Markovian Equilibrium in a Model of Investment Under Imperfect Competition

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Abstract

In this paper, we develop and analyze a classic dynamic model of irreversible investment under imperfect competition and stochastic demand. We characterize the markovian equilibrium when player's strategies are continuous in the state variable. At the equilibrium, firms invest as quickly as possible in order to join a zone in the space of capacities where there is no competition pressure. Furthermore, the equilibrium as an efficiency property: the point of this area which is reach by the firms is the point which minimizes the investment cost of the all industry.

Keywords: Capacity investment and disinvestment. Dynamic stochastic games. Markov-perfect equilibrium. Real option games.

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1 Introduction

Capacity expansion or reduction under uncertainty is one of the most important decisions that firms can made. It impacts their immediate profit and creates some long-run commitments. In a dynamic setting, the investment pattern by a monopoly is well known. Because of the uncertainty, the firm has incentives to delay a profitable project (in expectation) in order to wait for more information about the demand. This is the theory of real options. What happens in imperfect competition becomes the theory of real option games. (For recent surveys, see for instance Boyer, Gravel and Lasserre (2004), Azevedo and Paxson (2010), or Chevalier-Roignant et al (2011)).

In this literature several authors focused on capacity decision under uncertainty. In these models, at each time the profit made by a firm depends of its size (i.e. its capacity), the size of its opponents, and a parameter which evolves randomly with time (which can be a parameter of demand or cost, the important point being that its future evolution are unknown). As this parameter evolves, firms wish to adapt their sizes. Firms can either invest to increase their capacity, disinvest to reduce it, or let their capacity depreciates at its natural rate. We speak of perfectly reversible investment when the cost of investing is equal to the cost of disinvesting, in which case firms can perfectly adapt themselves to the stochastic evolution of the parameter. Such repeated game framework is classic in industrial organization. However, in reality, increasing its size implies hiring new employees, building new factories or office, buying new equipment, and so on... These investments are usually at least partially sunk, and the firm's size decisions are not perfectly reversible. We speak of totally irreversible investment when firms cannot reduce they sizes, and of partially irreversible investment when firms can decreases their size by disinvesting (but with a scrap value inferior to the cost of investing) or by depreciation. In these cases, the theory of real option game permits to link the hysteresis due to the irreversibility of investment and the imperfect competition.

In this domain, Baldursson (1998), Grenadier (2002), Back and Paulsen (2009) and Chevalier-Roignant, Huchzermeier and Trigeorgis (2011) focus on the same model of investment. Capacities are quasi-irreversible (with linear price), time is continuous and the parameter of demand follows a stochastic diffusion. Baldursson (1998) and Grenadier

(2002) exhibit an equilibrium, and show that the optimal investment is myopic, in the sense that firms can maximize their profit assuming that strategies of other players are constant. However, Back and Paulsen (2009) shows that this equilibrium is an open-loop one, which fails sub-game perfection. Then, Chevalier-Roignant, Huchzermeier and Trigeorgis (2011), focuses on markov-perfect equilibrium. The authors describe the optimal markovian best response of the firms. However, the linearity of the investment cost implies infinite value for the amount of investment (the capital jumps, as there is no interest to delay purchases and sells of capital). This prevents them to fully characterize the equilibrium. Our paper attempts to fill this gap.

To do so, we start studying the simplest investment game. In a one shot model, firm have initial capacities and can invest or disinvest in a quasi-irreversible way (firms can buy or sell capacities at linear but different prices). We exhibit an area in the space of capacities, that we name the no-move zone. If the capacities of the firms are inside this no-move zone, no firm will neither invest nor disinvest. So all point of this no-move zone is a possible equilibrium, given some initial capacities. We show that for some given initial capacities, the equilibrium is the point of the no-move zone which minimizes the costs of investment and disinvestment for the all industry. This efficiency result holds even if firms have a priori no interest to coordinate their decisions. Furthermore, as long as the prices of investment and disinvestment are not equal, the no-move zone is not reduced to one point, and an initial asymmetry in capacities can be preserved.

