Extended MAD for Real Option Valuation

A Case Study of Abandonment Option

Carol Alexander* Xi Chen† Charles Ward‡

Abstract

This paper extends the marketed asset disclaimer approach for real option valuation. In sharp contrast to the dominant real option valuation that assumes a stochastic process for an investment's capital value, this paper demonstrates the valuation of a real option assuming that cash flow follows a stochastic process. We show that this method is at least equally effective and sometimes more intuitive. We note that, in a discounted cash flow (DCF) framework, certain constraints must be met, and assuming capital value as a geometric Brownian motion (GBM) is compatible with simultaneously assuming cash flow as a GBM, We clarify the above argument with a simple textbook-standard case study.

Key words: real option, decision making, investment opportunity, geometric Brownian motion, abandonment option, marketed asset disclaimer, change of measure, lease, rental value, market utility, risk tolerance, risk aversion

*Professor of Finance, School of Business, Management and Economics, University of Sussex; Phone: +44 (0)1273 873950, Email: c.o.alexander@sussex.ac.uk.

†PhD Student in ICMA Centre, Henley Business School, University of Reading; Phone: +44 (0)118 378 8239, Email: x.chen@icmacentre.ac.uk.

‡Professor of Finance, ICMA Centre, Henley Business School, University of Reading; Phone: +44 (0)118 378 8239, Email: c.ward@icmacentre.ac.uk.

1 Introduction

This paper first extends the marketed asset disclaimer (MAD) approach proposed by Copeland and Antikarov (2001) and show that this approach can be applied regardless of which assumption(s) about the gross investment value or investment cash flows are made. We further address that, certain conditions must be met when we value a real option based on a DCF analysis on the base investment.

We illustrate the extended MAD approach in a numerical case reported by Azevedo-Pereira (2001) in Howell et al. (2001), hereafter referred to the Campeiro case. In particular, we value a real option to abandon a real estate lease that is not associated with a market price. In contrast to the original MAD approach which values the abandonment option based on assumptions about the gross lease value evolution, we take cash flows (rents) as the process that is driving the uncertainty in the gross lease value. We then explain how to derive a value for the abandonment option that is consistent with the value obtained under the original MAD approach.

Furthermore, when valuing a real option based on a DCF analysis of the base investment, there are several parameters in the valuation that we need approximate. An error in the real option value would be introduced if these approximations are flawed. For instance, irrespective of whether it is the investment value or cash flow that is taken as the driving source of uncertainty, we have to change from the physical ($\mathbb{P}$) measure for the investment valuation to the risk-neutral ($\mathbb{Q}$) measure for the real option valuation. This means that we need to derive the risk-neutral probability under which the gross investment value follows a $\mathbb{Q}$-martingale. The consistent valuation of the underlying investment in the $\mathbb{P}$ measure with the real option valuation in the $\mathbb{Q}$ measure presents some complexities which have been ignored by previous papers that consider the MAD approach. Indeed, Azevedo-Pereira applies a $\mathbb{Q}$ measure for the abandonment option valuation that is not consistent with the $\mathbb{P}$ measure for the lease value.*

In the following, Section 2 briefly describes the standard MAD approach and then explains our extension. We show that the real option can be valued using this method under any assumption about the investment value or cash flows. In order to numerically illustrate our findings, we introduce the Campeiro case in Section 3 and value the real option consistently in Section 4. Section 5 numerically and intuitively explain the problem involved in other ways of approximating (a) the risk-adjusted discount rate for cash flows if missing; (b) the drift of the gross investment value process if assumed to be a GBM; and (c) the dividend yield derived from the

DCF analysis on the base investment for the later valuation of the attached real option. Section 6 concludes.

2 An Extended MAD Approach

The original MAD approach comprises the separate valuation of the base investment and the attached real option; the former employs a conventional DCF analysis, and the latter adopts the standard risk-neutral financial option valuation technique. To justify this approach, CA (2001) argue that DCF analysis provides the best estimate of the current value for the base investment when the market price is unobservable, and that the attached real option can be perfectly hedged using the base investment, so its valuation should be carried out under the risk-neutral measure.[1]

CA's (2001) approach is limited to their investment cost assumption, and more importantly, to the case where data on the risk-adjusted discount rate for cash flows is known. The DCF analysis estimates the gross (net) investment value at time 0, and the risk-neutral valuation of the option requires the evolution of the gross investment value in the risk-neutral measure $\mathbb{Q}$. Given that the point value of this process under the physical measure $\mathbb{P}$ is the discounted sum of all future cash flows at that point, we would at least be able to specify the gross investment value process under $\mathbb{P}$ if we knew the risk-adjusted discount rate for cash flows. However, in practice, very often there are no data which allow this discount rate to be obtained, and instead, the risk-adjusted discount rate for net income is known and sometimes used as a proxy.[2]

We propose an extension of the MAD approach which is more generally applicable. To motivate our ideas we begin with a brief summary of the original MAD approach which also serves to introduce some notation.

2.1 The Original Approach of CA (2001)

CA (2001) first calculate q_0, the gross value of the base investment at time 0 excluding any attached real option, as the present value of the sum of all expected future cash flows from

[1]The argument seems widely accepted. For instance, in the asset pricing model by Jagannathan and Wang (1996). Another example can be found in enterprise valuation for mergers and acquisitions (M&A); an enterprise is valued as the portfolio of the asset-in-place and the attached real options (see Lambrecht, 2004).*

[2]For instance, in order to value at enterprise, one can download the company-specific weighted average cost of capital (WACC) from bloomberg, and use it to discount the earnings before interest after tax (EBIAT); where EBIAT is the net income and WACC is the risk-adjusted discount rate for net income.

the investment, each being discounted by its required risk-adjusted discount rate.[3] Hence, in continuous time,

$$q_0 = \int_0^T \mathbb{E}^{\mathbb{P}}[x_\tau] \exp\left(-\int_0^\tau r_s^x ds\right) d\tau, \tag{1}$$

where T denotes the investment maturity, x_τ is the investment cash flow for $0 \le \tau \le T$, and r_s^x represents the deterministic risk-adjusted discount rate for the cash flow at time s, $0 \le s \le \tau$.[4] Regarding the investment cost, c_0, CA(2001) assume that a lump-sum payment in advance is required.

