A General Decision-Tree Approach to Real Option Valuation

Abstract

The common paradigm for risk-neutral real-option pricing is a special case encompassed within our general model for valuing investment opportunities. Risk-neutral real option prices deviate from the risk-averse real option values that apply in an incomplete market, giving different rankings of investment opportunities and different optimal exercise strategies. Unlike risk-neutral prices, more general real option values often decrease with the volatility of the asset price. They also depend on the structure of fixed and variable investment costs, the expected return of the underlying asset, the frequency of decision opportunities, the price of the asset relative to initial wealth, the investor's risk tolerance and its sensitivity to wealth. We explain how these factors affect the ranking of real option values under a standard geometric Brownian motion for the asset price. Numerical examples also consider 'boom-bust' or mean-reverting price scenarios and investments with positive or negative cash flows.

Key words: CARA, CRRA, certain equivalent, development, divestment, displaced log utility, exponential utility, HARA, investment, mean-reversion, property boom, real estate, risk aversion, risk tolerance

1 Introduction

The term 'real option' is commonly applied to a decision opportunity for which the investment cost is predetermined, and the vast majority of the literature assumes the underlying asset is traded in a complete or partially complete market so that all (or at least the important) risks are hedgeable. Real options are typically regarded as tradable contracts with predetermined strikes, the standard risk-neutral valuation (RNV) principal is invoked, most commonly in a continuous-time setting, and the mathematical problem is no different to pricing an American option under the risk-neutral measure.

Using the RNV principle implies there is a unique market price for the real option that is positively related to the volatility of the underlying asset. This is in sharp contrast with the more traditional discounted cash flow (DCF) approach, where an increase in risk decreases the net present value of the investment's expected cash flows. But the RNV approach only applies to a very special type of real option. The original definition, first stated by Myers (1977), is a decision opportunity for a corporation or an individual. It is a right, rather than an obligation, whose value is contingent on the uncertain price(s) of some underlying asset(s). The RNV assumptions of perfectly hedgeable risks and predetermined strike are clearly inappropriate for many real options, particularly those in real estate, research & development or mergers & acquisitions. In many applications the market for the underlying of a real option is incomplete, the underlying investment is not highly liquid and the investment cost is not predetermined, indeed the transacted price is typically negotiated between individual buyers and sellers.

Under such conditions Grasselli (2011) proves that the real option to invest still has an intrinsic value, but it is purely subjective. Unlike the premium on a financial option, it has no absolute accounting value. It represents the dollar amount, net of financing costs, that the investor should receive for certain to obtain the same utility as the risky investment. The decision maker's utility function may be applied to value every opportunity but his subjective views about the evolution of the market price and any future cash flows may be project-specific. So real option values allow alternative investment opportunities to be ranked, just as financial investments are ranked using risk-adjusted performance measures. Thus, a pharmaceutical company may compare the values of real options to develop alternative products, or an oil exploration company may compare the values of drilling in different locations, or a private property development company may compare the values of opportunities to buy and develop plots of land in different locations. The solution also determines an optimal exercise strategy for each alternative.

We develop a general decision-tree approach for determining the value of an investment opportunity and its optimal exercise, in a market that need not be complete, where the solution is derived via maximisation of expected utility. To represent an investor's risk preferences using a utility is no greater assumption than using the mean-variance criterion or Sharpe ratio. Indeed, a utility is less restrictive than these criterion because they implicitly assume normally distributed uncertainties and an exponential utility. But exponential utilities make restrictive assumptions about the investor's preferences. More generally, in our approach the decision maker may have any utility in the hyperbolic absolute risk aversion (HARA) class, introduced by Mossin (1968) and Merton (1971). So we do not focus on analytic solutions, available only in the exponential case, and neither do we rely on expansion approximations, which can be unreliable.

Assumptions about costs are critical determinants of real option values, and of their sensitivities to expected return and risk. So we employ a general structure with both predetermined and stochastic components for investment costs. Thus, the general real option problem falls squarely into the realm of decision analysis. The standard RNV real option price only applies under a fixed cost assumption. Indeed, if the cost is stochastic and perfectly correlated with a martingale discounted asset price, for example, then the RNV price would be zero, yet a risk-averse investor would still place a positive value on such a decision opportunity.

We consider three different processes for the discounted market price: geometric Brownian motion (GBM), mean-reversion and boom-bust market price scenarios. Standard RNV real option prices correspond to the special case that the investment cost is predetermined, the forward asset price follows a GBM with total return equal to the risk-free rate, the price is derived from a linear utility and the investment decision may be taken at any point in time. More generally, in an incomplete market the optimal exercise path and the corresponding option value depend on the decision maker's risk preferences, on his subjective views about the market price process and cash flows, and on the frequency of decision opportunities (e.g. at a quarterly meeting of the board of directors).

The general properties of the model include a comparison of RNV real option prices with those attributed by risk-averse investors in incomplete markets. Moreover, our model provides answers to several important and relevant questions that have not previously been addressed in the real options literature. If the fixed or predetermined strike assumption is not valid, how does this change the option value – and what is the effect on its sensitivity to expected return and risks of the investment? How should a real option value reflect the frequency of decision opportunities? What is the effect of the investor's risk preferences on real option values, and how does this affect the ranking of different opportunities? And how is this ranking influenced by the structure of the cash flows, or by the price of the asset relative to the initial wealth of the investor?

We proceed follows: Section 2 places our work in the context of the real options literature; Section 3 describes the model; Section 4 describes its properties under a GBM price process; Section 5 considers investors that believe the price process is mean-reverting, or subject to regime shifts, and the case that the investment has associated cash flows; and Section 6 summarises and concludes.

2 Relevance to the Literature

Most of the literature on real options focuses on opportunities to enter a tradable contract with predetermined strike on an asset traded in a complete market. That is, the real option's pay-off distribution can be replicated using tradable assets, so that all risks are hedgeable. Also, the decision opportunity can be exercised continuously at any time over an infinite horizon. In this setting the option has the same value to all investors (Harrison and Kreps, 1981) and thus can be valued as if the investor is risk neutral, just like a financial option. This major strand of the literature regards a real option as a tradable contract, usually with a fixed or predetermined strike and a finite horizon: see Triantis and Hodder (1990), Capozza and Sick (1991), Trigeorgis (1993), Panayi and Trigeorgis (1998), Benaroch and Kauffman (2000), Boer (2000), Yeo and Qiu (2002), Shackleton and Wojakowski (2007) and many others. Some papers assume an infinite horizon – see Kogut (1991), Grenadier (1996), Smith and McCardle (1998) and Patel et al. (2005) for a review. Other studies include a stochastic strike, including McDonald and Siegel (1986), Quigg (1993) and Bowman and Moskowitz (2001).

However, in practice, many decision opportunities encompassed by Myer's original definition are not standardised, tradable securities and their risks are only partially hedgeable, if at all. For instance, if an oil exploration company must decide whether to drill in location A or location B, its views about the benefits of drilling in each location will depend on their subjective beliefs about the market prices of oil in the future, as well as their risk preferences. And investment costs are unlikely to be fixed or predetermined. For instance, when a pharmaceutical company decides which drug to research and develop, both the research costs and the subsequent profits tend to be positively correlated with the drug's potential market price. In this setting there is no unique value for a real option; it will be specific to the investor, depending upon his subjective views about the costs and benefits of investment, and upon his risk preferences.

Closed-form, continuous-time, deterministic-strike real option values in incomplete or partially complete markets have been considered for decision makers with an exponential utility by Henderson (2002, 2007), Miao and Wang (2007) and Grasselli (2011). When the underlying asset is correlated with a market price Henderson (2007) applies the closed-form solution derived by Henderson (2002) for the exponential option value and the investment threshold, showing that market incompleteness results in earlier exercise and a lower real option value; moreover the option value *de*creases with volatility. Henderson and Hobson (2002) employ a power utility function in the Merton (1969) investment model, proposing an approximate closed-form optimal reservation value for the option when the investment cost is small relative to initial wealth. They show that the exponential and power utility option values behave very differently as a function of risk aversion as it tends to zero, due to boundary constraints on the power utility value. Evans et al. (2008) compare power and logarithmic utility values in a similar setting. Most of these papers derive the optimal investment threshold and the indifference value for a finite-horizon, continuous choice, deterministic-strike investment opportunity using a two-factor GBM framework in which the value of the project to the investor is stochastic and possibly correlated with the price of a liquidly traded asset that may be used to hedge the investment risk. Both price processes are discounted to present value terms, so that a univariate time 0 utility function may be applied to maximise the expected utility of the maturity pay-off. Our work uses a similar discounted value process, but since we suppose from the outset that no risks are hedgeable by traded securities we utilise only a one-factor framework.

Like us, Grasselli (2011) considers the case where none of the risks of the investment are hedgeable. Importantly, he proves that the time-flexibility of the opportunity to invest still carries an option value for a risk-averse investor, so that the paradigm of real options can be applied to value a private investment decision. Employing an exponential utility he proves that the real option value converges to zero as risk tolerance decreases.

Early work on real-option problems employed a decision-tree approach with a constant, risk-adjusted discount rate in which the option value decreases with risk – see Mason and Merton (1985), Trigeorgis and Mason (1987), Copeland et al. (1990) and Copeland and Antikarov (2001). However, Copeland et al. (1990) dismissed the decision-tree approach and directed the mainstream of real options research towards risk-neutral valuation, where standard American option prices increase with the volatility of the underlying asset. Trigeorgis (1996), Smit (1996), Brandão and Dyer (2005), Brandão et al. (2005, 2008) and Smith (2005) have since employed a decision-tree approach, but also on the assumption of market-priced risk, thereby adopting a RNV framework.

The Integrated Valuation Procedure (IVP) introduced by Smith and Nau (1995) and extended by Smith and McCardle (1998) considered risk-averse decision makers in a decision-tree framework, this time explicitly endowed with a utility. A multivariate utility is defined as a sum of time-homogenous utilities of future cash flows and the decision tree applies backward induction on their certain equivalent (CE) value. The IVP approach is only valid for an exponential utility and when cash flows at different times are independent, because only exponential utilities have the unique property that their CE is additive over independent random variables. Although economic analysis is commonly based on inter-temporal consumption, it is standard in finance to base the utility of decisions on final wealth, with future values discounted to time 0 terms. This discounting is an important element of our approach because it greatly simplifies the decision analysis and allows the backward induction step to be defined on the expected utility relative to any univariate HARA utility function, not only on the CE values of an additively separable multivariate exponential utility.

3 The Model

We assume investment risks are unhedgeable, so the market for the underlying asset is incomplete; its forward market price measure is therefore subjective to the decision maker, with the risk-neutral measure arising as the special case of a linear utility and a risk-free expected return; investment costs may have predetermined and/or stochastic components; decision opportunities are discrete and are modelled using a binomial price tree with decision nodes placed at every k steps; the decision horizon T is finite; and the consequence of the decision is valued at some finite investment horizon, $T' > T$. We do not equate the investment horizon with the maturity of the investment, i.e. cash flows from the investment may continue beyond T', so the market price is not constrained to be the discounted expected value of the cash flows.[1] The decision maker holds subjective views not only about the evolution of the market price for $0 \leq t \leq T'$ but also about the stream of cash flows (if any) that would be realised if he enters the investment. In most applications cash flows would reflect the individual management style of the decision maker, e.g. aggressive, expansive, recessive etc. Additionally, the decision maker is characterised by his initial wealth, w_0 which represents the current net worth of all his assets, and a HARA utility function $U(w)$ which reflects his risk tolerance λ and how this changes with wealth.