In a continuous setting with demand uncertainty, we model the differential game in an unusual way to overcome the issue of infinite investment of Chevalier-Roignant, Huchzermeier and Trigeorgis (2011). We find the existence of no-move zone, depending of the demand parameter, such that firms behave as in the one shot game. At each time firms join the no-move zone minimizing the industry costs of investment and disinvestment. Furthermore, this equilibrium is unique in the class of continuous markovian equilibrium.

The plan of this paper is the following. Section 2, studies the one-shot model. Section 3 presents the dynamic model and characterizes the markovian equilibria. Section 4 concludes. All omitted proofs are reported in appendix.

2 Investment in the one-period game

2.1 The one-shot game model

The aim of this subsection is to abstract from dynamics and uncertainty issues, in order to focus on the effect of partial irreversibility of investment. To do so, we present a simple static model of competition in capacity.

More precisely, consider a market with n firms competing *à la Cournot* in capacities. Each firm i starts with some amount of capital k^i, which can be extended or reduced through buying or selling some assets. Purchases are made at a linear price p^+, and sales at a (also linear) price p^- (with $p^- \leq p^+$). We name K^i the capacity finally installed by firm i. Let k be the vector of industry's initial capacities and K the vector of installed capacities. For firm i, the cost of installing a new capacity is:

$$C(K^i, k^i) = \left\{ \begin{array}{l} p^+ \left(K^i - k^i\right) \text{ if } K^i \geq k^i \\ p^- \left(K^i - k^i\right) \text{ if } K^i < k^i \end{array} \right\}. \tag{1}$$

Firms produce and sell an homogenous good, at a price depending of the total quantity $\bar{q} = \sum_{i=1}^{n} q^i$. Each firm's production depends of its capacity, according to the technology $q^i = K^i$, [1] and has a cost, $c_i(q^i)$. Such technology is classic in dynamic investment models, and has been used by Fudenberg and Tirole (1983), Grenadier (2002), Merhi and Zervos (2007) among others[2]. So, by selling the quantity K^i, firm i obtains a payoff of:

$$\pi^i(K) = P\left(\bar{K}\right) K^i - c_i(K^i), \tag{2}$$

[1] In the one-shot model, this technology can be seen as the result of an endogenized game, in which firms buy capacities and then play a Cournot competition limited by the capacity previously bought. Indeed, no firms have interest to invest in capacity which will not be used to produce, as its opponents only react to the final quantity. (Except if the disinvestment price is negative. In this case a firm has interest to keep its unused capacity in order to avoid a disinvestment case. For example, this is the case for polluted production site, for which the cost of decontamination is more important than the cost of conservation of this asset.)

[2] As it was shown in Reynolds (1987), this technology assumption is also the result of a dynamic games with limited Cournot competition, without uncertainty. However, when there is uncertainty, firms have an incentive to keep their unused capacity for a possible further uses, when demands increases. Assuming that quantities are equal to capacities permits to avoid such adaptability effects and focus on the direct effect of uncertainty on capacity choice.

where $\bar{K} = \sum_{j=1}^{n} K^j$. The profit function of firm i is thus:

$$\Pi^i(K, k) = \pi^i(K) - C(K^i, k^i). \tag{3}$$

Note that if $p^- = p^+$, the investment decision is totally reversible, and the initial capacity variable has no impact. The smaller is p^-, the more irreversible are the capacities of the firm, and, at the limit, when $p^- = -\infty$, investment is totally irreversible, as in Grenadier (2002), Back and Paulsen (2009), Boyer, Lasserre and Moreaux (2012), and others.

In order to ensure the existence of the equilibrium we make the following usual hypothesis:

H1: For each $i = 1, .., n$, $c_i(.)$ is a twice-differentiable positive function such that $c_i' \geq 0$, $c_i'' \geq 0$. $P(.)$ is also a twice-differentiable positive function, with $P' < 0$, $P'' < 0$ when P is strictly positive.[3] Furthermore, for all $i = 1, .., n$, $P''(\bar{q})q^i < -c_i''(q^i)$.

