Sequential Investment, Capacity Sizing and Dividing Flexibility

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1 Introduction

We consider the problem of a typical investor who has discretion over not only the timing, but also the sizing of a new plant in sequential manner. We contrast the sequential investment strategies for different stage numbers in order to the value of flexibility. Additionally, we analyze the sequential investment for a case in which there exists a fixed cost in the investment one. The optimal stage numbers of sequential investment are obtained for various fixed costs.

2 The Model

We consider a firm which plans to build a new plant with capacity $\hat{Q}$ by multi-stage investments regarding to the market condition. Following Huisman and Kort [1], the price at time t in this market is given by

$$P_t = X_t(1 - \eta Q), \tag{1}$$

where Q is total market output, η a constant, and X_t an exogenous shock process. Since the price is strictly positive, $Q < \frac{1}{\eta}$ is required. We assume that X_t follows a geometric Brownian motion:

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \qquad X_0 = x, \tag{2}$$

which μ is the growth rate, dW_t is the increment of a Wiener process, and $\sigma > 0$ is the volatility. Suppose that the firm is risk neutral and the discount rate $r > \mu$ is a risk-free.

Let c be the operating costs for the plant. Instantaneous profit for a firm at time t is given by

$$\pi(t, X_t, Q) = (P_t - c)Q = (X_t - \eta Q X_t - c)Q. \tag{3}$$

We consider the problem as a sequence of optimal stopping problems. Let N denote the number of investment up to complete the construction. The firm decides the timing and the size of investment based on the market price (1). The sequence of the decision making is as follows: (i) At the time when the $k-1$th investment is made, the firm decides a level $\tilde{x}_k$ and the size of remaining capacity y_k after the k-th investment. Note that $y_{N+1} \equiv 0$. (ii) If the exogenous shock X_t exceeds the level $\tilde{x}_k$, the firm made k-th investment and the plant capacity is $\hat{Q} - y_k$. The investment needs a fixed cost $I > 0$ and proportional cost per unit capacity $\delta > 0$. Figure 1 illustrates the sequence for the case of $N = 4$.

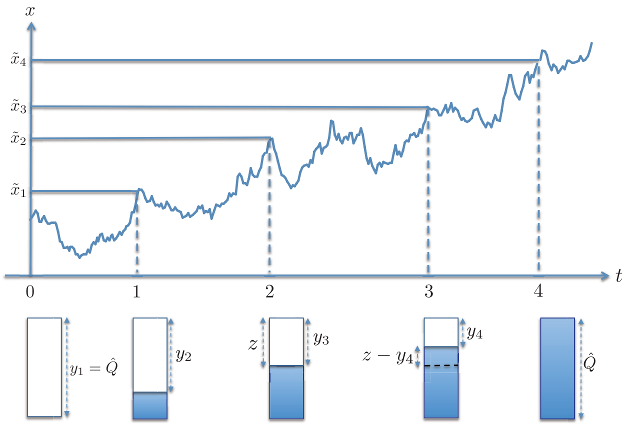

Figure 1: A time line of a stepwise investments for $N = 4$.

Define $V_k(x, z)$ be the value of the project when the exogenous shock level x and the remaining capacity z at time period k. The value function can be expressed as the discounted total expected profit:

$$\begin{aligned} V_k(x,z) &= \sup_{\tau} E\left[\int_0^{\tau} e^{-rt}\pi(t, X_t, \hat{Q} - z)dt \right. \\ &\qquad \left. + e^{-r\tau}\mathcal{M}V_{k+1}(X_\tau, z) \,\middle|\, X_0 = x\right], \end{aligned} \tag{4}$$

where $\mathcal{M}$ is the operator defined by

$$\mathcal{M}V_k(x,z) \equiv \sup_{y_k}\{V_k(x, y_k) - I - (z - y_k)\delta\}. \tag{5}$$

At the beginning of the period $N+1$, the construction of the plant have already been completed and the plant capacity is $Q_N = \hat{Q}$. In this period, it is not necessary to invest for the plant. Thus, the boundary condition is given by

$$\begin{aligned} V_{N+1}(x, 0) &= E_x\left[\int_0^{\infty} e^{-rt}\pi(t, X_t, \hat{Q})dt \,\middle|\, X_0 = x\right] \\ &= \hat{Q}(1 - \eta\hat{Q})K_1 x - K_2 c\hat{Q} \end{aligned} \tag{6}$$

where $K_1 = \frac{1}{r-\mu}$, $K_2 = \frac{1}{r}$.