3.1 Market Prices and Cash Flows

All future market prices and cash flows are expressed in time 0 terms by discounting at the decision-maker's borrowing rate r.[2] Thus the investor borrows funds at rate r to invest, rather than financing the cost from his initial wealth, which we suppose is not available for investment.[3] We suppose that r is a constant, risk-free rate.

First we allow the market price of the underlying asset to follow a GBM, as usual, so that the discounted forward market price p_t evolves over time according to the process:

$$\frac{dp_t}{p_t} = (\mu - r)dt + \sigma dW_t, \quad \text{for} \quad 0 < t \leq T', \tag{1}$$

[1]As in Kasanen and Trigeorgis (1995), the market price could be related to the utility of a representative decision maker, being the 'break-even' price at which the representative decision maker is indifferent between investing or not.

[2]This rate depends on the business risk of the investment as perceived by the financer, not as perceived by the decision maker. It may also depend on the decision-maker's credit rating but this assumption is not common in the real options literature.

[3]To avoid additional complexity we do not consider that investments could be financed from wealth, even though this would be rational if wealth is liquid and r is greater than the return on wealth, $\tilde{r}$.

where μ and σ are the decision-maker's subjective drift and volatility associated with p_t and W_t is a Wiener process. Then p_t has a lognormal distribution, $p_t \sim \log \mathrm{N}\left((\mu - r)\, t, \sigma^2 t\right)$.

It is convenient to use a binomial tree discretisation of (1) in which the price can move up or down by factors u and d, so that $p_{t+1} = p_t u$ with probability π and otherwise $p_{t+1} = p_t d$. No less than eleven different binomial parameterisations for GBM are reviewed by Chance (2008). Smith (2005), Brandão and Dyer (2005), Brandão et al. (2005, 2008), Smit and Ankum (1993) and others employ the 'CRR' parameterisation of Cox et al. (1979). However, the Jarrow and Rudd (1982) parameterisation, which is commonly used by option traders, is more stable for low levels of volatility and when there are only a few steps in the tree. Thus we set

$$m = \left[\mu - r - 0.5\sigma^2\right] \Delta t, \quad u = e^{m+\sigma\sqrt{\Delta t}}, \quad d = e^{m-\sigma\sqrt{\Delta t}} \quad \text{and} \quad \pi = 0.5. \tag{2}$$

We also consider a modification of (1) that represents a regime-dependent process, which trends upward with a low volatility for a sustained period and downward with a high volatility for another sustained period, replicating booms and busts in the market price. Thus we set:

$$\frac{dp_t}{p_t} = \begin{cases} (\mu_1 - r)dt + \sigma_1 dW_t, & \text{for} \quad 0 < t \leq T_1, \\ (\mu_2 - r)dt + \sigma_2 dW_t, & \text{for} \quad T_1 < t \leq T'. \end{cases} \tag{3}$$

Alternatively, the decision maker may believe the market price will mean-revert over a relatively short time horizon, and to replicate this we suppose the expected return decreases following a price increase but increases following a price fall, as in the Ornstein-Uhlenbeck process:

$$d \ln p_t = -\kappa \ln\left(\frac{p_t}{\bar{p}}\right) dt + \sigma dW_t \tag{4}$$

where κ denotes the rate of mean reversion to a long-term price level $\bar{p}$. Following Nelson and Ramaswamy (1990) (NR) we employ the following binomial tree parameterisation for the discretised Ornstein-Uhlenbeck process:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = u^{-1} \tag{5a}$$

$$\pi_{\mathbf{s}(t)} = \begin{cases} 1, & 0.5 + \nu_{\mathbf{s}(t)}\sqrt{\Delta t}/2\sigma > 1 \\ 0.5 + \nu_{\mathbf{s}(t)}\sqrt{\Delta t}/2\sigma, & 0 \leq 0.5 + \nu_{\mathbf{s}(t)}\sqrt{\Delta t}/2\sigma \leq 1 \\ 0, & 0.5 + \nu_{\mathbf{s}(t)}\sqrt{\Delta t}/2\sigma < 0 \end{cases} \tag{5b}$$

where

$$\nu_{\mathbf{s}(t)} = -\kappa \ln\left(\frac{p_{\mathbf{s}(t)u}}{\bar{p}}\right) \tag{5c}$$

is the local drift of the log price process, which decreases as κ increases. The corresponding price process thus has local drift:

$$\mu_{\mathbf{s}(t)} = -\kappa \ln\left(\frac{p_{\mathbf{s}(t)u}}{\bar{p}}\right) + 0.5\sigma^2 + r \tag{5d}$$

Note that when $\kappa = 0$ there is a constant transition probability of 0.5 and the NR parameterisation is equivalent to the parameterisation (2) with $m = 0$.

The cash flows, if any, may depend on the market price of the asset as, for instance, in rents from a property. Let $\mathbf{s}(t)$ denote the state of the market price at time t, i.e. a path of the market price from time 0 to time t. In the binomial tree framework $\mathbf{s}(t)$ may be written as a string of u's and d's with t elements, e.g. uud for $t = 3$. Now $\mathrm{CF}_{\mathbf{s}(t)}$ denotes the cash flow when the market price is in state $\mathbf{s}(t)$ at time t. Regarding cash flows as dividends we call the price excluding all cash flows before and at time t the 'ex-dividend' price, denoted $p^-_{\mathbf{s}(t)}$. At the time of a cash flow $\mathrm{CF}_{\mathbf{s}(t)}$ the market price follows a path which jumps from $p^+_{\mathbf{s}(t)} = p^-_{\mathbf{s}(t)} + \mathrm{CF}_{\mathbf{s}(t)}$ to $p^-_{\mathbf{s}(t)}$. We suppose that he receives the cash flow at time t if he divests in the project at time t but does not receive it if he invests in the project at time t. The alternative assumption that he receives the cash flow at time t only when he invests is also possible.[4] The dividend yield, also called dividend pay-out ratio, is defined as

$$\delta_{\mathbf{s}(t)} = \frac{p^+_{\mathbf{s}(t)} - p^-_{\mathbf{s}(t)}}{p^+_{\mathbf{s}(t)}}. \tag{6}$$

We assume that dividend yields are deterministic and time but not state dependent, using the simpler notation δ_t. Then the cash flows are both time and state dependent.[5]

3.2 Costs and Benefits of Investment and Divestment

The investment cost at time t, in time 0 terms, is:

$$K_t = \alpha K + (1 - \alpha)\, g(p^-_t), \quad 0 \le \alpha \le 1, \tag{7}$$

where K is a constant in time 0 terms and $g : \mathcal{R} \to \mathcal{R}$ is some real-valued function linking the investment cost to the market price, so that cost is perfectly correlated with price only when g is linear. When $\alpha = 1$ we have a standard real option with a predetermined strike K, such as might be employed for oil exploration decisions. When $\alpha = 0$ we have a variable cost linked to the market price p^-_t, such as might be employed for real estate or merger and acquisition options. The intermediate case, with $0 < \alpha < 1$ has an investment cost with both fixed and variable components.

We assume that initial wealth w_0 earns a constant, risk-free lending rate $\tilde{r}$, as do any cash flows paid out that are not re-invested. Any cash paid into the investment (e.g. development cost) is financed at the borrowing rate r. The financial benefit to the decision-maker on investing at time t is the sum of any cash flows paid out and not re-invested plus the terminal market price of the investment. Thus the wealth of the investor at T', in time 0 terms, following investment at time t

[4] The subscript $\mathbf{s}(t)$ denotes a particular realisation of the random variable that carries the subscript t, e.g. p^+_{uud} and p^-_{uud} are the left and right limits of a realisation of p_t when $t = 3$. From henceforth we only use the subscript $\mathbf{s}(t)$ when it is necessary to specify the state of the market – otherwise we simplify notation using the subscript t. Also, assuming cash flow is not received when divesting at time t, p^-_t is a limit of p_t from the right, not a limit from the left.

[5] If the cash flow is not state dependent, then the dividend yield must be state dependent. The state dependence of cash flows induces an autocorrelation in them because the market price is autocorrelated. For this reason, defining an additively separable multivariate utility over future cash flows as in Smith and Nau (1995) and Smith and McCardle (1998) may be problematic.

is

$$w^I_{t,T'} = e^{(\tilde{r}-r)T'} w_0 + \sum_{s=t+1}^{T'} e^{(\tilde{r}-r)(T'-s)} \mathrm{CF}_s + p^-_{T'} - K_t. \tag{8}$$

Some investments pay no cash flows, or any cash flows paid out are re-invested in the project. Then the financial benefit of investing at time t, in time 0 terms, is simply the cum-dividend price $\widetilde{p}_{t,T'}$ of the investment accruing from time t. If a decision to invest is made at time t, with $0 \le t \le T$, then $\widetilde{p}_{t,t} = p_t$ but the evolution of $\widetilde{p}_{t,s}$ for $t < s \le T'$ differs to that of p_s because $\widetilde{p}_{t,s}$ will gradually accumulate all future cash flows from time t onwards. In this case, when the decision maker chooses to invest at time t his wealth at time T' in time 0 terms is

$$\widetilde{w}^I_{t,T'} = e^{(\tilde{r}-r)T'} w_0 + \widetilde{p}_{t,T'} - K_t. \tag{9}$$

Note that if $\tilde{r} = r$ then also $\sum_{s=t+1}^{T'} e^{(\tilde{r}-r)(T'-s)} \mathrm{CF}_s + p^-_{T'} = \tilde{p}_{t,T'}$ in (8).

Similarly, if the decision maker already owns the investment at time 0 and chooses to sell it at time t, the time 0 value of his wealth at time T' is

$$w^S_{t,T'} = e^{(\tilde{r}-r)T'} w_0 + \sum_{s=1}^{t-1} e^{(\tilde{r}-r)(T'-s)} \mathrm{CF}_s + p^+_t - p_0. \tag{10}$$

Note that p_0 is subtracted here because we assume the investor has borrowed funds to invest in the property. Alternatively, if there are no cash flows, or they are re-invested,

$$\widetilde{w}^S_{t,T'} = e^{(\tilde{r}-r)T'} w_0 + \widetilde{p}_{0,t} - p_0. \tag{11}$$

The wealth $w^D_{t,T'}$ resulting from a defer decision at time t depends on whether he invests (or divests) later on. This must therefore be computed using backward induction as described in the next sub-section.

