Complementary Investments in Non-preemption Duopoly Markets under

Input Cost and Revenue Uncertainty

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Abstract

We study the combined effects of uncertainty and competition on the timing optimization of investments in complementarity inputs for non-preemption duopoly (leader-follower) markets with either a weak-patent system where spillover-knowledge is allowed or a strong-patent system where proprietary-knowledge holds. We find that, for some input-sequencing investment scenarios, ex-ante and (expected) ex-post market shares play an important role on firms' behaviour, and when uncertainty about the inputs cost and revenue are considered together with competition, the conventional wisdom which says that "when a production process requires two extremely complementary inputs firms should upgrade (or replace) them simultaneously", does not necessarily hold particularly for the follower. Some of the illustrated results show nonlinear and complex investment criteria for both firms.

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1. Introduction

Since the pioneering work of Smets (1993), the effect of profit or revenue uncertainty and duopoly competition on firms' investment behaviour has been extensively studied (e.g., Dixit and Pindyck, 1994, ch. 9; Grenadier, 1996; Huisman, 2001; Weeds, 2002; Lambrecht and Perraudin, 2003; Paxson and Pinto, 2005; Pawlina and Kort, 2006; Mason and Weeds, 2010; Femminis and Martini, 2011; Siddiqui and Takashima, 2012).[2] Yet, the existence of complementarity between the inputs of an investment has been neglected and, with few exceptions (e.g., Huisman, 2001, ch. 9; Siddiqui and Takashima, 2012), ex-ante, firms are assumed to hold one option to invest only.

However, firms often use inputs whose qualities are complements, such as computer and modem, equipment and structure, train and track, and transmitter and receiver, and, therefore, investment decisions on upgrades or replacements must consider the degree of complementarity between the inputs. In this work "complementarity" exists if the investment in one input increases the marginal or incremental return to other input in terms of "net cost savings" (NCS). More generally, in industrial organization contexts, complementarity exists if the implementation of one practice increases the marginal return to another practice (e.g., Carree et al., 2010). When the implementation of a technology/practice decreases the marginal return to the other technology/practice, there is "substitutability" (or subadditivity).[3]

The concept of complementarity has been used to study economic decisions in several contexts. For the context of a country, it is used to set innovation policies, for instance, the optimization of the balance between technology imports and in-house R&D (e.g., Braga and Wilmore, 1991, Cassiman and Veugelers, 2004), the allocation of financial resources to industries (e.g., Mohnen and Roller, 2000), to enhance innovation and/or to favour clustering (e.g., Anderson and Schmittlein, 1984), and to define production policies, for instance, the coordination between product and process innovation (e.g., Miravete and Pernías, 2006).

Research and development (R&D) is another area where the effect of complementarity is taken into account, since, when planning R&D activities, firms make strategic decisions regarding the degree of complementarity (sometimes called compatibility) between the new products they aim to launch in the future and the complement products that are already available in the market or they conjecture will be launched by their rivals in the near future, in the sense that the diffusion of an innovation depends, to some extent, on the diffusion of complement innovations which amplify its value.[4] Also, it has been

[2] Recent literature reviews on real option game models are provided by Chevalier-Roignant et al. (2011) and Azevedo and Paxson (2014).

[3] See Carree et al. (2010) for further details.

[4] In R&D contexts, firms who do not have a dominant market position and intend to grow rapidly tend to managing their R&D efforts so as to launch new products which are compatible with those of their rivals who have dominant market positions, firms who have dominant market positions tend to guide their R&D efforts in order to launch new products that are, as much as

argued that the pace of modernization of industries is often influenced by the degree of technological complementarity between the technologies adopted in the industries. For instance, Smith and Weil (2005) investigated how changes in retailing and manufacturing industries, together, affected the diffusion of new information technologies in the U.S. apparel industry between 1988 and 1992, and suggest that there is a significant effect of the degree of complementarity between the technologies that were adopted over time on the pace of modernization of interlinked industries.

The concept of complementarity is also considered in the works of Milgrom and Roberts (1990, 1995), which rely on the theory of supermodular optimization and games to analyse economic systems marked by complementarity, and Milgrom and Roberts (1994), who study the Japanese economy between 1940 and 1995 to interpret the characteristic features of Japanese economic organization in terms of the complementarity between some of the most important elements of its economic structure; and Colombo and Mosconi (1995), who examine the diffusion of flexible automation production and design/engineering technologies in the Italian metalworking industry, giving particular attention to the role of the technological complementarity and the learning effects associated with the experience of previously available technologies.

Conventional wisdom says that "when a production process requires two extremely complementary inputs, a firm should upgrade (or replace) them simultaneously", i.e., when raising the quality of one input it should upgrade its complements at the same time (see Jovanovic and Stolyarov, 2000, p. 15). From Milgrom and Robert (1990, 1995) models, we infer that it is relatively unprofitable to adopt only one part of the modern manufacturing technologies. Also, in Milgrom and Roberts (1990, p. 524), it is suggested that "we should not see an extended period of time during which there are substantial volumes of both highly flexible and highly specialized equipment being used side-by-side", and Cho and McCardle (2009) show that the economic dependence that inherently defines cost relationships inside a firm can significantly influence the timing of adoption, by expediting or delaying the adoption of an improved technology. A good survey on new technology adoption-related literature is provided by Hoppe (2002).

The conclusions above are drawn, however, for market structures where competition and investment costs uncertainty are absent. Smith (2005) considers the effects of complementarity and input cost uncertainty on firms' behaviour, yet she neglects competition. Her results suggest that for markets where investment costs are uncertain and there is not competition, the conventional wisdom stated above may not hold.

possible, not complement (compatible) with those of rivals. A practical illustration of the later strategy is, for instance, the nine-year battle between the European Union (EU) commission and Microsoft which culminated in October 2007 with a fine of €497 million due to a supposed misconduct in developing software that does not allow open-source software developers access to inter-operability information for work-group servers used by businesses and other large organizations (see Etro (2007), p. 221, and Financial Times, October 23, 2007, p. 1).

However, the assumption that there is not competition is unrealistic for most investment decisions. Hence, we added to Smith's model a (leader-follower) competition factor and study firms' investment behaviour for two specific market structures: one where spillover-knowledge is allowed and another where there is proprietary-knowledge.[5] Our results show that when the costs of two inputs, whose functions are complement, are uncertain, for instance decrease at different rates, it may pay to invest first in the input whose cost is falling more slowly and wait to invest in the input whose cost is falling more rapidly, being this behaviour more acute for the follower and slightly different across the two market structures.

Notice that, due to technological progress and improvements in productivity, input cost decreasing is quite common. For instance, the cost of solar power in the U.S. is now 60 percent cheaper than in early 2011, according to a joint report by the U.S. Solar Energy Industries Association (SEIA) and GTM Research[6], yet the cost of some (preferred) land in the U.S. where to install solar panel farms may have increased and, if so, it might have been optimal to invest in the land first and defer the investment in the solar panels. Also, wind towers comprise several components (e.g., the tower, rotor hub, blades, etc) whose cost growth rates per KW of energy power might be different and, if so, it might be optimal to replace the technological components of old wind farms asynchronously - i.e., first the components whose cost is decreasing more slowly and later the components whose cost is decreasing more rapidly.[7]

We also find that ex-ante and expected ex-post market shares play an important role on firms' behaviour for some input-sequencing investment scenarios. The effect of complementarity between the two investment inputs is incorporated in our model using the following inequality: $\gamma_{12} > \gamma_1 + \gamma_2$, where, γ_1 and γ_2 are the proportions of the firm's revenue that are expected to be saved if she operates with *input* 1 alone or *input* 2 alone, respectively, and γ_{12} is the proportion of the firm's revenue that is expected to be saved if she operates with the two inputs at the same time.

The rest of the paper is organized as follows. In section 2, we outline the model assumptions, introduce the methodology for the derivation of the base-model and characterize the two market structures. In section 3, we derive the value functions and investment thresholds for the two firms and each of the market structures, and provide some illustrative sensitivity analysis. In section 4, we show some further results. In section 5, we conclude and offer guidelines for further research.

[5] These market structures are characterized with detail in Section 2.
[6] See http://www.pv-magazine.com/, 20 September 2013.
[7] Se http://www.windmeasurementinternational.com/

2. The Model

Suppose a market comprised of two idle[8] firms, *i* and *j*, considering the investment in two (available) complementary inputs (*input* 1 and *input* 2), one after the other or both at the same time depending on which of these choices maximizes value. Firm *i*'s instantaneous NCS from the investment in input *k* is given by:

$$X(t)\gamma_k D_{k_i k_j} \tag{1}$$

where, $X(t)$ is the market revenue at time *t*; $\gamma_k \in (0,1)$ is the "proportion of the firm *i*'s market revenue that is expected to be saved if she invests in *input* k", with $k = \{0,1,2,12\}$, where "0" means that firm *i* is not yet active and "1", "2" and "12" mean that firm *i* operates with *input* 1, *input* 2, or the two inputs at the same time, respectively; $D_{k_i k_j}$ is the market share of firm *i* for when the two firms (*i* and *j*) operate with input(s) *k*, where $i, j = \{L, F\}$ with "*L*" meaning "leader" and "*F*" meaning "follower".

Market revenue, $X(t)$, follows a geometric Brownian motion (gBm) process given by: [9]

$$dX = \mu_X X dt + \sigma_X X dz_X \tag{2}$$

where, μ_X is the revenue growth rate, σ_X is the revenue volatility and dz_X is the increment of a standard Wiener process. For convergence reasons $r - \mu_X > 0$ holds.

Operating with *input* 1 provides a NCS (S_1) which is a fraction (γ_1) of the firm *i*'s revenue ($X.D_{k_i k_j}$):

$$S_1 = \gamma_1 X.D_{k_i k_j} \tag{4}$$

Since NCS is proportional to revenue and this follows a gBm process, so firm *i*'s NCS also follows a gBm process. Similarly, the use of *input* 2 alone provides NCS equal to:

$$S_2 = \gamma_2 X.D_{k_i k_j} \tag{5}$$

And the simultaneous use of the two inputs yields a NCS equal to: [10]

$$S_{12} = \gamma_{12} X.D_{k_i k_j} \tag{6}$$

The complementarity between the two inputs, ξ, with $\xi = \gamma_{12} - (\gamma_1 + \gamma_2)$ and $\xi \in (0,1)$, is ensured by:

$$\gamma_{12} > \gamma_1 + \gamma_2 \tag{7}$$

The cost of the *inputs* 1 and 2, respectively I_1 and I_2, also follow a gBm processes, given by:

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[8] In this research an idle firm means a firm which is inactive or that is active but operating without the most recent production input(s). For instance, a firm operating with an old rail train with old tracks is idle in not yet adopting high-speed trains and new tracks, if available.

[9] For simplicity of the notation, henceforth, we drop the "*t*".

[10] Suppose that a firm can get: a 10% reduction in operating costs per passenger if invests in a new train; 10% reduction in operating costs per passage if invests in a new track; and 30% reduction in operating cost if invests in both a new train and a new track. Consequently, within a given output range, the more it sells/produces, the more it saves.

$$dI_1 = \mu_{I_1} I_1 dt + \sigma_{I_1} I_1 dz_{I_1} \tag{8}$$

and

$$dI_2 = \mu_{I_2} I_2 dt + \sigma_{I_2} I_2 dz_{I_2} \tag{9}$$

where, μ_{I_1} and μ_{I_2} are the trend rates of growth of the cost of *input* 1 and *input* 2, respectively; σ_{I_1} and σ_{I_2} are the cost volatility of *inputs* 1 and 2, respectively; and dz_{I_1} and dz_{I_2} are the increments of the standard Wiener processes for the costs of *input* 1 and *input* 2, respectively.

Following the framework of Smets (1993), we impose the following constrains on the parameter $D_{k_i k_j}$, where *i* represents the "leader" and *j* the "follower":

$$(D_{12_L 0_F} = D_{1_L 0_F} = D_{2_L 0_F}) > (D_{12_L 1_F} = D_{12_L 2_F}) > (D_{12_L 12_F} = D_{1_L 1_F} = D_{2_L 2_F}) \tag{10}$$

with $D_{12_L 0_F} = D_{1_L 0_F} = D_{2_L 0_F} = 1.0$, $D_{12_L 12_F} = D_{1_L 1_F} = D_{2_L 2_F} = 0.5$ and $D_{12_L 1_F} = D_{12_L 2_F} \in (0.5, 1.0)$, which ensures that: (i) the leader gets 100% of the market share if active alone - regardless of the input(s) she operates with; (ii) the two firms get the same market share (50%) if active with the same input(s); (iii) the leader gets more than 50% of the market share if operates with the two inputs and the follower operates with one input only. [11] Additionally, the following condition holds: $D_{k_L k_F} + D_{k_F k_L} = 1.0$ - i.e., if both firms are active the sum of their market shares is 100%.

The "partial differential equation" (PDE) (11) describes the evolution of the value function of an inactive firm (*i, j*) that holds the option to invest in *input(s)* k:

$$\frac{1}{2}\sigma_X{}^2 X^2 \frac{\partial^2 F_k^{i,j}}{\partial X^2} + \frac{1}{2}\sigma_{I_k}^2 I_k{}^2 \frac{\partial^2 F_k^{i,j}}{\partial I_k{}^2} + X I_k \sigma_X \sigma_{I_k} \rho_{XI_k} \frac{\partial^2 F_k^{i,j}}{\partial X \partial I_k} + \mu_X X \frac{\partial F_k^{i,j}}{\partial X} + \mu_{I_k} I_k \frac{\partial F_k^{i,j}}{\partial I_k} - r F_k^{i,j} = 0 \tag{11}$$

where, ρ_{XI_k} is the correlation coefficient between the market revenue, *X*, and the cost of input(s) *k* (I_k) and *r* is the riskless interest rate. For convergence of the solution we assume that $r - \mu_X > 0$.

An useful analytical simplification of (11) is achieved by taking advantage of the natural homogeneity of degree one of the investment problem - i.e., $F_k^{i,j}(X, I_k) = I_k f_k^{i,j}(X / I_k)$, where $f_{12}^{i,j}$ is the variable to be

[11] The rationale for this assumption is that the leader gets higher cost savings due to the effect of complementarity between the two inputs and is able to use the cost savings advantage to earn a higher market share. Yet, notice that the cost savings advantage is eliminated at the moment the follower invests in the second input as well - henceforth the two firms operate the two inputs and each gets 50% of the market share. Also, for the sack of simplicity of our analysis in the next sections, we assume that the two inputs are symmetric in terms of generation of cost savings, although the model allows for other assumptions on this regard.

determined.[12] We reduce the dimensionality of the PDE (11) from two to one using the following variable change: $\phi_k = X / I_k$.[13]

Substituting ϕ_k in (10) yields (11)[14]

$$\frac{1}{2}\sigma_{m_k}(\phi_k)^2\frac{\partial^2 f_k^{i,j}(\phi_k)}{\partial \phi_k^{\,2}}+\left(\mu_X-\mu_{I_k}\right)(\phi_k)\frac{\partial f_k^{i,j}(\phi_k)}{\partial \phi_k}-(r-\mu_{I_k})f_k^{i,j}(\phi_k)=0 \tag{12}$$

where, $\sigma_{m_k}^2 = \sigma_X^2 + \sigma_{I_k}^2 - 2\rho_{XI_k}\sigma_X\sigma_{I_k}$.

Equation (12) is a homogeneous second-order linear ordinary differential equation (ODE) whose general solution has the form:

$$f_k^{i,j}(\phi_k) = A_k^{i,j}(\phi_k)^{\psi_1} + B_k^{i,j}(\phi_k)^{\psi_2} \tag{13}$$

where ψ_1 (ψ_2) is the positive (negative) solution of the characteristic quadratic function of the ODE (11): $0.5(\sigma_{m_k})^2\psi_1(\psi_1-1)+(\mu_X-\mu_{I_k})\psi_1-(r-\mu_{I_k})=0$. Solving this equation for ψ_1 (ψ_2) we get:

$$\psi_{1(2)} = \frac{1}{2}-\frac{\mu_X-\mu_{I_k}}{\sigma_{m_k}^2}+(-)\sqrt{\left(\frac{(\mu_X-\mu_{I_k})}{\sigma_{m_k}^2}-\frac{1}{2}\right)^2+\frac{2(r-\mu_{I_k})}{\sigma_{m_k}^2}} \tag{14}$$

Notice that as the ratio of market revenue over cost of *input* k, ϕ_k, approaches 0, the value of the option to invest in *input k* becomes worthless; therefore in (13) $B_k^{i,j}=0$. Using the appropriate “value matching” (VM) and “smooth pasting” (SP) conditions for each investment scenario we can determine in the next section the constants ($A_k^{i,j}$) and the investment thresholds for both firms.

2.1 Industry settings

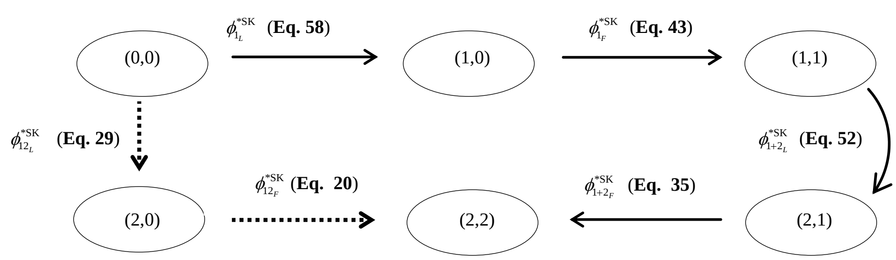
We formulate a leader-follower investment problem for two specific industry scenarios, following Siddiqui and Takashima (2010, p. 585): (i) symmetric non-preemptive duopoly with “spillover-knowledge” (SK); and (ii) symmetric non-preemptive duopoly with “proprietary-knowledge” (PK). The difference between these two scenarios is that, in the former, due to a weak patent-protection, the follower is allowed to proceed with her first stage investment (in *input* 1) immediately after the leader’s entry (with *input* 1) and, in the latter, due to a strong patent-protection, the leader invests in the two inputs sequentially (in *input* 1 first and *input* 2 afterwards, for instance) with the follower inactive.