For the real option valuation, CA (2001) let q_t, the gross investment value at time t, evolve over time according to a GBM under the $\mathbb{P}$ measure, *i.e.*

$$\frac{dq_t}{q_t} = (\mu_t^q - \delta_t)\, dt + \sigma_t^q dB_t^q, \tag{2}$$

with both μ_t^q and σ_t^q being deterministic, δ_t being the continuously compounded dividend yield and q_0 given by (1). In this case, the cash flow is generated from the investment as dividends, *i.e.*

$$x_t = q_t\tilde{\delta}_t, \quad \text{with} \quad \tilde{\delta}_t = 1 - \exp\left(-\delta_t\right)\ , \tag{3}$$

where $\tilde{\delta}_t$ is the discretely compounded equivalent of δ_t. CA (2001) then change from the $\mathbb{P}$ measure to the $\mathbb{Q}$ measure, under which the gross investment value process (2) becomes

$$\frac{dp_t}{p_t} = (r_t - \delta_t)dt + \sigma_t^q \left(dB_t^q\right)^{\mathbb{Q}}, \quad \text{with} \quad \left(dB_t^q\right)^{\mathbb{Q}} = dB_t^q + \frac{\mu_t^q - \delta_t}{\sigma_t^q}dt, \tag{4}$$

r_t denotes the deterministic risk-free rate. For this, they set

$$\tilde{\delta}_t = \frac{\mathbb{E}^{\mathbb{P}}[x_t]}{\mathbb{E}^{\mathbb{P}}[q_t]}, \tag{5}$$

where

$$\mathbb{E}^{\mathbb{P}}[q_t] = \int_t^T \mathbb{E}^{\mathbb{P}}[x_\tau] \exp\left(-\int_t^\tau r_s^x ds\right) d\tau, \tag{6}$$

[3] Note that, the risk-adjusted discount rate in (1) captures the uncertainty associated with only the future cash flows and not the investor's possible abandonment of these cash flows.

[4] Remark on notation: we write $\mathbb{E}^{\mathbb{P}}[x_\tau] := \mathbb{E}_0^{\mathbb{P}}[x_\tau]$, where $\mathbb{E}_i^{\mathbb{P}}[x_\tau]$ denotes the expected value of x_τ at time i, $0 \le i \le \tau$ under the $\mathbb{P}$ measure that is defined for the future value of x_τ.

and

$$\mathbb{E}^{\mathbb{P}}[x_t] = x_0 \exp\left(\int_0^t \mu_\tau^x d\tau\right), \tag{7}$$

with μ_t^x being deterministic. Under this $\mathbb{Q}$ measure, they then value the real option.

2.2 The Extended Approach

Our first generalisation is to replace the lump-sum cost assumption (c_0) to allow any cost payment function. The total investment cost at time 0 becomes the sum of all discounted future costs, *i.e.*

$$c_0 = \int_0^T \mathbb{E}^{\mathbb{P}}[k_\tau] \exp\left(-\int_0^\tau r_s^k ds\right) d\tau, \tag{8}$$

where k_s denotes the investment cost at time s and r_s^k is the deterministic risk-adjusted discount rate for cost at time s. Note that r_s^k would be a risk-free rate if k_s is certain, and setting $k_s = 0$ for all $s > 0$ and $k_0 = c_0$ yields the assumption in CA's (2001).

Let y_t be the net income, *i.e.* $y_t = x_t - k_t$. Hence, the investment is allowed to generate net incomes different from cash flows, periodically, before its expiry. So the net investment value at time 0 excluding any attached real option, is the present value of the sum of all expected future net income, *i.e.*

$$p_0 = \int_0^T \mathbb{E}^{\mathbb{P}}[y_\tau] \exp\left(-\int_0^\tau r_s^y ds\right) d\tau. \tag{9}$$

where r_s^y denotes the deterministic risk-adjusted discount rate for net income at time s. By no arbitrage, this value is equal to the gross investment value at time 0 less the total cost. Hence, when the risk-adjusted discount rate for cash flow is unavailable, we can compute the gross investment value as the net value (9) plus the total cost (8), instead of using (1), as in CA (2001).

Although the investment value is calculated under the $\mathbb{P}$ measure, for the real option valuation, we require a corresponding $\mathbb{Q}$ measure under which the gross investment value evolves as a martingale over time. So far we only have (two alternative) expressions for the investment value at time 0 under the $\mathbb{P}$ measure. First we model the gross investment value at time t under the $\mathbb{P}$ measure and then we find the corresponding $\mathbb{Q}$ measure. We can either (a) assume a cash flow process with the expectation (7), and then the gross investment value is a discounted sum of all expected cash flows in the future; or (b) follow CA (2001) by assuming a gross investment value process which guarantees the expected cash flows (7). Either way, the following relationship between the gross investment value and the cash flow holds,

$$q_t = x_t Y_t \ , \tag{10}$$

where Y_t requires knowing the risk-adjusted discount rate for cash flows, *i.e.*[5]

$$Y_t = \int_t^T \exp\left(\int_t^\tau (\mu_s^x - r_s^x)\, ds\right) d\tau. \tag{11}$$

To see this, we adopt assumption (a), and hence write

$$q_t = \int_t^T \mathbb{E}_t^{\mathbb{P}}[x_\tau] \exp\left(-\int_t^\tau r_s^x ds\right) d\tau.$$

Now (10) follows from (7). Alternatively, if we use assumption (b), (10) follows from (3), since by definition ($5 \sim 7$), $\tilde{\delta}_t$ is equal to the reciprocal of Y_t.[6] Hence, as long as we know the risk-adjusted discount rate for cash flow, we can pin down the gross investment value at time t using (10) under any assumption about the process for the gross investment value, or for the cash flows, not necessarily a GBM.