2.2 Best responses

In this part, we present the best response of firm i. Assume that for all $j \neq i$, firm j installs a capacity K^j. Obviously, the marginal revenue of capacity for firm i depends of the choices of its opponents, and we note $\frac{\partial \pi^i}{\partial K^i}^{-1}(x)$ the inverse function of the marginal revenue of capacity, so

$$\frac{\partial \pi^i}{\partial K^i}^{-1}(x) = K \Leftrightarrow \frac{\partial \pi^i}{\partial K^i}\left(K^1, .., K^i = K, .., K^n\right) = x.$$

By concavity of π in K^i, such inverse function is well defined and increasing. Then, firm i has three possible choices:

- invests $(K^i > k^i)$, and making a profit:

$$\Pi^i = \pi^i(K) - p^+\left(K^i - k^i\right),$$

[3] These conditions are not the more restrictive one could make in order to obtain theorem 1 (presented page 8). Indeed, our proof rests on the third theorem of Novshek (1985), and the linearity of cost of investment and disinvestment. However, as our point of interest is the dynamic game, and we need some regularity for the existence of the dynamic differentiable equations, we place our self in the assumption made by Szidarovszky and Yakowitz (1977).

leading to an optimal choice of capacity $\frac{\partial \pi^i}{\partial K^i}^{-1}(p_k^+)$. Hence, if $k^i > \frac{\partial \pi^i}{\partial K^i}^{-1}(p_k^+)$, the monopoly has no interest to invest.

∘ disinvests ($K^i < k^i$), and making a profit:

$$\Pi^i = \pi^i(K) + p^- \left(k^i - K^i\right),$$

leading to an optimal capacity $\frac{\partial \pi^i}{\partial K^i}^{-1}(p^-)$. Thus, the firm has no interest to disinvest if $k < \frac{\partial \pi^i}{\partial K^i}^{-1}(p^-)$ (which is higher than $\frac{\partial \pi^i}{\partial K^i}^{-1}(p^+)$).

∘ the last possibility is to do nothing (the firm neither invest nor disinvest). Indeed, as $\frac{\partial \pi^i}{\partial K^i}^{-1}(.)$ is increasing, the initial capacity of the firm can be greater than $\frac{\partial \pi^i}{\partial K^i}^{-1}(p_k^+)$, so the firm has no interest to invest more, but also smaller than $\frac{\partial \pi^i}{\partial K^i}^{-1}(p^-)$, so the firm has also no interest to disinvest.

Therefore there exists two thresholds, $\frac{\partial \pi^i}{\partial K^i}^{-1}(p^-)$ and $\frac{\partial \pi^i}{\partial K^i}^{-1}(p^+)$ such that the firm does not wish to invest nor disinvest if its capital is between this thresholds, invest until $\frac{\partial \pi^i}{\partial K^i}^{-1}(p^+)$ if its initial capacity is small, and disinvest until $\frac{\partial \pi^i}{\partial K^i}^{-1}(p^-)$ if its initial capacity is large. This can be summarized in the following proposition:

Proposition 1 *: The best response of firm i is:*

$$K^i_{BR} = \left\{ \begin{array}{c} \frac{\partial \pi^i}{\partial K^i}^{-1}(p^+) \text{ if } k < \frac{\partial \pi^i}{\partial K^i}^{-1}(p^+) \\ k^i \text{ if } k^i \in \left[\frac{\partial \pi^i}{\partial K^i}^{-1}(p^+), \frac{\partial \pi^i}{\partial K^i}^{-1}(p^-)\right] \\ \frac{\partial \pi^i}{\partial K^i}^{-1}(p^-) \text{ if } k > \frac{\partial \pi^i}{\partial K^i}^{-1}(p^-) \end{array} \right\}. \tag{4}$$

Of course, this best response depends of the capacities of other firms, as $\frac{\partial \tilde{\pi}^i}{\partial K^i}$ depends of the capacity of all firms. Graphic 1 represents the best response of firm 2 in the space of capacity, for a duopoly with linear demand and no production costs.