[12] See proof in section 1 of Appendix A.
[13] This analytical simplification leads to the following input-related ratios: $\phi_1 = X / I_1$, $\phi_2 = X / I_2$ and $\phi_{12} = X / I_{12}$, used in the next sections. Notice that $I_{12}(t)$ is the cost in the two input together at time *t*, which are assumed to follow a gBm process given by: $dI_{12} = \mu_{I_{12}} I_{12} dt + \sigma_{I_{12}} I_2 dz_{I_{12}}$, where $\mu_{I_{12}}$ is the instantaneous cost growth rate; $\sigma_{I_{12}}$ is the cost volatility; and $dz_{I_{12}}$ is the increment of the standard Wiener process.
[14] See full derivation in section 2 of Appendix A.

2.1.2 Non-preemptive duopoly with SK

This industry setting considers a duopoly where the leader cannot be pre-empted by the follower in her first move. After the leader's first move, the follower is allowed to proceed, since she obtains knowledge on the leader's investment (*input* 1) *via* spillover-knowledge. The diagram in Figure 1 indicates both investment approaches for the two firms: sequential-input investment (solid lines) and simultaneous-input investment (dotted lines). From state (1,1) the competition for establishing a dominant position resumes sequential until the investment cycle is completed in state (2,2). Similarly, in the direct approach (simultaneous-input investment - dotted line), the leader invests first, before the follower is allowed to proceed. We add to Siddiqui and Takashima (2010) investment problem the effect of cost uncertainty and complementarity between the investments of the two stages.

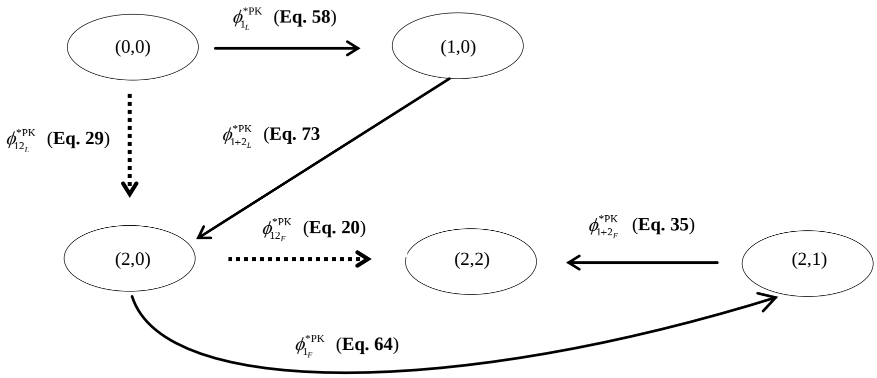
Figure 1 – Duopoly's state transition with SK. Due to SK the follower invests in *input* 1 before the leader's investment in *input* 2. The dotted lines in the diagram represent the *direct approach* where the two firms invest, one after the other, in the two *inputs* at the same time. The solid lines represent the sequential-input investment approach for the two firms. $\phi^*_{1_L}$ and $\phi^*_{1_F}$ are the leader's and the follower's thresholds to invest in *input* 1 alone, respectively; $\phi^*_{1+2_L}$ and $\phi^*_{1+2_F}$ are the leader's and the follower's threshold to invest in *input* 2 if active with *input* 1, respectively; and $\phi^*_{12_L}$ and $\phi^*_{12_F}$ are the leader's and the follower's thresholds to invest in the two inputs at the same time, respectively. The information in between brackets refers to the firms' threshold expressions for each stage of the investment, which are derived in section 3.



2.1.3 Non-preemptive duopoly with PK

In this scenario the leader is allowed to invest in *inputs* 1 and 2, sequentially or simultaneously, with the follower inactive, due to proprietary-knowledge.

Figure 2 – Duopoly's state transition with PK. The dotted lines refer to the direct approach where the two firms invest, one after the other, in the two inputs at the same time; the solid lines refer to the scenario where the two firms invest, one after the other, in the two inputs sequentially. The notation meanings of the thresholds above are the same as those described for Figure 1 above.



3. Analytical Results

3.1 Simultaneous-input investment

3.1.1 SK market

In this section we consider that both firms are inactive at stage (0,0) and invest, one after the other, in the two inputs at the same time if optimal to do so. I_{12} is the cost spent if firms invest in the two inputs at the same time.

3.1.1.1 Follower

ODE (12), with $i = F$ and $k = 12$, describes the follower's value if inactive in a SK market, whose general solution is given by:[15]

$$f_{12}^{\mathrm{F,SK}} = A_{12}^{\mathrm{F,SK}} \phi_{12}{}^{\eta_1} + B_{12}^{\mathrm{F,SK}} \phi_{12}{}^{\eta_2} \tag{15}$$

where $\eta_1\,(\eta_2)$ is the positive (negative) solution of the characteristic quadratic function of the ODE (12): $\frac{1}{2}(\sigma_{m_{12}})^2\eta(\eta-1)+(\mu_X-\mu_{I_{12}})\eta-(r-\mu_{I_{12}})=0$. Solving this equation for $\eta_1(\eta_2)$ leads to:

$$\eta_{1(2)} = \frac{1}{2} - \frac{\mu_X - \mu_{I_{12}}}{\sigma_{m_{12}}^2} + (-)\sqrt{\left(\frac{(\mu_X - \mu_{I_{12}})}{\sigma_{m_{12}}^2} - \frac{1}{2}\right)^2 + \frac{2(r - \mu_{I_{12}})}{\sigma_{m_{12}}^2}} \tag{16}$$

where, $\sigma_{m_{12}}^2 = \sigma_X^2 + \sigma_{I_{12}}^2 - 2\rho_{XI_{12}}\sigma_X\sigma_{I_{12}}$.

Notice that as ϕ_{12} approaches 0, the value of the option to invest in *inputs* 1 and *input* 2 at the same time becomes worthless. Therefore, in (15) $B_{12} = 0$.

Using the following VM condition:

[15] In (15) the superscripts "F" and "SK" stand for "follower" and "spillover-knowledge", respectively.

$$F_{12}^{\mathrm{F,SK}}(X^*, I_{12}^*) = \frac{X^* \gamma_{12} D_{12_F 12_L}}{r - \mu_X} - I_{12_F}^* \tag{17}$$

With the following economic interpretation: before investing in *input* 1 and *input* 2 at the same time the follower holds the option to invest whose value is given by the left-hand side of Eq. 17. This option will be exercised at the moment her value equals the present value of the cash flows she obtains from operating with the two inputs forever subtracted of the investment cost (right-hand side of Eq. 17).

Dividing (17) by $I_{12_F}^*$, replacing $f_{12}^{\mathrm{F,SK}}(\phi_{12}^*)$ by $A_{12}^{\mathrm{F,SK}} \phi_{12}^{*\,\eta_1}$ and rewriting gives,

$$A_{12}^{\mathrm{F,SK}} \phi_{12_F}^{*\mathrm{SK}\eta_1} = \frac{\phi_{12_F}^{*\mathrm{SK}} \gamma_{12} D_{12_F 12_L}}{r - \mu_X} - 1 \tag{18}$$

The SP condition is:

$$\eta_1 A_{12}^{\mathrm{F,SK}} \phi_{12_F}^{*\mathrm{SK}(\eta_1 - 1)} = \frac{\gamma_{12} D_{12_F 12_L}}{r - \mu_X} \tag{19}$$

Solving together equations (15), (18) and (19) and rearranging we get the follower's threshold to invest in the *inputs* 1 and 2 at the same time, $\phi_{12_F}^{*\mathrm{SK}}$, and the constant $A_{12}^{\mathrm{F,SK}}$, respectively:

$$\phi_{12_F}^{*\mathrm{SK}} = \frac{\eta_1}{\eta_1 - 1} \frac{r - \mu_X}{\gamma_{12} D_{12_F 12_L}} \tag{20}$$

$$A_{12}^{\mathrm{F,SK}} = \frac{\phi_{12}^{*\mathrm{SK}(1-\eta_1)}}{\eta_1} \frac{\gamma_{12} D_{12_F 12L}}{r - \mu_X} \tag{21}$$

The follower's value function is given by:

$$f_{12}^{\mathrm{F,SK}}(\phi_{12}) = \begin{cases} A_{12}^{\mathrm{F,SK}} \phi_{12}{}^{\eta_1} & \text{if } \phi_{12} < \phi_{12_F}^{*\mathrm{SK}} \\ \dfrac{\phi_{12} \gamma_{12} D_{12_F 12_L}}{r - \mu_X} - 1 & \text{if } \phi_{12} \geq \phi_{12_F}^{*\mathrm{SK}} \end{cases} \tag{22}$$

The first row of (22) represents the follower's option to invest in the two inputs at the same time; the second row is the payoff the follower attains from operating in the market with leader (both with the two inputs) from $\phi_{12_F}^*$ until infinity.

3.1.1.2 Leader

Assuming that the follower invests in the two inputs at the same time when $\phi_{12_F}^{*\mathrm{SK}}$ is reached, the leader's payoff at the time of her investment in the two inputs is:

$$F_{12}^{\mathrm{L,SK}}(X, I_{12}) = E\left[\int_{t=\tau}^{T_{12_F}} X_\tau\ \gamma_{12} D_{12_L 0_F} e^{-r\tau} d\tau - I_{12_L}^* + \int_{T_{12_F}}^{\infty} X_\tau\ \gamma_{12} D_{12_L 12_F} e^{-r\tau} d\tau\right] \tag{23}$$

where, the first integral represents the leader's payoff from the instant she invests in *inputs* 1 and 2 at the same time to the instant before the follower invests in *inputs* 1 and 2 at the same time; $I_{12_L}^*$ is the costs of the two inputs at the time of the investment; and the second integral is the leader's payoff from operating with the follower (both firms with the two inputs) from the moment the follower invests in *inputs* 1 and 2 at the same time until infinity.

The leader's value function is given by:

$$f_{12}^{\mathrm{L,SK}}(\phi_{12}) = \begin{cases} A_{12}^{\mathrm{L,SK}} \phi_{12}^{\ \eta_1} & \text{if } \phi_{12} < \phi_{12_L}^{*\mathrm{SK}} \\ \dfrac{\phi_{12}\gamma_{12} D_{12_L 0_F}}{r-\mu_X} - 1 + \dfrac{\phi_{12}\gamma_{12}(D_{12_L 12_F} - D_{12_L 0_F})}{r-\mu_X}\left(\dfrac{\phi_{12}}{\phi_{12_F}^{*\mathrm{SK}}}\right)^{\eta_1} & \text{if } \phi_{12} \in \left[\phi_{12_L}^{*\mathrm{SK}}, \phi_{12_F}^{*\mathrm{SK}}\right) \\ \dfrac{\phi_{12}\gamma_{12} D_{12_L 12_F}}{r-\mu_X} & \text{if } \phi_{12} \geq \phi_{12_F}^{*\mathrm{SK}} \end{cases} \tag{24}$$

The term in the first row is the leader's option value to invest in the two inputs at the same time; in the second row, the first two terms represent the leader's payoff from operating with the two inputs from $\phi_{12_L}^{*\mathrm{SK}}$ until infinity with the follower inactive subtracted of the investment cost, the third term is a correction factor which incorporates the fact that in future if $\phi_{12_F}^{*\mathrm{SK}}$ is reached the follower will invest in *input* 12 and the leader's payoff will be reduced - it is negative given that $(D_{12_L 12_F} - D_{12_L 0_F}) < 0$ (see inequality 10) [16]; the third row is the leader's payoff from operating with the follower (both firms with the two inputs) from the instant the follower invests in the two inputs until infinity.

This is a non-pre-emption game and, therefore, the leader enters the market at the moment her payoff is maximized. ODE (12), with $k=12$, describes the leader's value if inactive, whose solution is given by:

$$f_{12}^{\mathrm{L,SK}} = A_{12}^{\mathrm{L,SK}} \phi_{12}^{\ \eta_1} + B_{12}^{\mathrm{L,SK}} \phi_{12}^{\ \eta_2} \tag{25}$$

As ϕ_{12} approaches 0, the value of the option to invest in *inputs* 1 and 2 at the same time becomes worthless, hence $B_{12}^{\mathrm{L,SK}} = 0$. The constant $A_{12}^{\mathrm{L,SK}}$ and the leader's threshold, $\phi_{12_L}^{*\mathrm{SK}}$, are determined using the following VM and SP conditions:

VM condition:

$$A_{12}^{\mathrm{L,SK}} \phi_{12_L}^{*\mathrm{SK}\eta_1} = \frac{\phi_{12_L}^{*\mathrm{SK}} \gamma_{12} D_{12_L 0_F}}{r-\mu_X} - 1 \tag{26}$$

SP condition:

$$\eta_1 A_{12}^{\mathrm{L,SK}} \phi_{12_L}^{*\mathrm{SK}(\eta_1 - 1)} = \frac{\gamma_{12} D_{12_L 0_F}}{r-\mu_X} \tag{27}$$

Solving together (25), (26) and (27) we obtain $A_{12}^{L,\mathrm{SK}}$ and $\phi_{12_L}^{*\mathrm{SK}}$, given by: [17]

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[16] Notice that this term equals the leader's loss discounted back from the (random) time at which the follower invests in inputs 1 and 2. The term $(\phi_{12} / \phi_{12_F}^{*\mathrm{SK}})^{\eta_1}$ is interpreted as a stochastic discount factor which is equal to the present value of $1 received when the variable ϕ_{12} hits $\phi_{12_F}^{*\mathrm{SK}}$ (see Pawlina and Kort, 2006, p. 8).

[17] Equation (29) is derived from the equation system comprising (26) and (27) where $A_{12}^{\mathrm{L,SK}}$ and $\phi_{12_L}^{*\mathrm{SK}}$ as the unknown variables.

$$A_{12}^{\mathrm{L,SK}} = \frac{\gamma_{12} D_{12_L 0_F}}{(r-\mu_X)\eta_1 \phi_{12_L}^{*\mathrm{SK}(\eta_1 - 1)}} \tag{28}$$

$$\phi_{12_L}^{*\mathrm{SK}} = \frac{\eta_1 (r-\mu_X)}{\eta_1 \gamma_{12} D_{12_L 0_F} - \gamma_{12} D_{12_L 0_F}} \tag{29}$$

3.1.2 PK market

From Figures 1 and 2 we can easily see that "simultaneous-input" investments are one-shot games for the two firms and the PK and SK markets. Therefore, the behaviour of the leader and the follower should be the same for the two markets.[18] Consequently, the following proposition holds:

Proposition 1:

$$f_{12}^{\mathrm{L,PK}}(\phi_{12}) = f_{12}^{\mathrm{L,SK}}(\phi_{12}) \text{ - given by Expression (24)} \tag{29A}$$

$$f_{12}^{\mathrm{F,PK}}(\phi_{12}) = f_{12}^{\mathrm{F,SK}}(\phi_{12}) \text{ - given by Expression (22)} \tag{29B}$$

$$\phi_{12_L}^{*\mathrm{PK}} = \phi_{12_L}^{*SK} \text{ - given by Equation (29)} \tag{29C}$$

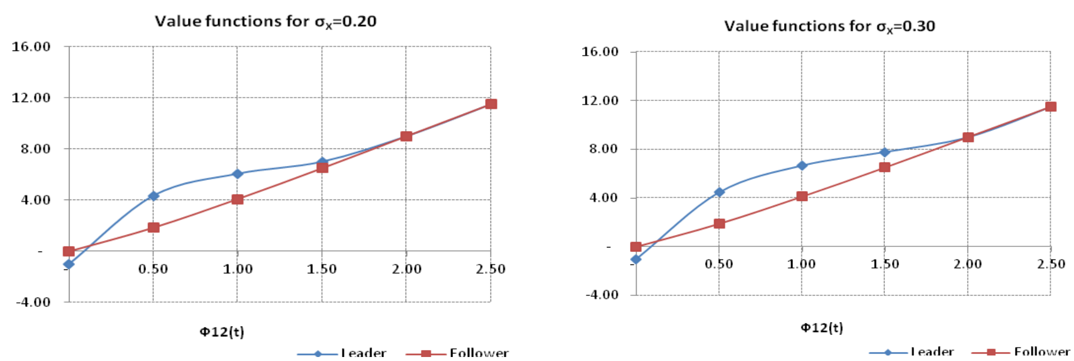
$$\phi_{12_F}^{*\mathrm{PK}} = \phi_{12_F}^{*SK} \text{ - given by Equation (20)} \tag{29D}$$

where, $f_{12}^{\mathrm{L,PK}}(\phi_{12})$ and $\phi_{12_L}^{*\mathrm{PK}}$ are the leader's value function and investment threshold for the PK market, respectively; and $f_{12}^{\mathrm{F,SK}}(\phi_{12})$, $\phi_{12_F}^{*SK}$ are the follower's value function and investment threshold for the SK market, respectively.

Proof: see Appendix B.

3.1.3 Illustrative Results

Figure 3 - this figure shows the value functions of the leader (expression 24) and the follower (expression 22) for the simultaneous-input investment scenario and the SK and PK markets, with two levels of market revenue volatility: 20% (left-hand side) and 30% (right-hand side).