3.3 Optimal Decisions and Real Option Value

As in any decision problem, we shall compare the expected utility of the outcomes resulting from investment with the utility of a base-case alternative, which in this case is to do nothing so that terminal wealth remains at w_0 in time 0 terms. For brevity, we describe the backward induction step only for the decision to invest – it is similar for the decision to divest, but replace I with S (for an existing investor that sells the property) and D with R (for an existing investor that remains invested). The option to invest at time t has time 0 utility value $U^I_{t,T'} = U\left(w^I_{t,T'}\right)$, but since $w^I_{t,T'}$ is random so is $U^I_{t,T'}$, and we use the expected utility

$$\mathrm{E}\left[U^I_{\mathbf{s}(t),T'}\right] = \mathrm{E}\left[U\left(w^I_{\mathbf{s}(t),T'}\right)\right],$$

as a point estimate. Then, given a specific decision node at time t, say when the market is in state $\mathbf{s}(t)$, the potential investor chooses to invest if and only if

$$\mathrm{E}\left[U^I_{\mathbf{s}(t),T'}\right] > \mathrm{E}\left[U^D_{\mathbf{s}(t),T'}\right],$$

and we set

$$\mathrm{E}\left[U_{\mathbf{s}(t),T'}\right] = \max\left\{\mathrm{E}\left[U^I_{\mathbf{s}(t),T'}\right], \mathrm{E}\left[U^D_{\mathbf{s}(t),T'}\right]\right\}. \tag{12}$$

Since there are no further decisions following a decision to invest, $\mathrm{E}\left[U^I_{\mathbf{s}(t),T'}\right]$ can be evaluated directly, using the utility of the terminal wealth values obtainable from state $\mathbf{s}(t)$ and their associated probabilities. However, $\mathrm{E}\left[U^D_{\mathbf{s}(t),T'}\right]$ depends on whether it is optimal to invest or defer at the decision nodes at time $t+1$. Thus, the expected utilities at each decision node must be computed via backward induction.

First we evaluate (12) at the last decision nodes in the tree, which are at the time T that option expires. These nodes are available only if the investor has deferred at every node up to this point. We associate each ultimate decision node with the maximum value (12) and select the corresponding optimal action, I or D. Now select a penultimate decision node; say it is at time $T - k\Delta t$. If we use a recombining binomial tree to model the market price evolution, it has 2^k successor decision nodes at time T.[6] Each market state $\mathbf{s}(T-k\Delta t)$ has an associated decision node. Each one of its successor nodes is at a market state $\mathbf{s}^*(T)$ that is attainable from state $\mathbf{s}(T-k\Delta t)$, and has an associated probability $\pi_{\mathbf{s}^*(T)}$ determined by the state transition probability of 0.5, given that we employ the parameterisation (2).[7] Using the expected utility associated with each attainable successor node, and their associated probabilities, we compute the expected utility of the decision to defer at time $T - k\Delta t$. More generally, assuming decision nodes occur at regular time intervals, the backward induction step is:

$$\mathrm{E}\left[U^D_{\mathbf{s}(t-k\Delta t),T'}\right] = \sum_{\mathbf{s}^*(t)} \pi_{\mathbf{s}^*(t)}\mathrm{E}\left[U_{\mathbf{s}^*(t),T'}\right], \quad t = k\Delta t, 2k\Delta t, \ldots, T-k\Delta t, T. \tag{13}$$

At each decision node we compute (13) and associate the node with the optimal action and its corresponding maximum expected utility. We repeat the backward induction until we arrive at a single expected utility value associated with the node at time 0. Finally, the option value is the CE of this expected utility, less the wealth resulting from the base-case alternative, i.e. w_0. By definition, $\mathrm{CE}(w) = U^{-1}\left(\mathrm{E}[U(w)]\right)$ for any monotonic increasing utility U.

3.4 Risk Preferences

Denote by $U : \mathcal{R} \to \mathcal{R}$ the decision-maker's utility function. Previous research on decision analysis of real options, reviewed in Section 2, employs either risk-neutrality or an exponential utility function, which may be written in the form

$$U(w) = -\lambda \exp\left(-\frac{w}{\lambda}\right), \tag{14}$$

[6] The recombining assumption simplifies the computation of expected utilities at the backward induction step. However, we do not require that the binomial tree is recombining so the number of decision nodes could proliferate as we advance through the tree. Note that the state price tree will recombine if cash flows are determined by a time-varying but not state-varying dividend yield.

[7] So if the tree recombines these probabilities are $0.5^k, k0.5^k, k!/(2!(k-2)!)0.5^k, ..., k0.5^k, 0.5^k$ under the JR parameterization (2). If the CRR parameterisation is employed instead the transition probabilities for a recombining tree would be more general binomial probabilities.

where w denotes the terminal (time T') wealth of the decision maker, expressed in time 0 terms and $\lambda > 0$ denotes his risk tolerance and $\gamma = \lambda^{-1}$ is his risk aversion. Note that w is a random variable taking values determined by the decision-maker's (subjective) views on the evolution of the market price and the decisions he takes before time T'. Under (14) we have

$$\mathrm{CE}(w) = -\lambda \log\left(-\frac{\mathrm{E}[U(w)]}{\lambda}\right). \tag{15}$$

This function is frequently employed because it has special properties that make it particularly tractable (see Davis et al., 2006, Chapter 6).

1. The exponential function (14) is the only utility with a CE that is independent of the decision-maker's initial wealth, w_0.

2. The Arrow-Pratt coefficient of risk aversion is constant, as $-U''(w)/U'(w) = \gamma$. Thus, the exponential utility (14) represents decision makers with constant absolute risk aversion (CARA) and λ in (14) is the absolute risk tolerance.

3. The CE of an exponential utility is additive over independent risks. When $x_t \sim$NID(μ, σ^2) and $w_T = x_1 + \ldots + x_T$ then $\mathrm{CE}(w_T) = \mu T - (2\lambda)^{-1}\sigma^2 T$.

Properties 1 and 2 are very restricting. CARA implies that decision makers leave unchanged the dollar amount allocated to a risky investment when their initial wealth changes, indeed the investor's wealth has no influence on his valuation of the option. Property 3 implies that when cash flows are normally and independently distributed (NID) the decision-maker's risk premium for the sum of cash flows at time t is $(2\lambda)^{-1}\sigma^2 t$, so it scales with time at rate $(2\lambda)^{-1}\sigma^2$. This could be used to derive the risk-adjustment term that is commonly applied to DCF models and in the influential book by Copeland et al. (1990).[8]

Exponential utilities also have an important time-homogeneity property which, unlike the properties above, is shared by other utility functions in the HARA class: Suppose $U_t : R \to R$ on $t = 0, 1, ..., T'$, is defined with risk tolerance $\lambda_t = e^{rt}\lambda$ and set $w_t^0 = e^{-rt}w_t$. Then:

$$U_t(w_t) = -\lambda_t \exp\left(-\frac{w_t}{\lambda_t}\right) = -e^{rt}\lambda \exp\left(-\frac{w_t^0}{\lambda}\right) = e^{rt}U(w_t^0), \tag{16}$$

where $U : R \to R$ is a time-invariant exponential utility function as in (14) defined on any future value of wealth discounted to time 0. This property makes any HARA utility particularly easy to employ in decision-tree analysis. In particular, on assuming the risk tolerance is time-varying and grows exponentially at the same rate as the discount rate, we can value any future uncertainty using the constant utility function (14) applied to time 0 values. This is much easier than discounting the expected values of time-varying utility functions applied to time t values at every step in the backward induction. Our framework is not constrained to exponential preferences over NID uncertainties; indeed, this would lead to a solution where the same decision (to invest, or to defer)

[8] Setting $\mu \exp[r^a] = \mu - (2\lambda)^{-1}\sigma^2$ gives $-r^a = \log[1 - (2\lambda\mu)^{-1}\sigma^2]$, so $r^a \approx (2\lambda\mu)^{-1}\sigma^2$.

would be reached at every node in the tree, because the uncertainties faced are just a scaled version of uncertainties at any other node.

The CARA property of the exponential utility is often criticised because it does not apply to investors that change the dollar amount allocated to risky investments as their wealth changes. For this reason we allow other utility functions from the HARA class, where both absolute and relative risk aversion can increase with wealth.[9] HARA utility functions have a local relative risk tolerance λ that increases linearly with wealth at the rate η, and are defined as:

$$U(w) = -[1 + \frac{\eta}{\lambda w_0}(w - w_0)]^{1-\eta^{-1}}(1-\eta)^{-1}, \quad \text{for } w > (1-\eta^{-1}\lambda)w_0. \tag{17}$$

We consider four special cases: when $\eta = 0$ we have the exponential utility; $\eta = 1$ corresponds to the displaced logarithmic utility; $\eta = 0.5$ gives the hyperbolic utility; and $\eta = \lambda$ gives the power utility. Note that an investor's risk tolerance and its sensitivity to wealth may be defined fairly accurately using the techniques introduced by Keeney and Reiffa (1993).

4 Properties under GBM

Our real option value represents the net present value that, if received with certainty, would give a risk-averse investor the same utility value as the expected utility of the uncertain investment. Such values enable the investor to rank alternative investment opportunities and the solution specifies an optimal time to exercise. The minimum value of zero applies when the investment would never be attractive whatever its future market price. The special case of RNV, while most commonly employed in the literature, only applies to a real option that is tradable on a secondary market.

A separate appendix presents a simple example, with full illustration of the decision tree and Matlab code, to help readers fix ideas. The decision maker has an exponential utility and the transacted price is the market price of the asset. Thus, the investment decision provides a concrete example of the zero-correlation case considered by Grasselli (2011), where the opportunity to invest still carries a positive value. A second example shows that the equivalent divest decision also has a positive value.

Properties of the general model are now described under the GBM assumption (1) for market prices. Unlike risk-neutral prices, general real option values may reflect: (i) the cost of the investment relative to the companies' net asset value/initial wealth; (ii) the scheduling of decision opportunities; and (iii) the sensitivity of risk tolerance to wealth.[10]

4.1 Investment Costs and the RNV Approach

Great care should be taken when making assumptions about investment costs. In some applications – for instance, when a licence to drill for oil has been purchased and the decision concerns whether the market price of oil is sufficient to warrant exploration – a fixed-strike or predetermined cost

[9] Relative risk tolerance is expressed as a percentage of wealth, not in dollar terms. So if, say, $\lambda = 0.4$ the decision maker is willing to take a gamble with approximately equal probability of winning 40% or losing 20% of his wealth, but he would not bet on a 50:50 chance (approximately) of winning $x\%$ or losing $x\%/2$ for any $x > 0.4$.

[10] From henceforth, unless otherwise stated, we set $\tilde{r} = r$; no additional insights to the model properties are provided by using different lending and borrowing rates.

assumption could be valid. However, in many cases the investment cost is linked to the market price. The assumption (7) about the investment cost has a crucial influence not only on the value of a real option and its optimal exercise strategy, but also on their sensitivity to changes in the input parameters.

A fixed cost, where $\alpha = 1$ in (7), may be regarded as the strike of an American call option, and the value is derived from the expected utility of a call option pay-off for which the upper part of the terminal wealth distribution above the strike matters. The opposite extreme ($\alpha = 0$) focuses on the lower part of the terminal wealth distribution below the current price p_0, where the investment costs are lowest. Although log returns are similar across the whole spectrum under the GBM assumption (1), P&L is in absolute terms and it is greater in the upper part of the distribution than in the lower part. For this reason an at-the-money fixed-strike assumption yields a greater real option value than the invest-at-market-price assumption.[11] Real option values for risk-averse investors always increase with risk tolerance, and can be greater than or less than the RNV option price depending on the investor's expected return and risk tolerance.