$[Insert\ G1]$

In this graphic, we can see the existence of an area in the space of capacities, Γ^2, limited by $\frac{\partial \pi^2}{\partial K^2}^{-1}(p_k^-)$ and $\frac{\partial \pi^2}{\partial K^2}^{-1}(p_k^+)$, such that, it is never optimal for firm 2 to be outside this area. Thus, if there is an equilibrium, it belongs to Γ^i for all firm i, so it belongs to the intersection of these areas. Let H be this intersection. We know that all equilibria belong to H. Furthermore, assume that the initial distribution of capacity belongs to H. Then, if all players except i keep their capacity constant, then the best response of i is to maintain its initial capacity. In the case of duopoly with linear demand and no production cost, this can be seen in graphic 2. We called H the no-move zone.

$[Insert\ G2]$

2.3 Characterization of the equilibrium

In the last subsection, we show the existence of an area in the set of space capacities, the no-move zone H, such that all equilibria belong to H. If the initial capacities of the firms are in H, the equilibrium is to keep the same capacities. This no-move zone is defined by,

$$H = \left\{ K \in \mathbb{R}_+^n : \forall i = 1, .., n, \ \frac{\partial \pi^i}{\partial K^i} \in [p^-, p^+] \right\}, \tag{5}$$

which can rewritten as

$$H = \left\{ K \in \mathbb{R}_+^n : \forall i = 1, .., n, \ P\left(\bar{K}\right) + P'\left(\bar{K}\right) K^i - c_i(K^i) \in [p^-, p^+] \right\} \tag{6}$$

However, we still do not know what the equilibrium is when the initial capacities does not belongs to H. Theorem 1 solve this point.

Theorem 1: Assume $H1$. Then, there exists only one Nash equilibrium K^*, which verifies:

$$K^* = \arg\min_{K \in H} \sum_{i=1}^{n} C(K^i, k^i). \tag{7}$$

This condition is equivalent to the distance condition,

$$K^* = \arg\min_{K \in H} \sum_{i=1}^{n} \left| K^i - k^i \right|. \tag{8}$$

Theorem 1 provides existence and uniqueness for the equilibrium, and its characterization. To understand this characterization, let focus on the no-move zone. In this area, according to (5), the marginal revenue of each firm is inferior to the price of adding a new capacity, but superior to the price of selling some capacity, so no firms wish to invest nor disinvest. The no-move zone is thus the set in the space of capacity such that no firms change its capacity at the equilibrium. By theorem 1, H can also be view as the set of all possible equilibria, in the meaning that all equilibrium belongs to H and all point of H can be an equilibrium for some initial value.

There exists another way to think of H. If we define the competitive pressure as the fact to react (for a firm) to a change in opponents' strategy, then the no-move zone is the set of capacities without any competitive pressure. Indeed, assume that firms decide to install a capacity which belongs to H and that one firm changes its strategy to implement another capacity. Then, if the new vector of capacity belongs to the no-move zone, the other firms have no new incentives to change their capacities. It does not mean that their strategy is optimal, but if for one firm the best strategy was to disinvest at the first place, its best strategy will still be to disinvest after the change of capacity.

In this light, the best response (4) can be reinterpret as the point of H which minimizes its cost of implementation, i.e. the cheapest vector of capacity without competitive pressure. In theorem 1, (7) establishes that the equilibrium is the vector of capacity in the no-move zone which minimizes its total cost of installation, for the all industry. So there exists of some efficiency in the market, in the meaning that the equilibrium will coordinate the individual cost minimization of each firm to a global cost minimization to reach the no-move zone, a position where there is no competition pressure.