[18] Notice that, in sequential-input investments, the difference between the SK and PK markets is that, in the former, the leader completes the two investment stages before the follower is allowed to proceed, whereas in the former, the follower is allowed to invests in the first stage (*input* 1) immediately after the leader has invested in the first stage (*input* 1).

The shapes of the value functions are standard within the real option games literature and, as expected, the value functions and investment thresholds of the leader and the follower increase with the volatility.

Figure 4 - this figure shows the effect on firms' investment thresholds of changes in the correlation between market revenue and the sum of the costs of the two inputs, $\rho_{X/I_{12}}$, (left-hand side) and the degree of complementarity between the two inputs, ξ, (right-hand side) for the SK and PK markets. According to proposition 1, the investment threshold expressions for the SK and PK markets are the same, we use $\Phi^{*}12L,SK\&PK$ and $\Phi^{*}12F,SK\&PK$ to represent the investment thresholds of the leader and the follower, respectively, for the two markets.

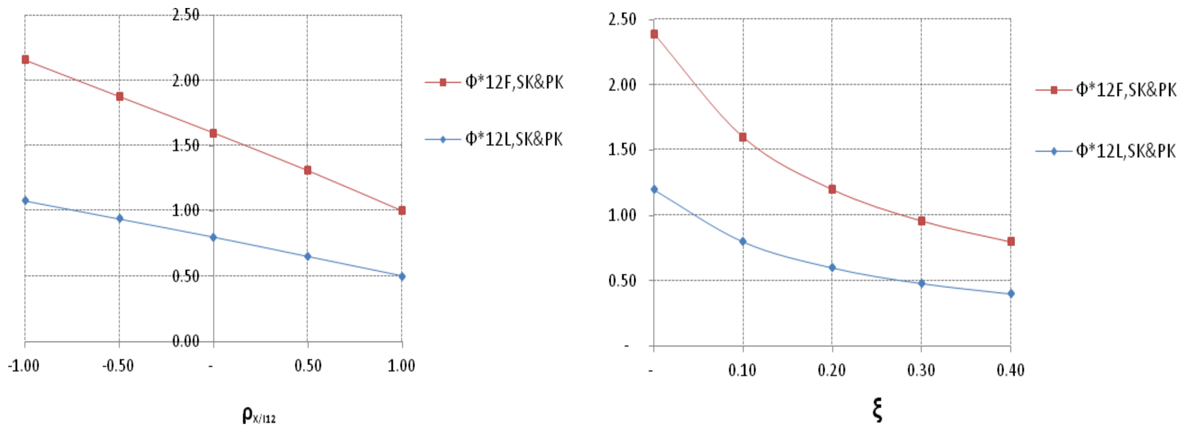
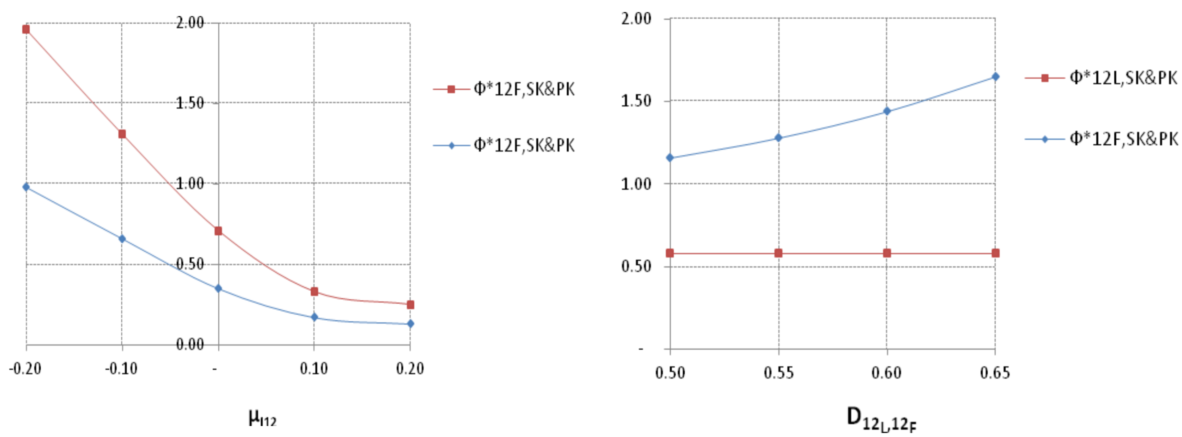


Figure 5 - this figure shows the effect on firms' investment thresholds of changes in the cost growth rate of the two inputs (left-hand side) and the leader's market share if active with the two inputs with the follower (right-hand side), for the SK and PK markets.



The results above show that for both markets, *ceteris paribus*: (i) an increase in $\rho_{X/I_{12}}$, or ξ, or $\mu_{I_{12}}$ accelerates the investment in the two inputs at the same time for both firms, being the follower more sensitive than the leader to these changes, particularly for negative levels of $\rho_{X/I_{12}}$ and $\mu_{I_{12}}$, and lower levels of ξ; and (ii) an increase in $D_{12_F 12_L}$ delays the investment of the follower and has no effect on the leader's behaviour.

3.2 "Sequential-input" investment

3.2.1 SK market

3.2.1.1 Terminal state: follower

We start by deriving the follower's value function and investment threshold for the state where she is active with *input* 1 and the leader is active with both inputs. At this stage the follower's value comprises of the option value to invest in *input* 2, $f_{1+2}^{\mathrm{F,SK}}(\phi_2)$, plus the cost savings attained from operating with *input* 1 forever, $X\gamma_1 D_{1_F 12_L}/(r-\mu_X)$. Following similar procedures as those described in the previous section, we get the homogeneous second-order linear ODE (12), with $k=2$, whose general solution for has the form:

$$f_{1+2}^{\mathrm{F,SK}}(\phi_2) = A_{1+2}^{\mathrm{F,SK}}\phi_2^{\psi_1} + B_{1+2}^{\mathrm{F,SK}}\phi_2^{\psi_2} \tag{30}$$

where ψ_1 (ψ_2) is the positive (negative) solution of the characteristic quadratic function of the homogeneous part of equation (12): $0.5(\sigma_{m_2})^2\psi(\psi-1)+(\mu_X-\mu_{I_2})\psi-(r-\mu_{I_2})=0$. Solving this equation for ψ_1 (ψ_2) we get:

$$\psi_{1(2)} = \frac{1}{2} - \frac{\mu_X-\mu_{I_2}}{\sigma_{m_2}^2} + (-)\sqrt{\left(\frac{(\mu_X-\mu_{I_2})}{\sigma_{m_2}^2} - \frac{1}{2}\right)^2 + \frac{2(r-\mu_{I_2})}{\sigma_{m_2}^2}} \tag{31}$$

where, $\sigma_{m_2}^2 = \sigma_X^2 + \sigma_{I_2}^2 - 2\rho_{XI_2}\sigma_X\sigma_{I_2}$.

As the ratio of market revenue over cost of *input 2*, ϕ_2, approaches 0, the value of the option to invest in *input 2* becomes worthless. Hence, in (30) $B_{1+2}^{\mathrm{F,SK}} = 0$.

Using the following VM condition:

$$F_{1+2}^{\mathrm{F,SK}}(X^*, I_2^*) + \frac{X^*\gamma_1 D_{1_F 12_L}}{r-\mu_X} = \frac{X^*\gamma_{12} D_{12_F 12_L}}{r-\mu_X} - I_{2_F}^* \tag{32}$$

With the following economic interpretation: before investing in *input* 2 the follower's payoff is equal to the value of the option to invest in *input* 2 plus the present value from operating with *input* 1 forever, with the leader active with the two inputs (left-hand side of 32). The option to invest in *input* 2 is exercised at the moment its value equals the present value of the cash flows the follower obtains from operating with the two inputs forever subtracted of the investment cost (right-hand side of 32).

Dividing (32) by $I^*_{2_F}$, replacing by $f^{\text{F,SK}}_{1+2}(\phi^*_2)$ and this by $A^{\text{F,SK}}_{1+2}\phi^{*\text{SK}\psi_1}_{1+2}$ and rewriting gives:[19]

$$A^{\text{F,SK}}_{1+2}\phi^{*\text{SK}\ \psi_1}_{1+2_F} = \frac{\phi^{*\text{SK}}_{1+2_F}\gamma_{12}D_{12_F12_L} - \phi_1\gamma_1 D_{1_F12_L}}{r-\mu_X} - 1 \tag{33}$$

The SP condition is given by:

$$\psi_1 A^{\text{F,SK}}_{1+2}\phi^{*\text{SK}\ (\psi_1-1)}_{1+2_F} = \frac{\gamma_{12}D_{12_F12_L}}{r-\mu_X} \tag{34}$$

Solving together equations (30), (33) and (34) we get the follower's threshold for investing in *input* 2 if active with *input* 1, $\phi^{*\text{SK}}_{1+2_F}$, and the constant $A^{\text{F,SK}}_{1+2}$, respectively:

$$\phi^{*\text{SK}}_{1+2_F} = \frac{\psi_1(\mu_X - r) - \psi_1\phi_1\gamma_1 D_{1_F12_L}}{\gamma_{12}D_{12_F12_L}(1-\psi_1)} \tag{35}$$

$$A^{\text{F,SK}}_{1+2} = \frac{\gamma_{12}D_{12_F12_L}}{(r-\mu_X)\psi_1\phi^{*\text{SK}\ (\psi_1-1)}_{1+2_F}} \tag{36}$$

The follower's value function at the instant she invests in *input* 1 if inactive (with the leader operating with the two inputs), is given by:

$$f^{\text{F,SK}}_{1+2}(\phi_2) = \begin{cases} \dfrac{\phi_1\gamma_1 D_{1_F12_L}}{r-\mu_X} + A^{\text{F,SK}}_{1+2}\phi_2^{\psi_1} & \text{if } \phi_2 < \phi^{*\text{SK}}_{1+2_F} \\ \dfrac{\phi_{12}\gamma_{12}D_{12_F12_L}}{r-\mu_X} - 1 & \text{if } \phi_2 \geq \phi^{*\text{SK}}_{1+2_F} \end{cases} \tag{37}$$

The first row of (37) is the follower's payoff from operating with *input* 1 forever. Its third term is the follower's option value to invest in *input* 2 if active with the *input* 1 and the leader is active with the two inputs. The second row is the follower's payoff from operating with the two inputs (with leader) from $\phi^{*\text{SK}}_{1+2_F}$ until infinity.

3.2.1.2 First state: follower

Now we derive the follower's value function and investment threshold for the state where she is inactive and the leader is active with *input* 1. Following similar procedures as those of the previous subsections we get the homogeneous second-order linear ODE (12), with $k=1$, whose general solution in this case has the form:

$$f^{\text{F,SK}}_1(\phi_1) = A^{\text{F,SK}}_1\phi_1^{\beta_1} + B^{\text{F,SK}}_1\phi_1^{\beta_2} \tag{38}$$

where, β_1 (β_2) is the positive (negative) solution of the characteristic quadratic function of the homogeneous part of equation (12): $0.5(\sigma_{m_1})^2\beta(\beta-1) + (\mu_X - \mu_{I_1})\beta - (r-\mu_{I_1}) = 0$. Solving this equation for β_1 (β_2) we get:

[19] Notice that ϕ^*_{1+2} is the threshold at which the follower invests in *input* 2 if active with *input* 1. Therefore, in the VM condition we replace the ϕ_2 of Eq. (30) by $\phi^*_{1+2_F}$.

$$\beta_{1(2)} = \frac{1}{2} - \frac{\mu_X - \mu_{I_1}}{\sigma_{m_1}^2} + (-)\sqrt{\left(\frac{(\mu_X - \mu_{I_1})}{\sigma_{m_1}^2} - \frac{1}{2}\right)^2 + \frac{2(r - \mu_{I_1})}{\sigma_{m_1}^2}} \tag{39}$$

where, $\sigma_{m_1}^2 = \sigma_X^2 + \sigma_{I_1}^2 - 2\rho_{XI_1}\sigma_X\sigma_{I_1}$.

As the ratio ϕ_1 approaches 0, the value of the option becomes worthless, so $B_1^{\mathrm{F,SK}} = 0$. Using the following VM condition:

$$F_1^{\mathrm{F,SK}}(X, I_1) = \frac{X^* \gamma_1 D_{1_F 1_L}}{r - \mu_X} - I_{1_F}^* \tag{40}$$

With the following economic interpretation: before investing in *input* 1 the follower's payoff is equal to the value of the option to invest in *input* 1 (left-hand side of Eq. 40). This option will be exercised at the moment its value equals the follower's present value of the cash flows from operating with *input* 1 forever subtracted of the investment cost (right-hand side of Eq. 40).

Dividing (40) by $I_{1_F}^*$, replacing $f_1^{\mathrm{F,SK}}(\phi_1)$ by $A_1^{\mathrm{F,SK}} \phi_1^{*\mathrm{SK}\beta_1}$ and rewriting gives:

$$A_1^{\mathrm{F,SK}} \phi_{1_F}^{*\mathrm{SK}\beta_1} = \frac{\phi_{1_F}^{*\mathrm{SK}} \gamma_1 D_{1_F 1_L}}{r - \mu_X} - 1 \tag{41}$$

The SP condition is:

$$\beta_1 A_1^{\mathrm{F,SK}} \phi_{1_F}^{*\mathrm{SK}(\beta_1 - 1)} = \frac{\gamma_1 D_{1_F 1_L}}{r - \mu_X} \tag{42}$$

Solving together equations (38), (41) and (42) we get the follower's threshold for investing in *input* 1 if inactive, $\phi_{1_F}^{*\mathrm{SK}}$, and the constant $A_1^{\mathrm{F,SK}}$:

$$\phi_{1_F}^{*\mathrm{SK}} = \frac{\beta_1}{\beta_1 - 1} \frac{r - \mu_X}{\gamma_1 D_{1_F 1_L}} \tag{43}$$

$$A_1^{\mathrm{F,SK}} = \frac{\phi_{1_F}^{*\mathrm{SK}(1-\beta_1)}}{\beta_1} \frac{\gamma_1 D_{1_F 1_L}}{r - \mu_X} \tag{44}$$

The follower's value function is given by:

$$f_1^{\mathrm{F,SK}}(\phi_1) = \begin{cases} A_1^{\mathrm{F,SK}} \phi_1^{\beta_1} & \text{if } \phi_1 < \phi_{1_F}^{*\mathrm{SK}} \\ \dfrac{\phi_1 \gamma_1 D_{1_F 1_L}}{r - \mu_X} - 1 & \text{if } \phi_1 \geq \phi_{1_F}^{*\mathrm{SK}} \end{cases} \tag{45}$$

The first row of (45) is the follower's option value to invest in *input* 1 if inactive. The second row is the follower's payoff from operating with the leader (both firms with *input* 1), from $\phi_{1_F}^{*\mathrm{SK}}$ until infinity.

3.2.1.3 Terminal state: leader

At the instant the leader invests in *input* 2, τ, her payoff is given by:

$$F_{1+2}^{\mathrm{L,SK}}(X, I_2) = E\left[\int_{t=\tau}^{T_{2_F}} X_t \ \gamma_{12} D_{12_L 1_F} e^{-r\tau} d\tau - I_{2_L}^{*} + \int_{T_{2_F}}^{\infty} X_t \ \gamma_{12} D_{12_L 12_F} e^{-r\tau} d\tau\right] \tag{46}$$

where the first integral represents the leader's payoff from the moment she invests in *input* 2 until the instant before the follower invests in *input* 2; and the second integral represents the leader's payoff from the moment the follower invests in *input* 2 until infinity.

The leader's value function is given by:

$$f_{1+2}^{\mathrm{L,SK}}(\phi_2) = \begin{cases} \dfrac{\phi_1 \gamma_1 D_{1_L 1_F}}{r-\mu_X} + A_{1+2}^{L,\mathrm{SK}} \phi_2^{\psi_1} & \text{if } \phi_2 < \phi_{1+2_L}^{*\mathrm{SK}} \\ \dfrac{\phi_{12}\gamma_{12} D_{12_L 1_F}}{r-\mu_X} - 1 + \dfrac{\phi_{12}\gamma_{12}(D_{12_L 12_F} - D_{12_L 1_F})}{r-\mu_X}\left(\dfrac{\phi_2}{\phi_{1+2_F}^{*\mathrm{SK}}}\right)^{\psi_1} & \text{if } \phi_2 \in \left[\phi_{1+2_L}^{*\mathrm{SK}}, \phi_{1+2_F}^{*\mathrm{SK}}\right) \\ \dfrac{\phi_{12}\gamma_{12} D_{12_L 12_F}}{r-\mu_X} & \text{if } \phi_2 \geq \phi_{1+2_F}^{*\mathrm{SK}} \end{cases} \tag{47}$$

In the first row, the first term represents the leader's value from operating with the follower (both firms with *input* 1) forever, the second term is the leader's option value to invest in *input* 2 if active with *input* 1; in the second row, the first two terms represent the leader's payoff from the instant she invests in *input* 2 until infinity subtracted of the investment cost; the third term is negative, given that $(D_{12_L 12_F} - D_{12_L 1_F}) < 0$ - see inequality 10-, and corresponds to a correction factor which incorporates the fact that in future if $\phi_{1+2_F}^{*\mathrm{SK}}$ is reached the follower will invest in *input* 2 reducing the leader's payoff; the third row represents the leader's payoff from operating with the follower (both with the two inputs) forever.

