The Joint Impact of Development Cost Uncertainty and Market Incompleteness on a Project that takes Time to Build

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1 Introduction

This paper considers the impact of development cost uncertainty on an investor's investment timing decision when the project in question takes time to build and the market in which the investor operates is incomplete. Additionally, the future revenue that will be generated from the project when it has been fully developed is also uncertain.

To illustrate the applicability of my framework, I give a few examples. Consider a real estate developer who pays to construct buildings on the land he owns. In this case, the work in progress is a residential or commercial property such as an apartment complex or a shopping centre. At any stage, the developer must speculate on (i) what rental income he will earn from the competed development and (ii) what the cost of carrying out its construction will be. The developer will begin investing when the expected rental income is sufficiently attractive and/or the expected construction cost is sufficiently low. He can also decide to suspend the construction process if the expected rental income deteriorates and/or the expected cost to completing its construction rises. However, he cannot perfectly hedge the risk from the fluctuating project value; in particular, the cost to completing the project has some diversifiable risk which cannot be hedged and, thus, he faces an incomplete market.

As an alternative example, consider a pharmaceutical company considering investing in the research and development dedicated to drug discovery for a particular disease. Such an investment begins with research which (with some probability) will lead to a new compound. The drug will require FDA approval

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and in order to obtain this, the pharmaceutical company must continue to extensively test the product through carrying out phase I to III trials. If FDA approval is obtained, then the company must commercialise the drug by investing in a production facility as well as in marketing the drug in the industry. Similar to the real estate example, if at any stage the expected revenue that the drug will generate falls to a sufficiently low level and/or if the expected cost to seeing the development through to completion rises, the pharmaceutical company can suspend investment and, if the expected revenue rises and/or the expected development cost falls at some later stage, it can resume investment at the point at which it left off. The risk associated with the cost of developing and commercialising the drug is not spanned by traded assets and, hence, the pharmaceutical company also operates in an incomplete market. Both the real estate developer and the pharmaceutical company could decide to go ahead with the early stages of their respective investments, and then temporarily halt development and wait before investing in later stages.

The investment projects that I consider in this paper are sequential because (i) each stage of the development takes time to build or complete and (ii) during any stage the investor can temporarily halt (or permanently abandon) development. Hence, the optimal timing problem must be solved in a similar manner to that considered by Majd and Pindyck [1] whereby investment occurs as a continuous flow; i.e., "each dollar spent gives the firm an option (which it may or may not exercise) to spend another dollar (given any arbitrary cumulative amount that has already been spent), which gives the firm a completed project".

2 The Model

Consider a program to invest in the research and development of some product that takes time to build or construct. The program involves a sequence of investment outlays which correspond to the various specific steps involved in its development. The payoff to completing the development of the product is uncertain and will not be received until its development is entirely complete. This payoff is the present value of the stream of uncertain future revenue from selling the product. I assume that once complete, it will produce a fixed flow of output forever. For convenience, I choose the units so that the quantity of output is one unit per year.

At this stage I assume that there are no costs to selling or storing the product once development is complete and, hence, the total revenue flow from selling the finished product is $P = YD(1)$ per year.[1] Hence, without further

[1] I also assume at this point that there is no option to temporarily suspend or completely abandon (i.e., sell the rights to) the product once completed. This assumption, along with the assumption over operating costs, will be relaxed later.

loss of generality, I let P be the exogenous stochastic variable.

I assume that the price of the product follows a geometric Brownian motion of the form:

$$dP = \alpha_P P dt + \sigma_P P dW.$$

The expected price grows at the rate α_P, σ_P denotes the price volatility and dW is the increment of a standard Brownian motion.

I assume that the fluctuations in price are spanned by the financial market. This implies that there is a traded asset (or that it is possible to construct a dynamic portfolio of assets) the price of which is perfectly correlated with P.

In this model I assume that the uncertainty inherent in the investment decision is not only related to the future revenue the product will generate for the firm, but that there is also some considerable uncertainty owing to the cost (i.e., the amount of capital) required to complete its development. In particular, the cost uncertainty is owing to two different types of cost risk which arise because of the length of time it takes to generate revenue after the initial investment payment is made. Since both types of risk affect the investment decision of firms in important ways, both ought to be incorporated into the analysis.