In fact, the gross investment value at time t can still be derived, even when the risk-adjusted discount rate for cash flow is unavailable. For this, note that the following condition must hold,

$$\mathbb{E}^{\mathbb{P}}[q_t] = \mathbb{E}^{\mathbb{P}}[p_t] + \mathbb{E}^{\mathbb{P}}[c_t], \tag{13}$$

where,

$$\mathbb{E}^{\mathbb{P}}[p_t] = \int_t^T \mathbb{E}^{\mathbb{P}}[y_\tau] \exp\left(-\int_t^\tau r_s^y ds\right) d\tau, \quad \text{and} \quad \mathbb{E}^{\mathbb{P}}[c_t] = \int_t^T \mathbb{E}^{\mathbb{P}}[k_\tau] \exp\left(-\int_t^\tau r_s^k ds\right) d\tau.$$

Therefore, either (i) Y_t in (10) takes the alternative form:

$$Y_t = \int_t^T \exp\left(\int_t^\tau (\mu_s^x - r_s^y)\, ds\right) d\tau + \int_t^T y_{t,\tau} \left[\exp\left(-\int_t^\tau r_s^k ds\right) - \exp\left(-\int_t^\tau r_s^y ds\right)\right] d\tau \ ,$$

[5] Note that, many papers set the required risk premium for cash flow, r_s^x, (or for net income, r_s^y) equal to the expected risk premium of the cash flow, μ_s^x (or of the net income, μ_s^y). (See for instance, McDonald and Siegel, 1986).* However, this sometimes may not be appropriate. For instance, in enterprise valuation, the growth rate of EBIAT is not necessarily equal to WACC.

[6] Therefore,

$$\delta_t = -\log\left(1 - \tilde{\delta}\right). \tag{12}$$

with $y_{t,\tau} = k_\tau^{-1} x_0 \exp\left(\int_0^t \mu_\tau^x d\tau\right)$, and can hence be computed without the risk-adjusted discount rate for cash flow; or (ii) we take Y_t (11) with the risk-adjusted discount rate for cash flow derived from that for net income, *i.e.*[7]

$$r_t^x = r_t^y - \left(r_t^y - r_t^k\right) \frac{\mathbb{E}^{\mathbb{P}}[c_t]}{\mathbb{E}^{\mathbb{P}}[q_t]}. \tag{14}$$

Both alternatives allow us to compute the gross investment value at time t when the risk-adjusted discount rate for cash flow is not known but that for net income is known, in contrast to CA's approach which can only be applied when the former is available.

Now, based on the process for the gross investment value (10) in the $\mathbb{P}$ measure, we look for an equivalent $\mathbb{Q}$ measure. For this, we keep the gross investment value at any time t the same under both measures (according to the law of one price) and derive the risk-neutral probabilities under which the expected gross investment value at any time t grows at the risk-free rate over time because the assumption is that this value can be perfectly hedged under the complete market assumption.* Thus,

$$\mathbb{E}_i^{\mathbb{Q}}[q_t] = q_i \exp\left(\int_i^t (r_\tau - \delta_\tau)\, d\tau\right), \quad 0 \le i \le t. \tag{15}$$

Under this $\mathbb{Q}$ measure we can then value the real option using the standard risk-neutral financial option valuation technique. In the following, we illustrate the extended MAD approach through a numerical case study. For this, we introduce the Campeiro case by AP (2001).

3 The Campeiro Case

The Campeiro case concerns an investor who holds a real estate lease for T years, during which time the investor receives regular rental payments from a tenant and has regular maintenance

[7] (14) is derived by solving (13) recursively. See Appendix A for further details.

(14) confirms that in general, the risk-adjusted discount rate for cash flow cannot be used as a proxy for the risk-adjusted discount rate for net income. An obvious case is when k_t is constant. Net income, by definition, becomes perfectly correlated with the cash flow, and the change in cash flow between time t_1 and t_2 is identical to that in net income in the same time period; however, the relative changes are different, and this is the relative change which determine the risk-adjusted discount rate. The exceptions happen only when the investment cost: 1) is close to zero ($\mathbb{E}^{\mathbb{P}}[c_t]/\mathbb{E}^{\mathbb{P}}[q_t] \to 0$), or 2) shares the same source of uncertainty with the net income ($r_t^k \to r_t^y$). In both cases, $r_t^x = r_t^y$.

Moreover, r_t^x may significantly diverge from r_t^y and get close to r_t^k for the investment whose total cost is close to its gross value ($\mathbb{E}^{\mathbb{P}}[c_t]/\mathbb{E}^{\mathbb{P}}[q_t] \to 1$). Indeed, when the expected future cash flows from the investment are only enough to cover the total cost, the risk involved in the investment would be no more than the risk associated with the costs, otherwise it would immediately be abandoned by the holder or unwanted in the market.

costs associated with the upkeep of the property. The annual maintenance cost is assumed to be constant, denoted by k, and the uncertainty which drives the lease value and the abandonment option value lies only in the amount of the rental income received during the year. After T years the lease expires and the investor has no further claim on the property.

At the beginning of each year $t \le T$. the investor has a real option, since he can abandon the lease at no cost, thus saving the annual maintenance cost but also forgoing the annual rent during years $(t+1), \ldots, T$. Clearly the investor would abandon the lease if all his future rents were less than the future maintenance costs, otherwise he would make a loss on this lease.

We refer to the rent received during year t as x_t, whose process has a drift $\mu^x = 16.848\%$ and a volatility $\sigma^x = 32.208\%$. At time 0, the rent $x_0 = 6.8182$. Furthermore, the risk-free rate $r = 9.531\%$, and the risk-adjusted discount rate associated with the net income $r^y = 25.254\%$.[8] In addition, we call the rent x_t less the maintenance cost $k = 3$ the net income y_t. In the Campeiro case, the lease is held for 10 years. Note that rent and cost are assumed to occur annually in advance so that $T = 9$ and we index years using $t = 0, 1, \ldots, T$.