This is a non-preemption game, hence the leader, if active with *input* 1, invests in *input* 2 at the point her payoff is maximized. ODE (12), with $k = 2$, describes the leader's value whose solution is given, in this case, by:

$$f_{1+2}^{L,\mathrm{SK}}(\phi_2) = A_{1+2}^{L,\mathrm{SK}} \phi_2^{\psi_1} + B_{1+2}^{L,\mathrm{SK}} \phi_2^{\psi_2} \tag{48}$$

Following similar rationale as those described in the previous subsections we conclude that $B_{1+2}^{L,\mathrm{SK}} = 0$. The constant $A_{1+2}^{L,\mathrm{SK}}$ and the leader's threshold are determined using the following VM and SP conditions:

VM condition:

$$\frac{\phi_1 \gamma_1 D_{1_L 1_F}}{r-\mu_X} + A_{1+2}^{L,\mathrm{SK}} \phi_{1+2_L}^{*\mathrm{SK}\,\psi_1} = \frac{\phi_{1+2_L}^{*\mathrm{SK}} \gamma_{12} D_{12_L 1_F}}{r-\mu_X} - 1 \tag{49}$$

with the following economic interpretation: the leader, , if active with *input* 1, should invest in *input* 2 at the moment the value she attains from operating with *input* 1 forever plus the value of the option to invest in *input* 2 equals the present value of the perpetual cash flow she obtains from operating with the two inputs forever subtracted of the investment cost (in *input* 2).

SP condition:

$$\psi_1 \; A_{1+2}^{L,\mathrm{SK}} \phi_{1+2_L}^{*\mathrm{SK}\,(\psi_1-1)} = \frac{\gamma_{12} D_{12_L 1_F}}{r-\mu_X} \tag{50}$$

Solving together (48), (49) and (50) we obtain the constant $A_{1+2}^{\mathrm{L,SK}}$:

$$A_{1+2}^{L,\mathrm{SK}} = \frac{\gamma_{12} D_{12_L 1_F}}{(r-\mu_X)\psi_1 \phi_{1+2_L}^{*\mathrm{SK}\,(\psi_1-1)}} \tag{51}$$

and the leader's threshold:

$$\phi_{1+2_L}^{*\mathrm{SK}} = \frac{\phi_1 \psi_1 \gamma_1 D_{1_L 1_F} + \psi_1 (r-\mu_X)}{\gamma_{12} D_{12_L 1_F} (\psi_1 - 1)} \tag{52}$$

3.2.1.4 First-state: leader

The leader's value function for this stage is given by:

$$f_1^{\mathrm{L,SK}}(\phi_1) = \begin{cases} A_1^{\mathrm{L,SK}} \phi_1^{\beta_1} & \text{if } \phi_1 < \phi_{1_L}^{*\mathrm{SK}} \\ \dfrac{\phi_1 \gamma_1 D_{1_L 0_F}}{r-\mu_X} - 1 + \dfrac{\phi_1 \gamma_1 (D_{1_L 1_F} - D_{1_L 0_F})}{r-\mu_X} \left(\dfrac{\phi_1}{\phi_{1_F}^{*\mathrm{SK}}} \right)^{\beta_1} & \text{if } \phi_1 \in \left[\phi_{1_L}^{*\mathrm{SK}}, \phi_{1_F}^{*\mathrm{SK}} \right) \\ \dfrac{\phi_1 \gamma_1 D_{1_L 1_F}}{r-\mu_X} & \text{if } \phi_1 \geq \phi_{1_F}^{*\mathrm{SK}} \end{cases} \tag{53}$$

The first row is the leader's option value to invest in *input* 1 if inactive; in the second row, the first two terms represent the leader's payoff from operating alone with the *input* 1 forever subtracted of the investment cost (in *input* 1), the third term is negative, given that $(D_{1_L 1_F} - D_{1_L 0_F}) < 0$ (see inequality 10), and corresponds to a correction factor which incorporates the fact that in future if $\phi_{1_F}^{*\mathrm{SK}}$ is reached the follower invest in *input* 1 and the leader's payoff will be reduced; the third row is the leader's payoff from operating with the follower (both with the *input* 1) forever.

The leader enters the market at the point her payoff is maximized. ODE (12) describes the leader's value if inactive, whose solution is given in this case by:

$$f_1^{\mathrm{L,SK}}(\phi_1) = A_1^{\mathrm{L,SK}} \phi_1^{\beta_1} + B_1^{\mathrm{L,SK}} \phi_1^{\beta_2} \tag{54}$$

Following standard procedures we find out that $B_1^{\mathrm{L,SK}} = 0$. The constant $A_1^{\mathrm{L,SK}}$ and the leader's threshold are determined using the following VM and SP conditions:

VM condition:

$$A_1^{\mathrm{L,SK}} \phi_{1_L}^{*\mathrm{SK}\,\beta_1} = \frac{\phi_{1_L}^{*\mathrm{SK}} \gamma_1 D_{1_L 0_F}}{r-\mu_X} - 1 \tag{55}$$

with the following economic interpretation: the leader should invest in *input* 1 at the moment the option value to invest in *input* 1 equals the value she obtains from operating with *input* 1 alone forever subtracted of the investment cost (cost of *input* 1).

SP condition:

$$A_1^{L,SK} \phi_{1_L}^{*SK\beta_1} = \frac{\gamma_1 D_{1_L 0_F}}{r-\mu_X} \tag{56}$$

Solving together (54), (55) and (56) we obtain the constant A_{1+2}^{L}, given by:

$$A_1^{L,SK} = \frac{\gamma_1 D_{1_L 0_F} \phi_{1_L}^{*SK(1-\beta_1)}}{\beta_1 (r-\mu_X)} \tag{57}$$

and the leader's threshold, given by:

$$\phi_{1_L}^{*SK} = \frac{\beta_1 (r-\mu_X)}{\beta_1 \gamma_1 D_{1_L 0_F} - \gamma_1 D_{1_L 0_F}} \tag{58}$$

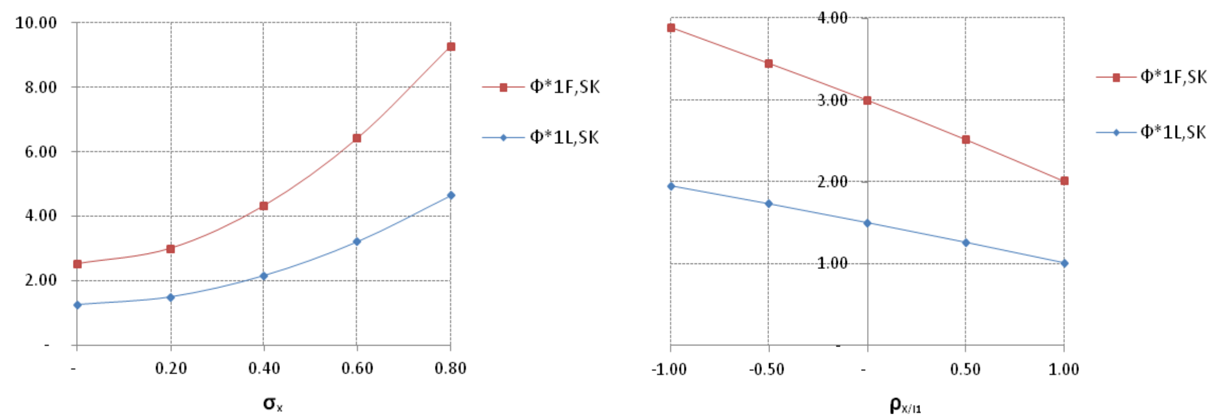
Notice that the firms' value function and investment threshold to invest in *input* 2 (rather than in *input* 1) if inactive is the same, only the notation "1" and "2" changes - to save space, we show here only the expressions for the case where both firms start the sequential-input investment by *input* 1.

Proposition 2: two inactive firms in a non-preemption duopoly (SK and PK) market invest in two complementary inputs (*input* 1 and *input* 2) sequentially if and only if there is a time t, $t \in [0, \infty)$, where $\phi_1(t)$ reaches $\phi_{1_L}^{*}(t)$ and $\phi_{1_F}^{*}(t)$ the first time with $\phi_{12}(t) < \phi_{12_L}^{*}(t)$ and $\phi_{12}(t) < \phi_{12_F}^{*}(t)$.

Proof: see Appendix B.

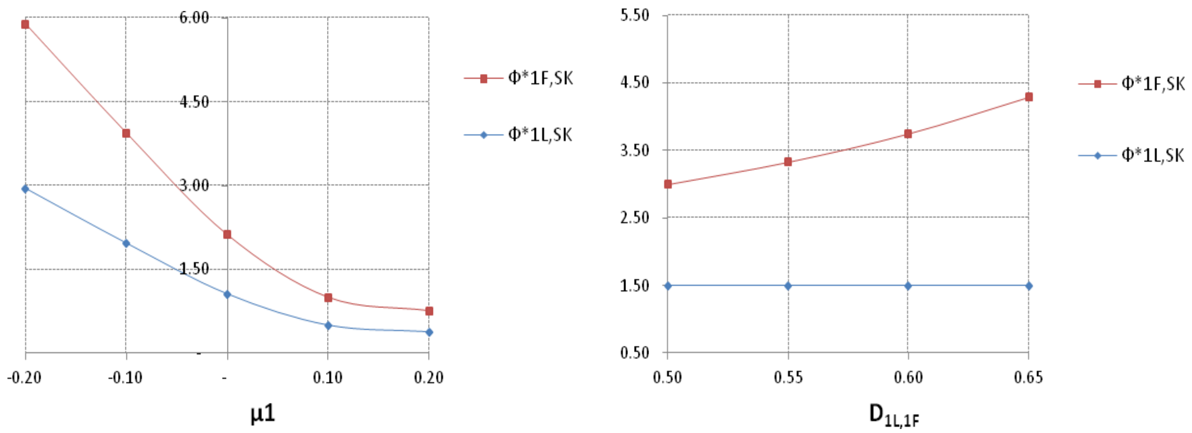
3.2.2 Illustrative Results

Figure 6 - this figure shows the sensitivity of the thresholds of the leader (Eq. 58) and the follower (Eq. 43) to invest in *input* 1 if in a SK market, to changes in the market revenue volatility (left-hand side) and the correlation between the market revenue and the cost of input 1 (right-hand side).



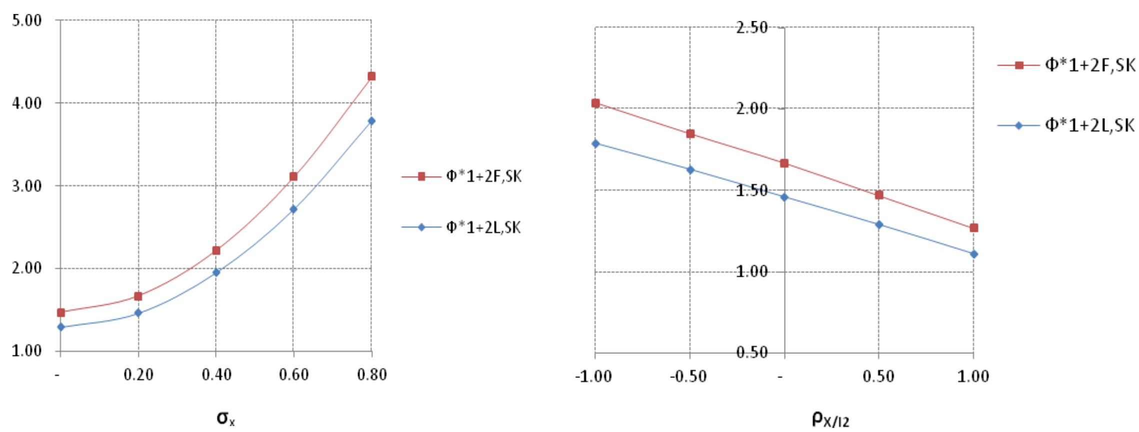
The above results show that an increase in the market revenue volatility delays the investment for both firms and an increase in the correlation between the market revenue and the cost of the *input* 1 accelerates the investment in *input* 1. Also, we find that the follower is more sensitive to changes in these variable than is the leader, particularly for higher levels of volatility and when the correlation is negative.

Figure 7 this figure shows the sensitivity of the thresholds of the leader (Eq. 58) and the follower (Eq. 43) to invest in *input* 1 if in a SK market, to changes in the cost growth rate of input 1 (left-hand side) and the leader's market share if active with the follower both firms with input 1 (right-hand side).



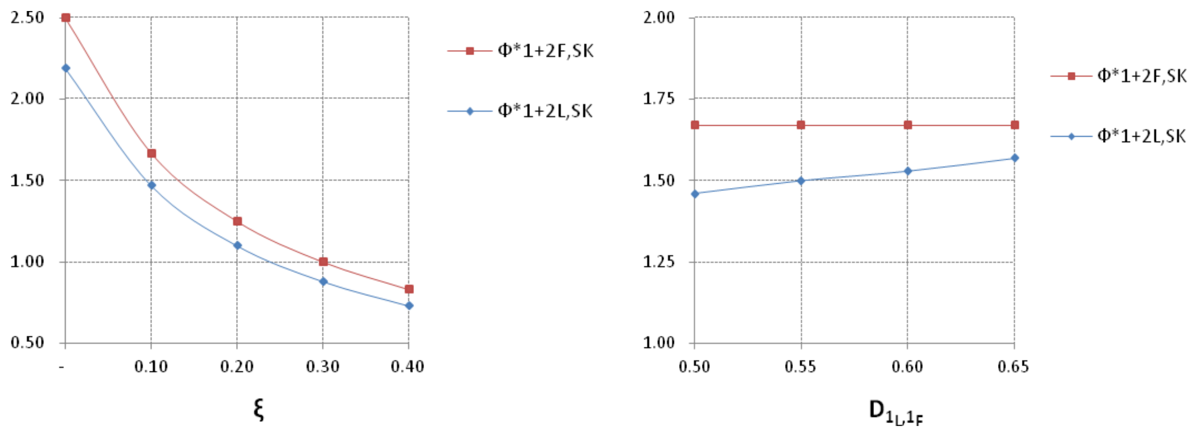
The above results show that an increase in the cost growth rate of *input* 1 accelerates the investment of both firms and an increase in the leader's expected market share if active with the follower with *input* 1, delays the investment of the follower in *input* 1 and has no effect on her threshold to invest in *input* 1. Also, firms' thresholds are more sensitive to changes in the cost growth rates of *input* 1 if this is negative.

Figure 8 - this figure shows the sensitivity of the thresholds of the leader (Eq. 52) and the follower (Eq. 35) to invest in *input* 2 if active with *input* 1 in the SK market, to changes in the market revenue volatility (left-hand side) and the correlation between the market revenue and the cost of *input* 2 (right-hand side).



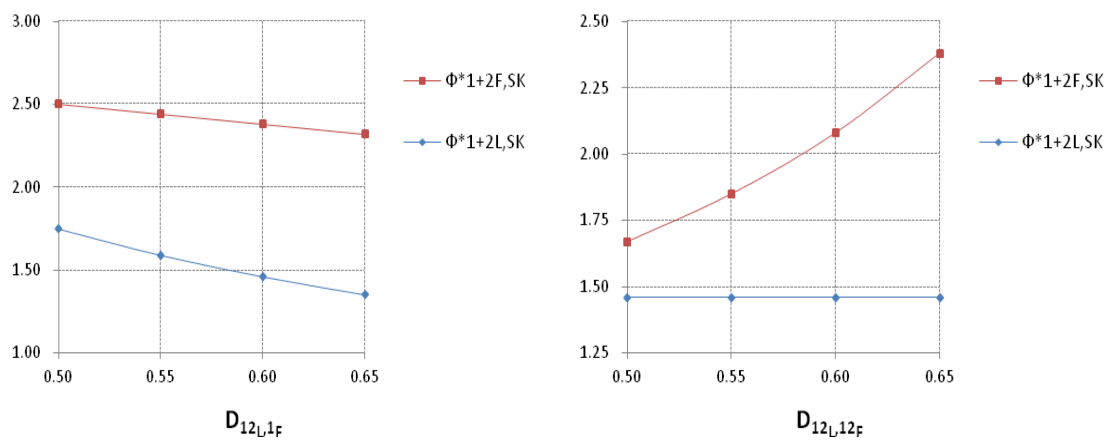
The results above are similar to those we described for the first-stage of the investment (Figure 6) but the differences in the sensitivity of the two firms to changes in σ_X and ρ_{X/I_2} are less notorious for this case.

Figure 9 - this figure shows the sensitivity of the thresholds of the leader (Eq. 52) and the follower (Eq. 35) to invest in *input* 2 if active with *input* 1, to changes in the degree of complementary (ξ) between the two inputs (left-hand side) and the leader's ex-ante market share if she is active with the follower with *input* 1 ($D_{1_L 1_F}$) - right-hand side.



The above results show that an increase in the degree of complementarity between the two inputs accelerates the investment in the second input for both firms and an increase in the leader's (ex-ante) market share if active with the follower with *input* 1, delays her investment in the second input and has no effect on the follower's threshold to invest in the second input.

Figure 10 - this figure shows the sensitivity of the thresholds of the leader (Eq. 52) and the follower (Eq. 35) to invest in *input* 2 if active with *input* 1, to changes in the leader's market share if active with the two inputs and the follower is active with *input* 1, $D_{12_L 1_F}$ (left-hand side) and if both firms are active with the two inputs, $D_{12_L 12_F}$ (right-hand side).