We illustrate these properties with a typical example of an option to purchase an asset that has no associated cash flows. The current price of the asset is \$1m and the investor believes this will evolve according to (1) with μ and σ as specified in Table 1. The risk-free lending and borrowing rates are both 5%. The investment horizon is $T' = 5$ years, investors have an exponential utility with risk tolerance λ, and the initial wealth is \$1m. As $\lambda \to \infty$ we have a linear utility, giving the option value for a risk-neutral investor, and further setting $\mu = r$ gives the RNV solution. Decisions are made once per year, so we set $\Delta t = k = 1$.[12]

The investment cost takes the general form (7) with $K = \$1\text{m}$. We set $g(x) = x$ and compare investment cost at market price ($\alpha = 0$) with fixed time 0 cost ($\alpha = 1$) and a mix of fixed and variable cost ($\alpha = 0.5$). We also consider $g(x) = x/2$ for variable cost at a fraction (in this case one-half) of the market price, plus a fixed cost if $\alpha > 0$, and set $g(x) = \sqrt{x}$ for a variable cost that increases non-linearly with market price, so the price and cost are not perfectly correlated. When $g(x) = \sqrt{x}$ the variable cost is less than (greater than) the current market price p_t^- if $p_t^- > \$1\text{m}$ ($p_t^- < \$1\text{m}$).

Table 1 reports the real option values under each cost assumption, for $\lambda = 0.1, 0.5$ and 1, the real option value corresponding to the linear utility of a risk-neutral investor ($\lambda = \infty$), and the RNV price. Under the option value we report the year and any market state that the optimal investment strategy is conditional upon. For instance, $1/d$ denotes invest at time 1 provided the market price moved down between time 0 and 1, and $4/uu$ denotes invest at time 4 provided the market price moved up between time 0 and 1, and again between time 1 and 2, irrespective of later market price moves. Investment at time 0 has no market state, $4/-$ denotes invest in year 4 irrespective of the price state, and never invest is marked simply $-$.

The risk-averse option value always increases with λ, in accordance with results of Henderson (2007). Clearly, the more risk tolerant the investor, the lower the risk premium required to invest.

[11]However, this property only holds under GBM views for market prices, see Section 5.1 for a counter example under different price processes.

[12]Similar properties are evident using other real option parameters, with results available on request.

Table 1: Exponential utility option values for different risk tolerance λ and different investment costs of the form (7). Risk-neutral values (linear utility, i.e. $\lambda = \infty$) and RNV prices (linear utility, $\mu = r$). Real option parameters: $T' = 5$ years, $\Delta t = k = 1$, $K = p_0 = \$1$m, $r = \tilde{r} = 5\%$, with μ and σ specified above.

$\mu = 10\%$		$g(x)$	-	x		$\sqrt{x}$		$x/2$	
$\sigma = 20\%$		α	1	0.5	0	0.5	0	0.5	0
λ	0.1	Value	47,387	29,349	0	36,260	32,222	115,185	390,296
		Year/State	$3/uu$	$4/uuu$	-	$4/uuu$	$4/uuu$	$4/u$	4/-
	0.5	Value	199,103	117,936	43,008	166,906	129,091	323,672	561,529
		Year/State	$2/uu$	$4/uu$	0	$4/uu$	$4/uu$	$4/u$	$2/dd$
	1	Value	261,839	157,221	141,249	218,048	170,077	393,401	641,249
		Year/State	$3/uuu$	$3/uud$	0	$4/uu$	$4/uu$	$2/ud$	0
	∞	Value	363,789	283,179	283,179	306,855	283,179	533,179	783,179
		Year/State	$2/uu$	0	0	$1/u$	0	0	0
	RNV	Value	160,122	80,030	0	124,891	89,661	270,281	499,604
		Year/State	$3/uuu$	$4/uuu$	-	$4/uuu$	$4/uuu$	$4/uu$	4/-

$\mu = 15\%$		$g(x)$	-	x		$\sqrt{x}$		$x/2$	
$\sigma = 50\%$		α	1	0.5	0	0.5	0	0.5	0
λ	0.1	Value	37,277	5,374	0	34,229	6,454	36,865	148,551
		Year/State	$3/uuu$	$4/uuuu$	-	$4/uuu$	$4/uuuu$	$4/uuu$	4/-
	0.5	Value	151,743	36,014	3,023	125,942	82,806	181,581	341,763
		Year/State	$3/uuu$	$4/uuu$	$3/ddd$	$4/uuu$	$4/uuu$	$4/uu$	$3/ddd$
	1	Value	254,963	112,602	17,221	216,766	166,120	312,741	461,455
		Year/State	$4/uuu$	$4/uuu$	$2/dd$	$4/uuu$	$4/uuu$	$4/uu$	$2/dd$
	∞	Value	823,324	608,935	608,935	720,471	624,340	879,799	1,108,935
		Year/State	$3/uuu$	0	0	$2/uu$	$1/u$	$1/u$	0
	RNV	Value	376,412	186,519	0	306,003	235,593	359,591	485,545
		Year/State	$3/uuu$	$4/uuu$	-	$4/uuu$	$4/uuu$	$4/uuu$	4/-

As $\lambda \to \infty$ the value converges to the risk-neutral (linear utility) value, which is greater than the standard RNV option price in this case because $\mu > r$, otherwise it would be less than the RNV price. The risk-averse option value can be much less than the RNV price especially when risk tolerance is low, or much greater than the RNV price especially when risk tolerance is high. Indeed when $\alpha = 0$ and $g(x) = x$, i.e. the investment cost is at the market price, the RNV price is always zero, because the CE of a linear utility is the expected value of terminal wealth, and since the discounted price is always a martingale under the risk-neutral measure, CE $= w_0$. So the value of any investment opportunity will be zero in this case. By contrast, the risk-averse decision maker places a positive value on the option in this case, except when $\lambda = 0.1$, when he would choose not to invest at any market price.

The cost structure also affects the optimal investment strategy. For a finite λ the optimal time to invest is never shorter than for the risk-neutral investor, again as shown by Henderson (2007). Further, when $\alpha = 0$ and $g(x) = x$, optimal investment is never conditional on a price rise, though it may be conditional on a price fall. Investment also becomes conditional on a price fall but not a rise when $\alpha = 0$ and $g(x) = kx$, for $0 < k < 1$, except that 'never invest' is not possible, since the last period pay-off is $x - kx > 0$ so an optimal strategy always invests at or before the last period of the option.[13] But when $g(x) = \sqrt{x}$ the last period pay-off $x - \sqrt{x} > 0$ only when $x > 1$, i.e. after price rise, and optimal investment becomes conditional on up moves.

When $\alpha = 0.5$ or 1 there is a fixed cost component with an at-the-money call option pay-off, and this pay-off is positive only if the price rises. Thus, we see in Table 1 that any condition on the optimal investment strategy is for up moves in price. Finally, the RNV approach typically (but not always) gives an optimal time of investment that is never less than, and often greater than the optimal investment time for a risk-neutral investor. We conclude that the use of RNV can lead to very different option values, and a later timing for optimal investment, compared with the more general solutions obtained using our methodology for risk-neutral investors.

4.2 Effect of Decision Frequency on Real Option Value

A real option value should not decrease when there is more flexibility to make decisions over the horizon of the option. If the decision never changes as a result of including more or less decision nodes, the option value will remain unchanged. Otherwise, the option value should increase as more decisions are allowed. Having fewer decision points places additional constraints on the opportunity so it becomes less attractive to the decision maker and the value of the opportunity to invest should decrease.

Table 2: Effect of number of decisions on real option value. Exponential utility, for different λ and k. $p_0 = \$1m, T' = 5\textbf{yrs}, \Delta t = 1/12, T = T' - k\Delta t, r = \tilde{r} = 5\%, \mu = 15\%, \sigma = 50\%$.

$\alpha = 0$: invest at market price

	λ	0.2	0.4	0.6	0.8	1	∞
k	12	109.5	1,132	3,712	8,197	14,480	645,167
	6	142.2	1,344	4,309	9,212	16,089	645,167
	3	163.9	1,471	4,618	9,803	17,022	645,167
	1	176.9	1,553	4,828	10,185	17,624	645,167

$\alpha = 1$: fixed strike with present value \$1m

	λ	0.2	0.4	0.6	0.8	1	∞
k	12	49,385	115,086	174,731	223,201	263,614	881,419
	6	71,365	144,638	204,078	252,243	292,632	908,333
	3	86,062	157,289	214,375	263,019	303,834	919,322
	1	93,115	166,450	225,568	273,619	313,888	926,058

Table 2 quantifies the effect of increasing the number of decision opportunities in a real option.[14] We again consider an opportunity to invest in an asset with no associated cash flows and current market price \$1m. The investor has an exponential utility and the real option is characterised by the parameter values given in the legend to the table. Thus, there are 60 monthly steps in the

[13] Here we only display results for $k = 0.5$, but a similar comment holds for other k with $0 < k < 1$.

[14] It is important that the trees are nested, i.e. no new decision nodes are inserted as their number decreases, because only in this way does reducing the number of nodes capture the effect of placing additional constraints on decision opportunities.

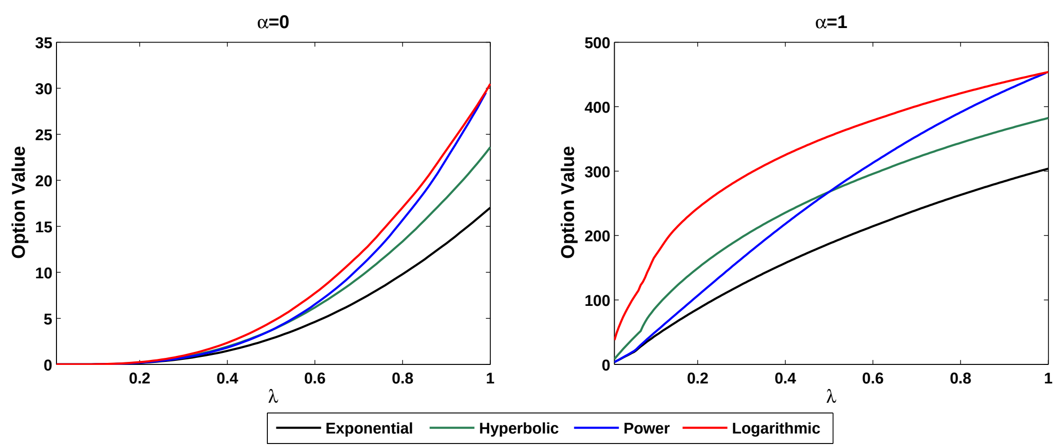
binomial tree for the market price, we place decision nodes every k steps in the backward induction algorithm (13), and the last decision is at $T = T' - k\Delta t$. For instance, if $k = 12$ the decision nodes occur only once per year and the last decision is taken at the fourth year. So that the four decision trees are nested the values considered for k are $12, 6, 3$ and 1 representing decision opportunities once per year and once every 6 months, 3 months and 1 month. The upper part of Table 2 reports the value of the option to invest at market price and the lower part reports the value of the option to invest at a fixed cost $K = \$1$m, for investors with different levels of risk tolerance λ. In both cases the option values increase when more decision nodes occur in the tree, i.e. as k decreases. The percentage increase in option value due to increased decision flexibility is greatest for investors with low λ, whereas the absolute increase in option value is greatest for high λ.