The first aspect of cost uncertainty is *technical cost risk* which relates to the physical difficulty of completing a project; i.e., how much time, effort, and materials will ultimately be required (see Pindyck [3]). This type of risk is largely independent from the overall economy and, thus, it cannot be hedged by the financial market. If stochastic changes in the cost to completing the project are all due to such technical costs, then when the rate of investment is zero, so too is the amount of technical cost risk present. Additionally, the higher the cost to completion, the greater this type of risk. The second aspect of cost uncertainty is known as *input cost risk* and relates to changes in wage rates, cost of materials, etc. Another source of this risk is the effect of a policy change, "such as the granting of an investment subsidy or tax benefit" (Pindyck [2]). This risk may be partly non-diversifiable since it is likely to be correlated with the economy. Hence, it can be partly hedged by the financial market. If stochastic changes in the cost to completion are all due to input costs, then even if the investor is not investing, the cost risk is still present. This is because it arises from changes that are external to changes made by the firm.

I let the cost to completion follow a controlled diffusion process of the form

$$dI = -\alpha_I dt + h(\alpha_I, I) dW_I, \tag{1}$$

where $\alpha_I > 0$ denotes the rate of investment (so that the cost to completion declines with ongoing investment), dW_I is the increment of a standard Brownian motion which is correlated to W with correlation $|\rho| < 1$, and $h_{\alpha_I} \geq 0$, $h_{\alpha_I \alpha_I} \leq 0$, and $h_I \geq 0$, where $h_x = \partial h / \partial x$ and $h_{xx} = \partial^2 h / \partial x^2$.

Since the cost to completing the project includes both diversifiable (technical and some input cost) and non-diversifiable (input cost) risk, it cannot be completely hedged by the financial market. Therefore, the cost process is not completely spanned by the existing assets in the economy and, hence, the market in which the investor operates is incomplete. This is an important point of departure from the model of Pindyck [3] because for ease of analysis he makes the unrealistic assumption that all risk is spanned by the financial market.

Since I assumed that the price process is perfectly spanned by the financial market, its risk is perfectly correlated with the risk of some market asset. This implies, therefore, that the risk associated with the cost of development is correlated with price risk, albeit not necessarily perfectly. This is intuitive because when a project has risk arising from two variables, the risk can be correlated due to some common macroeconomic shocks. Therefore, I express the risk of I in terms of the risk of P plus some additional diversifiable risk; i.e., $dW_I = \rho dW + \sqrt{1-\rho^2}dZ$. Thus, equation (1) can be re-written as

$$dI = -\alpha_I dt + h(\alpha_I, I)\rho dW + h(\alpha_I, I)\sqrt{1-\rho^2}dZ. \tag{2}$$

Equation (2) implies that stochastic changes in I can be due to technical costs, in which case all of the risk is independent from the economy so $\rho = 0$, $h(0, I) = 0$ and $h_{\alpha_I} > 0$, input costs, in which case $h(0, I) > 0$, or to both.

Since there is no revenue generated from the project until it is fully developed and all costs have been paid, the net present value of the project is given by

$$\begin{aligned} V(P_{\tau^*}, I_{\tau^*}) =& \int_{\widetilde{\tau}}^{\infty} e^{-\mu t} P(t)dt - \int_{\tau^*}^{\widetilde{\tau}} e^{-\mu t}\alpha_I(t)dt \\ =& e^{-\mu(\widetilde{\tau}-\tau*)}\mathbb{E}_{\widetilde{\tau}-\tau^*}\left[\int_{\tau^*}^{\infty} e^{-\mu t} P(t)dt\right] - \int_{\tau^*}^{\widetilde{\tau}} e^{-\mu t}\alpha_I(t)dt \\ =& e^{-\mu(\widetilde{\tau}-\tau*)}e^{-\delta_P\tau^*}\frac{P_{\widetilde{\tau}-\tau^*}}{\delta_P} - \int_{\tau^*}^{\widetilde{\tau}} e^{-\mu t}\alpha_I(t)dt \end{aligned} \tag{3}$$

subject to (2). μ denotes the equilibrium rate of return from investing in the project as determined by the market and includes an appropriate risk premium[2], and $\delta_P := \mu - \alpha_P > 0$ is the yield from selling the good and receiving the cash-flows (for example, the rental yield in a real estate framework). τ^* denotes the time of investment in the program and $\widetilde{\tau}$ is the stochastic time of project completion.

[2]It is straightforward to extend the analysis to allow for the discount rate to be stochastic and to be expressed in terms of the risk-free rate and the Sharpe ratio of some risky market asset which partially spans the project risk by using the no-arbitrage pricing approach (see, for example, Thijssen [4]). However, this would be at the expense of parsimony and no additional insight will be gained for the issue we investigate in this paper.