3 Optimal Investment Policy

In this section, we solve the problem (4) by using iterative procedure. To solve this problem, we suppose the continuation region for k-th investment takes the form $(0,\ \tilde{x}_k)$. The continuation region means that once the process X_t hits $\tilde{x}_k$, we should add the plant. Denote $F_k(x,z)$, $1 \le k \le N-1$ as a candidate for the value function $V_k(x,z)$, and it should solve the equation

$$\frac{1}{2}\sigma^2 x^2 F_k^{''}(x,z) + \mu x F_k^{'}(x,z) - rF_k(x,z) + x(\hat{Q}-z) - \eta(\hat{Q}-z)^2 x - c(\hat{Q}-z) = 0. \quad (7)$$

Thus, we can derive a option value of investment taking into account capacity sizing as follows;

$$F_k(x,z) = \begin{cases} A_k x^\beta + K_1\kappa(z)x - c(\hat{Q}-z)K_2, & \text{if } x < \tilde{x}_k, \\ \mathcal{M}F_{k+1}(x,z), & \text{if } x \ge \tilde{x}_k, \end{cases}$$

where $\kappa(z) = (\hat{Q}-z)(1-\eta(\hat{Q}-z))$, A_k is a constant to determine and $\beta > 1$ is a root of the equation

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0. \quad (8)$$

Proposition 3.1. *When the remaining capacity z at the beginning of the period k, $k = 1, \cdots, N-1$, the threshold $\tilde{x}_k$ and a value $\tilde{A}_k$ are given by*

$$\tilde{x}_k(z) = \frac{\beta\psi_k(z)}{K_1(\beta-1)\{R^2_{k+1}(y_{k+1}) - \kappa(z)\}}, \quad (9)$$

$$\tilde{A}_k(z) = \frac{1}{\beta-1}\psi_k(y_k)\tilde{x}_k^{-\beta}(z), \quad (10)$$

where

$$R^1_k(y_k) = D - \beta\psi_k(y_k)\frac{1}{\tilde{x}_k(y_k)}\frac{\partial\tilde{x}_k(y_k)}{\partial y_k},$$

$$R^2_k(y_k) = \psi_k(y_k)(1-2\eta(\hat{Q}-y_k))\frac{1}{R^1_k(y_k)} + \kappa(y_k),$$

$$\psi_k(z) = \left(z - y_{k+1} + \frac{1}{R^1_{k+1}(y_{k+1})}\psi_{k+1}(y_{k+1})\right)D + I,$$

$D = K_2c + \delta$ and $\psi_N(y_N) = y_N D + I$. Moreover, the protection level for period $k+1$, y_{k+1}, is a solution of the following equation;

$$\frac{1}{\beta-1}\left(\frac{\tilde{x}_k(z)}{\tilde{x}_{k+1}(y_{k+1})}\right)^\beta = \frac{K_1(1-2\eta(\hat{Q}-y_{k+1}))\tilde{x}_k(z) - D}{R^1_{k+1}(y_{k+1})}.$$

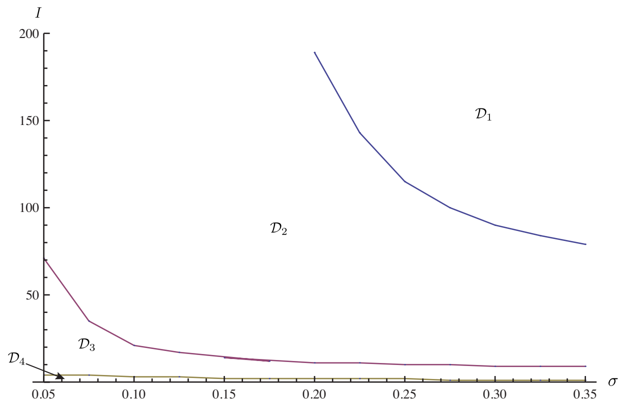

Figure 2: Region of the optimal number of investment for fixed cost and volatility when variable cost is $\delta = 100$.

Theorem 3.1. *The flexibility of the investment is attractive relative to the single investment, if and only if,*

$$\frac{x_L(\hat{Q})}{\tilde{x}_1(\hat{Q})} > \left(\frac{\hat{Q}D + I}{\psi_1(\hat{Q})}\right)^{\frac{1}{\beta}}. \quad (11)$$

Proposition 3.2. *If there are no operating and proportional costs, $c = \delta = 0$, then the value of single investment is larger than the one of stepwise investment.*

4 Numerical Examples

We investigate the effect of the size of fixed cost on the option value. We set the following base parameters: $r = 0.05$, $\mu = 0.01$, $\sigma = 0.2$, $\eta = 0.4$, $c = 20$, $\delta = 100$, $Q = 1$, $N = 4$. By comparing the option value for each N, we will find the optimal number of investment for each volatility σ and fixed cost I. Define $\mathcal{S}_+ := [0,\infty) \times [0,\infty)$. The region that the n-th investment is optimal for I and σ is defined as $\mathcal{D}_n = \{(I,\sigma) \in \mathcal{S}_+; F_n(x,\hat{Q};I,\sigma) > F_m(x,\hat{Q};I,\sigma),\ m \ne n\}$ for any x. Figure 2 shows the region with respect to I and σ. We can see that the higher volatility makes few investment attractive relative to the more stepwise investment. Moreover, the stepwise investment is preferable as the fixed cost I increases.

References

[1] Huisman, K. and Kort, P. (2009), Strategic capacity investment under uncertainty. Proceeding of 13th Annual International Real Options.