4 Abandonment Option Valuation

We assume that the market is complete. Now in order to value the option to abandon the lease, we require a $\mathbb{Q}$ measure corresponding to the $\mathbb{P}$ measure under which the lease is evaluated. For the change of measure, we need the gross lease value at time t under the $\mathbb{P}$ measure, starting from $t = 0$. Note that the risk-adjusted discount rate for rent is unknown. Hence, we compute the gross lease value at time 0 as the net value plus the sum of all discounted future maintenance costs, *i.e.*

$$q_0 = p_0 + c_0 = 35.7130 + 20.2771 = 55.9901, \tag{16}$$

where

$$p_0 = \sum_{\tau=0}^{T} \mathbb{E}^{\mathbb{P}}[y_\tau] \exp(-r^y \tau) = \sum_{\tau=0}^{T} (x_0 \exp(\mu\tau) - k) \exp(-r^y \tau) = 35.7130,$$

and

$$c_0 = \sum_{\tau=0}^{T} k \exp(-r\tau) = 20.2771.$$

[8] Although, following AP (2001), we employ a discrete-time framework, we remark that we use continuous-time compounding. The annual interest rates defined in the Campeiro case are appropriately re-defined here so that the effective annual rates are identical, for example a 10% annual rate is expressed as 9.531% continuously compounded.

To determine the gross leas value at time t for $0 < t \leq T$, we first assume a GBM evolution of the cash flow:

$$\frac{dx_t}{x_t} = \mu^x dt + \sigma^x dB_t. \tag{17}$$

Now, we follow the vast majority of papers on the MAD approach and present the above process in discrete time, using the Cox, Ross, and Rubinstein (1979) binomial tree parameterisation. In the binomial tree for rent, the time between successive nodes $\Delta t = 1$ since rents and costs occur annually. In each time period, the underlying can move up by a factor $u > 1$ or down by another factor $d < 1$, and we denote the state at time t by $s(t) = 0$, u, d, ud, uu, du, …, $uduu$ etc. In particular,

$$u = \exp\left(\sigma^x \sqrt{\Delta t}\right) = 1.38, \quad d = u^{-1} = 0.725, \quad \pi^x = \frac{\exp(\mu^x \Delta t) - d}{u - d} = 70\%, \tag{18}$$

where π^x denotes the physical transition probability of $x_{s(t)}$ moving up at time t.[9] Hence, we construct the binomial tree for rent as below,

$$x_{s(t)u} = x_{s(t)}u, \quad x_{s(t)d} = x_{s(t)}d.$$

Corresponding to this tree, we can build the binomial tree for the gross lease value based on (10) in discrete time, *i.e.*

$$q_{s(t)} = x_{s(t)}Y_t,$$

with Y_t taking the form (11) where the risk-adjusted discount rate for rent is given by (14).[10] Note that, the value of Y_t is not limited to the assumption(s) about the rent or the gross lease value process.

Now let us switch to CA's (2001) assumption (2). In order to build the binomial tree for gross lease value, we need the volatility of the GBM process (2) to calculate the values of u and d, and yet we only know the volatility of the rent process. To compute the former from the latter, we use (2) to derive the rent process and then set the volatility (which is a function of

[9]For instance, if the rent moves up at every step from time 0 until the option expiry year, then $x_{s(T)} = x_0 u^9 = 123.7600$, where $s(T) = \underbrace{uuu\ldots uu}_{9}$. At time 0, the probability of rent having this value at time T is $\pi_x^9 = 4.035\%$ under the $\mathbb{P}$ measure.

[10]To build a binomial tree for net lease value, denoted by $p_{s(t)}$, we deduct the total future costs at time t from each value in the gross lease value tree at the same time. That is, $p_{s(t)} = q_{s(t)} - k \int_t^T \exp(-r(\tau - t))d\tau$.

the former) equal to the latter. In fact, given (10), not only the rent under (2) also follows a GBM, but the processes for the rent and for the gross lease value have the same volatility, *i.e.*[11]

$$\sigma_t^q = \sigma^x = 32.208\%.$$

Therefore, regarding the binomial tree for the gross lease value, the values of u and d are identical to those in (18).[12] We thus construct the gross lease value tree as follows,

$$q_{s(t)u} = q_{s(t)} \exp\left(-\delta_t\right) u, \quad q_{s(t)d} = q_{s(t)} \exp\left(-\delta_t\right) d,$$

with δ_t given by (12).[13]

Based on the binomial tree for the gross lease value, we now seek a corresponding $\mathbb{Q}$ measure under which the gross lease value fits (15). In discrete time, that is to derive the risk-neutral probability, denoted by $\pi_t^{\mathbb{Q}}$, under which the following relationship holds:[14]

$$q_{s(t)} - x_{s(t)} = \mathbb{E}_{s(t)}^{\mathbb{Q}}\left[q_{s(t+1)}\right] \exp(-r), \quad \mathbb{E}_{s(t)}^{\mathbb{Q}}\left[q_{s(t+1)}\right] = \pi_t^{\mathbb{Q}} q_{s(t)u} + \left(1 - \pi_t^{\mathbb{Q}}\right) q_{s(t)d} \,.$$

where $s(t+1)$ denotes the succeeding states of $s(t)$: $s(t)u$ and $s(t)d$. We solve this equation for $\pi_t^{\mathbb{Q}}$, and thus obtain the expression below.

$$\pi_t^{\mathbb{Q}} = \frac{\left(q_{s(t)} - x_{s(t)}\right) \exp(r) - q_{s(t)d}}{q_{s(t)u} - q_{s(t)d}}.$$

By bringing in the values in the gross lease value tree, $\pi_t^{\mathbb{Q}}$ can be quantified. Note that, under (17), we can simplify the above formula and present $\pi_t^{\mathbb{Q}}$ as a function of the risk-adjusted discount rate for rent,

$$\pi_t^{\mathbb{Q}} = \frac{\exp(r + r_{t+1}^x - \mu) - d}{u - d}, \tag{19}$$

[11] In Appendix B, we prove the compatibility of assumption (2) and (17).

[12] Note that, the physical transition probability of $q_{s(t)}$ moving up at time t

$$\pi_{q,t} = \frac{\exp\left(\mu_t^q \Delta t\right) - d}{u - d} \neq \pi_x,$$

as μ_t^q is time dependent (see 23).

[13] Corresponding to the binomial tree for gross lease value, we can easily construct a binomial tree for rents using the definition of rent (3).