These results show that an increase in the leader's market share if active with the two inputs and the follower is active with *input* 1, accelerates (asymmetrically) the investment in the second input for both firms, and an increase in the leader's market share if active with the follower with the two inputs, delays significantly the investment of the follower in the second input and has no effect on the leader's investment in the second input. [20]

3.2.3 PK market

3.2.3.1 Terminal state: follower

From Figures 1 and 2 we can easily see that, for the *sequential-input* investment scenario, in the terminal state with the leader active with the two inputs, the follower's value function and investment threshold are the same for the PK and SK markets. Therefore, the following conditions hold:[21]

Proposition 3:

$$f_{1+2}^{\mathrm{F,PK}}(\phi_2) = f_{1+2}^{\mathrm{F,SK}}(\phi_2) \quad \text{- given by Expression (37)} \tag{59A}$$

$$\phi_{1+2_F}^{*\mathrm{PK}} = \phi_{1+2_F}^{*\mathrm{SK}} \quad \text{- given by (35)} \tag{59B}$$

where $f_{1+2}^{\mathrm{F,PK}}(\phi_2)$ and $f_{1+2}^{\mathrm{F,SK}}(\phi_2)$ are the follower's value function to invest in *input* 2 if active with *input* 1, for the PK and SK markets respectively; and $\phi_{1+2_F}^{*\mathrm{PK}}$ and $\phi_{1+2_F}^{*\mathrm{SK}}$ are the follower's investment thresholds to invest in *input* 2 if active with input 1, for the PK and the SK markets respectively.

Proof: See Appendix B.

3.2.3.2 First state: follower

In the first-state the follower optimizes the investment in *input* 1 with the leader active with the two inputs (notice that for the SK market, in the first-state, the follower optimizes the investment in *input* 1 with the leader active with *input* 1). Following similar procedures as those of the previous sections we get the homogeneous second-order linear ODE (12), with $k=1$, whose general solution has the form:

$$f_1^{\mathrm{F,PK}}(\phi_1) = A_1^{\mathrm{F,PK}} \phi_1^{\lambda_1} + B_1^{\mathrm{F,PK}} \phi_1^{\lambda_2} \tag{60}$$

where, λ_1 (λ_2) is the positive (negative) solution of the characteristic quadratic function of the homogeneous part of equation (12): $0.5(\sigma_{m_1})^2 \lambda(\lambda-1) + (\mu_X - \mu_{I_1})\lambda - (r - \mu_{I_1}) = 0$. Solving this equation for λ_1 (λ_2) we get:

---

[20] Notice that the leader's market share is a complement of the follower's market share, i.e., $D_{1_F 1_L} = 1 - D_{1_L 1_F}$, $D_{12_F 1_L} = 1 - D_{12_L 1_F}$ and $D_{12_F 12_L} = 1 - D_{12_L 12_F}$. Therefore, we could perform the analysis provided in Figures 5 to 10 as a function of the follower's market share.

[21] Notice that for both cases the leader is active with the two inputs and the follower is active with *input* 1, optimizing the time of the investment in *input* 2. Therefore, $f_{1+2}^{\mathrm{F,PK}}(\phi_2) = f_{1+2}^{\mathrm{F,SK}}(\phi_2)$, and given by Eq. (37), and $\phi_{1+2_F}^{*\mathrm{PK}} = \phi_{1+2_F}^{*\mathrm{SK}}$, and given by Eq. (35).

$$\lambda_{1(2)} = \frac{1}{2} - \frac{\mu_X - \mu_{I_1}}{\sigma_{m_1}^2} + (-)\sqrt{\left(\frac{(\mu_X - \mu_{I_1})}{\sigma_{m_1}^2} - \frac{1}{2}\right)^2 + \frac{2(r - \mu_{I_1})}{\sigma_{m_1}^2}} \tag{61}$$

where, $\sigma_{m_1}^2 = \sigma_X^2 + \sigma_{I_1}^2 - 2\rho_{XI_1}\sigma_X\sigma_{I_1}$.

As the ratio ϕ_1 approaches 0, the value of the option becomes worthless, so $B_1^{\mathrm{F,PK}} = 0$. Using the following VM condition:

$$A_1^{\mathrm{F,PK}} \phi_{1_F}^{*\mathrm{PK}\lambda_1} = \frac{\phi_{1_F}^{*\mathrm{PK}} \gamma_1 D_{1_F 12_L}}{r - \mu_X} - 1 \tag{62}$$

With the following economic interpretation: before investing in *input* 1 the follower's payoff is equal to the value of the option to invest in *input* 1 (left-hand side of Eq. 62). This option is exercised at the moment its value equals the present value of the follower's cash flows from operating with *input*1 forever (with the leader active with the two inputs) subtracted of the investment cost (right-hand side of Eq. 62).

The SP condition is:

$$\lambda_1 A_1^{\mathrm{F,PK}} \phi_{1_F}^{*\mathrm{PK}(\lambda_1 - 1)} = \frac{\gamma_1 D_{1_F 12_L}}{r - \mu_X} \tag{63}$$

Solving together equations (60), (62) and (63) we get the follower's threshold for investing in *input* 1 if inactive, $\phi_{1_F}^{*\mathrm{PK}}$, and the constant $A_1^{\mathrm{F,PK}}$:

$$\phi_{1_F}^{*\mathrm{PK}} = \frac{\lambda_1}{\lambda_1 - 1} \frac{r - \mu_X}{\gamma_1 D_{1_F 12_L}} \tag{64}$$

$$A_1^{\mathrm{F,PK}} = \frac{\phi_{1_F}^{*\mathrm{PK}(1-\lambda_1)}}{\lambda_1} \frac{\gamma_1 D_{1_F 12_L}}{r - \mu_X} \tag{65}$$

The follower's value function is given by:

$$f_1^{\mathrm{F,PK}}(\phi_1) = \begin{cases} A_1^{\mathrm{F,PK}} \phi_1^{\lambda_1} & \text{if } \phi_1 < \phi_{1_F}^{*\mathrm{PK}} \\ \dfrac{\phi_1 \gamma_1 D_{1_F 12_L}}{r - \mu_X} - 1 & \text{if } \phi_1 \geq \phi_{1_F}^{*\mathrm{PK}} \end{cases} \tag{66}$$

The first row of (66) is the follower's option value to invest in *input* 1 if inactive, with the leader active with the two inputs. The second row is the follower's payoff from operating with leader forever - the leader with both inputs and the follower with *input* 1.

3.2.3.3 Terminal state: leader

Expression (67) is the leader's value function for before and after investing in the second input: [22]

[22] As for the previous sections, we assume that the follower invests first in *input* 1 and then in *input* 2, although if the reverse happens the threshold expressions still hold, only the subscript "1" and "2" would change. We provide, however, sensitivity analysis for the case where firms invest in *input* 2 first and then in *input* 1 - see Figure 11.

$$f_K^{\mathrm{L,PK}}(\phi_k)=\begin{cases}\dfrac{\phi_1\gamma_1 D_{1_L 0_F}}{r-\mu_X}+A_{1+2}^{\mathrm{L,PK}}\phi_2^{\upsilon_1} & \text{if } \phi_1>\phi_{1_L}^{*\mathrm{PK}}\wedge\phi_2<\phi_{1+2_L}^{*\mathrm{PK}}\\ \dfrac{\phi_{12}\gamma_{12}D_{12_L 0_F}}{r-\mu_X}-1+\dfrac{\phi_1\gamma_1(D_{12_L 1_F}-D_{12_L 0_F})}{r-\mu_X}\left(\dfrac{\phi_1}{\phi_{1_F}^{*\mathrm{PK}}}\right)^{\upsilon_1} & \text{if } \phi_2\geq\phi_{1+2_L}^{*\mathrm{PK}}\wedge\phi_1<\phi_{1_F}^{*\mathrm{PK}}\\ \dfrac{\phi_{12}\gamma_{12}D_{12_L 1_F}}{r-\mu_X}+\dfrac{\phi_2\gamma_{12}(D_{12_L 12_F}-D_{12_L 1_F})}{r-\mu_X}\left(\dfrac{\phi_2}{\phi_{1+2_F}^{*\mathrm{PK}}}\right)^{\upsilon_1} & \text{if } \phi_1\geq\phi_{1_F}^{*\mathrm{PK}}\wedge\phi_2<\phi_{1+2_F}^{*\mathrm{PK}}\\ \dfrac{\phi_{12}\gamma_{12}D_{12_L 12_F}}{r-\mu_X} & \text{if } \phi_2\geq\phi_{1+2_F}^{*\mathrm{PK}}\end{cases} \quad (67)$$

In the first row, the first two terms represent the leader's payoff from operating alone with *input* 1until infinity, the second term is the leader's option value to invest in *input* 2 if active with *input* 1 - with the follower inactive (notice in the PK market the follower is allowed to invest only after the leader has invested in both inputs); in the second row, the first two terms represent the leader's payoff from the instant she invests in *input* 2 (operating henceforth with the two inputs) until infinity - with the follower inactive; the third term is negative, given that $(D_{12_L 1_F}-D_{12_L 0_F})<0$ (see inequality 10), and corresponds to a correction factor which incorporates the fact that in future if $\phi_{1_F}^{*\mathrm{PK}}$ is reached the follower invests in *input* 1 and the leader's payoff is reduced; in the third row, the first term represent the leader's payoff from operating with both inputs from the moment the follower invests in *input* 1 until infinity; the third term is negative, given that $(D_{12_L 12_F}-D_{12_L 1_F})<0$ (see inequality 10), and corresponds to the correction factor that incorporates the fact that in future if $\phi_{1+2_F}^{*\mathrm{PK}}$ is reached the follower invests in the second input and the leader's payoff is reduced. The fourth row represents the leader's payoff from operating with the follower (both with the two inputs) from the instant the follower invests in *input* 2 until infinity.

As this is a non-preemption game, the leader, if active with *input* 1, invests in *input* 2 at the point her payoff is maximized. ODE (12), with $k=2$, describes the leader's option value to invest in *input* 2 if active with *input* 1, whose solution is given by:

$$f_{1+2}^{\mathrm{L,PK}}(\phi_2)=A_{1+2}^{\mathrm{L,PK}}\phi_2^{\upsilon_1}+B_{1+2}^{\mathrm{L,PK}}\phi_2^{\upsilon_2} \quad (68)$$

where, υ_1 (υ_2) is the positive (negative) solution of the characteristic quadratic function of the homogeneous part of equation (15) given by: $\frac{1}{2}(\sigma_{m_2})^2\upsilon(\upsilon-1)+(\mu_X-\mu_{I_2})\upsilon-(r-\mu_{I_2})=0$. Solving the equation above for υ_1 leads to:

$$\upsilon_1=\frac{1}{2}-\frac{\mu_X-\mu_{I_2}}{\sigma_{m_2}^2}+\sqrt{\left(\frac{(\mu_X-\mu_{I_2})}{\sigma_{m_2}^2}-\frac{1}{2}\right)^2+\frac{2(r-\mu_{I_2})}{\sigma_{m_2}^2}} \quad (69)$$

Following the procedures described in the previous subsections we find that $B_{1+2}^{\mathrm{L,PK}}=0$. The constant $A_{1+2}^{\mathrm{L,PK}}$ and the leader's threshold are determined using the following VM and SP conditions:

VM condition:

$$\frac{\phi_1 \gamma_1 D_{1_L 0_F}}{r-\mu_X} + A_{1+2}^{\mathrm{L,PK}} \phi_{1+2_L}^{*\mathrm{SK}\,\upsilon_1} = \frac{\phi_{1+2_L}^{*\mathrm{PK}} \gamma_{12} D_{12_L 0_F}}{r-\mu_X} - 1 \tag{70}$$

with the following economic interpretation: the leader should invest in *input* 2 at the moment the value she attains from operating with *input* 1 alone forever plus the value of the option to invest in *input* 2 (left-hand side of Eq. 70) equals the present value of the cash flows she obtains from operating with the two inputs forever subtracted of the investment cost (right-hand side of Eq. 70).

SP condition:

$$\upsilon_1 \, A_{1+2}^{L,\mathrm{SK}} \phi_{1+2_L}^{*\mathrm{SK}\,(\upsilon_1 - 1)} = \frac{\gamma_{12} D_{12_L 0_F}}{r-\mu_X} \tag{71}$$

Solving together (68), (70) and (71) we obtain the constant $A_{1+2}^{\mathrm{L,SK}}$:

$$A_{1+2}^{\mathrm{L,PK}} = \frac{\gamma_{12} D_{12_L 0_F}}{(r-\mu_X)\upsilon_1 \phi_{1+2_L}^{*\mathrm{PK}\,(\upsilon_1 - 1)}} \tag{72}$$

and the leader's threshold:

$$\phi_{1+2_L}^{*\mathrm{PK}} = \frac{\phi_1 \upsilon_1 \gamma_1 D_{1_L 0_F} + \upsilon_1 (r-\mu_X)}{\gamma_{12} D_{12_L 0_F} (\upsilon_1 - 1)} \tag{73}$$

3.2.3.4 First state: leader

As we can easily see from Figures 1 and 2, in the "sequential-input investment" scenario, in the first-state, the value function and investment thresholds of the leader for the PK and SK markets are the same. Therefore, the following proposition holds:[23]

Proposition 4:

$$f_1^{\mathrm{L,PK}}(\phi_1) = f_1^{\mathrm{L,SK}}(\phi_1) \quad \textit{- given by Expression (53)} \tag{74A}$$

$$\phi_{1_L}^{*\mathrm{PK}} = \phi_{1_L}^{*\mathrm{SK}} \quad \textit{- given by Equation (58)} \tag{74B}$$

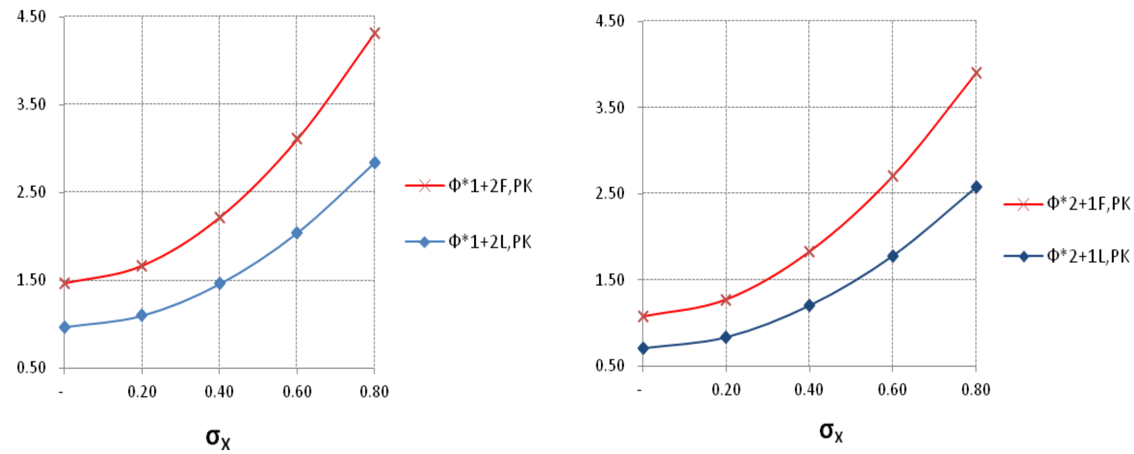
where $f_1^{\mathrm{L,PK}}(\phi_1)$ and $f_1^{\mathrm{L,SK}}(\phi_1)$ are the leader's value function to invest in *input* 1 if inactive in the PK and SK markets, respectively; $\phi_{1_L}^{*\mathrm{PK}}$ and $\phi_{1_L}^{*\mathrm{SK}}$ are the leader's investment threshold to invest in *input* 1 if inactive in the PK and SK markets, respectively.

Proof: See Appendix B.

[23] Notice that for both the SK and SP markets the leader is active with the two inputs and the follower is active with *input* 1 only.

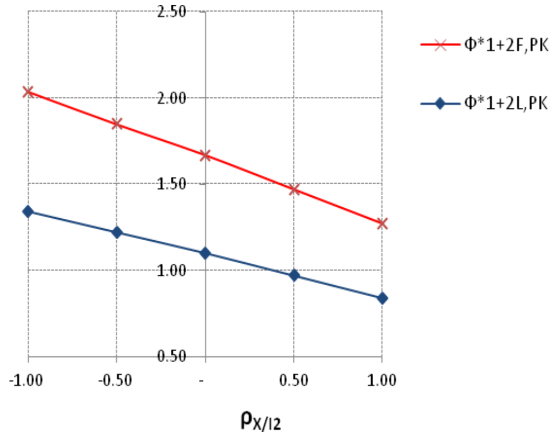
3.2.4 Illustrative results

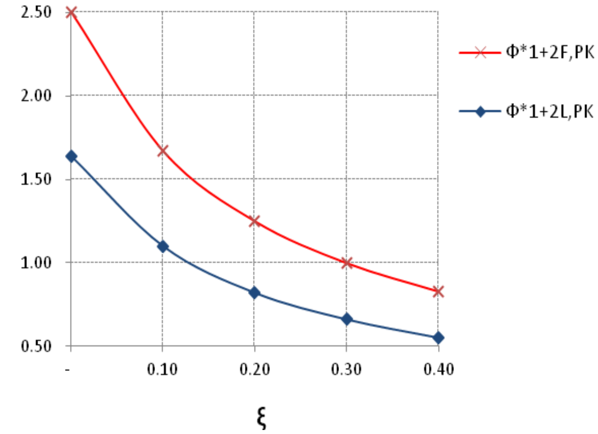
Figure 11 - this figure shows the sensitivity of the investment threshold of the leader and the follower to invest in *input* 2 if active with *input* 1 (left-hand side) or in *input* 1 if active with *input* 2 (right-hand side) to changes in the market revenue volatility, for the PK market.