4.3 Which Utility?

The local absolute risk tolerance coefficient is λw_0 for general HARA utilities but just λ for the exponential utility. So to compare HARA utility option values with exponential utility values we set $w_0 = \$1$m so that the initial risk tolerance is the same. Figure 1 graphs the exponential, logarithmic, hyperbolic and power utility real option values as a function of λ, with $0 < \lambda \leq 1$, for the options to invest (a) at market price and (b) at a fixed time 0 cost equal to the current market price. The current asset price is $p_0 = \$1$m and the other real option characteristics are:

$$T' = 5\text{yrs}, K = \$1\text{m}, g(x) = x, \Delta t = 1/12, k = 3, r = 5\%, \mu = 15\%, \sigma = 50\%. \tag{18}$$

Figure 1: Comparison of invest option values under exponential, logarithmic, power and hyperbolic utilities as a function of risk tolerance. Real option values on the vertical scale have been multiplied by 1000 for clarity. Parameters are as in (18).



For typical risk tolerance $(0 < \lambda < 1)$ the exponential and logarithmic utility values provide lower and upper bounds for the real option values derived from other HARA utilities. For very high risk tolerance, i.e. $\lambda > 1$, hyperbolic utility values still lie between the exponential and logarithmic values, but the power utility values exceed the logarithmic values, and as λ increases further the power values can become very large indeed because the risk tolerance increases extremely rapidly with wealth.

Because of the boundary in (17) HARA utilities are not always well-behaved,unlike exponential utilities that even yield analytic solutions in some cases. But exponential utility values will be too low if the decision maker's risk tolerance increases with wealth (generally regarded as a better assumption than CARA). In that case, power utilities produce the most reliable real option values for typical values of risk tolerance, but with very high risk tolerance logarithmic or hyperbolic utility representations would be more appropriate, the former giving real option values that are greater than the latter.

4.4 Effect of Asset Price on Option Value

The higher the underlying asset price at time 0 the more variable the terminal P&L. Its effect on the option value depends on risk tolerance (and how it changes with wealth). To investigate this we compute both exponential and logarithmic real option values, supposing the time 0 asset price is either \$0.1m, \$1m or 10\$m, fixing the investor's initial wealth at \$1m and keeping all other real option characteristics fixed, as in (18).

Figure 2: Real option values under exponential and logarithmic utilities as a function of risk tolerance λ, $T' = 5$, $\Delta t = 1/12$, $k = 3$, $r = \tilde{r} = 5\%$, $\mu = 15\%$, $\sigma = 50\%$, $K = p_0 = \$0.1, \$1, \$10$m, $w_0 = 1$m, $0.1 \leq \lambda \leq 100$, CE in \$m. Both axes are in $\log_{10}$ scale.

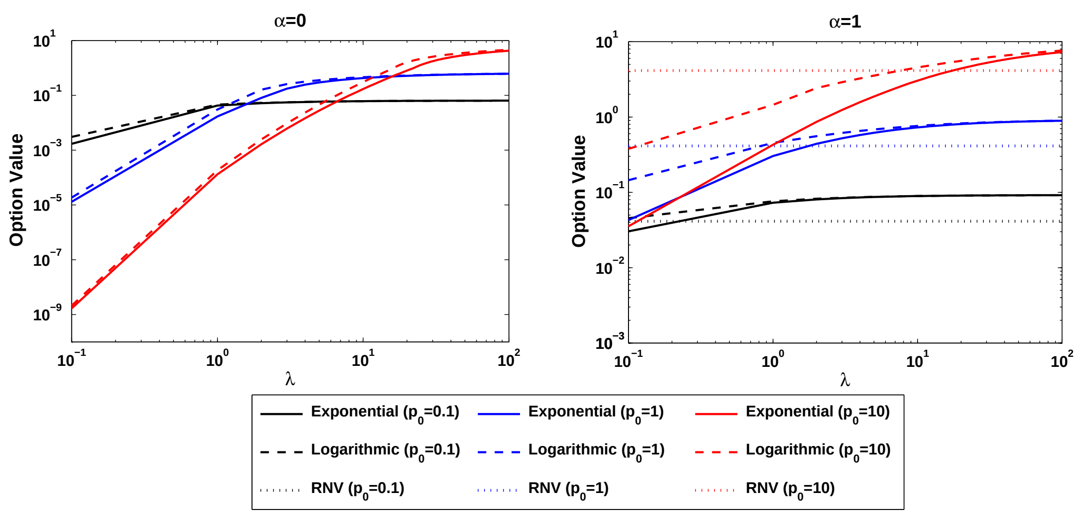


Figure 2 displays the results for different values of λ with both option value and λ represented on a base 10 log scale. We depict option values for $\alpha = 0$ (invest at market price) and $\alpha = 1$ (fixed ATM strike at p_0). The values for the high-priced asset, represented by red lines, are most sensitive to λ and the values for the low-priced asset, represented by black lines, are least sensitive to λ. Usually, the smaller (greater) the risk tolerance of the investor, the higher he ranks the option to invest in the relatively low-priced (high-priced) asset, given that the asset-price dynamics follow the same GBM process. In each case the option value converges to the value obtained for a risk-neutral investor as $\lambda \to \infty$, and this value increases with p_0. As $\lambda \to 0$ the option value decreases with p_0, except for an investor in a fixed-strike option with an exponential utility. For intermediate values of lambda there is no general rule for ranking, especially when invest cost is linked to market price. Hence, risk-neutral investors rank investments according to the underlying price, ceteris paribus, but this need not be the case risk-averse investors.

The RNV option price is is zero when $K_t = p_t^-$ ($\alpha = 0$) and when K_t is fixed at \$1m ($\alpha = 1$) it is marked by the dotted horizontal lines in Figure 2. In this case it is less than the risk-neutral subjective value because $\mu > r$, but would exceed that value if $\mu < r$. The exponential option values (solid lines) never exceed the logarithmic utility values (dashed lines) for the same initial risk tolerance, and they are much lower for the fixed-strike option to invest in a high-priced asset by a highly risk-averse investor.

4.5 Sensitivity to μ and σ

When the decision maker has low confidence in his views about the price process his subjective values for μ and σ may be highly uncertain. We describe the sensitivity of real option values to the expected returns and risks of the investment. The RNV principle yield option prices that increase with volatility under the fixed time 0 cost assumption, yet the DCF approach implies the opposite. In our approach option values can decrease or increase with volatility depending on the cost structure and the utility.

Figure 3: Value of an investment option under exponential and logarithmic utilities as a function of the investor's subjective views on expected return μ and volatility σ, for $\lambda = 0.2$. CE value in \$m for $p_0 = \$1\text{m}, w_0 = \$10\text{m}, r = 5\%, T' = 5, \Delta t = 1/12, k = 3$.

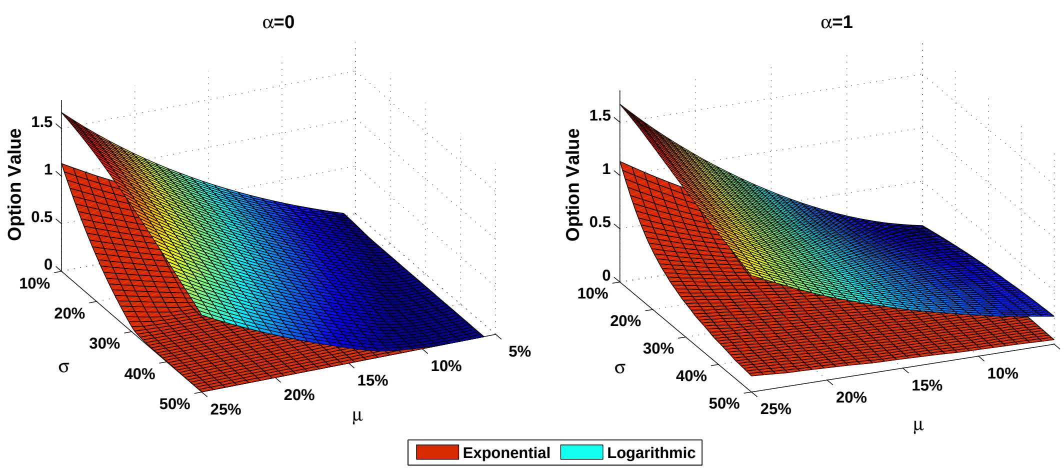


Figure 3 depicts the values of a real option to invest as a function of these expected return and volatility of the GBM price process, with other parameters fixed as stated in the legend. When $\alpha = 0$ the option value always decreases as uncertainty increases, for any given expected return, due to the risk aversion of the decision maker. When there is high uncertainty (σ greater than about 30%) the exponential utility values are zero, i.e. the investment opportunity is valueless as the price would never fall far enough (in the decision maker's opinion) for investment to be profitable. By contrast, the logarithmic utility always yields a positive value provided the expected return is greater than about 10%, but again the investor becomes more likely to defer investment as the volatility increases. Indeed, the option values are monotonically decreasing as σ for every μ, and monotonically increasing with μ for every σ. Volatility sensitivity can be different for the fixed-strike option, $\alpha = 1$. For instance, with the logarithmic utility the option values can decrease as

σ increases, when the expected return is low. In particular, when $\mu = r$ the logarithmic option values increase monotonically with volatility, as they also do in the RNV case.

The sensitivity of the option value to μ and σ also decreases as risk tolerance increases. To illustrate this we give a simple numerical example, for the case $\alpha = 0$ and an exponential utility. Suppose $\mu = 20\%$. If $\lambda = 0.2$ the option value is \$1,148 when $\sigma = 40\%$ and \$241,868 (almost 211 times larger) when $\sigma = 20\%$. When $\lambda = 0.8$ the option value is \$167,716 when $\sigma = 40\%$; but now when $\sigma = 20\%$ it only increases by a multiple of about 4, to \$713,812. Similarly, fixing $\sigma = 20\%$ but now decreasing μ from 20% to 10% the value changes from \$241,867 to \$109 (2,210 times smaller) for $\lambda = 0.2$, but from \$713,812 to \$113,959 (only about 6 times smaller) when $\lambda = 0.8$. Hence, the option value's sensitivities to μ and σ are much greater for low levels of risk tolerance. Similar effects are present with the logarithmic utility but they are much less pronounced.

5 Cash Flows and non-GBM Price Processes

Given the generality and flexibility of our approach it has applications to a wide range of investment or divestment decisions. Here we use examples of decision problems commonly encountered by private real-estate companies or housing trusts, from large-scale land development to individual residential property transactions. Many real estate real options have been considered in the literature, including: the option to abandon, e.g. Smith (2005), the option to defer a land development, e.g. Brandão and Dyer (2005) and Brandão et al. (2005), and the option to divest, e.g. Brandão et al. (2008) and Smith (2005). But all these papers employ the RNV approach.