Of course, in case of totally reversible investment (when $p^+ = p^-$) the no-move zone is reduced to a unique point, and no industry efficiency appears. This is the usual Cournot competition[4]. When investment is not totally reversible, the no-move zone is a set, none reduced to a singleton, and each point of this set can be an equilibrium for some initial values. Graphic 3 presents which points of the no-move zone will be an equilibrium, in function of the initial capacities, for a duopoly with linear demand and no production costs. As it can be seen on the graphic, firms with different initial capacities can still be asymmetric at the equilibrium, and there are different possible symmetric equilibria, even when firms has the same profit function.

[4] With a cost $c_i(K^i) + p^+ K^i$.

$[Insert\ G3]$

3 Investment in a dynamic game

3.1 The dynamic model

In this section, we use a variation of a classic model, as used in Baldursson (1998), Grenadier (2002), Back and Paulsen (2009), and Chevalier-Roignant, Huchzermeier and Trigeorgis (2011), to study the effect of dynamic and uncertainty on the previous results. Baldursson (1998) and Grenadier (2002) exhibit open-loop equilibrium for oligopoly. However, Back and Paulsen (2009) shows that this equilibrium fails sub-game perfection. Thus, Chevalier-Roignant, Huchzermeier and Trigeorgis (2011) focus on markovian equilibrium, and find interesting properties. In this part, we take a further step, and fully characterize the markovian equilibrium.

Let K_t^i be the capital of firm i at time t, time is continuous and capital is partially reversible. Let $\pi^i(A_t, K_t^1, .., K_t^n)$ be the instantaneous payoff of firm i, A_t being the parameter of uncertainty, following a diffusion process:

$$dA_t = \beta(A_t)dt + \sigma(A_t)dW_t, \tag{9}$$

where W_t is a standard Wiener process. We assume Cournot competition. Let $P_{A_t}(.)$ be the inverse demand function, depending of the level of demand A_t, and some production cost for each firm, $c^i(.)$ such that

$$\pi^i(A_t, K_t^1, .., K_t^n) = P_{A_t}\left(\bar{K}_t\right) K_t^i - c^i(K_t^i). \tag{10}$$

In the following, we assume that the price is a continuous function of A_t, and $H1$. The interest rate is r and, as previously, the purchase price of capital is p^+, and the selling price p^-. The total expected profit of firm i at time 0 is thus:

$$\Pi^i = E\left[\int_0^{+\infty} e^{-rt}\pi^i(A_t, K_t^1, .., K_t^n)dt - p^+\int_0^{+\infty} e^{-rt}dK_t^{i+} + p^-\int_0^{+\infty} e^{-rt}dK_t^{i-}|A_0\right]. \tag{11}$$

The objective of each firm is to maximize its own expected profit, given the initial levels of capital and demand[5]. In this framework, the usual method is to introduced I_t^i, the investment done by firm i at date t, so that the capital of each firm is determined by the following differential equation:

$$\frac{\partial K_t^i}{\partial t} = I_t^i. \tag{12}$$

Chevalier-Roignant, Huchzermeier and Trigeorgis (2011) assume that investment is markovian, so firm i chooses a function of the demand level and the capital of each firm, and at each time, it invests according to the value of the state variable (demand level, capital), $I_t^i = \tilde{I}^i(A_t, K_t^1, .., K_t^n)$. In this case, the Bellman formula gives:

$$r\Pi^i(A,K) = \sup_{I_t^i} \left\{ \begin{array}{c} P_{A_t}\left(\bar{K}\right) K^i - p^+ \left(I_t^i\right)_+ - p^- \left(I_t^i\right)_- + I_t^i \frac{\partial \Pi^i}{\partial K^i} + b(A)\frac{\partial \Pi^i}{\partial A} + \frac{\sigma^2(A)}{2}\frac{\partial^2 \Pi^i}{(\partial A)^2} \\ + \sum_{\substack{j=1 \\ j\neq i}}^n I^j(A,K)\frac{\partial \Pi^i}{\partial K^j} + \sum_{\substack{j=1 \\ j\neq i}}^n \sum_{\substack{h=1 \\ j\neq i}}^n I^j(A,K) I^h(A,K) \frac{\partial^2 \Pi^i}{\partial K^j \partial K^h} \end{array} \right\}. \tag{13}$$