[14] Note that, Azevedo-Pereira made the assumption (17) but computed the risk-neutral probabilities with which the discounted rent (rather than the discounted gross lease value) was a martingale. See Section 5.1 for further discussion of this important point.

where $\pi_t^{\mathbb{Q}}$ is time dependent, since the risk-adjusted discount rate for rent varies over time and other parameters are constants. On the other hand, when the gross lease value follows a GBM (2), $\pi_t^{\mathbb{Q}}$ becomes a fixed number: $\pi_t^{\mathbb{Q}} = 57.252\%$.[15]

Under these risk-neutral probabilities, let us now value the abandonment option under RNV principle. Irrespective of whether we assume (2) or (17), the valuation process of the abandonment option is as follows. We can value it either alone or together with the lease. Intuitively, if the future rent is less than the future cost, we would abandon the lease, in which case we save the costs but lose the further rents; otherwise we do not exercise the option, then it would become valueless. Hence, we can present the pay-off of the option with the lease as

$$\max\{q_t - c_t, 0\} = \max\{p_t, 0\},$$

and without the lease as

$$\max\{0, c_t - q_t\} = \max\{0, -p_t\}.$$

That is, the option with and without the lease are analogous to an American call and a put respectively.

To calculate the net lease value with the option at time 0 under the $\mathbb{Q}$ measure, we apply the following backward induction.

$$C_{s(t)} = \max\left\{y_{s(t)} + \mathbb{E}^{\mathbb{Q}}_{s(t)}\left[C_{s(t+1)}\right]\exp(-r), 0\right\}, \quad \mathbb{E}^{\mathbb{Q}}_{s(t)}\left[C_{s(t+1)}\right] = \pi_t^{\mathbb{Q}} C_{s(t)u} + \left(1 - \pi_t^{\mathbb{Q}}\right) C_{s(t)d}\ , \tag{20}$$

where $C_{s(t)}$ is the net value of the lease and the option in state $s(t)$ under the $\mathbb{Q}$ measure. This backward induction starts from the last step at $t = 9$ with $C_{s(10)} := 0$ and goes backwards in time.[16] The abandonment option will have the value $P_0 = C_0 - p_0$. Alternatively, we value the abandonment option alone using backward induction:

$$P_{s(t)} = \max\left\{-p_{s(t)}, \mathbb{E}^{\mathbb{Q}}_{s(t)}\left[P_{s(t+1)}\right]\exp(-r)\right\}, \quad \mathbb{E}^{\mathbb{Q}}_{s(t)}\left[P_{s(t+1)}\right] = \pi_t^{\mathbb{Q}} P_{s(t)u} + \left(1 - \pi_t^{\mathbb{Q}}\right) P_{s(t)d}\ , \tag{21}$$

[15] Note that, under either assumption, the rent process is not a $\mathbb{Q}$-martingale.

[16] For instance, $C_{s(9)} = \max\left\{y_{s(9)}, 0\right\}$. It is then simple to use; $C_{s(8)} = \max\left\{y_{s(8)} + \mathbb{E}^{\mathbb{Q}}_{s(8)}\left[C_{s(9)}\right]\exp(-r), 0\right\}$ where $\mathbb{E}^{\mathbb{Q}}_{s(8)}\left[C_{s(9)}\right] = \pi_8^{\mathbb{Q}} C_{s(8)u} + \left(1 - \pi_8^{\mathbb{Q}}\right) C_{s(8)d}$. The rest of this backward induction follows.

where $P_{s(t)}$ denotes the value of the option only at state $s(t)$ under the $\mathbb{Q}$ measure. This backward induction starts from $t = 9$ with $P_{s(10)} := 0$.

According to the put-call parity (PCP), the above two valuation processes are equivalent and hence, under the same assumption, they would generate identical abandonment option values (see Appendix C for further discussion). We present our calculation results in Table 1 along with the corresponding assumptions. There is a minor difference between the option values under (2) and (17). We believe that it is only a discretisation error. With more steps in the binomial trees that we construct, the two option values should be the same.

Table 1: A summary of the assumptions (2) and (17) and the corresponding abandonment option values. The relative option values are in percentage and equal to the abandonment option values proportional to the net lease value (16).

	Main Assumption	Option price	Relative option price
(17)	x_t follows a GBM	0.3998	1.120%
(2)	q_t follows a GBM	0.4040	1.131%

So far we have illustrated the extended MAD approach and valued a real option with an assumed stochastic process for the evolution of either the cash flow or the investment value. The impression that practitioners are sometimes given is that only the latter assumption should be used to value the real option. In contrast, we hold that either assumption can be appropriate to use in specific instances. In an efficient rental market (or any market) where the tenants (investors) are predominantly interested in rents (investment cash flows), we might recommend to value real options assuming (17). In the conditions where the cash flows are significant and observable, this assumption may also be preferred. On the other hand, one would prefer to value the real option by employing (2) if, referring to a lease (or any other investments), one has not only an abandonment option but other real options related to the lease (investments) price such as an option to sell the lease, to share the ownership of the lease, or to redevelop the lease for other use.

5 Other Approximations of MAD Parameters

We hold the relationship among the investment cost and the net and gross investment values (13) and extend the MAD approach so that any assumption about the process of the investment value and cash flows can be used to value a real option. Apart from our approximation method of the MAD parameters, $\pi_t^{\mathbb{Q}}$, r_t^x, and μ_t^q, there are other ways to approximate them which, however, may fail the relationship (13). Consequently, the option value may be biased. For instance, in AP's (2001) study of the Campeiro case, $\pi_t^{\mathbb{Q}}$ was mis-calculated and hence, it led to a different and obviously incorrect value of the abandonment option.

In the following, we first explain AP's (2001) valuation of the abandonment option. This allows us to illustrate the importance of changing the measure consistently in the valuation. We later choose different approximations of r_t^x and μ_t^q. It will then be clear that our approximations regarding these parameters are the most advanced ones.

5.1 A Discussion of AP's (2001) Abandonment Option Valuation

In the original study of Campeiro case, AP (2001) obtained the value 12.3505 for the abandonment option. That this is different from our option value is due to the inconsistent change-of-measure in his calculation. In this subsection, we exhibit AP's (2001) assumptions and calculations, so as to illustrate the impact on the real option valuation regarding the change-of-measure.

