$. The merit of the GBM is that its option elasticities are constant which simplifies the formation of $\boldsymbol{\beta}_U, \boldsymbol{\beta}_W$ etc. For $0 < \gamma < 1$, it also permits the same option solutions $V_g(P)$ for all γ powers of values and flows P^γ, π^γ.

5 Conclusion

Flexibility to time the launch, suspension or other transformation within a multi threshold policy can be valued using discounted claims on the underlying uncertainty process. Values at the beginning and end of each option's life can be separately identified on a network graph and these lend themselves to such discounting.

Discount factors, with functional forms and betas dependent on assumed diffusion dynamics, have been used before (Dixit, Pindyck and Sødal [11] and Sødal [23]) but with limited interaction. By capturing the discounts value and beta characteristics within matrices, we extend their use allowing many options within an investment network to interact.

For a set of policy thresholds and a network of transitions between them, a diffusion choice determines a set of option or timing values. The diffusion and its discount function also does this via its betas which are incorporated into the necessary smooth pasting conditions.

Assuming that thresholds are optimally placed to maximise option values, matrices are used to conduct a linear algebra solution for these values and the implied option costs.

[32] Baldwin and Clark (00) [3] and Gamba Fusari (09) [14] motivate and value modularity. The design rests on six principles; splitting, substituting, augmenting, excluding, inverting, porting and the implementation is via multivariate least squares Monte Carlo. These are open to network or graphical investment approaches.

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RN diffusion $d\pi$	Call/put discount functions	β quadratic for c, p
$(r-\delta)\,\pi dt$ $+\sigma\pi dB$	$D_c(\pi,\pi_4) = (\pi/\pi_4)^c : \pi \leqslant \pi_4$ $D_p(\pi,\pi_1) = (\pi/\pi_1)^p : \pi \geqslant \pi_1$	$\frac{\sigma^2}{2}\beta(\beta-1)+$ $(r-\delta)\,\beta - r = 0$
$\alpha dt + \sigma dB$	$D_c(\pi,\pi_4) = e^{c(\pi-\pi_4)} : \pi \leqslant \pi_4$ $D_p(\pi,\pi_1) = e^{p(\pi-\pi_1)} : \pi \geqslant \pi_1$	$\frac{\sigma^2}{2}\beta^2+$ $\alpha\beta - r = 0$
$\kappa(\overline{\pi}-\pi)\,dt$ $+\sigma\pi dB$	$D_p(\pi,\pi_1) = \frac{\left(\frac{\pi}{\pi_1}\right)^p M\left(-p, 2+2p+2\kappa/\sigma^2, \pi\frac{2\kappa\overline{\pi}}{\sigma^2}\right)}{M\left(-p, 2+2p+2\kappa/\sigma^2, \pi_1\frac{2\kappa\overline{\pi}}{\sigma^2}\right)}$	$\frac{\sigma^2}{2}\beta(\beta-1)$ $-\kappa\beta - r = 0$

Table 6: Discount factors for Geometric, Arithmetic and Mean Reverting processes.

6 Appendix: Other processes and discount factors

The same discount outcomes can be applied to other stochastic processes whose stopping time densities are available; Table 6 shows standardised discount factors for Arithmetic and Mean Reverting flows π along with the GBM claims. Whilst these processes have similar solutions c, p to a fundamental quadratic, unlike GBM they are not iso-elastic (constant beta). However the techniques developed here are robust for processes other than GBM (e.g. the mean reversion process used in Sarkar and Zapatero [20]).