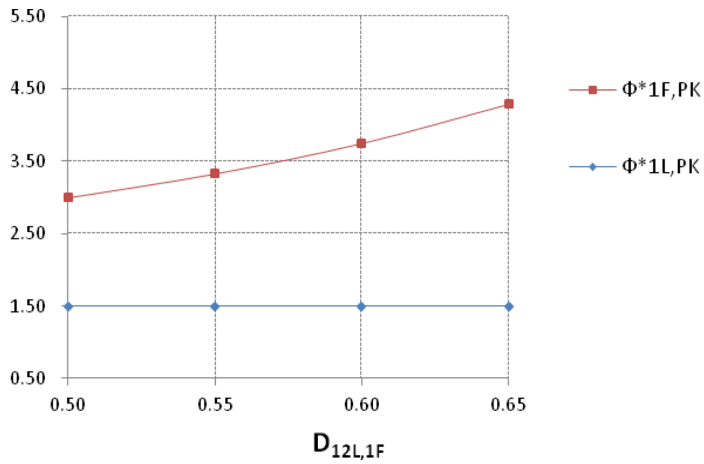


The above results show that the investment threshold of the leader is lower than the investment threshold of the follower, as we expect, and both thrsholds increase with the market revenue volatility. Comparing the firms' thresholds of the left-hand side with those of the right-hand side, we conlude that for both firms the investment threshold to invest in *input* 1 if active with *input* 2 is lower than the investment threshold to invest in *input* 2 if active with *input* 1. This is because in our base case we use $\mu_{I_1} = -0.05$ and $\mu_{I_2} = -0.10$, i.e. the cost of *input* 2 is assumed to decrease more rapidly than the cost of *input* 1 (see Table 1A), and confirms the general intuition whish says that if the cost of two complementary production inputs decrease at different rates, it might be optimal to invests sequentially, first in the input whose cost is decreasing more slowly (*input* 1) and then in the input whose cost decrease more rapidly (*input* 2).

Figure 12 - this figure shows our results for the PK market. At the top are our results for the sensitivity of the thresholds of the leader and follower to invest in *input* 2 if active with *input* 1, to changes in the correlation between market revenue and cost of *input* 2 (left-hand side) and degree of complementarity between the two inputs (right-hand side). At the bottom is our result for the sensitivity of the thresholds of the leader and the follower to invest in *input* 1 if inactive, to changes in the leader's market share if she is active with the two inputs and the follower is active with *input* 1 ($D_{12_L 1_F}$).







The results for the sensitivity of frims' investment thresholds to changes in ρ_{X/I_1} and ξ are similar to those described in previous sections for the SK market, and therefore similar comments apply. The results for the sensitivity of the firms' threshold to invest to changes in $D_{12_L 1_F}$ show that an increase in the leader's expected market share if active with te two inputs and the follower is active with one input, delays the investment of the follower in *input* 1 but has no effect on her investment threshold in *input* 1.

4. Results

In this section we provide further sensitivity analysis regarding the most relevant model parameters. As for the previous results, we use the following base inputs: [24]

[24] For simplicity of notation we use $\delta = \mu_{I_1} - \mu_{I_2}$.

Table 1A: market variables

X	I_1	I_2	μ_X	μ_{I_1}	μ_{I_2}	$\mu_{I_{12}}$	δ	r	σ_X	σ_{I_1}	σ_{I_2}	$\sigma_{I_{12}}$	ρ_{XI_1}	ρ_{XI_2}	$\rho_{XI_{12}}$
1.0	5.0	5.0	0.02	-0.05	-0.10	-0.75	0.05	0.05	0.20	0.20	0.20	0.20	0.00	0.00	0.00

Table 1B: Firms' market revenue share, $D_{k_i k_j}$

Leader						Follower			
$D_{1_L 0_F}$	$D_{12_L 0_F}$	$D_{12_L 1_F}$	$D_{12_L 12_F}$	$D_{1_L 1_F}$	$D_{2_L 2_F}$	$D_{12_F 12_L}$	$D_{1_F 1_L}$	$D_{2_F 2_L}$	$D_{1_F 12_L}$
1.0	1.0	0.6	0.5	0.5	0.5	0.5	0.5	0.5	0.4

Table 1C: complementarity factors, ξ and γ_k

γ_2	γ_2	γ_{12}	$\xi = \gamma_{12} - \gamma_1 - \gamma_2$
0.10	0.10	0.30	***0.10***

Table 2 - model inputs used in Figure 10 and Table 3

μ_{I_1}	-0.05	-0.05	-0.05	-0.05	-0.05	-0.05
μ_{I_2}	-0.05	-0.10	-0.15	-0.20	-0.25	-0.30
δ	***0.00***	***0.05***	***0.10***	***0.15***	***0.20***	***0.25***

γ_1	0.10	0.10	0.10	0.10	0.10	0.10
γ_2	0.10	0.10	0.10	0.10	0.10	0.10
γ_{12}	0.20	0.25	0.30	0.35	0.40	0.45
ξ	***0.00***	***0.05***	***0.10***	***0.15***	***0.20***	***0.25***

Table 3 - this table provides complementary results for the combined effect of changes in both ζ and δ (ranging from 0.00 to 0.25) on the threshold to invest in the two inputs at the same time (for the SK and PK markets), for the leader (table at the top) and the follower (table at the bottom).

$\phi_{12_L}^{*PK\&SK}$		ξ					
		0.00	*0.05*	*0.10*	*0.15*	*0.20*	*0.25*
δ	*0.00*	*0.75*	0.60	0.50	0.43	0.38	0.33
	0.05	0.87	*0.69*	0.58	0.50	0.43	0.39
	0.10	0.99	0.79	*0.66*	0.56	0.49	0.44
	0.15	1.11	0.89	0.74	*0.63*	0.55	0.49
	0.20	1.23	0.98	0.82	0.70	*0.61*	0.55
	0.25	1.35	1.08	0.90	0.77	0.68	*0.60*

$\phi_{12_F}^{*PK\&SK}$		ξ					
		0.00	*0.05*	*0.10*	*0.15*	*0.20*	*0.25*
δ	*0.00*	*1.50*	1.20	1.00	0.86	0.75	0.67
	0.05	1.73	*1.39*	1.16	0.99	0.87	0.77
	0.10	1.97	1.58	*1.31*	1.13	0.99	0.88
	0.15	2.21	1.77	1.48	*1.26*	1.11	0.98
	0.20	2.46	1.96	1.64	1.40	*1.23*	1.09
	0.25	2.70	2.16	1.80	1.54	1.35	*1.20*

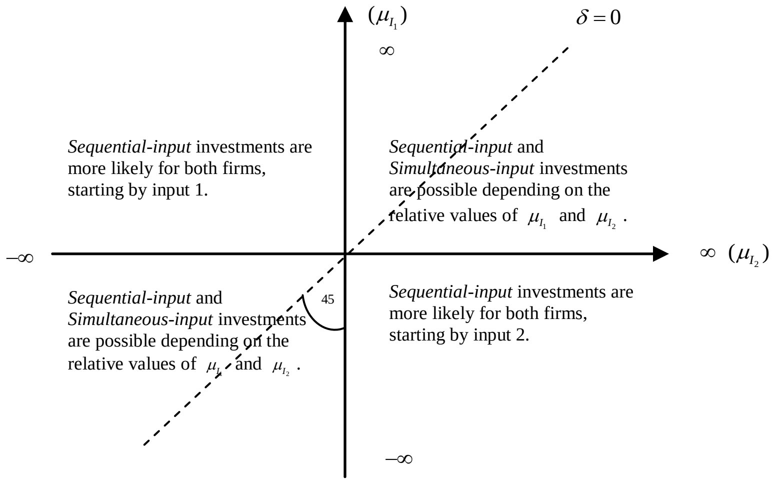
The above results show that ξ and δ have opposite effects on the investment thresholds of the leader and the follower to invest in the two inputs at the same time. More specifically, an increase in the degree of complementarity (ξ) accelerates the investment for both firms, whereas an increase in the difference between the cost growth rates of the two inputs (δ) delays the investment for both firms. These counteracting effects of ξ and δ make the conventional wisdom, which says that "when a production process requires two extremely complementary inputs, a firm should upgrade (or replace) them simultaneously…" less likely to hold. Indeed, if the costs of the two inputs decrease at very different rates, it might be optimal to invest first in the input whose cost decreases more slowly and delay the investment in the input whose cost decreases more rapidly.

In Table 3 the marginal changes in ξ and δ are of the same size - i.e., $\Delta\xi = \Delta\delta = 0.05$ -, yet the threshold values stated in the diagonals of the up and down matrix of Table 3 decrease slightly. Therefore, we conclude that, *ceteris paribus*, for a given marginal change in ξ and δ the effect on firms' investment thresholds of the former dominates that of the latter. We provide further insights on this regard below.

4.1 Further analysis

Contrary to previous works (e.g., Siddiqui and Takashima, 2012), where the options value depends on the same underlying variable, our work uses a modelling setting where the options value (to invest in *input* 1 alone, or *input* 2 alone or *inputs* 1 and 2 at the same time) are driven by independent (and possibly correlated) underlying variables - $\phi_1(t)$, $\phi_1(t)$ and $\phi_{12}(t)$ respectively. This turns more difficult the characterization of the market conditions that lead to *sequential-input vs simultaneous-input* investments for both firms, since the investment thresholds related to $\phi_1(t)$, $\phi_1(t)$ and $\phi_{12}(t)$ are not comparable. Nevertheless, we summarize below some further insights from our model results.

Figure 13 - this figure illustrates the most likely investment behaviours (sequential-input *vs* simultaneous-input) for the leader and the follower according to the relative values of μ_{I_1} and μ_{I_2}, for the SK and PK markets.



We start our analysis by the scenarios where the (μ_{I_1} , μ_{I_2}) point sets are in the *down-right* or *top-left* quadrants, and conclude, respectively: if $\mu_{I_1} \downarrow$, with $\mu_{I_1} \in [0,-\infty)$, and $\mu_{I_2} \uparrow$, with $\mu_{I_2} \in [0,+\infty)$, $\phi^*_{1_L} \uparrow$, $\phi^*_{1_F} \uparrow$, $\phi^*_{2_L} \downarrow$ and $\phi^*_{2_F} \downarrow$, and *sequential-input* investments (starting by input 2) are more likely for both firms; and if $\mu_{I_1} \uparrow$, with $\mu_{I_1} \in [0,+\infty)$, and $\mu_{I_2} \downarrow$, with $\mu_{I_2} \in [0,-\infty)$, $\phi^*_{1_L} \downarrow$, $\phi^*_{1_F} \downarrow$, $\phi^*_{2_L} \uparrow$ and $\phi^*_{2_F} \uparrow$, and *sequential-input* investments (starting by input 1) are more likely for both firms. If the (μ_{I_1} , μ_{I_2}) point sets are in the *top-right* or *down-left* quadrants, we conclude, respectively: if $\mu_{I_1} \downarrow$, with $\mu_{I_1} \in [0,+\infty)$, and $\mu_{I_2} \uparrow$, with $\mu_{I_2} \in [0,+\infty)$, $\phi^*_{1_L} \uparrow$, $\phi^*_{1_F} \uparrow$, $\phi^*_{2_L} \downarrow$ and $\phi^*_{2_F} \downarrow$, and *sequential-input* investments (starting by input 2) are more likely for both firms; and if $\mu_{I_1} \uparrow$, with $\mu_{I_1} \in [0,-\infty)$, and $\mu_{I_2} \downarrow$, with $\mu_{I_2} \in [0,-\infty)$, $\phi^*_{1_L} \downarrow$, $\phi^*_{1_F} \downarrow$, $\phi^*_{2_L} \uparrow$ and $\phi^*_{2_F} \uparrow$, and *sequential-input* investments (starting by input 1) are more likely for both firms.

Notice that if the (μ_{I_1} , μ_{I_2}) point sets are in the *top-left* or *down-right* quadrants, *sequential-input investments* are more likely than if the (μ_{I_1}, μ_{I_2}) point sets are in the *down-left* or *top-right* quadrants. This is because, in the former cases, μ_{I_1} and μ_{I_2} have different signs which makes *sequential-input* investments, starting by the input whose growth rate is positive, more likely. Finally, if the (μ_{I_1}, μ_{I_2}) point sets are on the 45 degrees dotted line, $\delta = 0$ and *simultaneous-input* investments are more likely regardless of the relative values of μ_{I_1} and μ_{I_2}, since these represent cases where the cost growth rates of the two inputs are the same (either positive or negative).[25]

[25] Notice that $\delta = \mu_{I_1} - \mu_{I_2}$ and, for instance, if: (i) $\mu_{I_1} = 0.05$ and $\mu_{I_2} = 0.05$ or $\mu_{I_1} = -0.05$ and $\mu_{I_2} = -0.05$, $\delta = 0$; (ii) if $\mu_{I_1} = -0.05$ and $\mu_{I_2} = -0.10$, $\delta = 0.05$ - *down-left quadrant*; (iii) if $\mu_{I_1} = -0.05$ and $\mu_{I_2} = 0.10$, $\delta = -0.15$ - *down-right quadrant*; (iv) if $\mu_{I_1} = 0.05$ and $\mu_{I_2} = 0.10$, $\delta = -0.05$ - *top-right quadrant*; (v) if $\mu_{I_1} = 0.05$ and $\mu_{I_2} = -0.10$, $\delta = 0.15$ - *top-left quadrant*.

We note the fact that, although in general the quadrant where the (μ_{I_1}, μ_{I_2}) point sets are determine to some extent the investment behaviour of both firms (*sequential-input* or *simultaneous-input* investments), the investment thresholds of the leader and the follower have different sensitivities to changes in the inputs cost growth rates - see section 3.

Proposition 5*: in non-pre-emption duopoly SK and PK markets, ceteris paribus, an increase in the difference between the cost growth rates of the two inputs (δ), turns more likely sequential-input investments for both firms.*

Proof: See Appendix B.

Corollary 5.1: *for an inactive leader (follower), ceteris paribus, if there is a $+\Delta\mu_{I_1}$ and a $-\Delta\mu_{I_2}$ so as $\mu_{I_{12}}$ is kept unchanged, $\phi^*_{1_L}\downarrow$ ($\phi^*_{1_F}\downarrow$), $\phi^*_{2_L}\uparrow$ ($\phi^*_{2_F}\uparrow$) and $\phi^*_{12_L}$ ($\phi^*_{12_F}$) is kept unchanged - and sequential-input investments, starting by input 1, are more likely for both firms.*

Proof: See Appendix B.

Corollary 5.2: *for an inactive leader (follower), ceteris paribus, if there is a $-\Delta\mu_{I_1}$ and a $+\Delta\mu_{I_2}$ so as $\mu_{I_{12}}$ is kept unchanged, $\phi^*_{1_L}\uparrow$ ($\phi^*_{1_F}\uparrow$), $\phi^*_{2_L}\downarrow$ ($\phi^*_{2_F}\downarrow$) and $\phi^*_{12_L}$ ($\phi^*_{12_F}$) is kept unchanged - and sequential-input investments, starting by input 2, are more likely for both firms.*

Proof: See Appendix B.

Corollary 5.3: *for an inactive leader (follower), ceteris paribus, if there is a $+\Delta\mu_{I_1}$ and a $-\Delta\mu_{I_2}$ so as there is a $+\Delta\mu_{I_{12}}$, $\phi^*_{1_L}\downarrow$ ($\phi^*_{1_F}\downarrow$), $\phi^*_{2_L}\uparrow$ ($\phi^*_{2_F}\uparrow$) and $\phi^*_{12_L}\downarrow$ ($\phi^*_{12_F}\downarrow$) - and both sequential-input investments, staring by input 1, and simultaneous-input investments are possible, being the predominant investment behaviour dependent of the (ex-ante) relative values of μ_{I_1} and $\mu_{I_{12}}$ and (ex-ante) how far away $\phi_1(t)$ and $\phi_{12}(t)$ are from $\phi^*_{1_L}(t)$ and $\phi^*_{12_L}(t)$, respectively.*

Proof: See Appendix B.

Proposition 6: *for the SK and PK markets, ceteris paribus: (i) an increase in the degree of complementarity between the two inputs (ξ) accelerates the investment of both firms in the two inputs at the same time and the investment of the follower in input 2 if active with input 1, yet the sensitivity of the follower's threshold to invest in the two inputs at the same time to changes ξ is more acute than that of the leader.*

Proof: See Appendix B.

Corollary 6.1*: as $\gamma_{12}\to 0$, the follower delays infinitely the investment in input 2 if active with input 1, and the two firms delay infinitely the investment in the two inputs at the same time if inactive.*

Proof: See Appendix B.

Corollary 6.2*: as $\gamma_{12}\to 1$ (i.e., cost savings if operating with the two inputs tends to a maximum): (i) the leader behaves as if she was in a monopoly-like regarding the investment in the two inputs at the same* *time; (ii) if ex-post market shares of the two firms when active with the two inputs are expected to be symmetric (i.e., $D_{12_L 12_F} = 0.5$), the threshold of the follower to invest in the two inputs at the same time tends to be twice that of the leader; (iii) if the leader is active with the two inputs, the closer the market share of the follower if active with input 1 ($D_{1_F 12_L}$) to her market share if operating with the two inputs ($D_{12_F 12_L}$), the later she invests in input 2 if active with input 1.*

Proof: See Appendix B.

***Corollary 6.3**: (i) as $\gamma_1(\gamma_2) \to 0$ (i.e., cost savings if operating with input 1(2) tends to zero), the leader delay infinitely her investment in input 1(2); (ii) as $\gamma_1(\gamma_2) \to 1$ (i.e., cost savings if operating with input 1(2) tends to a maximum) the leader tends to behave as if she was in a monopoly-like regarding the investment in input 1(2); (iii) an increase in γ_1 (γ_2) accelerates the follower's first-stage investment in input 1(2) if inactive, and delays the second-stage investment in input 2 (1) if active with input 1 (2).*

Proof: See Appendix B.