Under the $\mathbb{P}$ measure, AP (2001) firstly assumed (17) and then employed the Cox, Ross, and Rubinstein binomial tree parameterisation to construct the tree for rents. The values that he chose for u, d and π_x were identical with ours (18), so was his calculation of the net lease value at time 0 (16). However, for the option valuation, in contrast with $\pi_t^{\mathbb{Q}}$ (19), he applied a different risk-neutral probability.[17]

$$\pi^{\mathbb{Q}'} = \frac{\exp(r) - d}{u - d} = 57.252\% \ ,$$

where $\mathbb{Q}'$ denotes a risk-neutral measure which differs from $\mathbb{Q}$.

Now the problem is that the $\mathbb{Q}'$ measure is inconsistent with the $\mathbb{P}$ measure. To see this, we show that the gross lease value at time 0 diverges from q_0 (16), which is against the law of one price.[18]

$$q_0' = \sum_{\tau=0}^{T} \mathbb{E}^{\mathbb{Q}'}[x_\tau] \exp(-r\tau) = (T+1)x_0 = 68.1818 \ .$$

We therefore argue that it is inappropriate to value the abandonment option under the $\mathbb{Q}'$

[17] With this transition probability, clearly the discounted rent would be a $\mathbb{Q}'$-martingale over time.

[18] We can also calculate the net lease value at time 0: $p_0' = \sum_{\tau=0}^{T} \mathbb{E}^{\mathbb{Q}'}[y_\tau] \exp(-r\tau) = 47.9047$, which is also different from (16).

measure. We illustrate this point by analysing AP's (2001) option valuation. He applied (20) with $\pi^{\mathbb{Q}'}$ to the tree of rents and obtained the net lease value with option as $C_0^{\mathbb{Q}'} = 48.0635$. He then claimed the option value to be $C_0^{\mathbb{Q}'} - p_0 = 48.0635 - 35.7130 = 12.3505$.

Yet this option value is too high to be correct, as the abandonment option is a deep out-of-the-money put. The option strike is the total costs of all time periods $c_0 = 20.2771$ (16), whereas the current underlying price is the gross lease value at time 0, either 55.9901 or 68.1818; the latter value is much higher. Intuitively, it can hardly seem plausible that this option is worth roughly 20% of the current underlying price.[19]

5.2 Assuming r_t^x under (17) Is Constant

Under (17), in order to value the abandonment option, recall that we apply (14) to calculate the time-varying risk-adjusted discount rate for rent such that we obtain the process of the gross lease value. A simpler way is to derive a constant risk-adjusted discount rate, denoted by r^x, that sets the gross lease value at time 0 equal to 55.9901 (16) equal. Hence,

$$(\exp(-r^x))^{T+1-t} - \tilde{\delta}_t^{-1}(\exp(-r^x)) + \tilde{\delta}_t^{-1} - 1 = 0, t = 0 \ .$$

But this approximation is flawed, because for any time $t > 0$, this equation would not hold for a fixed value of r^x and consequently, (13) no longer holds. Hence, the abandonment option would be mis-priced.

Let us calculate the abandonment option value so that we can observe how much difference this approximation can make. Bringing in the inputs of the Campeiro case, we obtain $r^x = 21.416\%$. The risk-neutral probability (16) is then $\pi^{\mathbb{Q}} = 49.752\%$ and hence, the value of the abandonment option is 0.3774. This value is almost 10% lower than that calculated with the time-varying r_t^x (14), 0.3998. In fact, r^x over-values the rents from $t = 4$ to maturity, which already have higher expectations than the short term rents ($t = 1$, 2 and 3), whilst the current rent x_0 remains the same. Hence, if the investor abandon the lease, he would lose the long term rents and only obtain the short term ones. He would therefore tend to wait for future

[19] In fact, the number 12.3505 comprises not only the abandonment option value but also the change in the current lease value due to AP's (2001) inconsistent change-of-measure from $\mathbb{P}$ to $\mathbb{Q}'$. This value change can be calculated as the difference between the current gross (net) lease values under the two measures $q_0' - q_0 = p_0' - p_0 = 12.1917$. Note that, since the maintenance costs are risk-free under any measure, they would always have the same values regardless of the measures they are under. Therefore $q_0' - q_0 = (p_0' - c_0) - (p_0 - c_0) = p_0' - p_0$, and we can use either the gross or net lease value to determine the change in the current lease value due to the change-of-measure.

rents to cover the maintenance costs than abandoning the lease early. The abandonment option therefore becomes less attractive and consequently under-valued.

5.3 Assuming μ_t^q under (2) Is Fixed

To simplify the calculation under (2), we may assume a fixed drift, denoted by μ^q, instead of a time-varying μ_t^q (23). However, with this assumption, (13) does not hold and therefore the abandonment option would be slightly biased. To see this, let us calculate the abandonment option value first. Given (7), we solve for the dividend yield $\tilde{\delta}_t$ and μ^q the system of non-linear equations presented as below.

$$\tilde{\delta}_t = \frac{\tilde{\delta}_{t-1}}{1 - \tilde{\delta}_{t-1}} \exp(\mu^x - \mu^q). \tag{22}$$

with $\tilde{\delta}_0 = x_0/q_0$ and $\tilde{\delta}_9 = 1$. Since the associated risk-neutral probability (19) under (2) applies here, we can thus compute the abandonment option value as $0.3796 < 0.4040$.

This bias on the option value occurs because the gross lease value is in general over-valued. In particular, for the same expected rent at time t, the corresponding dividend yield computed using (22) is lower than that given by (12). This means the gross lease value that we calculate here, as the base of the dividend yield, is higher than that calculated in the previous valuation; hence, the investor would believe in a higher profitability of the lease and therefore be less attempted to abandon the lease, and the abandonment option calculated here is lower than in our previous valuation.