Similar to Pindyck [3], I assume that there is a maximum rate, k, at which the firm can productively invest and, hence, $0 \leq \alpha_I(t) \leq k$. Furthermore, at the time the product is fully developed, $I(\widetilde{\tau}) = 0$.

Returning to equation (2): I adopt the specification of Pindyck [3] by letting $h(\alpha_I, I) = \kappa \alpha_I^{\nu} I^{1-\nu}$ for $\nu \in [0, 1/2]$. This specification ensures that (i) the value of the investment opportunity, $V(P, I)$, declines as the actual cost to completing the development rises, (ii) the instantaneous variance of dI is bounded for all $I < \infty$ and tends to zero with I, and (iii) if the firm invests at the maximum rate k until the project is complete, $\int_t^{\widetilde{\tau}} k ds = I(t)$ (cf. Pindyck [3] for the proof).

If stochastic changes in I are due to pure input cost risk, then it must be the case that $h(0, I) > 0$. This can only hold if $\nu = 0$. Thus, if only input cost risk is present, $h(\alpha_I, I) = \kappa I$. This would imply equation (2) becomes

$$dI = -\alpha_I dt + \rho \kappa I dW + \kappa I \sqrt{1 - \rho^2} dZ. \tag{4}$$

On the other hand, if only technical cost risk is present, then $\nu = \frac{1}{2}$. Furthermore, (essentially) all of this risk is diversifiable and uncorrelated with the market. Thus, intuitively, if all cost volatility is owing to technical cost risk, then $\rho = 0$. Hence, equation (2) becomes

$$dI = -\alpha_I dt + \kappa \sqrt{\alpha_I I} dZ. \tag{5}$$

To allow for both types of cost risk to be present, I combine equations (4) and (5) into a single equation to give[3]

$$dI = -\alpha_I dt + \rho \beta I dW + \gamma \sqrt{\alpha_I I (1 - \rho^2)} dZ. \tag{6}$$

The specification given by (6) is slightly ad hoc in the sense that if $\beta = 0$, then this implies that only technical cost risk is present. It further implies that *all* of the risk associated with the cost is diversifiable and, hence, $\rho = 0$. However, as Pindyck [3] points out, input cost risk "may be partly non-diversifiable". Therefore, there is some diversifiable risk which is owing to input costs which is not accounted for when $\rho = 0$. On the other hand, when $\gamma = 0$, then all risk is non-diversifiable implying $\rho = 1$. Since only input costs are associated with non-diversifiable risk, then all risk associated with the cost is owing to input cost risk. The issue surrounding $\rho = 0$ is not important. In particular, it is useful to specify the cost equation in this way so that we can identify input cost risk separate from technical cost, and vice versa. **A more general specification will be attempted towards the end of the paper, but this will lead to more cumbersome technicalities**.

[3]Note that the parameters β and γ respectively represent the input cost and technical cost risk. They are not necessarily equal when both types of uncertainty are present. Hence I replace κ with β and γ

As previously mentioned, the market for this investment project is incomplete and, hence, risk-preferences for the investor must be introduced. I assume that his utility function displays constant relative risk aversion (CRRA) so that $U(y) = y^\eta$, for $0 < \eta \leq 1$. His problem is to choose an investment time τ^* so that his expected discounted utility from the completed investment payoff is maximised. Thus, the investor must solve the following problem:

$$\begin{aligned}
&\sup_{\alpha_I(t)} \sup_{\tau^* \geq 0} \mathbb{E}_0 \left[e^{-\zeta\tau^*} U \left[V(P_{\tau^*}, I_{\tau^*}) \right] \mid P_0 = P; I_0 = I \right] \\
&= \sup_{\alpha_I(t)} \sup_{\tau^* \geq 0} \mathbb{E}_0 \left[e^{-\zeta\tau^*} \left(e^{-\mu(\widetilde{\tau} - \tau^*)} e^{-\delta_P \tau^*} \frac{P_{\widetilde{\tau} - \tau^*}}{\delta_P} - \int_{\tau^*}^{\widetilde{\tau}} e^{-\mu t} \alpha_I(t) dt \right)^\eta \right],
\end{aligned} \tag{7}$$

where ζ denotes the investor's subjective discount rate.