Our model permits risk-averse private companies, publicly-funded entities, or individuals to compute a real option value that is tailored to the decision maker, and which could be very different from the risk-neutral price obtained under the standard but (typically) invalid assumption of perfectly-hedgeable risk and fixed costs. This is significant because such decisions can have profound implications for the decision maker's economic welfare. For instance, an individual's investment in housing may represent a major component of his wealth and should not be viewed simply in expected net present value terms, nor should all uncertainties be based on systematic risk because they are largely unhedgeable. Private companies typically generate returns and risks that have a utility value that is specific to the owners' outlook. Similarly, charitable and publicly-funded entities may have objectives that are far removed from wealth maximisation under risk-neutrality.

First we consider an option to purchase a residential property when the investor's views are captured by a process (3) where a long period of property price momentum could be followed by a crash. Then we analyse real options for investors that believe in a mean-reverting price process (4). After this we focus on the inclusion of cash flows, considering both positive cash flows for modelling buy-to-let real options and negative cash flows for modelling construction costs in an on-going development.

5.1 Property Price Recessions and Booms

Many property markets are subject to booms and recessions. For example, the average annualised return computed from monthly data on the Vanguard REIT exchanged traded fund (VQN) from

January 2005 to December 2006 was 21% with a volatility of 15%. However, from January 2007 to December 2009 the property market crashed, and the VQN had an average annualised return of −13% with a volatility of 58%. But from January 2010 to June 2011 its average annualised return was 22% with volatility 24%. Clearly, when the investment horizon is several years a property investor may wish to take account of both booms and busts in his views about expected returns. We now give a numerical example of an option to purchase a residential property, i.e. with zero cash flows, under such scenarios.

Consider a simple boom-bust scenario over a 10 year horizon. The expected return is negative, $\mu_1 < 0$ for the first n years and positive, $\mu_2 > 0$ for the remaining $10-n$ years. Following the above observations about VQN we set $\mu_1 = -10\%, \sigma_1 = 50\%$ and $\mu_2 = 10\%, \sigma_2 = 30\%$. We suppose decisions are taken every quarter with $\Delta t = 0.25$ and set $r = 5\%$ in the price evolution tree. The real option values given in Table 3 are for investors having exponential utility, with varying levels of risk tolerance between 0.2 and 1. The property price recession is believed to last $n = 0, 2, 4, 6, 8$ or 10 years.[15]

Table 3: Effect of a time-varying drift for the market price, with downward trending price for first n years followed by upward trending price for remaining $10-n$ years. Exponential utility with different levels of risk tolerance, with $\lambda = \infty$ corresponding to the risk-neutral (linear utility) value. Real option values in bold are the maximum values, for given λ. $p_0 = \$1$ million, $w_0 = \$1$ million, $T' = 10$, $\Delta t = 0.25$, $T = T' - \Delta t$, $r = 5\%$, $\mu_1 = -10\%$, $\mu_2 = 10\%$, $\sigma_1 = 50\%$ and $\sigma_2 = 20\%$.

$\alpha = 0$

	n	0	2	4	6	8	10
λ	0.2	866	382,918	**727,922**	438,482	174,637	0
	0.8	211,457	1,230,035	**2,297,909**	1,148,695	370,354	0
	1.0	266,752	1,349,295	**2,552,434**	1,300,724	406,567	0
	∞	648,173	2,234,361	4,777,950	**5,308,672**	1,045,800	0

$\alpha = 1$

	n	0	2	4	6	8	10
λ	0.2	164,721	391,060	**552,578**	194,371	51,040	5,022
	0.8	392,866	961,279	**1,704,007**	692,529	159,663	14,943
	1.0	432,280	1,064,416	**1,938,157**	821,733	186,257	17,191
	∞	740,493	1,989,551	4,339,659	**4,860,376**	841,706	50,787

When $n = 0$ the investor expects the boom to last the entire period, but since $\sigma_2 = 30\%$ there is still uncertainty about the evolution of the market price, and with $\mu_2 = 10\%$ the price might still fall. The case $n = 10$ corresponds to the view that the market price could fall by $\mu_1 = -10\%$ each year for the entire 10 years, but with $\sigma_2 = 50\%$ this view is held with considerable uncertainty. At these two extreme values for n we have a standard GBM price process, so for any given λ the invest-at-market-price option has a lower value than the fixed-strike option. However, for intermediate values of n the invest-at-market-price option often has a higher value than the fixed-strike option. This ordering becomes more pronounced as n increases, because the investment cost

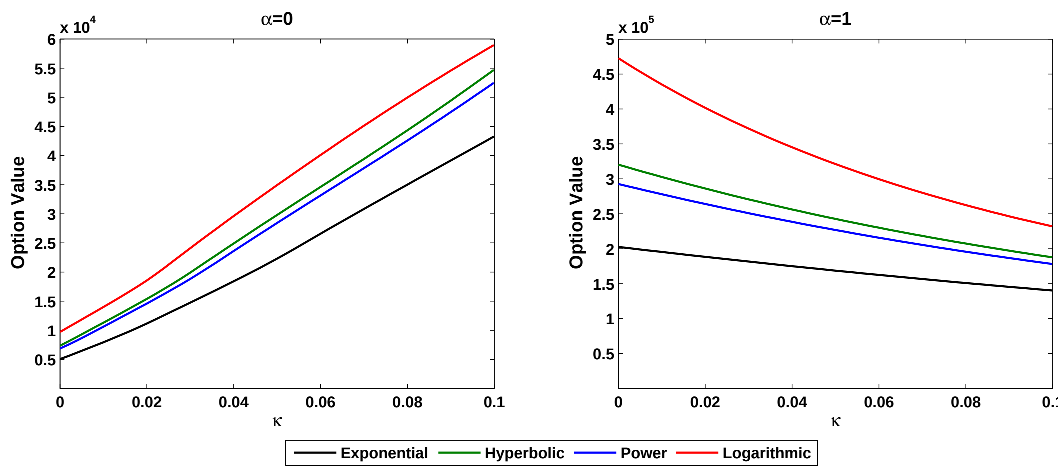
[15] Results for the divest option, or for the invest option under different utilities are not presented, for brevity, but are available on request. The qualitative conclusions are similar.

decreases when $\alpha = 0$ and, provided that the price rises after the property is purchased, the profits would be greater than they are under the fixed-cost option. For risk-averse investors the maximum value arises when the length of the boom and bust periods are approximately the same. However, a risk-neutral investor would place the greatest value on a property location where the recession period lasts longer ($n = 6$ in our example).

5.2 Property Price Mean-Reversion

Momentum is not the only well-documented effects in property prices; they may also mean-revert over a relatively short horizon. We now investigate how an investor's belief in price mean-reversion would influence his value of the option to invest in a residential property. We suppose the market price follows the OU process (4) and we set $\bar{p} = p_0$ for simplicity. To model the effect of mean reversion on the option value we employ the NR parameterisation (5), allowing κ to vary between 0 and 0.1, the case $\kappa = 0$ corresponding to GBM and $\kappa = 0.1$ giving the fastest characteristic time to mean revert of 10 time-steps (so assuming these are quarterly this represents 2.5 years).[16] The other parameters are fixed, as stated in the legend to Figure 4, which displays the real option values for $\alpha = 0$ and $\alpha = 1$ as a function of κ.

Figure 4: Comparison of option values under HARA utilities with respect to mean-reversion rate κ. $w_0 = \$1$m, $r = 5\%$, $k = 1$ and $\lambda = 0.4$ $T' = 10$, $\Delta t = 1/4$, $K = p_0 = \$1$m, $T = T' - \Delta t$, $\sigma = 40\%$. Characteristic time to mean-revert $\phi = \Delta t/\kappa$ in years, e.g. with $\Delta t = 1/4$ then $\kappa = 0.02 \to \phi = 12.5$yrs, $\kappa = 0.1 \to \phi = 2.5$yrs.



Increasing the speed of mean-reversion has a similar effect to decreasing volatility. Hence, fixed-cost option values can decrease with κ, as they do on the right graph of Figure 4 ($\alpha = 1$) especially when the investor has logarithmic utility. By contrast, the invest-at-market-price ($\alpha = 0$) option displays values that increase with κ. Note that for fixed κ these values could *in*crease with σ, due to the positive effect of σ in the local drift (5d).[17]

[16] Lower values for κ have slower mean-reversion, e.g. $\kappa = 0.02$ corresponds to a characteristic time to mean revert of $\phi = 0.02^{-1}/4 = 12.5$ years if time-steps are quarterly.

[17] This effect is only evident for values of κ below a certain bound, depending on the utility function and other real option parameters. For the parameter choice of Figure 4 the invest-at-market-price option values have their usual

Now consider a risk-neutral investor who wishes to rank two property investment opportunities, A and B. Both have mean-reverting price processes but different κ and σ: property A has a relatively rapid mean-reversion in its price ($\kappa = 1/10$, $\phi = 2.5$ years) with a low volatility ($\sigma = 20\%$) and property B has a relatively slow mean-reversion in its price ($\kappa = 1/40$, $\phi = 10$ years) with a higher volatility ($\sigma = 40\%$). Such an investor would have a great preference for property B since he is indifferent to the high price risk and has regard only for the local drift. Given the slow rate of mean-reversion, he sees only the possibility of a sharp fall in price – at which point he would invest – followed by a long upward trend in its price. Indeed, assuming $p_0 = \bar{p} = \$1$m the risk-neutral value of option B is \$797,486, but the corresponding value for option A is only \$98,077.

However, once we relax the risk-neutral assumption the ranking of these two investment opportunities may change, depending on the utility of the investor. The real option values for risk-averse investors with high risk tolerance ($\lambda = 0.8$) or low risk tolerance ($\lambda = 0.4$) are shown in Table 4. In each case the greater option value is depicted in bold. This shows that an investor with a logarithmic utility would also prefer property B, as would an investor with a relatively high risk tolerance ($\lambda = 0.8$) and a hyperbolic or a power utility. In all other cases the investor would prefer property A.

Table 4: Comparison of invest option values for property A ($\kappa = 1/10$, $\sigma = 20\%$) and proporty B ($\kappa = 1/40$, $\sigma = 40\%$) with $\lambda = 0.4$ or 0.8 . All other parameter values are the same as in Figure 4. For each utility we highlight the preferred property in bold.

λ	0.4		0.8		∞	
	A	B	A	B	A	B
Exponential	**30,421**	12,952	**52,732**	43,947	98,077	**797,486**
Hyperbolic	**33,526**	17,563	55,230	**58,909**		
Power	**33,045**	16,651	56,365	**68,051**		
Logarithmic	19,890	**21,260**	56,986	**73,131**		

5.3 Positive Cash Flows: Buy-to-Let Options

Short-horizon decision trees for the invest and divest options on an investment that yields positive cash flows are depicted in Figures 5 and 6. A typical real-esate investment with regular positive cash flows is a buy-to-let residential property or office block, or a property such as a car park where fees accrue to its owner for its usage. Rents, denoted $x_{\mathbf{s}(t)}$ in the trees, are captured using a positive dividend yield defined by (6) that may vary over time. Even a constant dividend yield would capture rents that increase/decrease in line with the market price. We suppose cash flows not re-invested, otherwise we could employ the cum-dividend price approach that has previously been considered. Instead, cash flows are assumed to earn the risk-free rate, as to suppose they are invested in another risky project would introduce an additional source of uncertainty which is beyond the scope of this paper.