6 Conclusion

In our extended MAD framework, in contrast to the lack of enthusiasm in practice to value real options using MAD based on an assumed evolution of cash flows, this paper shows such a valuation process of a real option is equivalent and consistent with the traditional implementation of MAD, *i.e.* valuing a real option according to an assumed stochastic process of the investment values. More precisely, when cash flows and investment value share one source of uncertainty, the evolution of the cash flows (investment values) would determine the process of the investment values (cash flows). For instance, when the cash flows follow a simple GBM, we show how this leads to a straightforward derivation of the implied investment value. Based on either of these two processes, we also demonstrate the valuation of the real option attached to the investment.

By valuing the real option with various assumptions, this paper demonstrates the link between the value of the real option and the valuation of the underlying investment *via* the change-of-measure. The most classic and popular operation in practice regarding the valuation of an investment is conducted in the DCF framework, whilst the paradigm of the real option valuation literature and practice is within RNV framework. Switching from the DCF framework to the risk-neutral world requires a change-of-measure, mainly the derivation of the appropriate risk-neutral probabilities.

Yet the change-of-measure may be problematic to achieve in practice. For instance, before an acquisition, an investment bank may fund the acquirer having calculated the value of the target company, produced in the DCF framework (albeit heavily adjusted or extended). Then the acquirer may simply take this single value and price its own real option to acquire without running through and being consistent with the assumptions made in the valuation of the target company conducted by the investment bank.

On the other hand, the danger of using inappropriate approximation of the parameters in the real option valuation is significant and can result in a mis-estimated real option value. A typical example is the real option valuation in the original case study by AP (2001), which is examined in Section 5.1. The mis-estimation is shown to be too large to be ignored - AP (2001) valued the abandonment option as worth 12.3505 which is roughly a third of the net lease value. This is an implausible result for a deep out-of-the-money American put option and is shown here to be more than twenty times larger than the correct value(s). To conclude, the main key here is to compute the real option value strictly following the assumptions made in both the DCF analysis on the base investment and the real option valuation process, and also to be consistent among assumptions, so that the model involves less model risks.

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A Proof of (14)

To derive (14), we assume that the cash flow and the cost happen periodically at the beginning of every time period $(t-\Delta t, t)$ for $0 \leq t \leq T$ and Δt is very small. By definition,

$$q_{t-\Delta t} = x_{t-\Delta t} + \mathbb{E}^{\mathbb{P}}_{t-\Delta t}[q_t] \exp\left(-r^x_t \Delta t\right).$$

By taking the time 0 expectation on both side and applying the tower propery, we obtain

$$\mathbb{E}^{\mathbb{P}}[q_{t-\Delta t}] = \mathbb{E}^{\mathbb{P}}[x_{t-\Delta t}] + \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}_{t-\Delta t}[q_t] \exp\left(-r^x_t \Delta t\right)] = \mathbb{E}^{\mathbb{P}}[x_{t-\Delta t}] + \mathbb{E}^{\mathbb{P}}[q_t] \exp\left(-r^x_t \Delta t\right).$$

Similarly,

$$\mathbb{E}^{\mathbb{P}}[p_{t-\Delta t}] = \mathbb{E}^{\mathbb{P}}[x_{t-\Delta t}] - \mathbb{E}^{\mathbb{P}}[k_{t-\Delta t}] + \mathbb{E}^{\mathbb{P}}[p_t] \exp\left(-r^y_t \Delta t\right),$$

$$\mathbb{E}^{\mathbb{P}}[c_{t-\Delta t}] = \mathbb{E}^{\mathbb{P}}[k_{t-\Delta t}] + \mathbb{E}^{\mathbb{P}}[c_t] \exp\left(-r^k_t \Delta t\right).$$

Now, given (13),

$$\mathbb{E}^{\mathbb{P}}[q_{t-\Delta t}] = \mathbb{E}^{\mathbb{P}}[p_{t-\Delta t}] + \mathbb{E}^{\mathbb{P}}[c_{t-\Delta t}],$$

we therefore can write

$$\mathbb{E}^{\mathbb{P}}[q_t] \exp\left(-r^x_t \Delta t\right) = \mathbb{E}^{\mathbb{P}}[p_t] \exp\left(-r^y_t \Delta t\right) + \mathbb{E}^{\mathbb{P}}[c_t] \exp\left(-r^k_t \Delta t\right).$$

That is,

$$\exp\left(-r_t^x \Delta t\right) = \exp\left(-r_t^y \Delta t\right) + \left(\exp\left(-r_t^k \Delta t\right) - \exp\left(-r_t^y \Delta t\right)\right) \frac{\mathbb{E}^{\mathbb{P}}[c_t]}{\mathbb{E}^{\mathbb{P}}[q_t]}.$$

By extracting $\exp(-r_t^y \Delta t)$ from both terms on the right, we obtain

$$\exp\left(-r_t^x \Delta t\right) = \exp\left(-r_t^y \Delta t\right) \left(1 + \left(\exp\left(\left(r_t^y - r_t^k\right) \Delta t\right) - 1\right) \frac{\mathbb{E}^{\mathbb{P}}[c_t]}{\mathbb{E}^{\mathbb{P}}[q_t]}\right).$$

Now, we take the log on both sides and hence,

$$r_t^x \Delta t = r_t^y \Delta t - \log\left(1 + \left(\exp\left(\left(r_t^y - r_t^k\right) \Delta t\right) - 1\right) \frac{\mathbb{E}^{\mathbb{P}}[c_t]}{\mathbb{E}^{\mathbb{P}}[q_t]}\right).$$

Divided by Δt, the above equation becomes,

$$r_t^x = r_t^y - \Delta t^{-1} \log\left(1 + \left(\exp\left(\left(r_t^y - r_t^k\right) \Delta t\right) - 1\right) \frac{\mathbb{E}^{\mathbb{P}}[c_t]}{\mathbb{E}^{\mathbb{P}}[q_t]}\right).$$