3 Model Solution

Letting $\widetilde{F}(P, I)$ denote the value of the *entire* investment opportunity, standard dynamic programming techniques imply that the Bellman equation associated with the developer's utility maximising problem is given by

$$\begin{aligned}
&\frac{1}{2}\sigma_P^2 P^2 \widetilde{F}_{PP} + \alpha_P P \widetilde{F}_P + \frac{1}{2}\beta^2\rho^2 I^2 \widetilde{F}_{II} + \beta\rho\sigma_P IP \widetilde{F}_{PI} - \zeta \widetilde{F} \\
&+ \sup_{\alpha_I} \{ \frac{1}{2}\gamma^2 \alpha_I (1 - \rho^2) I \widetilde{F}_{II} - \alpha_I \widetilde{F}_I - \alpha_I \} = 0.
\end{aligned} \tag{8}$$

Equation (8) is linear in α_I implying that the rate of investment which maximises $\widetilde{F}(P, I)$ is either 0 or the maximum rate, k, at which the firm can productively invest; i.e.,

$$\alpha_I = \begin{cases} k & \text{for } \frac{1}{2}\gamma^2(1 - \rho^2) I \widetilde{F}_{II} - \widetilde{F}_I - 1 \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

In other words, the problem has a bang-bang solution. Therefore, equation (8) has a free boundary at a point P^*, such that $\alpha_I = k$ for $P \geq P^*$ and $\alpha_I = 0$ otherwise. Similar to Majd and Pindyck [1], I assume that if no investment is being made, any capital which has been previously installed does not decay. Thus I denote the value of the investment program in the region where $P \geq P^*$ by $F(P, I)$ and the value of the program for $P < P^*$ by $f(P, I)$. $F(P, I)$ and $f(P, I)$ satisfy the following PDEs:

$$\begin{aligned}
&\frac{1}{2}\sigma_P^2 P^2 F_{PP} + \alpha_P P F_P + \frac{1}{2}\beta^2\rho^2 I^2 F_{II} + \beta\rho\sigma_P IP F_{PI} - \zeta F \\
&+ \frac{1}{2}\gamma^2 k (1 - \rho^2) I F_{II} - k F_I - k = 0.
\end{aligned} \tag{10}$$

and

$$\frac{1}{2}\sigma_P^2 P^2 f_{PP} + \alpha_P P f_P + \frac{1}{2}\beta^2\rho^2 I^2 f_{II} + \beta\rho\sigma_P IP f_{PI} - \zeta f = 0. \tag{11}$$

Note that the solution to equation (10) is the value of the investment program while investing; i.e., while development is taking place. It represents the expected PV of the completed project during this time, *plus* the option to abandon its development to avoid losses should P fall or I rise in the future. The solution to equation (11) is the option to invest in the project (when development is not occurring) if and when P increases and/or I falls in the future.

These equations satisfy the following boundary conditions:

$$f(0, I) = 0, \tag{12}$$

$$F(P, 0) = \left(\frac{P}{\delta_P}\right)^{\eta}, \tag{13}$$

and

$$\lim_{P\to\infty} F(P, I) = \lim_{P\to\infty}\left(\frac{P}{\delta_P}e^{-\mu\widetilde{\tau}} + \frac{k}{\mu}\left(e^{-\mu\widetilde{\tau}} - 1\right)\right)^{\eta}. \tag{14}$$

Equation (12) implies that when the revenue from the completed project is zero, the value of the investment program to the risk averse investor is also zero. Equation (13) says that when the product is fully developed and all costs have been paid, the value of the project tends to the utility that the investor derives from the revenue he obtains from sales. Finally, equation (14) states that as the value of the completed product becomes very large relative to the costs of development, the value of the option to suspend investment during the development stage becomes negligible and the value of the project to the manager is simply the utility obtained from investing in the project which will be carried through to completion.

As well as the boundary conditions, equations (10) and (11) satisfy the following value-matching and smooth pasting conditions at the free boundary, P^*:

$$\frac{1}{2}\gamma^2(1-\rho^2)IF_{II}(P^*, I) - F_I(P^*, I) - 1 = 0, \tag{15}$$

$$F_P(P^*, I) = f_P(P^*, I), \tag{16}$$

and

$$F_I(P^*, I) = f_I(P^*, I). \tag{17}$$

Condition (15) follows from (9) and essentially implies that $F(P^*, I) = f(P^*, I)$. Hence, I refer to it as the value-matching condition.