***Proposition 7:** for both the SK and PK markets where the leader is active with the two inputs, ceteris paribus: (i) if the follower is inactive, an increase in the market share she expects to attain after investing in the two inputs ($D_{12_F 12_L}$) as well, accelerates her investment in the two inputs at the same time; (ii) if the follower is active with input 1(2), an increase in the market share she attains after investing in input 1(2), $D_{1_F 12_L}$ ($D_{2_F 12_L}$), accelerates her second-stage investment in input 2(1); (iii) if the follower is active with input 1(2), an increase in the market share the follower expect to attain if active with the two inputs ($D_{12_F 12_L}$), accelerates her second-stage investment in input 2(1).*

Proof: See Appendix B.

***Proposition 8:** (i) for the SK and PK markets, ceteris paribus, an increase in γ_{12} accelerates the leader's investment in the second stage (input 2) if active with input 1; (ii) the leader's threshold to invest in the second stage (input 2) if active with one input (input 1) is more sensitive to changes in the γ_{12} if in a SK market.*

***Proposition 9:** (i) for the SK and PK markets, ceteris paribus, an increase in $\gamma_1(\gamma_2)$ accelerates the leader's investment in the second-stage (input 2) if active with one input (input 1); (ii) the leader's threshold to invest in the second input (input 2) if active with one input (input 1) is more sensitive to changes in γ_1 if in a SK market.*

Proof: See Appendix B.

***Proposition 10:** (i) for the SK and PK markets, an increase in γ_1 accelerates the leader's investment in the first-stage (input 1) if inactive; (ii) the leader's threshold to invest in input 1 if inactive is more sensitive to changes in γ_1 if in a PK market.*

Proof: See Appendix B.

Table 2 - this table provides information on firms' investment thresholds for all the investment scenarios considered in this research. This analysis provides us with the opportunity to test empirically *Propositions* 1, 3 and 4 stated in previous sections - see comment made in the right hand-side column of the table. The results were computed using the base parameters stated in Tables 1A, 1B and 1C.

				SK market	PK market	Note
Simultaneous-input investment		Leader:	Real option value	0.67	0.67	Proposition 1, page 12.
			$\phi^*_{12_L}$	0.58	0.58	
		Follower:	Real option value	0.25	0.25	
			$\phi^*_{12_F}$	1.16	1.16	
Sequential-input investment	First-stage, starting by input 1	Leader:	Real option value	0.32	0.32	Proposition 4, page 25.
			$\phi^*_{1_L}$	1.50	1.50	
		Follower:	Real option value	0.14	0.10	*Follower invests earlier in the first input if in a PK market.*
			$\phi^*_{1_F}$	**3.00**	**3.75**	
	First-stage, starting by input 2	Leader:	Real option value	0.37	0.37	Proposition 4, page 25.
			$\phi^*_{2_L}$	1,97	1,97	
		Follower:	Real option value	0.17	0.13	*Follower invests earlier in the first input if in a PK market.*
			$\phi^*_{2_F}$	**3.94**	**4.93**	
	Second-stage, with input 2	Leader:	Real option value	0.99	1.25	*Leader invests earlier in the second input if in a PK market.*
			$\phi^*_{1+2_L}$	**1.46**	**1.10**	
		Follower:	Real option value	0.58	0.58	Proposition 3, page 22
			$\phi^*_{1+2_F}$	**1.67**	**1.67**	
	Second-stage, with input 1	Leader:	Real option value	0.98	1.12	*Leader invests earlier in the second input if in a PK market.*
			$\phi^*_{2+1_L}$	**1.11**	**0.83**	
		Follower:	Real option value	0.50	0.50	Proposition 3, page 22.
			$\phi^*_{2+1_F}$	**1.27**	**1.27**	

These results show that for both the SK and PK market, in *sequential-input* investments, the thresholds to invest in *input* 2 if active with *input* 1 are higher than the thresholds to invest in *input* 1 if active with *input* 2 for both firms (i.e., $\phi^{*SK}_{1+2_L} > \phi^{*SK}_{2+1_L}$ or $1.46 > 1.11$, $\phi^{*PK}_{1+2_L} > \phi^{*PK}_{2+1_L}$ or $1.10 > 0.83$, $\phi^{*SK}_{1+2_F} = \phi^{*PK}_{1+2_F} > \phi^{*SK}_{2+1_F} = \phi^{*PK}_{2+1_F}$ or $1.67 > 1.27$). This behaviour of the firms is a consequence of the fact that we assume in our base model inputs that the cost growth rate of *input* 1 decreases more slowly than the cost growth rates of *input* 2 ($\mu_{I_1} = -0.05$, $\mu_{I_1} = -0.10$), and is a more vividly illustration of how sensitivity the two firms are to changes in the inputs cost growth rates. These results also confirm Propositions 1, 3 and 4 - see our comment provided in the column of the right hand-side of Table 2.

5. Conclusions

We study a multi-input investment problem for two market structures (SK and PK markets), where there is (a leader-follower) competition, complementarity between the investment inputs and uncertainty about both the input-related investment cost and the market revenue. This is a more realistic modelling setting for some investment decisions, and enabled us to produce a rich set of analysis and make some insightful conclusions regarding firms' optimal investment behaviours for this investment context.

Amongst other results, we find that the degree of complementarity between the inputs of an investment has a nonlinear and asymmetric effect on the investment behaviour of the leader and the follower and across the market structures. More specifically, in *simultaneous-input* investments, the behaviour of the two firms is the same for the SK and PK markets; in *sequential-input* investments, the behaviour of the leader regarding the investment in the first input is the same for the both markets, but she invests later in the second input if in a SK market. Also, the follower if in a SK market invests earlier in the first input and later in the second input as compared to when she is in PK market.

Furthermore, ex-ante and ex-post market shares have a significant effect on firms' investment behaviour for some sequencing-investment scenarios. For instance, for the SK and PK markets, an increase in the leader's ex-post market share for when she is active with the follower with the two inputs, has no effect on her behaviour to invest in the two inputs at the same time, but delays significantly the investment of the follower in the two inputs at the same time; an increase in the leader's expected market share for when she is active with the follower with the two inputs, delays significantly the investment of the follower in the second input if active with one input and has no effect on the leader's behaviour to invest in the second input if active with one input; and, for the SK market, an increase in the leader's expected market share for when she is active with the two inputs and the follower is active with one input, accelerates the investment in the second input for the two firms.

Additionally, we show that there are specific markets conditions where the conventional wisdom which says that "when a production process requires two extremely complementary inputs, a firm should upgrade (or replace) them simultaneously" is less likely to hold, particularly for the follower. More specifically, we find that f the cost growth rates of two inputs differ considerably and the ex-ante and or ex-post market shares of the leader and the follower are very asymmetric, it is very likely that at least the follower will invest in the two inputs sequentially. We also find that the degree of complementarity between the two inputs and the difference between the cost growth rates of the two inputs have opposite effects on firms' thresholds to invest in the two inputs at the same time. The former accelerates the investment whereas the later delays the investment, being the effect of the degree of complementarity slightly dominant over the effect of the difference between the cost growth rates of the two inputs. Finally, an increase in the correlation between market revenue and in the investment cost accelerates asymmetrically the investment of both firms for all scenarios.

This research can be extended in several ways. For instance, it would be interesting to consider markets where pre-emption is allowed, or there is a second-mover advantage, or the degree of complementarity between inputs changes over time, or ex-post market shares are stochastic.

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Appendix A

1. Proof of homogeneity of degree-one

If the value-matching relationship can be expressed as the equality between the option value denoted by $F_{12}^{\mathrm{F,SK}}(\bar{x},\bar{I}_2)$ and the difference between the two functions, $f_2^{\mathrm{F,SK}}(\bar{X})$ and $f_3^{\mathrm{F,SK}}(\bar{I}_2)$, representing the net value generated from exercising the option, where the vectors $\bar{x}$ and $\bar{I}_2$, of size n and m respectively are defined by $\bar{X}=\{X_1,X_2,\ldots,X_n\}$ and $\bar{I}_2=\{I_2^1,I_2^2,\ldots,I_2^m\}$, then Euler's theorem on homogenous functions applies (see Sydsaeter and Hammond, 2006). The value matching relationship is:

$$F_{12}^{\mathrm{F,SK}}\left(\bar{X},\bar{I}_2\right)=f_2^{\mathrm{F,SK}}(\bar{X})-f_3^{\mathrm{F,SK}}(\bar{I}_2)$$

The associated smooth pasting conditions are:

$$\frac{\partial F_{12}^{\mathrm{F,SK}}}{\partial X_i}=\frac{\partial f_2^{\mathrm{F,SK}}}{\partial X_i}\ \forall i$$
$$\frac{\partial F_{12}^{\mathrm{F,SK}}}{\partial I_{2j}}=-\frac{\partial f_3^{\mathrm{F,SK}}}{\partial I_{2j}}\ \forall j$$

These conditions imply:

$$\sum_{i=1}^{n}X_i\frac{\partial F_{12}^{\mathrm{F,SK}}}{\partial X_i}+\sum_{j=1}^{m}I_{2j}\frac{\partial F_{12}^{\mathrm{F,SK}}}{\partial I_{2j}}=\sum_{i=1}^{n}X_i\frac{\partial f_2^{\mathrm{F,SK}}}{\partial X_i}-\sum_{j=1}^{m}I_{2j}\frac{\partial f_3^{\mathrm{F,SK}}}{\partial I_{2j}}$$

If the two functions, $f_2^{\mathrm{SK}}(\bar{X})$ and $f_3^{\mathrm{SK}}(\bar{I}_2)$, possess the homogeneity of degree-one property, then by Euler's theorem:

$$\sum_{i=1}^{n}X_i\frac{\partial F_{12}^{\mathrm{F,SK}}}{\partial X_i}+\sum_{j=1}^{m}I_j\frac{\partial F_{12}^{\mathrm{F,SK}}}{\partial I_{2j}}=f_2^{\mathrm{F,SK}}-f_3^{\mathrm{F,SK}}=F_{12}^{\mathrm{F,SK}}$$

which implies that $F_{12}^{\mathrm{F,SK}}$ is a homogenous function of degree one. The assertion that the option value is represented by a homogenous degree-one function can be tested by the value matching relationship and its associated smooth pasting conditions. Examining the value "matching conditions" we can easily prove that homogeneity exists. Taking the value matching condition given by Eq. (17), reproduced here as Equation A3,

$$F_{12}^{\mathrm{F,SK}}(X^*,I_2^*)=\frac{\gamma_{12}X^*.D_{12_F12_L}}{r-\mu_X}-I_{2_F}^* \quad \text{(A1)}$$

if the option value is $F_{12}^{\mathrm{F,SK}}(X, I_2)$ and the value after exercising the option is $\gamma_{12} X^{*}.D_{12_F 12_L} / (r-\mu_X) - I_{2_F}^{*}$, with both X and I_2 stochastic, then if $F_{12}^{\mathrm{F,SK}}(X, I_2) = \gamma_{12} X^{*}.D_{12_F 12_L} / (r-\mu_X) - I_{2_F}^{*}$ holds, doubling X^{*} and $I_{2_F}^{*}$ doubles $F_{12}^{\mathrm{F,SK}}(X, I_2)$, if so, there is homogeneity of degree-one. If the "value matching" relationship exhibits homogeneity of degree-one, then the two variables (X, I_2) can be replaced by, in this case, the ratio $\phi_2 = X / I_2$. [26]

2. Derivation of ODE (12)

Rewriting Equation (11) as (A4): [27]

$$\frac{1}{2}\frac{\partial^2 F_{12}}{\partial X^2}\sigma_X{}^2 X^2 + \frac{1}{2}\frac{\partial^2 F_{12}}{\partial I_2{}^2}\sigma_{I_2}^2 I_2{}^2 + \frac{\partial^2 F_{12}}{\partial X \partial I_2} X I_2 \sigma_X \sigma_{I_2} \rho_{XI_2} + \frac{\partial F_{12}}{\partial X}\mu_X X + \frac{\partial F_{12}}{\partial I_2}\mu_{I_2} I_2 - rF_{12} = 0 \qquad \text{(A4)}$$

In order to reduce the homogeneity of degree two in the underlying variables to homogeneity of degree one, similarity methods can be used. Let $\phi_2 = X / I_2$, so:

$$F(X, I_2) = f\left(X / I_2\right) I_2 = f(\phi_2) I_2$$
$$\frac{\partial F(X, I_2)}{\partial I_2} = f(\phi_2) - \frac{X}{I_2}\frac{\partial f(\phi_2)}{\partial \phi_2}$$
$$\frac{\partial F(X, I_2)}{\partial X} = \frac{\partial f(\phi_2)}{\partial \phi_2}$$
$$\frac{\partial^2 F(X, I_2)}{\partial I^2} = \frac{\partial^2 f(\phi_2)}{(\partial \phi)^2}\frac{X^2}{(I_2)^3}$$
$$\frac{\partial^2 F(X, I_2)}{\partial X^2} = \frac{\partial^2 f(\phi_2)}{\partial \phi_2{}^2}\frac{1}{I_2}$$
$$\frac{\partial^2 F(X, I_2)}{\partial X\, \partial I_2} = -\frac{\partial^2 f(\phi_2)}{\partial \phi_2{}^2}\frac{X}{(I_2)^2}$$

Substituting back to Equation (A4) we obtain Equation (12), rewritten here as (A5):

$$\frac{1}{2}\sigma_{m_2}^2 (\phi_2)^2 \frac{\partial^2 f_{12}(\phi_2)}{\partial \phi_2{}^2} + \left(\mu_X - \mu_{I_2}\right)(\phi_2)\frac{\partial f_{12}(\phi_2)}{\partial \phi_2} + \gamma_1 X.D_{1_L 1_F} - (r - \mu_{I_2}) f_{12}(\phi_2) = 0 \qquad \text{(A5)}$$

where, $\sigma_{m_2}^2 = \sigma_X^2 + \sigma_{I_2}^2 - 2\rho_{XI_2}\sigma_X \sigma_{I_2}$.

Appendix B

Proof of proposition 1*: in simultaneous-input investments both firms play a "one-shot" game, regardless of the market structure, i.e. the investment game ends for the two firms at the moment they exercise the option to invest in the two inputs at the same time. Consequently, firms' value, if inactive, equals the option value to invest in the two inputs at the same time - which is the same for both markets - and, if active with the two inputs, equals the present value of the cost savings she attains from operating with the two inputs forever - which is the same for both markets. Therefore, conditions 29A and 29B are proven. From the above rationale we also conclude that the boundary conditions used to derive the investment thresholds expressions are the same for the SK and PK markets, and so conditions (29C) and (29D) are also proved.* □

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[26] This property can be easily proved empirically using the above VM condition and the base inputs of section 4, i.e., ceteris paribus, doubling both X^{*} and $I_{2_F}^{*}$ in (A3) doubles $F_{12}^{\mathrm{F,SK}}(X, I_2)$ - proof can be provided under request.

[27] For simplicity of notation we drop the upper script on F_{12} and $f_{12}(\phi_2)$.

Proof of proposition 2*: in our framework the investment thresholds are trigger points which, if reached the first time, advises firms to invest, otherwise instruct firms to defer the investment. It also results from our modelling setting that firms lose the option to invest in the two inputs at the same time if they exercise the option to invest in one of the inputs alone (input 1 or input 2). Consequently, if the thresholds which advise the leader and the follower to invest in input 1 alone ($\phi^*_{1_L}(t)$ and $\phi^*_{1_F}(t)$ respectively) or input 2 alone ($\phi^*_{2_L}(t)$ and $\phi^*_{2_F}(t)$ respectively) are reached before that which advises the leader and the follower to invest in the two inputs at the same time ($\phi^*_{12_L}(t)$ and $\phi^*_{12_F}(t)$ respectively), both firms invest in the two inputs sequentially.* □

Proof of proposition 3*: in sequential-input investments, in the terminal-state where the follower operates with input 1 and the leader operates with the two inputs, the follower plays a "one-shot" game regarding the investment in the second input, regardless of the market structure (i.e., the investment game ends at the moment the follower exercises the option to invest in input 2). Before investing in the second input, the follower's value equals the value of the option to invest in input 2 plus the cost savings she attains from operating with input 1 forever. Therefore, condition (59A) is proved. From the above rationale we also conclude that the boundary conditions used to derive the follower's investment threshold expression are the same for the SK and PK markets, and therefore condition (59B) is also proved.* □

Proof of proposition 4*: in sequential-input investments the leader invests before the follower in one of the inputs (input 1 or input 2), regardless of the market structure. Therefore, the leader's value, if inactive, equals the value of the option to invest in input 1 (2), and, if active with input 1 (2), equals the value of the option to invest in input 2 (1) plus the cost savings she attains from operating with input 1 (2) forever. Therefore, condition (74A) is proved. Following the same rationale as that used in the proofs of propositions 1 and 2 above, we also prove condition (74B).* □

Proof of proposition 5 and corollaries 5.1 and 5.2*: in order to prove proposition 5 we have to prove that $\phi^*_{1_L}$ and $\phi^*_{1_F}$ decrease with δ, and or $\phi^*_{12_L}$ and $\phi^*_{12_F}$ increase with δ. The most obvious way to do this would be to determine the first derivative of the threshold expressions with respect to δ (i.e., $\partial\phi^*_{k_{i,j}}/\partial\delta$). Yet, δ is not in the threshold expressions and, therefore, this approach is not feasible. Nevertheless, from section 3 we know that, ceteris paribus, if $\mu_{I_1}\uparrow$ and $\mu_{I_2}\downarrow$ so as $\delta\uparrow$, $\phi^*_{1_L}\downarrow$, $\phi^*_{1_F}\downarrow$, $\phi^*_{2_L}\uparrow$ and $\phi^*_{2_F}\uparrow$, both firms invest earlier in input 1 and later in input 2 - and therefore sequential-input investments (starting by input 1) are more likely. Also, if $\mu_{I_1}\downarrow$ and $\mu_{I_2}\uparrow$ so as $\delta\uparrow$, $\phi^*_{1_L}\uparrow$, $\phi^*_{1_F}\uparrow$, $\phi^*_{2_L}\downarrow$ and $\phi^*_{2_F}\downarrow$, both firms invest earlier in input 2 and later in input 1 - and therefore sequential-input investments (starting by input 2) are more likely.* □