The optimal investment policy maximizes $\frac{\partial \Pi^i}{\partial K^i} I_t^i - p^+ \left(I_t^i\right)_+ - p^- \left(I_t^i\right)_-$. So, if $\frac{\partial \Pi^i}{\partial K^i} \in [p^-, p^+]$, the firm i has no interest to invest nor disinvest. Otherwise, the optimal flow of investment I_t^i is infinite: the firm installs its optimal capital (capital in the region ${\frac{\partial \Pi^i}{\partial K^i}}^{-1}([p^-, p^+])$) instantly. The optimal capital policy of the firm is thus to jump in the area ${\frac{\partial \Pi^i}{\partial K^i}}^{-1}\left(\left[p_K^-, p_K^+\right]\right)$, and to do nothing as long as the capital stays in this area. So the optimal strategy cannot be defined by the investment variable, as the linearity of the cost of investment implies non-continuous capital strategies.

To avoid such difficulty, we focus on the choice of capacity instead of investment, assuming that K_t^i is the markovian control of firm i. If time was discrete, then the choice of capacity at time t is just a function of the level of capital at time $t-1$. In this case, the optimal control at time t (K_t^i) depends on the state variable at time t (which is K_{t-1}^i), which is the markovian control of the previous period. To mimic this construction in continuous time, the choice of capital at time t should depend on a state variable representing the level

[5] As usual, this maximization is done in the set of left-continuous A_t-adapted stochastic process. Furthermore, in order to ensure the existence of (11) we also assume that the process has finite variation. This assumption is natural with our cost function. Indeed, if a firm have a infinite variation of its capital, it will pay an infinite cost of investment and disinvestment (as $p^+ > p^-$). However, its future revenue is finite (due to hypothesis $H4$), which leads to a negative and infinite profit!

of capital of the industry just before time t. As for all s there exists another s closer to t, the state variable will be the left limit of industry capacities. This permits to define a different modelization.

Definition: The investment game previously considered is in its markovian state-control form if:

(i) the state variable at time t is $k_t = (k_t^1, .., k_t^n)$, as defined by

$$k_t^i = \lim_{\substack{s \to t \\ s < t}} K_t^i, \tag{14}$$

where k_0 is the given initial level of capital;

(ii) for each player i, the strategic variable is its capacity, and the strategy is markovian, i.e. the firm choose a function, $\tilde{K}^i$, of the state variable (industry's capacities and level of demand) and $K_t^i = \tilde{K}^i(A_t, k_t)$.

In such framework, a markov perfect equilibrium is defined as usually, by the vector of functions $\tilde{K}(.,.) = \left(\tilde{K}^{*1}(.,.), .., \tilde{K}^{*n}(.,.)\right)$ such that, for all $(A, k) \in \mathbb{R}_+^{n+1}$,

$$\forall i \in \{1, .., n\}, \tilde{K}^{*i}(A, k) \in \arg\max_{K^i(.,.)} E[\Pi^i(A, k, K^i, (\tilde{K}^{*j})_{j \neq i}) \mid A]. \tag{15}$$

Furthermore, a continuous markovian equilibrium is defined as a markovian equilibrium in which the functions $\tilde{K}^{*1}(.,.)$,..,$\tilde{K}^{*n}(.,.)$ are continuous.

To our knowledge, this is a new way to model markovian strategy. In our problem, such definition allows to properly define the best responses of the firms. In the next section, proposition 2 verifies that the best responses are the same in both model. In addition, this definition allows us to characterize the markov perfect equilibrium when we assume that the strategies are continuous functions of the state variable. Theorem 2 presents the parallel with the one-shot game in a general framework.