Equation (11) has the analytical solution

$$f(P, I) = B\left(\frac{P}{I}\right)^{\varphi_1},$$

where B is some constant and φ_1 is the positive root of the quadratic equation

$$\Psi(\varphi) = (\sigma_P - \beta\rho)^2\varphi^2 + (2\alpha_P - \sigma_P^2 + \beta^2\rho^2)\varphi - 2\zeta = 0.$$

Hence

$$\varphi_1 = \frac{\sigma_P^2 - 2\alpha_P - \beta^2\rho^2}{2(\sigma_P - \beta\rho)^2} + \frac{1}{2(\sigma_P - \beta\rho)^2}\sqrt{(2\alpha_P - \sigma_P^2 + \beta^2\rho^2)^2 + 8\zeta(\sigma_P - \beta\rho)^2}. \tag{18}$$

The positivity of φ_1 ensures that condition (12) is satisfied.

Conditions (15) through (17) imply

$$F_P(P^*, I) = -\frac{I}{P^*}F_I(P^*, I) \tag{19}$$

which is the free-boundary condition which must be adhered to.

It is not possible to obtain an analytical solution for (10) and, thus, it must be solved numerically in conjunction with (19) using a finite difference approximation. See Appendix A for details.

4 Results

References

[1] S. Majd and R.S. Pindyck. Time to build, option value, and investment decisions. *Journal of Financial Economics*, 18(1):7–27, 1987.

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Appendix

A Finite Difference Transformation

Equation (10) is transformed by letting $F(P,I) = e^{-\zeta I/k} H(W,I)$ where $W = \ln P$. This yields

$$\begin{aligned}
&\frac{1}{2}\sigma_P^2 H_{WW} + \left(\alpha_P - \frac{1}{2}\sigma_P^2 - \frac{\zeta}{k}\beta\rho\sigma_P I\right) H_W \\
&+ \frac{1}{2}\left(\beta^2\rho^2 I + \gamma^2 k(1-\rho^2)\right)\left(H_{II} - 2\frac{\zeta}{k}H_I + \frac{\zeta}{k}H\right) I \\
&+ \beta\rho\sigma_P I H_{WI} - kH_I - ke^{\zeta I/k} = 0.
\end{aligned} \tag{A.1}$$

The boundary conditions (13) and (14) as well as the free-boundary condition (19) are also transformed accordingly and become, respectively,

$$F(P,0) = H(W,0) = \left(\frac{1}{\delta_P}e^W\right)^\eta, \tag{A.2}$$

$$\lim_{W\to\infty} H_W(W,I) = \lim_{W\to\infty}\left[\frac{\eta}{\delta_P}e^{W+(\zeta-\mu)\widetilde{\tau}}\left(\frac{e^{W-\mu\widetilde{\tau}}}{\delta_P} + \frac{k}{\mu}\left(e^{-\mu\widetilde{\tau}} - 1\right)\right)^{\eta-1}\right], \tag{A.3}$$

and

$$H_W = I\left(\frac{\zeta}{k}H - H_I\right) \tag{A.4}$$

Since the rate of investment is bang-bang, then when the firm invests and the payoff from completion is infinitely large relative to the development cost, then the stochastic time of project completion is given by $\widetilde{\tau} = I/k$. In solving for the model numerically, it is necessary to make use of this fact.

I adopt the explicit form of the finite difference method and I let $H(W,I) = H(i\Delta W, j\Delta I) \equiv H_{i,j}$ for $-a \le i \le m$ and $0 \le j \le n$.

Now we make the following finite-difference substitutions in equation (A.1):

$$H_W \approx \frac{H_{i+1,j} - H_{i-1,j}}{2\Delta W}$$

$$H_{WW} \approx \frac{H_{i+1,j} - 2H_{i,j} + H_{i-1,j}}{(\Delta W)^2}$$

$$H_I \approx \frac{H_{i,j+1} - H_{i,j-1}}{2\Delta I}$$

$$H_{II} \approx \frac{H_{i,j+1} - 2H_{i,j} + H_{i,j-1}}{(\Delta I)^2}$$

$$H_{WI} \approx \frac{H_{i+1,j+1} - H_{i+1,j-1} - H_{i-1,j+1} + H_{i-1,j-1}}{4\Delta W\Delta I}$$