Proof of corollaries 5.3*: following the rationale used above for corollaries 5.1 and 5.2 we conclude that when simultaneous changes in the cost growth rates of the two inputs lead to a decrease in both the threshold to invest in one input alone (input 1 or input 2) and the threshold to invest in the two inputs at the same time, sequential-input investments (staring by input 1 or input 2) and simultaneous-input investments are possible. The predominant investment behaviour is not possible to generalise since both the investments in input 1 alone, or input 2 alone, or inputs 1 and 2 at the same time are all earlier and, therefore, the occurrence of sequential-input or simultaneous-input investments depend on the relative* *values and changes in* μ_{I_1}, $\mu_{I_{12}}$ *and (ex-ante) how far away* $\phi_1(t)$ *and* $\phi_{12}(t)$ *are from* $\phi_{1_L}^*(t)$ *and* $\phi_{12_L}^*(t)$, *respectively.* □

Proof of proposition 6*: proposition 1 shows that* $\phi_{12_L}^{*SK} = \phi_{12_L}^{*PK} \equiv \phi_{12_L}^{*SK\&PK}$ *and* $\phi_{12_F}^{*SK} = \phi_{12_F}^{*PK} \equiv \phi_{12_F}^{*SK\&PK}$. *To study the relationship between* ξ *and the above thresholds we determine the first derivative of* $\phi_{12_L}^{*SK\&PK}$ *and* $\phi_{12_F}^{*SK\&PK}$ *with respect to* ξ *- see Eqs. (20) and (29). Yet, since* ξ *is not in the above threshold expressions, we determine the first derivatives of (29) and (20) with respect to the complementarity-related parameter(s) that is in the threshold expression (* γ_{12} *), which yield:* [28]

$$\frac{\partial \phi_{12_L}^{*SK\&PK}}{\partial \gamma_{12}} = -\frac{\eta_1 (r-\mu_X)\left(\eta_1 D_{12_L 0_F} - D_{12_L 0_F}\right)}{\left(\eta_1 \gamma_{12} D_{12_L 0_F} - \gamma_{12} D_{12_L 0_F}\right)^2} < 0 \tag{1B}$$

$$\frac{\partial \phi_{12_F}^{*SK\&PK}}{\partial \gamma_{12}} = -\frac{\eta_1 (r-\mu_X)(\eta_1 - 1) D_{12_F 12_L}}{\left[(\eta_1 - 1)\gamma_{12} D_{12_F 12_L}\right]^2} < 0 \tag{2B}$$

$$\frac{\partial \phi_{1+2_F}^{*SK\&SP}}{\partial \gamma_{12}} = -\frac{\left[\psi_1(\mu_X - r) - \psi_1 \phi_1 \gamma_1 D_{1_F 12_L}\right]\left[D_{12_F 12_L}(1-\psi_1)\right]}{\left[\gamma_{12} D_{12_F 12_L}(1-\psi_1)\right]^2} < 0 \tag{3B}$$

1B, 2B and 3B are all negative, therefore (i) is proved. To prove that the follower's threshold is more sensitive to changes in ξ *than the leader's threshold we have to show that* $\frac{\partial \phi_{12_L}^{*SK\&PK} / \partial \xi}{\partial \phi_{12_F}^{*SK\&PK} / \partial \xi} = \Delta < 1$, $\forall\, \gamma_{12}$. *Using the above expression and simplifying, yields:* $\Delta = \frac{\left(\eta_1 D_{12_L 0_F} - D_{12_L 0_F}\right)(\eta_1 - 1)\gamma_{12}{}^2 D_{12_F 12_L}}{\left(\eta_1 \gamma_{12} D_{12_L 0_F} - \gamma_{12} D_{12_L 0_F}\right)^2}$, *with* $\gamma_{12} \in (0,1)$. *Taking the limits yields:* $\underset{\gamma_{12} \to 0}{Lim}\, \Delta = 0$ *and* $\underset{\gamma_{12} \to 1}{Lim}\, \Delta = \frac{\eta_1 D_{12_F 12_L} - D_{12_F 12_L}}{\eta_1 D_{12_L 0_F} - D_{12_L 0_F}}$. *Since* $D_{12_L 0_F} = 1$ *and* $D_{12_F 12_L} \in (0, 0.5]$, $\Delta < 1$, $\forall\, \gamma_{12}$, *and therefore (ii) is also proved.*

Proof of corollary 6.1*:* $\gamma_{12} \in (0,1)$, $\underset{\gamma_{12} \to 0}{Lim}\, \phi_{1+2_F}^{*SK\&PK} = \infty$ *and* $\underset{\gamma_{12} \to 0}{Lim}\, \phi_{12_L}^{*SK\&PK} = \infty$, *therefore, corollary 6.1 is proved.*

Proof of corollary 6.2*:* $\gamma_{12} \in (0,1)$, $\underset{\gamma_{12} \to 1}{Lim}\, \phi_{12_L}^{*SK\&PK} = \frac{\eta_1 (r-\mu_X)}{\eta_1 - 1}$, $\underset{\gamma_{12} \to 1}{Lim}\, \phi_{12_F}^{*SK\&PK} = \frac{\eta_1 (r-\mu_X)}{(\eta_1 - 1) D_{12_F 12_L}}$ *and* $\underset{\gamma_{12} \to 1}{Lim}\, \phi_{1+2_F}^{*SK\&PK} = \frac{\psi_1(\mu_X - r) - \psi_1 \phi_1 \gamma_1 D_{1_F 12_L}}{D_{12_F 12_L}(1-\psi_1)}$. *Notice that* $\eta_1 (r-\mu_X)/(\eta_1 - 1)$ *is the threshold mark up normalized by the investment cost for a monopoly market, therefore (i) is proved. The follower's threshold differs from that of the leader by a factor:* $1/D_{12_F 12_L}$. *Note that* $D_{12_F 12_L} \in (0, 0.5]$ *and when* $D_{12_F 12_L} = 0.5$ *(i.e., the ex-post market shares of the two firms if active with the two inputs are symmetric) the above expression for* $\underset{\gamma_{12} \to 1}{Lim}\, \phi_{12_F}^{*SK\&PK}$ *is twice that of* $\underset{\gamma_{12} \to 1}{Lim}\, \phi_{12_L}^{*SK\&PK}$, *and therefore (ii) is proved. Finally, from* $\underset{\gamma_{12} \to 1}{Lim}\, \phi_{1+2_F}^{*SK\&PK}$ *we conclude that as* $\gamma_{12} \to 1$, *the follower's threshold to invest in input 2 if active with input 1 increases with the ratio* $D_{1_F 12_L} / D_{12_F 12_L}$, *therefore, (iii) is also proved.*

Proof of corollary 6.3: *note that* $\gamma_1 \in (0,1)$ *and* $\underset{\gamma_1 \to 0}{Lim}\, \phi_{1_L}^{*SK\&PK} = \infty$, *therefore (i) is proved;* $\underset{\gamma_1 \to 1}{Lim}\, \phi_{1_L}^{*SK\&PK} = \frac{\beta_1 (r-\mu_X)}{\beta_1 - 1}$ *and corresponds to the threshold mark up normalized by the investment cost for a monopoly market,*

[28] The proofs for the signs of the first derivative are available under request.

therefore (ii) is proved; to prove (iii) we need to prove that the first derivatives of $\phi_{1_L}^{*\mathrm{SK\&PK}}$ *and* $\phi_{1+2_F}^{*\mathrm{SK\&PK}}$ *with respect to* γ_1 *are negative and positive, respectively, which we show below.* [29] *Therefore (iii) is also proved.*

$$\frac{\partial \phi_{1_L}^{*\mathrm{SK\&PK}}}{\partial \gamma_1} = -\frac{\beta_1(r-\mu_X)\left(\beta_1 D_{1_L 0_F} - D_{1_L 0_F}\right)}{\left(\beta_1\gamma_1 D_{1_L 0_F} - \gamma_1 D_{1_L 0_F}\right)^2} < 0 \tag{4B}$$

$$\frac{\partial \phi_{1+2_F}^{*\mathrm{SK\&SP}}}{\partial \gamma_1} = -\frac{\psi_1\phi_1 D_{1_F 12_L}}{\gamma_{12} D_{12_F 12_L}(1-\psi_1)} > 0 \tag{5B}$$

□

Proof of proposition 7: $D_{12_F 12_L}$ *is the follower's market share if both firms are active with the two inputs and* $D_{1_F 12_L}$ *is the follower's market share if she is active with input 1 and the leader is active with the two inputs. To prove (i) and (ii) we need to show that the first derivative of* $\phi_{12_F}^{*\mathrm{SK\&PK}}$ *with respect to* $D_{12_F 12_L}$ *and the first derivative of* $\phi_{1+2_F}^{*\mathrm{SK\&SP}}$ *with respect to* $D_{1_F 12_L}$*, respectively, are both negative. This is shown by (6B) and (7B) below and therefore (i) and (ii) are proved. To prove (iii) we need to show that the first derivative of* $\phi_{1+2_F}^{*\mathrm{SK\&SP}}$ *with respect to* $D_{12_F 12_L}$ *is negative. This is shown by (8B), and therefore (iii) is also proved.*

$$\frac{\partial \phi_{12_F}^{*\mathrm{SK\&PK}}}{\partial D_{12_F 12_L}} = -\frac{\eta_1(r-\mu_X)(\eta_1-1)\gamma_{12}}{\left[(\eta_1-1)\gamma_{12} D_{12_F 12_L}\right]^2} < 0 \tag{6B}$$

$$\frac{\partial \phi_{1+2_F}^{*\mathrm{SK\&SP}}}{\partial D_{1_F 12_L}} = \frac{-\psi_1\phi_1 D_{1_F 12_L}}{\left[\gamma_{12} D_{12_F 12_L}(1-\psi_1)\right]^2} < 0 \tag{7B}$$

$$\frac{\partial \phi_{1+2_F}^{*\mathrm{SK\&SP}}}{\partial D_{12_F 12_L}} = -\frac{\left[\psi_1(\mu_X - r) - \psi_1\phi_1\gamma_1 D_{1_F 12_L}\right]\gamma_{12}(1-\psi_1)}{\left[\gamma_{12} D_{12_F 12_L}(1-\psi_1)\right]^2} < 0 \tag{8B}$$

□

Proof of proposition 8: *to prove (i) we need to prove that the first derivatives of* $\phi_{1+2_L}^{*\mathrm{SK}}$ *and* $\phi_{1+2_L}^{*\mathrm{PK}}$ *with respect* γ_{12} *is negative, which is shown by (9B) and (10B) below. Therefore (i) is proved.*

$$\frac{\partial \phi_{1+2_L}^{*\mathrm{SK}}}{\partial \gamma_{12}} = -\frac{\phi_1\psi_1\gamma_1 D_{1_L 1_F} + \psi_1(r-\mu_X) + D_{12_L 0_F}(\psi_1 - 1)}{\left[\gamma_{12} D_{12_L 0_F}(\psi_1 - 1)\right]^2} < 0 \tag{9B}$$

$$\frac{\partial \phi_{1+2_L}^{*\mathrm{PK}}}{\partial \gamma_{12}} = -\frac{\phi_1\upsilon_1\gamma_1 D_{1_L 0_F} + \upsilon_1(r-\mu_X) + D_{12_L 0_F}(\upsilon_1 - 1)}{\left[\gamma_{12} D_{12_L 0_F}(\upsilon_1 - 1)\right]^2} < 0 \tag{10B}$$

To prove (ii) we need to compare (9B) with (10B) and show that $\frac{\partial \phi_{1+2_L}^{*\mathrm{PK}}}{\partial \gamma_{12}} > \frac{\partial \phi_{1+2_L}^{*\mathrm{SK}}}{\partial \gamma_{12}}, \forall \gamma_{12}$ *, or that* $\frac{\partial \phi_{1+2_L}^{*\mathrm{PK}}}{\partial \gamma_{12}} / \frac{\partial \phi_{1+2_L}^{*\mathrm{SK}}}{\partial \gamma_{12}} = \Pi < 1,\ \forall \gamma_{12}$*. Replacing the terms of this inequality by their respective expressions and simplifying yields:* $\frac{(\upsilon_1-1)^2\phi_1\psi_1\gamma_1 D_{1_L 1_F} + (\upsilon_1-1)^2\psi_1(r-\mu_X) + (\upsilon_1-1)^2 D_{12_L 0_F}(\psi_1-1)}{(\psi_1-1)^2\phi_1\upsilon_1\gamma_1 D_{1_L 0_F} + (\psi_1-1)^2\upsilon_1(r-\mu_X) + (\psi_1-1)^2 D_{12_L 0_F}(\upsilon_1-1)} = \Pi$*. If the market conditions (revenue volatility, inputs cost volatility, etc) are the same for the SK and PK markets,* $\psi_1 = \upsilon_1$ *and the*

[29] The proofs for the signs of the first derivative are available under request.

above expression yields: $\frac{\phi_1\psi_1\gamma_1 D_{1_L 1_F} + \psi_1(r-\mu_X) + D_{12_L 0_F}(\psi_1 - 1)}{\phi_1\psi_1\gamma_1 D_{1_L 0_F} + \psi_1(r-\mu_X) + D_{12_L 0_F}(\psi_1 - 1)} = \Pi$ *. Since* $D_{1_L 1_F} < D_{12_L 0_F}$ *and* $\gamma_{12} \in (0,1)$, $\Pi < 1$, $\forall \gamma_{12}$. *Therefore, (ii) is also proved.* □

Proof of proposition 9: *to prove (i) we need to prove that the first derivative of* $\phi^{*SK}_{1+2_L}$ *and* $\phi^{*PK}_{1+2_L}$ *with respect to the ex-ante cost savings the leader attains while operating with the input1 (* γ_1 *) is negative. This is shown below by (9B) and (10B) and therefore (i) is proved.*

$$\frac{\partial \phi^{*SK}_{1+2_L}}{\partial \gamma_1} = -\frac{\phi_1\psi_1\gamma_1 D_{1_L 1_F}}{\gamma_{12} D_{12_L 0_F}(\psi_1 - 1)} < 0 \tag{11B}$$

$$\frac{\partial \phi^{*PK}_{1+2_L}}{\partial \gamma_1} = -\frac{\phi_1\upsilon_1 D_{1_L 0_F}}{\gamma_{12} D_{12_L 0_F}(\upsilon_1 - 1)} < 0 \tag{12B}$$

To prove (ii) we need to compare (11B) with (12B) and show that $\frac{\partial \phi^{*PK}_{1+2_L}}{\partial \gamma_1} > \frac{\partial \phi^{*SK}_{1+2_L}}{\partial \gamma_1}, \forall \gamma_1$ *, or that* $\frac{\partial \phi^{*PK}_{1+2_L}}{\partial \gamma_1} / \frac{\partial \phi^{*SK}_{1+2_L}}{\partial \gamma_1} = \Omega < 1,\ \forall \gamma_1$. *Replacing the terms of the inequality by the respective expressions and proceeding as for proposition 8 (i.e.,* $\psi_1 = \upsilon_1$ *if the same market conditions hold for the SK and PK markets) we obtain:* $\frac{\gamma_1 D_{1_L 1_F}}{D_{1_L 0_F}} = \Omega$. *Since* $D_{1_L 0_F} > D_{1_L 1_F}$ *and* $\gamma_1 \in (0,1)$, $\Omega < 1, \forall \gamma_1$, *and therefore (ii) is proved.* □

Proof of proposition 10: *to prove (i) we need to prove that the first derivatives of* $\phi^{*SK}_{1+2_L}$ *and* $\phi^{*PK}_{1+2_L}$ *with respect to* γ_1 *is negative. This is shown below by (13B) and (14B), therefore (i) is proved.*

$$\frac{\partial \phi^{*SK}_{1_F}}{\partial \gamma_1} = -\frac{\beta_1(r-\mu_X)(\beta_1 - 1) D_{1_F 1_L}}{\left[(\beta_1 - 1)\gamma_1 D_{1_F 1_L}\right]^2} < 0 \tag{13B}$$

$$\frac{\partial \phi^{*PK}_{1_F}}{\partial \gamma_1} = -\frac{\lambda_1(r-\mu_X)(\lambda_1 - 1) D_{1_F 12_L}}{\left[(\lambda_1 - 1)\gamma_1 D_{1_F 12_L}\right]^2} < 0 \tag{14B}$$

To prove (ii) we need to compare (13B) with (14B) and show that $\frac{\partial \phi^{*PK}_{1_F}}{\partial \gamma_1} > \frac{\partial \phi^{*SK}_{1_F}}{\partial \gamma_1}, \forall \gamma_1$ *, or that* $\frac{\partial \phi^{*PK}_{1_F}}{\partial \gamma_1} / \frac{\partial \phi^{*SK}_{1_F}}{\partial \gamma_1} = \Upsilon < 1,\ \forall \gamma_1$. *Replacing the terms of the inequality by the respective expressions and proceeding as for propositions 8 and 9 (i.e.,* $\beta_1 = \lambda_1$ *if the same market conditions hold for the SK and PK markets) we obtain:* $\frac{D_{1_F 1_L}}{D_{1_F 12_L}} = \Upsilon$ *. Since* $D_{1_L 1_F} > D_{1_F 12_L}$ *(due to the effect of complementarity),* $\Upsilon > 1$ *, and (ii) is proved.* □

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