3.2 Characterization of the continuous markov equilibrium

In this subsection, we characterize the continuous markovian equilibria. We start by introducing technical assumptions. H2 is needed to prove proposition 2 (in order to use Ito's Lemma, to inverse the Ito's Lemma results and to apply theorem 1). H3 is classic to ensure the existence of strong solution to (9). H4 ensures the existence of the stochastic integral determining the profit of the firms.

H2: For each $i = 1, .., n$, $c_i(.)$ is a four times differentiable positive function such that $c_i' \geq 0$, $c_i'' \geq 0$, and for all $A \in \mathbb{R}_+^*$, $P(.,.)$ is also four times differentiable positive and strictly concave function in each variable. Furthermore, for all $i = 1, .., n$, $P''(A, \bar{q})q_i < -c_i''(q_i)$.

H3: $\beta(A)$ and $\sigma(A)$ are continuous functions, and verify the Lipschitz conditions.

H4: There exists a function $G : \mathbb{R}_+ \to \mathbb{R}_+$, such that, $\forall (A, x) \in \mathbb{R}_+^2$, $xP(A, x) < G(A)$, and $\int_0^{+\infty} e^{-rt} G(A_t) dt < +\infty$.

These assumptions allow to state proposition 2, which gives the form of the best response in the markovian state-control form of the game.

Proposition 2: Assume **H2, H3** and **H4**. Let $i \in \{1, .., n\}$. In the markovian state-control form of the game, assume that for all $j \neq i$, the strategy of firm j, $\tilde{K}^j(.,.)$ is a continuous function of the state variable. Then, for all $(A, k) \in \mathbb{R}_+^{n+1}$ there exists some[6] continuous decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that the best response of firm i is:

$$\tilde{K}^i(A, k) = \left\{ \begin{array}{c} \phi_{A,k}(p^-) \text{ if } k^i > \phi_{A,k}(p^-) \\ k^i \text{ if } k^i \in \left[\phi_{A,k}(p^+), \phi_{A,k}(p^-)\right] \\ \phi_{A,k}(p^+) \text{ if } k^i < \phi_{A,k}(p^+) \end{array} \right\}. \tag{16}$$

Furthermore, the best response of firm i is continuous in the state variable.

This proposition shows that the optimal capacity of firm i can jump: if at some time t, k_t^i is strictly smaller than $\phi_{A,k}(p^+)$, then the firm has interest in investing instantly to $\phi_{A,k}(p^+)$. In this case the investment in period t is infinite, so the markovian state-control form gives the same result as the regular form. However it also allows to go a step further and to characterize the equilibria, as presented in theorem 2. In fact, at each time, everything happens as in the one-shot game presented in the last section (with of course some modification of the no-move zone H in order to take into account the future profit). Firms always want to invest or disinvest forthwith in order to reach the no-move zone. As

[6]The implicit definition of ϕ is given in the proof in appendix but, for simplicity, is not presented here. In particular, we have the property than ϕ is continuous and differentiable in a and k.

long as they are in the no-move zone, no firms change its capacity. The shape of the no-move zone depends on the expected amount of money earned over time by unit of capacity when firms keep their capacity constant,

$$\forall x > 0,\ v(A,x) = E\left[\int_0^{+\infty} P\left(A_t, x\right) e^{-rt} dt | A\right]. \tag{17}$$

This value v is the finite solution to the following differential equation[7]:

$$\forall x > 0,\ rv(A,x) = P\left(A,x\right) + \beta(A)\frac{\partial v}{\partial A}(A,x) + \frac{\sigma^2(A)}{2}\frac{\partial^2 v}{\left(\partial A\right)^2}(A,x). \tag{18}$$

The no-move zone is then defined by:

$$H_v(A) = \left\{k \in \mathbb{R}_+^n | \forall i \in \{1,..,n\} : v\left(\bar{k}\right) + v'_A\left(\bar{k}\right) k^i - \frac{1}{r}c'_i(k^i) \in [p^-, p^+]\right\}. \tag{19}$$

Now, we can present theorem 2.