When $\Delta t \to 0$, the limit of the second term on the right can be calculated using l′Hôpital's rule as below,

$$\begin{aligned}
&\lim_{\Delta t \to 0} \Delta t^{-1} \log\left(1 + \left(\exp\left(\left(r_t^y - r_t^k\right) \Delta t\right) - 1\right) \frac{\mathbb{E}^{\mathbb{P}}[c_t]}{\mathbb{E}^{\mathbb{P}}[q_t]}\right) \\
&= \lim_{\Delta t \to 0} \frac{d\left(\log\left(1 + \left(\exp\left(\left(r_t^y - r_t^k\right) \Delta t\right) - 1\right) \frac{\mathbb{E}^{\mathbb{P}}[c_t]}{\mathbb{E}^{\mathbb{P}}[q_t]}\right)\right)}{d\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\frac{\mathbb{E}^{\mathbb{P}}[q_t]}{\mathbb{E}^{\mathbb{P}}[c_t]} \left(r_t^y - r_t^k\right) \exp\left(\left(r_t^y - r_t^k\right) \Delta t\right)}{1 + \left(\exp\left(\left(r_t^y - r_t^k\right) \Delta t\right) - 1\right) \frac{\mathbb{E}^{\mathbb{P}}[c_t]}{\mathbb{E}^{\mathbb{P}}[q_t]}} \\
&= \left(r_t^y - r_t^k\right) \frac{\mathbb{E}^{\mathbb{P}}[c_t]}{\mathbb{E}^{\mathbb{P}}[q_t]},
\end{aligned}$$

We now bring this result back to the previous equation and thus have (14).

B The Compatibility of Assuming a GBM Evolution of the Gross Investment Value or the Cash flow

CA (2001) assume a GBM (2) evolution of the gross investment value. From the relationship between the gross investment value and the cash flow (10), we can derive the process of the cash flow as below,

$$dx_t = d\left(q_t Y_t^{-1}\right) = q_t Y_t^{-1}\frac{dq_t}{q_t} - q_t Y_t^{-1}\frac{dY_t}{Y_t} = \left(\mu_t^q - \delta_t - \frac{Y_t'}{Y_t}\right)x_t dt + \sigma_t^q x_t dB_t^q,$$

where $Y_t' = dY_t/dt$. Given the expectation (7) of the future cash flows, the drift of the above process is equal to μ_t^x. Hence, we can specify the drift of the process (2), *i.e.*

$$\mu_t^q - \delta_t = \mu_t^x + \frac{Y_t'}{Y_t} \; . \tag{23}$$

Alternatively, under the assumption that the cash flow follows a GBM (17), the evolution of the gross investment value can be derived from (10), *i.e.*

$$dq_t = d\left(x_t Y_t\right) = x_t Y_t \frac{dx_t}{x_t} + x_t Y_t \frac{dY_t}{Y_t} = \left(\mu_t^x + \frac{Y_t'}{Y_t}\right) q_t dt + \sigma_t^x q_t dB_t^x.$$

That is, the gross investment value process is also a GBM with the same drift as in (23):

$$\frac{dq_t}{q_t} = \left(\mu_t^x + \frac{Y_t'}{Y_t}\right)dt + \sigma_t^q dB_t^q. \tag{24}$$

Also note that, the dispersions of the cash flow process and the gross investment value process are identical, *i.e.* for the same investment,

$$\sigma_t^q = \sigma_t^x, \quad dB_t^q = dB_t^x. \tag{25}$$

C The Equivalence of the Option Valuation Processes (20) and (21)

The option valuation processes (20) and (21) are equivalent according to put-call parity (PCP):

$$P_{s(t)} + p_{s(t)} = C_{s(t)} \; . \tag{26}$$

The proof is as follows.

We start with backward induction (21) at time 0. That is,

$$P_0 = \max\left\{(-p_0)\,, \mathbb{E}^{\mathbb{Q}}\left[P_{s(1)}\right]\exp(-r)\right\} \; .$$

Adding p_0 to both sides of the equation,

$$P_0 + p_0 = \max\left\{(-p_0 + p_0), \mathbb{E}^{\mathbb{Q}}\left[P_{s(1)}\right]\exp(-r) + p_0\right\} \ . \tag{27}$$

We now rewrite the second term in the maximisation,

$$\mathbb{E}^{\mathbb{Q}}\left[P_{s(1)}\right]\exp(-r) + p_0 = \mathbb{E}^{\mathbb{Q}}\left[P_{s(1)} + p_{s(1)}\right]\exp(-r) - \mathbb{E}^{\mathbb{Q}}\left[p_{s(1)}\right]\exp(-r) + p_0 \ ,$$

where, bringing back the relationship between the gross and net lease value given in Footnote 10,

$$-\mathbb{E}^{\mathbb{Q}}\left[p_{s(1)}\right]\exp(-r) + p_0 = -\mathbb{E}^{\mathbb{Q}}\left[q_{s(1)} - k\sum_{\tau=1}^{T} e^{-r(\tau-1)}\right]\exp(-r) + \left(q_0 - k\sum_{\tau=0}^{T}\exp(-r\tau)\right) \ .$$

After some simple algebra, we rewrite this expression as,

$$-\mathbb{E}^{\mathbb{Q}}\left[q_{s(1)} - k\sum_{\tau=1}^{T}\exp(-r\tau)\right]\exp(-r) + \left(q_0 - k\sum_{\tau=0}^{T}\exp(-r\tau)\right) = \left(q_0 - \mathbb{E}^{\mathbb{Q}}\left[q_{s(1)}\right]\exp(-r)\right) - k \ ,$$

where, assuming it is q_t that follows a $\mathbb{Q}$ martingale (**??**),

$$q_0 - \mathbb{E}^{\mathbb{Q}}\left[q_{s(1)}\right]\exp(-r) = x_0 \ .$$

With these decompositions, we can rearrange (27) as following,

$$P_0 + p_0 = \max\left\{0, \mathbb{E}^{\mathbb{Q}}\left[P_{s(1)} + p_{s(1)}\right] + (x_0 - k)\right\} \ . \tag{28}$$

Now apply the put-call parity (26) to (28), we reach to

$$C_0 = \max\left\{0, \left(\mathbb{E}^{\mathbb{Q}}\left[C_{s(1)}\right] + x_0 - k\right)\right\},$$

which is indeed backward induction (20) at time 0.

This proof applies at any $t \in [0, T]$.