Each time a cash flow is paid the market price jumps down from p_t^+ to $p_t^- = p_t^+ - x_t$. Between

negative sensitivity to σ once κ exceeds approximately 2, where the characteristic time to mean revert is 1/8th of a year or less. Detailed results are not reported for lack of space, but are available from the authors on request.

payments the decision maker expects the discounted market price to grow at rate $\mu - r$, and based on the discretisation (2) we have $p_{t+1}^{+} = up_t^{-}$ or $p_{t+1}^{+} = dp_t^{-}$ with equal probability. The terminal nodes of the tree are associated with the increment in wealth $w - w_0$ where the final wealth w is given (8) for the option to invest in Figure 5, and by (10) for the option to divest in Figure 6, now setting CF$= x$.

The decision tree in Figure 5 is now used to rank the options to buy two different buy-to-let properties. Both properties have current market value $p_0 = 1$, the initial wealth of the investor is $w_0 = 1$ and the risk-free rate $r = 5\%$. In each case rents are paid every six months, and are set at a constant percentage δ of the market price at the time the rent is paid. But the investor has different views about the future market price and rents on each property, as specified in the first pair of columns in Table 5. Similarly, Figure 6 is used to rank the options to sell two different buy-to-let properties, with views on market prices and rents as specified in the second pair of columns in Table 5. Note that the CE values given for the divest option include the expected utility value of capital gains on the property itself, as well as the expected utility value of the opportunity to divest. In each case the value of the preferred property is marked in bold.

Table 5: Columns 2 and 3 compare the values of 2 real options, each to purchase a buy-to-let property based on the decision tree shown in Figure 5. Columns 4 and 5 compare the values of 2 real options, each for buy-to-let property with the option to sell, based on the decision tree in Figure 6. For the decision maker, in each case $\lambda = 0.4$, $r = 5\%$, $w_0 = \$1$ million and for each property $p_0 = \$1$ million. The decision maker's beliefs about μ, σ and δ depend on the property's location. For each utility we highlight in bold the preferred location for buying (or selling) the property.

	Invest		Divest	
σ	40%	25%	30%	20%
μ	15%	10%	15%	10%
δ	10%	10%	20%	10%
Exponential	289	**316**	**6,775**	6,052
Hyperbolic	**348**	339	**6,610**	6,103
Power	**340**	336	**7,283**	6,296
Logarithmic	**358**	345	4,286	**5,330**

Again the form assumed for the utility function investor has material consequences for decision making. An investor with exponential (CARA) utility would prefer the option to buy the second property, whilst investors with any of the other HARA utilities having the same risk tolerance at time 0 would prefer the option to buy the first property. Similarly, regarding the divest real option values of two other properties, both currently owned by the decision maker, all investors except those with a logarithmic utility would favour selling the first property.

5.4 Negative Cash Flows: Buy-to-Develop Options

Setting a negative dividend yield is a straightforward way to capture cash that is paid into the land or property to cover development costs. But there are other important differences between the buy-to-develop and buy-to-rent option above. In the buy-to-develop case there are no cash flows until the land or property is purchased, and thereafter these cash flows are included in the market price. Hence, the market price following an invest decision is cum-dividend and prior to this the market price evolves as in the zero cash flow case. Also, now the investment horizon T' is path dependent because it depends on the time of the investment (e.g. it takes 2 years to develop the property after purchasing the land).

A simple decision tree for the buy-to-develop option is depicted in Figure 7, in which the development cost is $y_{\mathbf{s}(t)} > 0$ and $\tilde{p}_t$ is the (cum-dividend) market price. The option maturity T is 2 periods, and so is the development time, so T' varies from 2 to 4 periods depending on the time of investment. To keep the tree simple we suppose that development costs are paid only once, after 1 period, to allow for planning time. For example, consider the node labelled D_u that arises if the investor does not purchase the land or property at $t = 0$ and subsequently the market price moves up at $t = 1$. A decision to invest at this time leads to four possible P&L's. For instance, following the dotted red lines, if the price moves up again at $t = 2$ the development cost at this time is y_{uu}, based on the market price of uup_0. But if the price subsequently moves down at $t = 3$, the terminal value of the property is $\tilde{p}_{1,uud} = d(uup_0 + y_{uu})$ and the costs are the sum of the price paid for the land and the development cost, i.e. $up_0 + y_{uu}$.

Table 6: Value comparison of real options to buy two different properties for development based on decision tree shown in Figure 7. Here $\lambda = 0.4$, $r = 5\%$ and $w_0 = \$1$ million. Each property has $p_0 = \$1$ million but the decision maker's views on μ, σ and development costs δ differ for each property as shown in the table. The preferred option is indicated by the value in bold.

	A	B
σ	25%	15%
μ	10%	35%
δ	20%	40%
Exponential	**289**	182
Hyperbolic	299	**326**
Power	315	**350**
Logarithmic	**41**	0

Table 6 displays some numerical results for the decision tree in Figure 7, reporting the value of two options to buy-to-develop land, each with initial market price \$1 million and $r = 5\%$, but the options have different μ, σ and development costs δ. For each option we suppose the development takes one year in total with the costs paid six months after purchase. The investors all have initial wealth \$10 million and $\lambda = 0.4$.

In general, a higher development cost for given μ and σ decreases the buy-to-develop option value, and the investor becomes more likely to defer investment until the market price falls. But the option value also increases with μ and decreases with σ. We find that option A is preferred by an investor with an exponential or logarithmic utility whereas option B is preferred by an investor with a hyperbolic or a power utility. Hence, different investors that have identical wealth, share the same initial risk tolerance, and hold the same views about development costs and the evolution of market prices may again rank the values of two land-development options differently, just because their risk tolerance has different sensitivity to changes in wealth.

6 Summary and Conclusion

This paper introduces a general and flexible decision-tree framework for valuing real options which can encompass a great variety of real-world applications. In addition to the predetermined strike real options that are tradable in complete markets – the RNV assumption that is most commonly employed in the literature – we model options to invest at a cost that has both fixed and variable components. The variable component is determined by (but not necessarily perfectly correlated with) the market price of the investment asset, and the market for this asset need not be complete. The decision maker may be risk-neutral or have a utility in the HARA class, and his belief in the evolution of the market price is not constrained to be a standard GBM. Many price processes are possible and we have used real-estate investments and divestments to illustrate our model when the price is believed to follow a regime-switching GBM or a mean-reversion, and also when investments have positive or negative cash flows.

The real option price that is obtained using standard RNV assumptions can be very much greater than or less than the value that would be found using a more realistic assumptions about investment costs in an incomplete market, and very often the RNV approach would specify a later investment time. After demonstrating this we answer several important questions relating to real options that have not previously been addressed, finding: (i) The assumption about the investment cost, whether it is predetermined or stochastic, has a significant influence on the real option value. The predetermined-strike assumption can significantly over-estimate the value of a real option when the more-appropriate assumption is that the investment cost is positively related to the market price, or has both a fixed and variable components; (ii) It is important to account for the flexibility of the decision-making process in real option analysis because the real option value increases with the frequency of decision opportunities; (iii) The price of the investment relative to the decision maker's wealth matters: the smaller (greater) the risk tolerance of the investor, the higher he ranks the option to invest in a relatively low-priced (high-priced) asset, given that the asset-price dynamics follow the same GBM process; (iv) Under the GBM assumption the sensitivity of a fixed-strike real option to the volatility of the underlying asset price may be positive, as in the RNV approach, whereas the invest-at-market-price real option always has a negative sensitivity to the asset-price volatility; (v) The decision maker's ranking of different real options depends on how his risk tolerance changes with wealth.

References

Benaroch, M., R. J. Kauffman. 2000. Justifying electronic banking network expansion using real options analysis. *MIS Quarterly* **24** 197–225.

Boer, P. F. 2000. Valuation of technology using "real options". *Research Technology Management* **43** 26–30.

Bowman, E. H., G. T. Moskowitz. 2001. Real options analysis and strategic decision making. *Organization Science* **12** 772–777.

Brandão, L. E., J. S. Dyer. 2005. Decision analysis and real options: A discrete time approach to real option valuation. *Annals of Operations Research* **135**(1) 21–39.

Brandão, L. E., J. S. Dyer, W. J. Hahn. 2005. Using binomial decision trees to solve real-option valuation problems. *Decision Analysis* **2** 69–88.

Brandão, L. E., J. S. Dyer, W. J. Hahn. 2008. Response to comments on brandão et al. (2005). *Decision Analysis* **2** 103–109.

Capozza, D. R., G. A. Sick. 1991. Valuing long-term leases: the option to redevelop. *Journal of Real Estate Economics and Finance* **4** 209–233.

Chance, D. M. 2008. *Convergence of the Binomial to the Black-Scholes Model*. E. J. Ourso College of Business, Lousiana State University.

Copeland, T., V. Antikarov. 2001. *Real Options: A Practitioner's Guide*. Texere, New York.

Copeland, T., T. Koller, J. Murrin. 1990. *Valuation: Measuring and Managing the Value of Companies*. John Wiley & Sons.

Cox, J. C., S. A. Ross, M. Rubinstein. 1979. Option pricing: A simplified approach. *Journal of Financial Economics* **7**(3) 229–263.

Davis, M. H. A., Y. Kabanov, R. Lipster, J. Stoyanov. 2006. Optimal hedging with basis risk. *Stochastic Calculus to Mathematical Finance: The Shiryaev Festschrift* **1** 169–187.

Evans, J., V. Henderson, D. G. Hobson. 2008. Optimal timing for an indivisible asset sale. *Mathematical Finance* **18**(4) 545–567.

Grasselli, M. R. 2011. Getting real with real options: A utility-based approach for finite-time investment in incomplete markets. *Journal of Business Finance and Accounting* **38**(5-6) 740–764.

Grenadier, S. R. 1996. The strategic exercise of options: development cascades and overbuilding in real estate markets. *The Journal of Finance* **51** 1653–1679.

Harrison, J. M., D. M. Kreps. 1981. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* **20** 381–408.

Henderson, V. 2002. Valuation of claims on nontraded assets using utility maximization. *Mathematical Finance* **12**(4) 351–373.

Henderson, V. 2007. Valuing the option to invest in an incomplete market. *Mathematics and Financial Economics* **1**(2) 103–128.

Henderson, V., D. G. Hobson. 2002. Real options with constant relative risk aversion. *Journal of Economic Dynamics and Control* **27**(2) 329–355.

Jarrow, R., A. Rudd. 1982. Approximate option valuation for arbitrary stochastic processes. *Journal of Financial Economics* **10** 347–369.

Kasanen, E., L. Trigeorgis. 1995. *Real Options in Capital Investment: Models, Strategies, and Applications*. 47-68. Praeger.

Keeney, R., H. Reiffa. 1993. *Decisions with Multiple Objectives*. Cambridge University Press, New York. Original edition published by Wiley, New York, 1976.

Kogut, B. 1991. Joint ventures and the option to expand and acquire. *Management Science* **37** 19–33.

Mason, S. P., R. C. Merton. 1985. *Recent Advances in Corporate Finance*. 7-54. Irwin, Boston, MA.

McDonald, R., D. Siegel. 1986. The value of waiting to invest. *Quarterly Journal of Economics* **101**(4) 707–727.

Merton, R. C. 1969. Lifetime portfolio selection under uncertainty: the continuous time case. *Review of Economics and Statistics* **51** 247–257.