Theorem 2: Assume **H2, H3** and **H4**. Then, there exists at most one continuous markov perfect equilibrium, $\tilde{K}^*\left(.,.\right) = \left(\tilde{K}^{*1}\left(.,.\right),..,\tilde{K}^{*n}\left(.,.\right)\right)$. Furthermore, for all $(A,k) \in \mathbb{R}_+^{n+1}$, $i \in \{1,..,n\}$, this equilibrium verifies:

$$\tilde{K}^*\left(A,k\right) = \arg\min_{K \in H_v(A)} \sum_{i=1}^n C(K^i, k^i). \tag{20}$$

This condition is equivalent to the distance condition,

$$\tilde{K}^*\left(A,k\right) = \arg\min_{K \in H_v(A)} \sum_{i=1}^n \left|K^i - k^i\right|. \tag{21}$$

Theorem 2 is the analog of theorem 1, but in a continuous time setting. It ensures the uniqueness of the continuous markovian equilibrium, and characterizes it. At each time, firms invest (or disinvest) in order to join the no-move zone at the smallest possible cost for the industry. However, this efficiency result is time-myopic. When demand evolves, firms face investment and disinvestment period, leading to a costly path of investment. This path

[7] Notes that (18) is not the classic Bellman differential equation. Indeed, in our proof, we just need to characterize the profit inside the no-move zone, as we already know what happens outside the no-move zone. So (18) is just due to Ito's lemma application to the evolution of uncertainty.

of investment verify the dynamic competitive pressure, and belongs to $H_v(A_t)$ at each time t. But there is no reason to think that another path compatible with the dynamic competitive pressure, less sensitive to the evolution of demand, which can not be less costly on the all period.

This theorem focuses on the continuous markovian equilibrium. Nevertheless, one can ask about other markovian equilibrium. Indeed, there is a priori no reason that collusion can not be sustain by markovian strategy in a differential game. In a companion paper on the same investment game, (Fagart (2013)), we exhibit a markovian equilibrium with tit-for-tat strategy implementing the monopoly profit.[8]

4 Conclusion

In this work, we characterize the continuous markovian equilibrium of a classic model of investment under uncertainty whith Cournot competition. We establish the existence of an area in the space of firms' capacities, the no-move zone, such that firms invest or disinvest in order to join this area as soon as possible, and keep their capacity constant when they are inside this area. At the equilibrium, the firms reach the point of the no-move zone which minimizes the cost of investment of the all industry. The intuition on this result is that the no-move is the area where other firms' actions do not impact the action of a firm, so the efficiency of the equilibrium comes from the absence of competitive pressure inside the no-move zone.

The existence of this area is due to the irreversibility of investment, and when investment is perfectly reversible, the no-move zone is reduces to a unique point, as in usual Cournot competition. In the other case, the no-move zone is a set of vector of capacity, each

[8]Such collusive equilibria arise because we assume that the strategy of the players rests on the state variable. In the classic modelization of differential games, the strategy of the players leads on the derivative of the state variable, which imposes that the state variable will evolves continuously with time. As shown by theorem 2 in our case, this continuity assumption provides the finding of collusive equilibria when the strategy of the player rests on the derivative of the state variable. If the finding of collusive equilibria is an argument in favor of our modelization, it also brings the question to know why the monopoly profit can not be implemented by continuous strategy.

point of it can be an equilibrium for some initial value. It implies that the asymmetry of an industry can be preserved at the equilibrium, even if firms have the same profit function. However, when the time goes by, the parameter of uncertainty evolves, and the no-move zone evolves with it. The issue of the evolution of an industry will be study in our companion paper (Fagart (2013)). In particular, we show that even if firm's asymmetry is preserved in the short run, it disappears in the long run, and that shocks of demand have an impact which depends of the size of the firm considered.

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