which gives

$$
\begin{aligned}
&\left(1+\frac{(j-1)\Delta I}{k}\left[(j-1)\Delta I\frac{\zeta\beta^2\rho^2}{k}+\gamma^2\zeta(1-\rho^2)+\frac{\beta\rho\sigma_P}{\Delta W}\right]\right)H_{i,j}\\
&=\left(p^+-(j-1)A\left(1+\frac{k}{\zeta\Delta I}\right)\right)H_{i+1,j-1}+\left(p^-+(j-1)A\left(1+\frac{k}{\zeta\Delta I}\right)\right)H_{i-1,j-1}\\
&\quad+\left(p_0+\frac{\zeta(j-1)\Delta I}{2k^2}\left(2+\Delta I\right)\left[(j-1)\Delta I\beta^2\rho^2+\gamma^2k(1-\rho^2)\right]\right)H_{i,j-1}\\
&\quad+(j-1)\frac{\Delta I}{k\Delta W}\beta\rho\sigma_P H_{i+1,j}-\Delta I e^{\zeta(j-1)\Delta I/k}.
\end{aligned}
\tag{A.5}
$$

where

$$
\begin{aligned}
p^+ &= \frac{\Delta I}{2k\Delta W}\left(\frac{\sigma_P^2}{\Delta W}-\frac{1}{2}\sigma_P^2+\alpha_P\right),\\
p^- &= \frac{\Delta I}{2k\Delta W}\left(\frac{\sigma_P^2}{\Delta W}+\frac{1}{2}\sigma_P^2-\alpha_P\right),\\
p_0 &= 1-\frac{\sigma_P^2\Delta I}{k(\Delta W)^2}.\\
A &= \frac{(\Delta I)^2}{2k^2\Delta W}\zeta\beta\rho\sigma_P
\end{aligned}
$$

Note that in deriving equation (A.5), I use the fact that

$$
H_I=\frac{H_{i,j}-H_{i,j-2}}{2\Delta I}=\frac{H_{i,j}-H_{i,j-1}}{\Delta I}
$$

by definition, to eliminate $H_{i,j-2}$.

The terminal boundary condition becomes

$$
H_{i,0}=\left(\frac{e^{i\Delta W}}{\delta_P}\right)^{\eta} \tag{A.6}
$$

and the upper boundary condition becomes

$$
H_{m+1,j}=2\Delta W\left[\frac{\eta}{\delta_P}e^{m\Delta W+(\zeta-\mu)(j\Delta I)/k}\left(\frac{e^{m\Delta W-\mu(j\Delta I)/k}}{\delta_P}+\frac{k}{\mu}\left(e^{-\mu(j\Delta I)/k}-1\right)\right)^{\eta-1}\right]+H_{m-1,j}
$$

Substituting this latter expression for $H_{m+1,j}$ and $H_{m+1,j-1}$ in equation (A.5) (by setting $i = m$) yields

$$\left(1 + \frac{\zeta(j-1)\Delta I}{k^2}\left[(j-1)\Delta I\beta^2\rho^2 + \gamma^2 k(1-\rho^2)\right]\right) H_{m,j}$$
$$= \left(p^+ - (j-1)A\right) 2\Delta W \frac{\eta}{\delta_P} e^{m\Delta W + (\zeta-\mu)(j-1)\Delta I/k} \left(\frac{e^{m\Delta W - \mu(j-1)\Delta I/k}}{\delta_P} + \frac{k}{\mu}\left(e^{-\mu(j-1)\Delta I/k} - 1\right)\right)^{\eta-1}$$
$$+ \left(p^+ + p^-\right) H_{m-1,j-1}$$
$$+ \left(p_0 + \frac{\zeta(j-1)\Delta I}{2k^2}(2+\Delta I)\left[(j-1)\Delta I\beta^2\rho^2 + \gamma^2 k(1-\rho^2)\right]\right) H_{m,j-1}$$
$$+ (j-1)\frac{\eta\Delta I}{\delta_P k}\beta\rho\sigma_P e^{m\Delta W}\Bigg[e^{(\zeta-\mu)(j\Delta I)/k}\left(\frac{e^{m\Delta W - \mu(j\Delta I)/k}}{\delta_P} + \frac{k}{\mu}\left(e^{-\mu(j\Delta I)/k} - 1\right)\right)^{\eta-1}$$
$$- e^{(\zeta-\mu)(j-1)\Delta I/k}\left(\frac{e^{m\Delta W - \mu(j-1)\Delta I/k}}{\delta_P} + \frac{k}{\mu}\left(e^{-\mu(j-1)\Delta I/k} - 1\right)\right)^{\eta-1}\Bigg] - \Delta I e^{(j-1)\Delta I\zeta/k}. \tag{A.7}$$

Finally, the free boundary condition becomes

$$H_{i^*,j} = \left(\frac{j\Delta W}{k}(\zeta\Delta I - k) + 1\right) H_{i^*-1,j} + j\Delta W H_{i^*-1,j-1}. \tag{A.8}$$