Merton, R. C. 1971. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory* **3** 373–413.

Miao, J., N. Wang. 2007. Investment, consumption, and hedging under incomplete markets. *Journal of Financial Economics* **86**(3) 608–642.

Mossin, J. 1968. Optimal multiperiod portfolio policies. *Journal of Business* **41** 215–229.

Myers, S. C. 1977. Determinantes of corporate borrowing. *Journal of Financial Economics* **5**(2) 147–175.

Nelson, D. B., K. Ramaswamy. 1990. Simple binomial processes as diffusion approximations in financial models. *The Review of Financial Studies* **3** 393–430.

Panayi, S., L. Trigeorgis. 1998. Multi-stage real options: the cases of information technology infrastructure and international bank expansion. *The Quarterly Review of Economics and Finance* **38** 675–682.

Patel, K., D. Paxson, T. F. Sing. 2005. A review of the practical uses of real property options. RICS Research Paper Series, 5:1.

Quigg, L. 1993. Empirical testing of real option-pricing models. *The Journal of Finance* **48** 621–640.

Shackleton, M. B., R. Wojakowski. 2007. Finite maturity caps and floors on continuous flows. *Journal of Economic Dynamics and Control* **31**(12) 3843–3859.

Smit, H. T. J. 1996. The valuation of offshore concessions in the netherlands. *Financial Management* **26**(2) 5–17.

Smit, H. T. J., L. A. Ankum. 1993. A real option and game-theoretic approach to corporate investment strategy under competition. *Financial Management* **22**(3) 241–250.

Smith, J. E. 2005. Alternative approaches for solving real-options problems. *Deicison Analysis* **2**(2) 89–102.

Smith, J. E., K. F. McCardle. 1998. Valuing oil properties: Integrating option pricing and decision analysis approaches. *Operations Research* **46**(2) 198–217.

Smith, J. E., R. F. Nau. 1995. Valuing risky projects - option pricing theory and decision-analysis. *Management Science* **41**(5) 795–816.

Triantis, A. J., J. E. Hodder. 1990. Valuing flexibility as a complex option. *The Journal of Finance* **45** 549–565.

Trigeorgis, L. 1993. The nature of option interactions and the valuation of investments with multiple real options. *The Journal of Financial and Quantitative Analysis* **28** 1–20.

Trigeorgis, L. 1996. *Real Options: Managerial Flexibility and Strategy in Resource Allocation*. The MIT Press, Cambridge, MA.

Trigeorgis, L., P. Mason. 1987. Valuing managerial flexibility. *Midland Corporate Finance Journal* **5** 14–21.

Yeo, K. T., F. Qiu. 2002. The value of management flexibility - a real option approach to investment evaluation. *International Journal of Project Management* **21** 243–250.

Figure 5: Decision tree depicting the option to invest in a property that pays rents, $x_{\mathbf{s}(t)}$. $T' = 3, T = 2, k = 1$. Terminal nodes labelled with P&L $(w - w_0)$, given by (8) following decision to invest (I) and 0 otherwise, with CF $= x$.

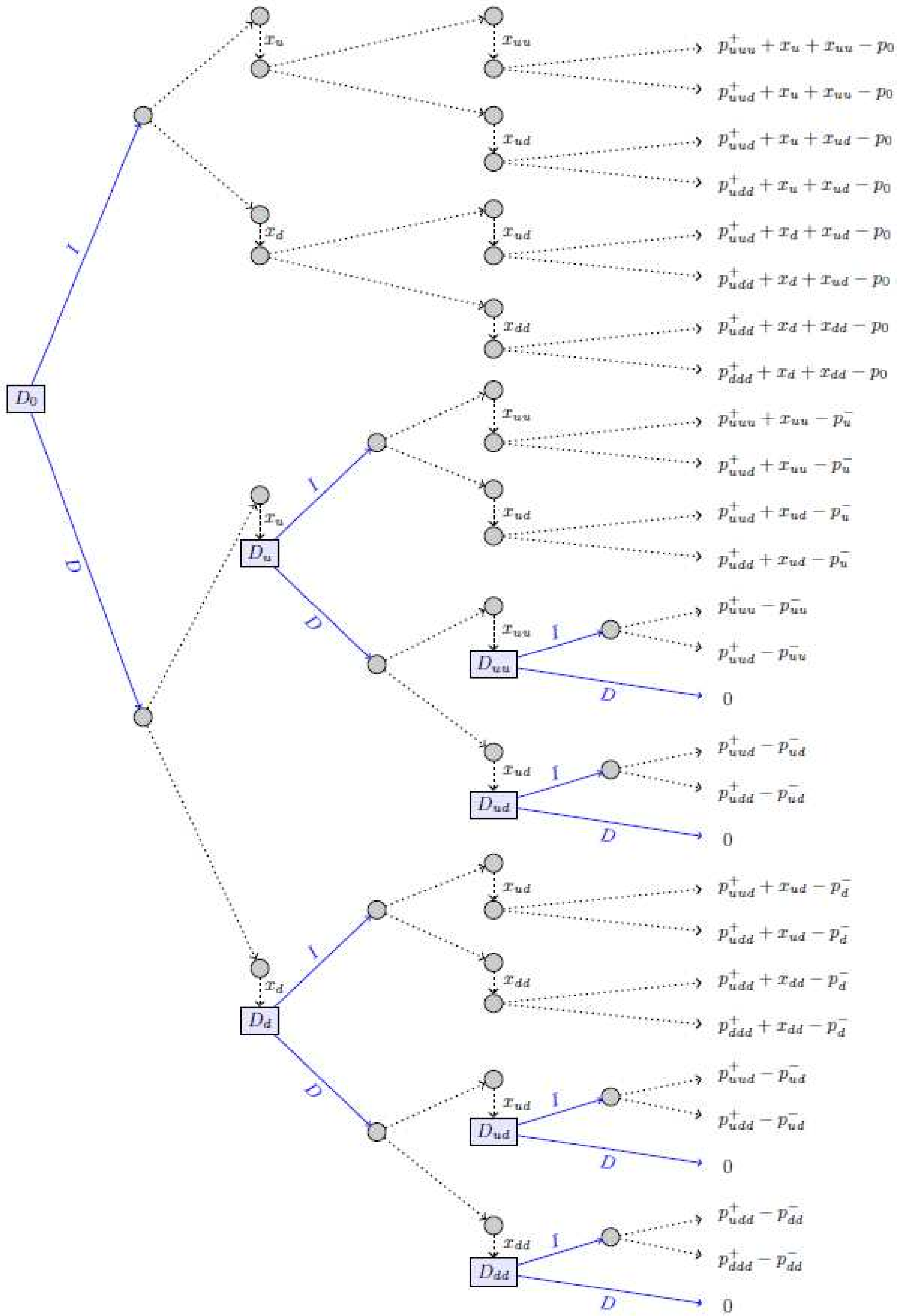


Figure 6: Decision tree depicting the option to sell a property that pays rents, $x_{\mathrm{s}(t)}$. $T' = 3, T = 2, k = 1$. Terminal nodes labelled with P&L $(w - w_0)$, given by (10) with CF $= x$ if the owner remains invested (R), or if the owner sells the property (S) by the difference between the selling price and initial price.

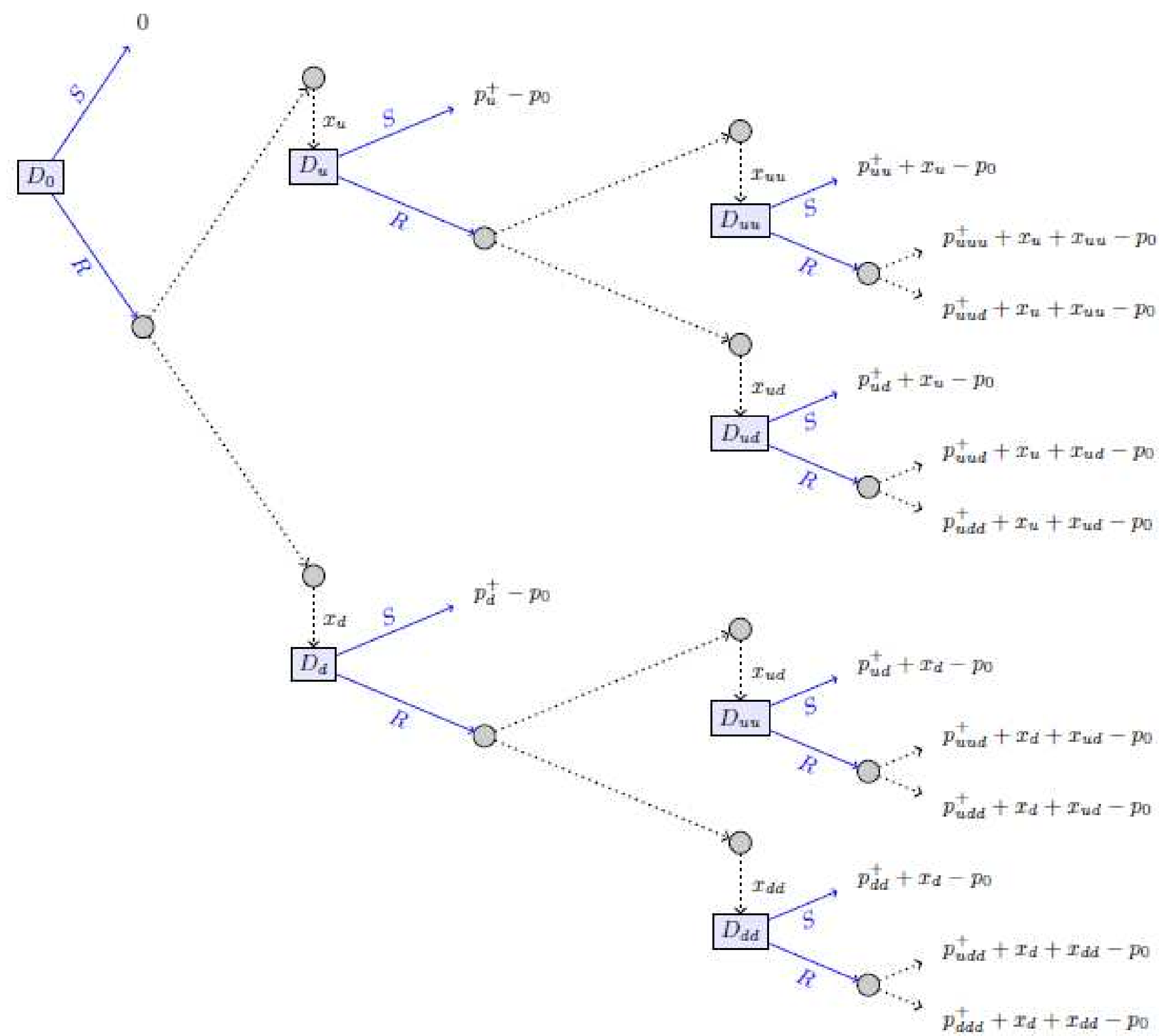


Figure 7: Decision tree depicting the option to invest in land for development. If the land is acquired (I) the development takes 2 periods and development costs occur only after the first period. Terminal nodes are associated with the P&L, $w - w_0$ resulting from the decision.